ESSAYS IN ECONOMIC NETWORKS

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Abstract

Essay one: A Simple Application of the Independent Link Formation Model

[1]

In the networks literature, the E-R model stands for a class of random graph models whose link formation between any two nodes is independent. Based on this assumption, I investigates an entry problem where the welfare of the entrant depends on specific modes of interaction upon entering an existing group of agents of heterogeneous types. First I characterize the one-step best strategies for the entrant under two different modes of interactions, collective approval and independent approval, and found that the latter deviates positively from the former. Then I consider the dynamic version of this problem where entry is allowed to extend to infinity. Even though the best strategies differ under two different modes of interactions when there is only one entrant, they coincide in the limit.

[1] It has an independent table of contents, same for the second essay.

Essay two: A Modified Connection Model

In this essay, I study the efficiency and stability of a modified version of the social communication model introduced by Jackson and Wolinsky (1996). Inspired by empirical evidence on how having strong vs weak ties affects job matching and referral hiring, I introduce two types of links, a "strong link" and a "weak link", to the original model. I characterize the efficient and stable networks. Similar to previous findings, there does not always exist a stable network that is efficient. Furthermore, there exists efficient networks that are not stable, but such problems might be remedied by redistributing utility values among players.

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A Simple Application of Independent Link Formation Model

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Abstract

In the networks literature, the E-R model stands for a class of random graph models whose link formation between any two nodes is independent. Based on this assumption, I investigates an entry problem where the welfare of the entrant depends on specific modes of interaction upon entering an existing group of agents of heterogeneous types. First I characterize the one-step best strategies for the entrant under two different modes of interactions, collective approval and independent approval, and found that the latter deviates positively from the former. Then I consider the dynamic version of this problem where entry is allowed to extend to infinity. Even though the best strategies differ under two different modes of interactions when there is only one entrant, they coincide in the limit.

This is the first paper in my master thesis *Essays in Economic Networks*, under the supervision of Professor Ioannides.

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1. Introduction

The seminal work of Erdos and Rényi (1960) laid the groundwork for random graphs analysis. Their paper, based on the assumption that links between nodes are formed uniformly at random, characterizes asymptotic structures of random graphs as the number of nodes goes to infinity. If we allow nodes to be added to existing network sequentially, we obtain the benchmark growing random graphs model. However, this baseline random graph model is not satisfactory when it comes to explaining degree distributions obtained in the real world. Numerous research efforts have sought to provide better fits. One such successful attempt is to modify the link formation probability based on properties of the existing network. For example, Barabsi and Albert (1999) shows that if the link formation probability is proportional to node degree, i.e., well connected nodes are more likely to form a link with the incoming node (called "preferential attachment"), one can generate a degree distribution that resembles Pareto distribution, in contrast to the exponential distribution obtained from the baseline E-R growing random graph model. This modification provides a good description of real world degree distributions such as the distribution of number of coauthors in scientific articles (See chapter 3 of Jackson (2008), which gives a summary of Goyal et al. (2006) and many others). It is worth noting that since the incoming node can only form a fixed number of links with existing nodes, even though link formation events are still random, link formation events are no longer independent. Nonetheless, these random graph models, among many others, are quite successful in explaining some key features in our physical and societal world.

However, elegant as they are, these models certainly have their limitations. For instance, even though the preferential attachment model might shed light on why the upper tail of city size distribution is Pareto (Simon, 1955), it cannot explain why the body of city size distribution is log normal (Ioannides and Skouras, 2013). One reason for their limitations is that these models are physical processes essentially and their predictions do not emanate from strategic considerations. Thus, it is both necessary and desirable to incorporate game-theoretic decision making and network analysis, especially when studying systems consisting of agents capable of strategic interactions. Numerous research have sought to combine strategic decision making and networks, with topics ranging from more traditional ones such as trade (Gagnon and Goyal, 2015), inequality (Kets et al., 2011), and employment (Calvo - Armengol and Jackson, 2004; 2007) to favor exchange (Jackson et al., 2012) and military alliance (Jackson and Nei, 2014). Jackson and Zenou (2015) presented an excellent compendium of the rapidly growing literature in games on networks. Much of the discussion aimed at explaining equilibrium concepts, such as network structure, stability and efficiency

(Jackson and Wolinsky, 1996). There is also some discussion on the evolution of strategic interaction (Jackson, 2003; Jackson and Watts, 2002;Bala and Goyal, 2000). Another example, although not quite suitable as strategic interaction, is the DeGroot model of social learning (Golub and Jackson, 2010). The success of games on networks in predicting behaviors is not surprising: Game theory is regarded as the language of incentives, while networks provide channels for incentives to take effect.

It is thus interesting to study strategic decision making in a dynamic setting using growing graph models, especially their evolution. Specifically, based on the key assumption that link formation between agents is independent, this essay investigates an entry problem where the welfare of the entrant depends on specific modes of interaction upon entering an existing group of networked agents of heterogeneous types. The entrant is asked to report his or her type in order to play a game with the existing agents. Type-dependent links are formed probabilistically between the entrant and each of the existing agents. In the collective approval case, the entrant gets a payoff equal to one, if all links are formed, and a payoff of zero otherwise. In the individual approval case, if link is formed with agent j, the entrant gets this agents type as payoff, and zero otherwise. I show that, in the one-step case, the best strategy for the entrant is to obey the social norm (defined as the mean type of the population) if collective approval is assumed, and deviation from the social norm towards higher types, conditional on the magnitude of social norm as well as on the dispersion of types if we allow the entrant to collect payoff independently. Furthermore, I show that if entry is allowed to extend to infinity, the incentive for entrants to deviate declines over time so that in the limit the two cases coincide.

This essay proceeds as follows. The next section describes the entry problem under two modes of interaction and characterizes the best strategy for the entrant in each situation. Section three extends the one-step in the preceding section and characterizes the long run behaviors of entrants' best strategy if we allow entry to be extended to infinity. The last section compares and discusses these results.

2. The Entry Problem: One - Step

2.1. Homophily

In social network analysis, *homophily* is a commonly observed pattern that people of the same or similar types are more likely to connect to or benefit each other, all else being equal. Granovetter (1973) elaborates this phenomenon in terms of strength of ties: :"...if strong ties connect A to B and B to C, both B and C, being similar to A, are probably similar to one another, increasing the likelihood of a friendship once they have met". Schlenker et al. (1975) found that people who are similar are more likely to reward and less likely to harm each other than otherwise. Using high school friendships data from National Longitudinal Survey of Adolescent Health, Currarini et al. (2009) found that students of the same race have friends predominantly from their own group. Furthermore, efforts have been made to analytically capture homophily using a particular functional form. For an example, see section 2.3 in Golub and Jackson (2012).

