a. Because triangles ZQR, ZTP, and SPA are similar, $RP^2/QT^2 = SA^2/SP^2$, where $RP^2 = QR*LR$ (by Euclid's Prop. 36 again)

b. Applying Theorem 3, $\lim(QT^2/QR) = SP^3/SA^2$

c. Hence force varies as $1/SP^5$

4. (My guess is that Newton had first solved for the Ptolemaic eccentric circle, discovering that force varies as the product of two varying geometric magnitudes and then decided to present this result which follows from that one when the eccentricity becomes 1; see Appendix)

5. Problem 2: the rule of force for a body in an ellipse under centripetal forces directed to the center of the ellipse, not a focus

6. Solution: force varies with distance from center -- a linear relationship

a. Since an ellipse, $PV*VG/QV^2 = PC^2/CD^2$ (the counterpart to Euclid's Book 3 Prop. 36 for ellipses) and $QV^2/Qt^2 = PC^2/PF^2$ -- facts about conjugate diameters from Apollonius (derived in the Appendix)

b. Therefore, $PV*VG/Qt^2 = (PC^2/CD^2)*(PC^2/PF^2)$

c. But QR=PV and BC∗CA = CD∗PF, and in the limit 2PC = VG

d. Thus, $Qt^2*PC^2/QR = 2*BC^2 * CA^2/PC$ -- q.e.d.

III. The Results on Keplerian Elliptical Orbits

A. Problem 3: The Inverse-Square Rule of Force

1. Suppose now that the trajectory is an ellipse governed by centripetal forces aimed at a focus -- i.e. the converse of the problem Hooke posed in his letter

a. Solution turns on proving that $\lim(QT^2/QR)$ for any such ellipse varies simply as the latus rectum of the ellipse, and hence a constant

b. But then the force varies as $1/SP^2$ -- i.e. as $1/r^2$, or more fully as $1/(L*r^2)$

2. Newton's proof is comparatively intricate, turning on a number of properties of ellipses

a. EP = AC, the semi-major axis (something Newton "discovered" in solving this problem)

b. By a complex chain of ratios,

$$L*QR/QT^2 = (2*PC/GV)*(M/N), \text{ where } M/N = QV^2/QX^2$$

(1) Using $(GV*VP/QV^2) = CP^2/CD^2$ -- as above

(2) And using lemma ii to obtain 4∗CD∗PF = 4∗CB∗CA

c. But as Q approaches P, (2∗PC/GV) and (M/N) approach 1, so that $(SP^2 * QT^2)/QR = L * SP^2$

3. Newton's solution is more general and perhaps easier in analytical form, as developed by Johann Bernoulli around 1710

a. Equation for any conic: $1/r = A + B*\cos(\theta)$, where $e = B/A$, $B = CS/CB^2$

b. An ellipse so long as $0 < B/A < 1$; a parabola when A=B; and a hyperbola when $B/A > 1$

c. $\lim(QR/QT^2) = 1/2[(A+B*\cos(\theta)) + d^2/d\theta^2\,(A+B\cos(\theta))] = A/2$

e. But the latus rectum is just 2/A

f. Therefore, for any conic section trajectory governed by centripetal forces aimed at the focus, the force along the trajectory varies as $1/r^2$

4. The Scholium immediately following Problem 3 has mystified people ever since -- vide Wilson

"The major planets orbit, therefore [sic!], in ellipses having a focus at the centre of the Sun, and with their radii drawn to the Sun describe areas proportional to the times, exactly as Kepler supposed"

a. To infer that the area rule holds exactly, need to establish that the only forces acting on the planets are aimed to the center of the sun

b. To infer that the orbits are exactly elliptical, then need the converse of Problem 3 -- i.e. the problem Hooke originally wanted solved, and Newton told Halley he had solved: if a body is governed by inverse-square centripetal forces, then its only closed circuit trajectory is an ellipse

c. And even then need the claim that the centripetal force is inverse-square, something that has so far been shown for the planets only under the assumption of uniform motion in perfectly circular orbits

d. The seeming logical lacuna persists right through the first edition of the *Principia*, to be filled in the second edition following some pointed comments by Johann Bernoulli

5. My suspicion is that Newton was here engaging in a type of evidential reasoning (hence appropriate to a Scholium) that he employs widely (provoking much controversy) in Book III of the *Principia*

a. From the rough phenomena alone -- e.g. circular orbit -- can conclude from Theorem 1 and 2 that, at least to a first approximation, an inverse-square centripetal force aimed at the sun is the dominant factor keeping the planets in their orbits

b. Problem 3 shows that the next level of refinement of the phenomena, to a roughly elliptical orbit sweeping equal areas vis-a-vis the focus, requires no revision of that conclusion

c. Licenses the inference that, at the second level of approximation, the planets obey Kepler's first two rules exactly, for nothing beyond forces inferred from the first approximation is required for them to do so

6. Whether such evidential reasoning is legitimate -- or more to the point, exactly what conclusion is to be drawn from it -- I leave as an open question for now

a. But the interpretation I offer at least has the virtue of not saddling Newton with a glaring blunder in logic

b. I find the idea that Newton would fall trap to such a blunder in simple 'if-then' reasoning beyond belief

B. Theorem 4: The Keplerian 3/2 Power Rule

1. Theorem 4 provides a generalization of one half of Corollary 5 of Theorem 2: Kepler's 3/2 power rule holds for bodies in elliptical orbits governed by centripetal forces aimed at (a common) focus when those forces vary as $1/r^2$ -- i.e. P^2 varies as a^3