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Review of *Projective Representations of Finite Groups* by Gregory Karpilovsky

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Projective representations take their name from projective geometry. To be specific, let G be a finite group, K a field, and V a finite-dimensional vector space over K. Let h be a homomorphism of G into the projective general linear group $\mathrm{PGL}(V)$, i. e., the group of all projective transformations of the projective space whose points are the one-dimensional subspaces of V. Since many of the finite simple groups are defined as subgroups of groups $\mathrm{PGL}(V)$ for finite K, their natural injections into $\mathrm{PGL}(V)$ furnish important examples. $\mathrm{PGL}(V)$ can be identified with the quotient group of the group $\mathrm{GL}(V)$ of all invertible linear transformations of V by the normal subgroup Z consisting of scalar multiples of the identity $1_{\mathrm{GL}(V)}$ by the elements of $K^{\times} = K - \{0\}$. Accordingly, h can be studied as follows: for each $g \in G$ choose a representative $\rho(g)$ of the coset $h(g)Z = h(g)K^{\times}$; then ρ is a mapping of G into $\mathrm{GL}(V)$ such that

$$\rho(g_1)\rho(g_2) = \alpha(g_1, g_2)\rho(g_1g_2) \tag{1}$$

for some mapping α of $G \times G$ to $K^{\times}$; we can suppose that

$$\rho(1_G) = 1_{\mathrm{GL}(V)}. \tag{2}$$

Then ρ can be studied in place of h; this replaces a projective situation by a more familiar linear one, though at the price that ρ depends on arbitrary choices. Any mapping ρ of G to $\mathrm{GL}(V)$ that satisfies (1) and (2) for any α is called a *projective representation* of G; if α is specified, ρ is called an *α-representation.*

Many examples can be constructed as follows: let

$$1 \to A \to H \xrightarrow{f} G \to 1 \tag{3}$$

be a *central extension*, i. e., an epimorphism $H \to G$ of finite groups with $\ker f \cong A$ contained in the center of H; thus $G \cong H/A$ if we identify $\ker f$ and A. (The group H is also called a central extension of G.) For each $g \in G$ choose an inverse image $\mu(g) \in H$ such that $f(\mu(g)) = g$, with $\mu(1_G) = 1_H$. Then for each linear representation r of H, the rule

$$\rho(g) = r(\mu(g)) \tag{4}$$

defines a projective representation ρ of G. For example, if H is either of the nonabelian groups of order 8, A its center, and f the natural map to

$G = H/A$, the 2-dimensional irreducible complex representation of H yields a projective representation of the Klein four-group. Central extensions play an important role in the proof of the classification of finite simple groups [**2**], [**7**, pp. 295–303]; furthermore, attempts to use the classification to prove a conjecture for arbitrary finite groups sometimes reduce the conjecture to the case of central extensions of simple groups. (Projective representations of Lie groups can also be defined and are used in quantum theory, in particular in connection with spinors.)

The associative law for G implies that α is a 2-cocycle in $Z^2(G, K^\times)$, where $K^\times$ is considered a G-module with multiplication as operation and trivial action. If we replace ρ by another choice ρ' with corresponding 2-cocycle α', we find that α and α' are cohomologous; so it is the cohomology class of α in $H^2(G, K^\times)$ that matters. If α is the trivial cocycle, the α-representations of G are just the linear representations of G; this means that projective representation theory is a generalization of linear representation theory. On the other hand, we shall see that the study of linear representations leads inevitably to the introduction of projective representations.

One way to do projective theory is to go through the linear theory, generalizing each definition and proof as you go along. For example, the group algebra KG is generalized by the *twisted group algebra* $K^\alpha G$: this is an associative K-algebra possessing a basis $\{b_g | g \in G\}$ with multiplication determined by $b_{g_1} b_{g_2} = \alpha(g_1, g_2) b_{g_1 g_2}$. Then a bijection between all α-representations ρ of G and all linear representations R of $K^\alpha G$ is defined by $\rho(g) = R(b_g)$. Thus all the representation theory of this algebra becomes available to study the α-representations of G. In particular we can talk in terms of $K^\alpha G$-modules. (But beware: the adjective "projective" has completely different meanings for modules and for representations.)

A second way is to use central extensions to reduce questions about projective representations to corresponding questions about linear representations. For example, suppose that α has finite order m in $Z^2(G, K^\times)$ and that K contains a primitive mth root ζ of unity. Let

$$G^\alpha = \{\, \zeta^i b_g \mid i \in \mathbf{Z},\, g \in G \,\} \subseteq K^\alpha G.$$

Then G^α is a finite group under multiplication, and for each α-representation ρ of G, the restriction r to G^α of the corresponding R is a linear representation of G^α with $r(b_g) = \rho(g)$. This is a reverse of the construction that led to (4) applied to the central extension $1 \to \{\zeta^i\} \to G^\alpha \to G \to 1$. This lifting of ρ to r reduces much of the theory of α-representations to the study of some of the linear representations of G^α, which is called the *α-covering group* (or *α-representation group*) of G.