2.2. Entry Problem With Collective Approval

Consider an environment of n ($n \geq 1$) agents with observable types $\theta_i \in (\theta_{low}, \theta_{high})$, where $0 < \theta_{low} < \theta_{high} < 1$. A risk neutral entrant wants to join the group and plays the following game with the existing agents. He or she needs to establish links, or seek approval, from each existing agents individually. In order to do so, the entrant is asked to report its type $\hat{\theta}$, where the probability of forming a link with an agent with type θ_j is given by

$$P\{link\ is\ formed\} = e^{-(\hat{\theta}-\theta_j)^2}.$$

This probability captures the notion of homophily: the closer the entrant's reported type with the jth agent's type, the larger the chance that they can form a link. Furthermore, this quantity is always between zero and one. The entrant receives a payoff equal to one, if all the links are established, and a payoff of zero otherwise. He or she seeks to maximize the expected payoff, where the expected payoff is

$$\begin{aligned} &1 \cdot P\{all\ links\ are\ formed\} + 0 \cdot P\{at\ least\ one\ link\ is\ not\ formed\} \\ =&1 \cdot P\{all\ links\ are\ formed\} \\ =&\prod_{j=1}^{n} e^{-(\hat{\theta}-\theta_j)^2}. \end{aligned}$$

Thus, the entrant solves

$$max_{\hat{\theta}} : \{\prod_{j=1}^{n} e^{-(\hat{\theta}-\theta_j)^2}\}.$$

Taking log, this becomes

$$max_{\hat{\theta}} : \{-\sum_{j=1}^{n} (\hat{\theta}-\theta_j)^2\},$$

which amounts to solving

$$min_{\hat{\theta}} : \{\sum_{j=1}^{n} (\hat{\theta} - \theta_j)^2\}.$$

It is well known that population mean minimizes this sum of squares, therefore

$$\hat{\theta}^* = \frac{1}{n}\sum_{j=1}^{n} \theta_j.$$

That is, a potential entrant chooses to report his or her type as $\hat{\theta}^*$ upon entry.

2.3. Entry Problem With Independent Approval

Now consider another mode of interaction. The entrant still form links with existing agents independently with above setting, but the payoff he or she gets is θ_j if the link is formed with the jth agent, zero otherwise. The expected payoff with this particular agent is therefore

$$\begin{aligned} &\theta_j \cdot P\{link\ with\ j\ is\ formed\} + 0 \cdot P\{link\ with\ j\ is\ not\ formed\} \\ =&\theta_j \cdot P\{linkwith\ j\ is\ formed\} \\ =&\theta_j. \end{aligned}$$

Thus the expected payoff upon entry is

$$\sum_{j=1}^{n} \theta_j e^{-(\hat{\theta}-\theta_j)^2}.$$

The entrant now solves

$$max_{\hat{\theta}} : \{\sum_{j=1}^{n} \theta_j e^{-(\hat{\theta}-\theta_j)^2}\}.$$

The first order condition does not have a close form solution when $n > 2$. However, as we are interested in understanding the behaviors qualitatively, recall that if $(\hat{\theta} - \theta_j)^2$ is small, the first order Taylor Approximation says that $e^{-(\hat{\theta}-\theta_j)^2} \approx 1 - (\hat{\theta} - \theta_j)^2$. Furthermore, it is easy to verify that this is always between zero and one. For mathematical tractability, I will assume that this is the link formation probability of the independent approval case, but it also gives an intuitive explanation of the best strategy. Therefore, the entrant's problem becomes

$$max_{\hat{\theta}} : \{\sum_{j=1}^{n} \theta_j (1 - (\hat{\theta} - \theta_j)^2)\}.$$

The first order condition is

$$\sum_{j=1}^{n} \theta_j(\hat{\theta} - \theta_j) = 0,$$

which gives the solution

$$\hat{\theta}^{**} = \frac{\sum_{j=1}^{n} \theta_j^2}{\sum_{j=1}^{n} \theta_j}.$$

This expression can be rewritten as

$$\begin{aligned}
\hat{\theta}^{**} =& \frac{\sum_{j=1}^{n} \theta_j^2}{\sum_{j=1}^{n} \theta_j} \\
=& \frac{\sum_{j=1}^{n} (\theta_j - \hat{\theta}^* + \hat{\theta}^*)^2}{\sum_{j=1}^{n} \theta_j} \\
=& \frac{\sum_{j=1}^{n} (\theta_j - \hat{\theta}^*)^2}{\sum_{j=1}^{n} \theta_j} + \frac{\sum_{j=1}^{n} \hat{\theta}^{*2}}{\sum_{j=1}^{n} \theta_j} + \frac{2\hat{\theta}^*(\theta_j - \hat{\theta}^*)}{\sum_{j=1}^{n} \theta_j} \\
=& \frac{\sigma^2}{\hat{\theta}^*} + \hat{\theta}^*.
\end{aligned}$$

3. Discussion of One - Step Results

3.1. Interpretation of Results

The types in these two models should be understood as one's observable characteristics, such as appearance, use of language, or measurable preference (for example, the fraction of time spent with a particular agent). Thus, link formation in the current context is equivalent to making an impression upon contact. Furthermore, this interpretation justifies the assumption that there is no cost to individuals who form links, as what is required for the entrant is simply to display the manner that is most appropriate for the current scenario (*Learning* how and when to display appropriate manners, however, is another matter and is usually associated with heavy cost). This is a drawback of the models, as it fails to demonstrate how unobservable characteristics of individuals, such as thoughts, morals, etc., affect link formation. Besides, more serious social relationships require both parties to invest a significant portion of their resources and thus cannot be studied using these two models. Therefore, the spectrum of behaviors that can be understood by these two models must be categorized as casual or informal.

The best response of the entrant in the collective approval case is to report $\hat{\theta}^* = \frac{1}{n}\sum_{j=1}^{n} \theta_j$, which maximizes the probability that all links are formed and so thus the expected payoff. The intuition is quite clear: when the entrant enters a new party, it is better to pretend be average, even that means concealing what he or she usually behaves.

Another extension is immediate. Instead of having complete information, consider the scenario where agents having i.i.d unobservable types and the distribution of types is common knowledge. Then the ex ante best response is just $E(\theta)$, regardless of the number of agents in the system (assumed to be finite) and their true types (they even might not realize what it is). Intuitively speaking, it is better for everybody to behave exactly the same as a normal person, if they have only minimum knowledge about the state of the world. Therefore, the population mean and has the interpretation of *social norm*.

On the other hand, the best response in the independent approval case for the entrant is not surprising. Given that payoff is collected independently from each existing individual, the entrant has to strike a more delicate balance. Moving towards an agent with higher type will increase the expected payoff given by this particular relation, but will also decrease the expected payoff from other agents with lower types. The optimal decision for the entrant is reached when the marginal benefit of moving towards higher types is equal to its marginal cost. Thus, the best response will be the type that achieves this balance, not just the average of the existing population. The first term of $\hat{\theta}^{**}$ captures this deviation qualitatively. If the mean of the existing population is fixed, the deviation from it will be determined by variance of the existing population. When the variance is large, that is, there is more dispersion among existing types, it is expected that the entrant would be leaning towards the portion with high types, thus the best response of the entrant will be large. Intuitively, this says the entrant has more room to move around and suffer less from the restriction on its behaviors. If the variance of the existing population is fixed, then the deviation is determined by its mean and a larger mean will lead to a smaller deviation. Intuitively, if the average caliber of the existing group is already quite high, than there is no much incentive for the entrant to deviate from its norm.

3.2. Two Examples

The best response in the collective approval case is the best response when there is serious need for conformity, especially in time of extreme political events such as the Cultural Revolution in China or Stalin's Soviet era, or maybe today's North Korea. Under a oppressive regime, everyone is required to swear absolute allegiance to the supreme leader and the state, not only verbally, but also in every detail of daily lives: dress, use of language, preference of arts, books read, etc. Anyone who dared to defy these expectations will be labeled as "pests" or "enemy of the people" and suffer humiliating punishment. Furthermore, everybody is encouraged to check up on their friends, neighbours, and even officials to see if they behave rightly. If a person walks into a conference room and recognizes that his or her behaviors is going to be watched closely by everybody else present, it is to this person's best interest to make a good impression. So this person does the following. He or she infers the type of each person by observing the use of languages and dress codes. For simplicity, let's say there are two types of people, the reformists and the hardliners. For further simplicity, let's say the only difference between these two groups of people is that the former call for economic reform and the latter want to stall it. If the conference room is full of a single type of people and this person is asked about his or her opinion on economic reform, even though this person might have opposing views, it is of his or her best interests to give most of the credits to the other side. If the room has a mixed of two types, then it might want to say nothing significant at all because if he or she did so, people from opposing side will use his or her opinion as an excuse for future persecution.

By contrast, the best strategy for the entrant in the independent approval case can apply to situations where the entrant has heterogeneous preference towards the existing agents. Consider the situation where one is hired to babysit two kids, of whom kid A he or she likes very much and kid B not so much and he or she needs to decide how much time or attention should be given to one child versus the other. The best strategy (in expectation, assuming risk neutrality) for the this person should just to spend most attention on kid A and little on kid B, just enough so that kid B would not tell their mother that this person is doing a bad job, while enjoying his or her time with kid A.

4. The Entry Problem: Long - Run

4.1. Goals

Upon having these static (one - step) results, it is natural to investigate the long run behaviors of these quantities. The extension of one - step result to allowing infinite entry should not be surprising, as we see frequently short from run vs long run type of analysis in the theory of the firm as well as steady state results from state - space models in macroeconomics. Another example is Ioannides (2016) where he studies the long run behaviors of intergenerational transfers based on the simultaneous game developed by Cabrales et al. (2011).

Specifically, this question is formulated as follows. Given an initial population of n types $\theta_j, j = 1, 2, \ldots, n$, with mean $\overline{\theta}_0$ and variance σ_0^2 with infinite entry, how will the sequences (indexed by m ($m \geq 0$)) of population mean $\{\overline{\theta}_m\}$, variance $\{\sigma_m^2\}$, and entrant best response $\{\hat{\theta}_m^*\}$ ($\{\hat{\theta}_m^{**}\}$ for the second case) evolve through time. This section is devoted to this question.

Notice that in either case, all of these quantities are non-negative. Furthermore, we assume that entrants are indexed by their order of entry.

4.2. Collective Approval

In this case, the evolution of states can be described by the following system of three equations:

$$\begin{cases} \hat{\theta}^*_m = \overline{\theta}_{m-1}; \\ \overline{\theta}_m = \frac{\hat{\theta}^*_m + (n+m-1)\overline{\theta}_{m-1}}{n+m}; \\ \sigma^2_m = \frac{\sum_1^{n+m-1}(\theta_j - \overline{\theta}_m)^2 + (\hat{\theta}^*_m - \overline{\theta}_m)^2}{n+m}. \end{cases}$$

Then it follows immediately that $\hat{\theta}^*_m = \overline{\theta}_m = \overline{\theta}_0$ and $\sigma^2_m = \frac{n\sigma^2_0}{n+m}$, since adding an observation of mean to the current population does not change the mean and thus makes no contribution to the initial sum of squared deviations. Thus, in the limit (which exists), the population mean and best response will be the same as the initial mean and the variance will go to zero.

4.3. Independent Approval

The independent welfare case is more interesting. Albeit not having been able to obtain a close formed solution, it turns out that the long run behaviors (i.e., whether limits exist or not) of these quantities of interest can be completely characterized.

Notice that the evolution in this case is described by the following system of equations:

$$\begin{cases} \hat{\theta}^{**}_m = \overline{\theta}_{m-1} + \frac{\sigma^2_{m-1}}{\overline{\theta}_{m-1}} & (a); \\ \overline{\theta}_m = \frac{\hat{\theta}^{**}_m + (n+m-1)\overline{\theta}_{m-1}}{n+m} & (b); \\ \sigma^2_m = \frac{\sum_1^{n+m-1}(\theta_j - \overline{\theta}_m)^2 + (\hat{\theta}^{**}_m - \overline{\theta}_m)^2}{n+m} & (c). \end{cases}$$

The following three propositions provide a somewhat lengthy but rigorous account to the convergence of $\{\hat{\theta}^{**}_m\}$, $\{\overline{\theta}_m\}$, and $\{\sigma^2_m\}$. However, the convergence of these quantities are easy to see because, intuitively, the "room" available for the entrants to deviate will eventually disappear as the number of agents in the system goes to infinity.