If K is algebraically closed, a more elaborate construction due to Schur yields a finite group H such that *all* the projective representations of G

can be lifted to representations of H (apart from a shift to cohomologous cocycles). H is called a *representation group* of G; it is "almost unique". For example, any finite simple group has a unique representation group, but both nonabelian groups of order 8 are representation groups for the four-group. In the corresponding exact sequence (3), A is isomorphic to $H^2(G, K^\times)$. If $K = \mathbf{C}$, this is called the *Schur multiplier* of G. The calculation of the Schur multipliers of the simple groups is a difficult part of the above-mentioned work related to the classification; for example, the Schur multiplier of the Mathieu group M_{22} is cyclic of order 12.

Projective representations come into linear representation theory as follows. Let N be a normal subgroup of G and σ an irreducible linear representation of N, with K algebraically closed and σ stable (up to equivalence) under conjugation by all the elements of G. It is natural to ask whether σ can always be extended to a representation of G. It turns out that the answer is "no" if we demand a linear representation, but "yes" if we will settle for a β-representation ρ for a certain β; furthermore β is inflated from a cocycle ω of G/N, and the irreducible linear representations of G whose restrictions to N contain σ are precisely the tensor products of ρ with the irreducible ω^{-1}-representations of G/A (inflated to G). This result, proved by W. H. Clifford in 1937, shows that projective representations are needed to study linear ones. In 1958 Mackey proved a similar result with σ replaced by a projective representation, so that the projective theory is self-contained in a way that the linear theory is not. (There are deep theorems giving cases in which $\beta = 1$.)

Schur created the central part of this theory in 1904 in a remarkably mature form. Many results on linear representations have projective analogues; the proof may be the same as in the linear case or may involve messy calculations with cocycles, but it is most interesting when the result itself is different. For example, if $K = \mathbf{C}$, the number of irreducible α-representations of G is the number of conjugacy classes of G that satisfy a certain condition, namely that $\alpha(g, x) = \alpha(x, g)$ whenever g is in the class and $gx = xg$; thus in the example above, the four-group has only one α-representation but four linear representations. The same example shows that irreducible projective representations of abelian groups over $\mathbf{C}$ need not be one-dimensional. While Brauer showed that all the linear representations of G in $\mathbf{C}$ can be written over the field of eth roots of unity where e is the least common multiple of the orders of the elements of G, this is false for projective representations but becomes true if we take the field of $|G|$th roots instead [**11**], [**12**]. The characters of projective representations need not be constant on conjugacy classes; and while this difficulty can be circumvented for a single group, it arises again once subgroups are considered.

Now for Karpilovsky's book. This is the first book devoted entirely to projective representations, and it is a big one. The contents have been well described in Humphreys' review [**8**]. It gives a comprehensive presentation, mostly in the first of the two ways I have described. As a result, it includes a large part of the linear theory as a special case, and even some theory of algebras. Characteristics 0 and p are treated together when possible. There are some nice touches, such as the definition of "α-covering group" that I have stated here and a proof in full generality of a theorem of Noether (Theorem 8.1.7) on the behavior of representations of algebras under field extensions, the only such proof I know of in print except for Noether's original paper. However, no reference to a proof of Proposition 1.1.13, essential here, is given. (The author has pointed out to me that this proposition is proved on pp. 176-177 of [**9a**]; this book appears in his bibliography.) The book treats Mackey's results (originally done for **C**) for general fields. It also contains a number of projective analogues of results that were previously known only in the linear case, although some of these are easy generalizations. Some noteworthy omissions are the p-adic theory and projective Brauer characters. The generalization of twisted group algebras to fully G-graded rings made in the 1960's independently by Dade, Fell, Kanzaki, and Ward [**4**] is not discussed.

Unfortunately the book has serious drawbacks. It is in the dry style of much mathematical writing, with too little explanation of what is important and why; this fault is especially bad because of the book's size. Often the author duplicates a proof from a research paper with little change and no new insights, when it would have been better and briefer to explain the significance of the result and give a reference for the proof. Sometimes the exposition is inferior to that in the original papers. For example, the proof of Proposition 4.1.14 contains two inaccuracies. For another, the author adopts on p. 305 a definition of "projective splitting field" that refers to irreducible (rather than arbitrary, or completely reducible) projective representations; as a result he states some of the succeeding theorems, including the above-mentioned one involving eth roots, in an unnecessarily weakened form. These examples may not be typical, but it would probably be a good idea to have another reference handy when using this book.

The book is photographically reproduced from a typescript. Apart from a considerable number of misprints, which may not be the author's fault, the format would be fine for something like the Lecture Notes in Mathematics, but in this age of TeX it is not satisfactory in such a high-priced book.

Because it contains so much material in self-contained form, and an excellent 24-page bibliography, this will be a useful reference for anyone working in the field. Although it originated in a set of lecture notes for a graduate course, my own preference would be to have a graduate student begin to learn about representations from such books as [**1**], [**3**], [**5**], [**6**], [**9**], and [**10**], rather than to study projective representations single-mindedly; in particular, [**9**] contains a good introduction to the projective theory. Afterwards, or concurrently, the student could make selective use of Karpilovsky's book.

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