We first show that $\{\overline{\theta}_m\}$converges. However, we need a lemma for this.

Lemma 1. $\{\hat{\theta}^{**}_m\}$ cannot exceed θ_{max}, where θ_{max} is the maximum of the initial population of types. That is, $\{\hat{\theta}^{**}_m\}$ is bounded above.

Proof. It suffices to show $\{\hat{\theta}_1^{**}\} < \theta_{max}$, i.e., $\overline{\theta}_0 + \frac{\sigma_0^2}{\overline{\theta}_0} < \theta_{max}$. It is straightforward to verify that

$$\begin{aligned}
&\overline{\theta}_0 + \frac{\sigma_0^2}{\overline{\theta}_0} < \theta_{max} \\
\Leftarrow&\overline{\theta}_0^2 + \sigma_0^2 < \overline{\theta}_0\theta_{max} \\
\Leftarrow&n\overline{\theta}_0^2 + \sum_{j=1}^{n}(\theta_j - \overline{\theta}_0)^2 < \sum_{j=1}^{n}\theta_j\theta_{max} \\
\Leftarrow&n\overline{\theta}_0^2 + \sum_{j=1}^{n}\theta_j^2 - 2\overline{\theta}_0\sum_{j=1}^{n}\theta_j + \sum_{j=1}^{n}\overline{\theta}_0^2 < \sum_{j=1}^{n}\overline{\theta}_0\theta_{max} \\
\Leftarrow&\sum_{j=1}^{n}\theta_j^2 < \sum_{j=1}^{n}\overline{\theta}_0\theta_{max}.
\end{aligned}$$

Which is obviously true. Since one step best response will not exceed the maximum type, the maximum will remain so throughout. This proves that $\{\hat{\theta}_m^{**}\}$ is bounded above.

There is also an economic argument for this. Suppose for some reason some entrant chooses $\hat{\theta}_m^{**} = \theta_{max} + \epsilon$, where $\epsilon < \theta_{max} - \theta_{second\ max}$. Then if this entrant shifts to $\hat{\theta}_m^{**} = \theta_{max} - \epsilon$, he or she will be able to maintain the same payoff with the agent with the highest type, while increasing payoffs from all other lower types. This comes from the fact that expected payoff between the entrant and a particular existing agent reflects homophily. Intuitively, even though the entrant has incentive to deviate from the norm, he or she will not take an extreme position because that will cause aversion to the majority. Thus, any strategy beyond $\theta_{max}$is not efficient and the entrant will deviate, which proves the claim.

Proposition 1. $\{\overline{\theta}_m\}$ *converges.*

Proof. Substitute (a) into (b) yields

$$\overline{\theta}_m = \overline{\theta}_{m-1} + \frac{\sigma_{m-1}^2}{(n+m)\overline{\theta}_{m-1}} > \overline{\theta}_{m-1}.$$

This shows $\overline{\theta}_m$ is increasing. Since $\overline{\theta}_{m-1} < \hat{\theta}_m^{**}$ and $\{\hat{\theta}_m^{**}\}$ is bounded above by lemma 1, $\overline{\theta}_m < sup\{\hat{\theta}_m^{**}\} < \theta_{max}$ for all m. This shows that $\{\overline{\theta}_m\}$ is bounded above. Since bounded monotone sequence must converge, it follows that limit of $\{\overline{\theta}_m\}$ exists. □

Next I show that $\{\sigma_m^2\}$ and $\{\hat{\theta}_m^{**}\}$ both converges. However, it turns out that proving either one will suffice.

Lemma 2. $\{\sigma_m^2\}$ converges if and only if $\{\hat{\theta}_m^{**}\}$ converges.

Proof. (⇒) Suppose $\{\sigma_m^2\}$ converges. Then from (a) we know that $\lim_{m\to\infty}\hat{\theta}_m^{**} = \lim_{m\to\infty}\overline{\theta}_{m-1} + \frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}} = \lim_{m\to\infty}\overline{\theta}_{m-1} + \frac{\lim_{m\to\infty}\sigma_{m-1}^2}{\lim_{m\to\infty}\overline{\theta}_{m-1}} = \lim_{m\to\infty}\overline{\theta}_m + \frac{\lim_{m\to\infty}\sigma_m^2}{\lim_{m\to\infty}\overline{\theta}_m}$. Since $\lim_{m\to\infty}\overline{\theta}_m \neq 0$, this limit exists.

(⇐) Suppose $\{\hat{\theta}_m^{**}\}$ converges. Then from (a) we have

$$\sigma_{m-1}^2 = \overline{\theta}_{m-1}\hat{\theta}_m^{**} - \overline{\theta}_{m-1}^2.$$

Thus, $\lim_{m\to\infty}\sigma_{m-1}^2 = \lim_{m\to\infty}\overline{\theta}_{m-1}\hat{\theta}_m^{**} - \overline{\theta}_{m-1}^2 = \lim_{m\to\infty}\overline{\theta}_{m-1}\lim_{m\to\infty}\hat{\theta}_m^{**} - \lim_{m\to\infty}\overline{\theta}_{m-1}^2 = \lim_{m\to\infty}\overline{\theta}_{m-1}\lim_{m\to\infty}\hat{\theta}_m^{**} - (\lim_{m\to\infty}\overline{\theta}_{m-1})^2$, which exists.

Now I tackle the case for $\{\sigma_m^2\}$. We need a lemma for this purpose.

Lemma 3. $\{\sigma_m^2\}$is bounded.

Proof. By the $AM - GM$[1] inequality,

$$\hat{\theta}_m^{**} = \overline{\theta}_{m-1} + \frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}} \geq \sqrt{\overline{\theta}_{m-1}\frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}}} = 2\sigma_{m-1},$$

which is equivalent to $\sigma_{m-1}^2 \leq \frac{\hat{\theta}_m^{**}}{4}$. Since $\{\hat{\theta}_m^{**}\}$ is bounded above, we know that $\{\sigma_m^2\}$ is bounded above. It is also bounded below, since it is non-negative.

Proposition 2. *$\{\sigma_m^2\}$ and $\{\hat{\theta}_m^{**}\}$ both converge.*

Proof. We first obtain a recursive expression for $\{\sigma_m^2\}$. From (c) we know that

$$\begin{aligned}\sigma_m^2 &= \frac{\sum_{j=1}^{n+m-1}(\theta_j - \overline{\theta}_m)^2 + (\hat{\theta}_m^{**} - \overline{\theta}_m)^2}{n+m} \\ \Rightarrow (n+m)\sigma_m^2 &= \sum_{j=1}^{n+m-1}(\theta_j - \overline{\theta}_m)^2 + (\hat{\theta}_m^{**} - \overline{\theta}_m)^2.\end{aligned}$$

Note that

$$\begin{aligned}(\theta_j - \overline{\theta}_m)^2 &= (\theta_j - (\overline{\theta}_{m-1} + \frac{\sigma_{m-1}^2}{(n+m)\overline{\theta}_{m-1}}))^2 \\ &= (\theta_j - \overline{\theta}_{m-1})^2 - 2\frac{\sigma_{m-1}^2}{(n+m)\overline{\theta}_{m-1}}(\theta_j - \overline{\theta}_{m-1}) + \frac{\sigma_{m-1}^4}{(n+m)^2\overline{\theta}_{m-1}},\end{aligned}$$

and

$$(\hat{\theta}^{**} - \overline{\theta}_{m-1})^2 = ((1 - \frac{1}{n+m})\frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}})^2 = (\frac{n+m-1}{n+m})^2\frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2}.$$

[1]The inequality of arithmetic and geometric means.

Thus,

$$
\begin{aligned}
&\sum_{j=1}^{n+m-1} (\theta_j - \overline{\theta}_m)^2 + (\hat{\theta}^{**} - \overline{\theta}_{m-1})^2 \\
= &\sum_{j=1}^{n+m-1} (\theta_j - \overline{\theta}_{m-1})^2 - 2\frac{\sigma_{m-1}^2}{(n+m)\overline{\theta}_{m-1}} \sum_{j=1}^{n+m-1} (\theta_j - \overline{\theta}_{m-1}) + \sum_{j=1}^{n+m-1} \frac{\sigma_{m-1}^4}{(n+m)^2\overline{\theta}_{m-1}^2} + (\frac{n+m-1}{n+m})^2 \frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2} \\
= &\sum_{j=1}^{n+m-1} (\theta_j - \overline{\theta}_m)^2 + \frac{n+m-1}{(n+m)^2}\frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2} + (\frac{n+m-1}{(n+m)})^2 \frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2}.
\end{aligned}
$$

By dividing both sides by $n+m-1$, we have

$$
\begin{aligned}
\frac{n+m}{n+m-1}\sigma_m^2 &= \frac{\sum_{j=1}^{n+m-1} (\theta_j - \overline{\theta}_m)^2}{n+m-1} + \frac{1}{(n+m)^2}\frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2} + \frac{n+m-1}{(n+m)^2}\frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2} \\
&= \sigma_{m-1}^2 + \frac{1}{(n+m)^2}\frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2} + \frac{n+m-1}{(n+m)^2}\frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}^2}.
\end{aligned}
$$

Then we have the recursive representation of $\{\sigma_m^2\}$:

$$
\sigma_m^2 = \frac{n+m-1}{n+m}\sigma_{m-1}^2 + \frac{n+m-1}{(n+m)^2}\frac{\sigma_{m-1}^4}{\overline{\theta}_{m-1}}
$$

But, $\overline{\theta}_m = \overline{\theta}_{m-1} + \frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}} \Rightarrow \sigma_m^2 = (n+m)(\overline{\theta}_m - \overline{\theta}_{m-1})\overline{\theta}_{m-1}$. Thus,

$$
\begin{aligned}
\sigma_m^2 &= (n+m-1)(\overline{\theta}_m - \overline{\theta}_{m-1})\overline{\theta}_{m-1} + (n+m-1)(\overline{\theta}_m - \overline{\theta}_{m-1})^2 \\
&= (n+m)(\overline{\theta}_m - \overline{\theta}_{m-1})\overline{\theta}_{m-1} + (n+m)(\overline{\theta}_m - \overline{\theta}_{m-1})^2 - (\overline{\theta}_m - \overline{\theta}_{m-1})\overline{\theta}_{m-1} - (\overline{\theta}_m - \overline{\theta}_{m-1})^2 \\
&= \sigma_{m-1}^2 + (n+m)(\overline{\theta}_m - \overline{\theta}_{m-1})^2 - (\overline{\theta}_m - \overline{\theta}_{m-1})\overline{\theta}_m \\
&= \sigma_{m-1}^2 + \frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}}(\overline{\theta}_m - \overline{\theta}_{m-1}) - (\overline{\theta}_m - \overline{\theta}_{m-1})\overline{\theta}_m \\
&= \sigma_{m-1}^2 + (\frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}} - \overline{\theta}_m)(\overline{\theta}_m - \overline{\theta}_{m-1})
\end{aligned}
$$

Thus, we have the following:

$$
\sigma_m^2 - \sigma_{m-1}^2 = (\frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}} - \overline{\theta}_m)(\overline{\theta}_m - \overline{\theta}_{m-1}).
$$

From Lemma 1 and 3 we know that $\{\sigma_m^2\}$ and $\{\overline{\theta}_m\}$ are bounded and $\{\overline{\theta}_m\}$ is increasing.

Then we have

$$|\sigma_m^2 - \sigma_{m-1}^2| \le (|\frac{\sigma_{m-1}^2}{\overline{\theta}_{m-1}}| + |\overline{\theta}_m|)|(\overline{\theta}_m - \overline{\theta}_{m-1})| \le (|\frac{\theta_{max}}{\overline{\theta}_0}| + |\overline{\theta}_0|)|(\overline{\theta}_m - \overline{\theta}_{m-1})|.$$

If we let $(|\frac{\theta_{max}}{\overline{\theta}_0}| + |\overline{\theta}_0|) = C$, which is a finite number, we have

$$|\sigma_m^2 - \sigma_{m-1}^2| \le C|\overline{\theta}_m - \overline{\theta}_{m-1}|.$$

Recall that (in a complete metric space) a sequence converges if and only if it is Cauchy. I now show that $\{\sigma_m^2\}$ is Cauchy. For $\forall \epsilon > 0$, since $\{\overline{\theta}_m\}$ is Cauchy, there $\exists N$, $\forall p, q$, $p \ge q > N$, we have

$$|\overline{\theta}_p - \overline{\theta}_q| < \frac{\epsilon}{C}.$$

Then (recall $\{\overline{\theta}_m\}$is increasing)

$$\begin{aligned}
\epsilon &= \frac{\epsilon}{C} \cdot C > C|\overline{\theta}_p - \overline{\theta}_q| \\
&= C|\overline{\theta}_p - \overline{\theta}_{p-1} + \overline{\theta}_{p-1} - \overline{\theta}_{p-2} + \cdots + \overline{\theta}_{q+1} - \overline{\theta}_q| \\
&= C(|\overline{\theta}_p - \overline{\theta}_{p-1}| + |\overline{\theta}_{p-1} - \overline{\theta}_{p-2}| + \cdots + |\overline{\theta}_{q+1} - \overline{\theta}_q|) \\
&\ge (|\sigma_p^2 - \sigma_{p-1}^2| + |\sigma_{p-1}^2 - \sigma_{p-2}^2| + \cdots + |\sigma_{q+1}^2 - \sigma_q^2|) \\
&\ge |\sigma_p^2 - \sigma_{p-1}^2 + \sigma_{p-1}^2 - \sigma_{p-2}^2 + \cdots + \sigma_{q+1}^2 - \sigma_q^2| \\
&= |\sigma_p^2 - \sigma_q^2|.
\end{aligned}$$

which shows $\{\sigma_m^2\}$ is Cauchy. Then It follows from lemma 2 that $\hat{\theta}_m^{**}$ also converges. □

The result of one step transition shows that best response is strictly larger than the previous population mean. It is thus tempting to inquire whether this is true in the limit. It turns out that the answer is no.

Proposition 3. *$\{\hat{\theta}_m^{**}\}$ and $\{\overline{\theta}_m\}$ have the same limit. Therefore, $\{\sigma_m^2\}$converges to zero.*

Proof. I will prove this by contradiction. Since $\hat{\theta}_m^{**} \ge \overline{\theta}_m$ for $\forall m > 0$ (this is because if we add a number to a population that has average smaller than this number, the resulted new average is still smaller than this number), a standard theorem in undergraduate analysis yields $\lim_{m\to\infty} \hat{\theta}_m^{**} \ge \lim_{m\to\infty} \overline{\theta}_m$. Let $\lim_{m\to\infty} \hat{\theta}_m^{**} = \hat{\theta}^{**}$ and $\lim_{m\to\infty} \overline{\theta}_m = \overline{\theta}$ and assume $\hat{\theta}^{**} > \overline{\theta}$ with $l = \hat{\theta}^{**} - \overline{\theta} > 0$. By the definition of limit, for $\forall \epsilon < \frac{l}{2}$, there $\exists M_1$ such that

$$m > M_1 \Rightarrow |\hat{\theta}_m^{**} - \hat{\theta}^{**}| < \frac{\epsilon}{2}.$$

Similarly, there $\exists M_2$ such that

$$m > M_2 \Rightarrow |\overline{\theta}_m - \overline{\theta}| < \epsilon,$$

and let $M = max\{M_1, M_2\}$ so that the above two conditions both hold.

Recall that we index agents based on their order of entry. Note that, by the convergence of $\hat{\theta}^{**}$, there can be at most $m+M$ types lie outside the open ball $br(\hat{\theta}^{**}, \frac{\epsilon}{2}) = (\hat{\theta}^{**} - \frac{\epsilon}{2}, \hat{\theta}^{**} + \frac{\epsilon}{2})$ (the number of types outside this $br(\hat{\theta}^{**}, \frac{\epsilon}{2})$ can be exact $n+M$ if we manage to choose the smallest M). Then beyond the Mth entry (and let the increment of entry be indexed by m'), the existing population mean can be expressed as (including the $(M+m)$th type)

$$\begin{aligned}
\overline{\theta}_{M+m'} &= \frac{\sum \{\theta_j | \theta_j \notin (\hat{\theta}^{**} - \frac{\epsilon}{2}, \hat{\theta}^{**} + \frac{\epsilon}{2})\} + \sum \{\theta_j | \theta_j \in (\hat{\theta}^{**} - \frac{\epsilon}{2}, \hat{\theta}^{**} + \frac{\epsilon}{2})\}}{n+M+m'} \\
&\approx \frac{\sum_{j=1}^{n+M} \theta_j + \sum_{j=n+M+1}^{n+M+m'} \theta_j}{n+M+m'} \\
&= \frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'} + \frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{n+M+m'}
\end{aligned}$$

Subtract $\hat{\theta}^{**}$ on both sides, we have

$$\overline{\theta}_{M+m'} - \hat{\theta}^{**} = \frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'} + \frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{n+M+m'} - \hat{\theta}^{**},$$

which implies

$$\begin{aligned}
|\overline{\theta}_{M+m'} - \hat{\theta}^{**}| &= |\frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'} + \frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{n+M+m'} - \hat{\theta}^{**}| \\
&\leq |\frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'}| + |\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{n+M+m'} - \hat{\theta}^{**}| \\
&= |\frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'}| + |\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'} - \frac{(n+M)\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'(n+M+m')} - \hat{\theta}^{**}| \\
&\leq |\frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'}| + |\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'} - \hat{\theta}^{**}| + |\frac{(n+M)\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'(n+M+m')}| \\
&= |\frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'}| + |\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'} - \hat{\theta}^{**}| + \frac{n+M}{n+M+m'}|\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'}|.
\end{aligned}$$

Since $\theta_j \in br(\hat{\theta}^{**}, \frac{\epsilon}{2})$, $\forall j > n+M$, $\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'} \in br(\hat{\theta}^{**}, \frac{\epsilon}{2})$ for $\forall m' > 0$. Thus,

$$|\overline{\theta}_{M+m'} - \hat{\theta}^{**}| < |\frac{\sum_{j=1}^{n+M} \theta_j}{n+M+m'}| + \frac{n+M}{n+M+m'}|\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'}| + \frac{\epsilon}{2}$$

Since $\sum_{j=1}^{n+M} \theta_j$ is finite, there $\exists M_3$ such that (again, let m' be the increment of index beyond $n+M+M_3$)

$$|\frac{\sum_{j=1}^{n+M} \theta_j}{n+M+M_3+m'}| < \frac{\epsilon}{4}.$$

Similarly, notice that $|\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'}|$ is bounded (for example, by θ_{max}. It is bounded because $\theta_j \in br(\hat{\theta}^{**}, \frac{\epsilon}{2})$, $\forall j > n+M$), there $\exists M_4$ such that

$$\frac{n+M}{n+M+M_4+m'}|\frac{\sum_{j=n+M+1}^{n+M+m'} \theta_j}{m'}| < \frac{\epsilon}{4}.$$

Let $M' = max\{M_3, M_4\}$ so that these two conditions both meet. Then for $\forall m' > 0$, we have

$$|\overline{\theta}_{M+M'+m'} - \hat{\theta}^{**}| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

However, this shows that $\overline{\theta}_{M+M_3+m'} \in br(\hat{\theta}^{**}, \epsilon)$ for $\forall m' > 0$, which is a contradiction to

$$|\overline{\theta}_m - \overline{\theta}| < \epsilon$$

when m is sufficiently large. Thus, we must conclude that $\hat{\theta}^{**} = \overline{\theta}$. It follows immediately from Lemma 2 that $\lim_{m\to\infty} \sigma_m^2 = 0$.

□

5. Discussion of Long - Run Results

5.1. *Intuition Behind the Convergence*

From the one step results we know that in the first case, each entrant has no incentive to deviate from social norm, whereas in the second case, entrants have incentives to deviate towards higher types, but just enough so that marginal benefits (gains from moving closer to higher types) equals marginal cost (lost from moving away from lower types). However, as the number of agents in the system increase, even though each entrant still has incentive to move closer to the high types, it becomes increasingly costly to do so because the average caliber or standards of the existing population is high and there are a huge number of relationships that the entrant has to contemplate. Thus, in the limit the entrant is indifferent between any deviation whatsoever and the social norm. Adam Smith envisioned in his Wealth of Nations that as the division of labor deepens in a society, *alienation* of individuals occurs. This essay provides some analogy to this concept. When the number of existing agents becomes increasingly large but every entrant is still required to form relationship with them, the marginal cost of deviating from the norm mounts while the "room" for deviation gets smaller. Thus, the incentives for entrants gradually decrease due to the need to handle the ever completed social relationships.

5.2. Monotonicity of Variance

The variance of the existing population of types can be interpreted as a measure of conformity and consensus of the group. We have seen that in both the collective approval case and the independent approval case, in the limit the society is in complete consensus. It is also easy to see that the variance of the existing population strictly decreases in the collective approval case. However, I found that this is not the case for the independent approval case.

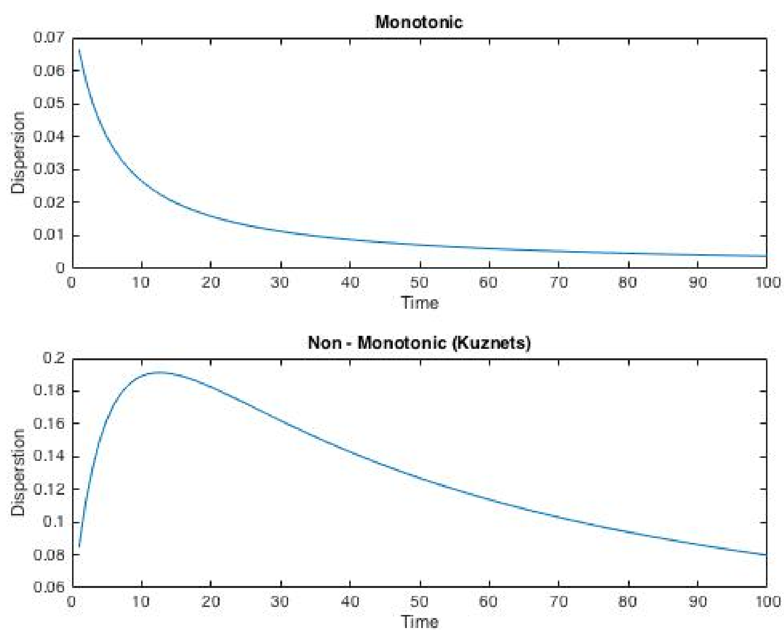

Fig. 1. Monotonicity Comparison
evolution of variance comparison for the independent approval case

We see in Figure 1. that in some situations, the evolution of variance is monotone, but in some other cases, especially the ones with a large number of very low types, the evolution of variance displays an inverse U - shape pattern. The program that generates this figure is presented in appendix A.

One possible explanation is that agents of high types serve as "models" or "leaders" for the entrants at the early stage of the evolution. The intuition is explained as follows. Entrants coming in deviate from the norm (which is low due to a large number of agents with low types) towards the "models", agents with high types, in the existing group. However, the self-interest behaviors of entrants also affects the existing group not only by raising the group average, but also by introducing agents that have types much higher than previous group averages. This disrupts the previous level of conformity (the majority of the society are of much lower types) and as a result, we observe an increase in the dispersion of types. However, this stage cannot last forever. When the number of agents in the system reaches a certain level, entrants will find themselves in a position where there are many existing agents with similar types as the "room" for deviation gradually drops. Entrants are now forced to pay relatively more attention to existing members of the group rather than only those "models", so the new consensus begin to form and we observe a decrease in the dispersion of types.

6. Conclusion

I study an entry problem using a model that combines certain features of growing random graph models and strategic decision making. An entrant is forced to form relationships with all existing agents upon entry, but he or she can choose a position based on the observed information and specific modes of interactions, collective approval vs. independent approval. The former rationalizes the norm of the existing group and can be used to understand behaviors when conformity is desired, whereas the latter justifies positive deviation from such norm if the entrant has heterogeneous preferences towards existing agents. However, such difference will disappear if infinite entry is allowed, intuitively, because entrants need to balance increasingly intricate social relationships and the incentives for the entrants to deviate drop gradually. As a result, the level of consensus decreases over time and complete conformity is reached in the limit.

Appendix A. Program That Generates Figure 1.

```
ini = [0.2,0.6,0.5,0.7,0.99];
%initial population with a relative distribution of types
for i = 1:100
    si = var([ini,mean(ini)]);
    x(i) = si;
    me = mean(ini) + var([ini,mean(ini)])/mean(ini);
    ini = [ini,me];
end

ins = ...
    [0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.01,0.99];
%initial population with a lot of low types and few high types
for i = 1:100
    sl = var([ins,mean(ini)]);
    y(i) = sl;
    me = mean(ins) + var([ins,mean(ins)])/mean(ins);
    ins = [ins,me];
end

%draw comparison graph
z = linspace(1,100);
figure;
subplot(2,1,1);
plot(z,x);
xlabel('Time');
ylabel('Dispersion');
title('Monotonic');

subplot(2,1,2);
plot(z,y);
xlabel('Time');
ylabel('Disperstion');
title('Non - Monotonic (Kuznets)');
```

References

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

Barabsi, A.-L. and Albert, R. (1999). Emergence of scaling in random networks. *Science*, 286(5439):509–512.

Cabrales, A., Calvo-Armengol, A., and Zenou, Y. (2011). Social interactions and spillovers. *Games and Economic Behavior*, 72(2):339–360.

Calvo-Armengol, A. and Jackson, M. O. (2004). The effects of social networks on employment and inequality. *American Economic Review*, 94(3):426–454.

Calvo-Armengol, A. and Jackson, M. O. (2007). Networks in labor markets: Wage and employment dynamics and inequality. *Journal of Economic Theory*, 132(1):27–46.

Currarini, S., Jackson, M. O., and Pin, P. (2009). An economic model of friendship: Homophily, minorities, and segregation. *Econometrica*, 77(4):1003–1045.

Erdos, P. and Rényi, A. (1960). On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5(1):17–60.

Gagnon, J. and Goyal, S. (2015). Social structure, markets and inequality.

Golub, B. and Jackson, M. O. (2010). Naive learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, 2(1):112–149.

Golub, B. and Jackson, M. O. (2012). Does homophily predict consensus times? testing a model of network structure via a dynamic process. *Review of Network Economics*, 11(3):81–95.

Goyal, S., van der Leij, M. J., and Moraga-Gonzalez, J. L. (2006). Economics: An emerging small world. *Journal of Political Economy*, 114(2):403–412.

Granovetter, M. S. (1973). The strength of weak ties. *American Journal of Sociology*, 78(6):1360–1380.

Ioannides, Y. (2016). Endogenous social networks and inequality in an intergenerational setting. *Working Paper*.

Ioannides, Y. and Skouras, S. (2013). Us city size distribution: Robustly pareto, but only in the tail. *Journal of Urban Economics*, 73(1):18–29.

Jackson, M. O. (2003). *The Stability and Efficiency of Economic and Social Networks*. Studies in Economic Design. Heidelberg and New York: Springer.

Jackson, M. O. (2008). *Social and Economic Networks*. Princeton and Oxford: Princeton University Press.

Jackson, M. O. and Nei, S. (2014). Networks of military alliances, wars, and international trade.

Jackson, M. O., Rodriguez-Barraquer, T., and Tan, X. (2012). Social capital and social quilts: Network patterns of favor exchange. *American Economic Review*, 102(5):1857–1897.

Jackson, M. O. and Watts, A. (2002). The evolution of social and economic networks. *Journal of Economic Theory*, 106(2):265–295.

Jackson, M. O. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74.

Jackson, M. O. and Zenou, Y. (2015). Games on networks. *Handbook of Game Theory with Economic Applications*, 4(1):95–163.

Kets, W., Iyengar, G., Sethi, R., and Bowles, S. (2011). Inequality and network structure. *Games and Economic Behavior*, 73(1):215–226.

Schlenker, B. R., Brown, R. C., and Tedeschi, J. T. (1975). Attraction and expectations of harm and benefits. *Journal of personality and social psychology*, 32(4):664–670.

Simon, H. A. (1955). On a class of skew distribution functions. *Biometrika*, 42(3):425–440.

A Modified Connection Model

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Abstract

In this essay, I study the efficiency and stability of a modified version of the social communication model introduced by Jackson and Wolinsky (1996). Inspired by empirical evidence on how having strong vs weak ties affects job matching and referral hiring, I introduce two types of links, a "strong link" and a "weak link", to the original model. I characterize the efficient and stable networks. Similar to previous findings, there does not always exist a stable network that is efficient. Furthermore, there exists efficient networks that are not stable, but such problems might be remedied by redistributing utility values among players.

This is the second paper in my master thesis *Essays in Economic Networks*, under the supervision of Professor Ioannides.

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1. Introduction

Network structures, as many recent studies have found, play a significant role in understanding economic relationships among individuals and within organizations. In particular, Jackson and Wolinsky (1996) introduced a symmetric connections network that models social communications among individuals. In this model, connected individuals not only benefit from their costly direct connections, but also from their indirect connections with attenuation. It provides useful insight in understanding social interaction, particularly how individuals benefit from mutual relationships.

However, it is highly unlikely that interactions among individuals are homogeneous. Social interactions, even for the same purpose, vary tremendously in their intensity. For example, one can categorize relationships into either strong or weak conditional on the frequency of interactions from the past, whereas in some literature the strength of an individual's social relationships is assumed to be positively associated with his or her significance in society. Maintaining social relationships not only requires significant investment of one's time, emotion, and often finances, but also the calculation needed to allocate those resources. As a result of this heterogeneity, the economic outcome is fairly expected to be different.

One area where such difference is well documented is job finding. Granovetter (1973) found that weak ties are the "bridges" of information diffusion between groups and they play a important role in finding a job. More recent studies (Kramarz and Skans, 2014) found that strong ties are also useful in job finding. Defining a person's parents' social ties as strong ties, they discovered that these social relationships is a key determinant of his or her initial employment prospect and such effect is even stronger when this person is in a disadvantageous position. Gee et al. (2017) provides an explanation to these seemingly opposing views. Using Facebook data, they found that weak ties are indeed more useful than strong ties only because weak ones are more numerous. When compared one-to-one, it is found that strong ties do have higher marginal benefit than weak ties in terms of finding a job. These empirical findings motivates the possibility of incorporating heterogeneous types of links in the original model.

The main goal of this essay is to characterize which network structure achieves the maximum *efficiency* when "strong ties" are allowed in the original Jackson and Wolinsky (1996) model. *Stability* of networks is also examined. I find that these two desired goals of economic design are often not compatible in a symmetric communication network, but might be remedied by redistributing utility values among players.

This essay proceeds as follows. In the next section I introduce definitions needed for discussion. The third section review the models and presents empirical evidence that justifies the use of ”strong ties” in the modified model. Section four characterizes efficient networks and section five discusses the stability of networks. The last section discusses and summarizes the results. I follow the original paper for the description of the model and game-theoretic notations and the textbook by West (2001)[1] for graph-related notations.

2. Definitions

2.1. Graphs

A graph G is a pair $(V(G), E(G))$ where $V(G)$ is the *vertex set* and $E(G)$ is the *edge set*. In the following discussion I use graphs and networks interchangeably. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An edge $e = e((i,j),t) \in E(G)$ consist of an unordered vertex pair $(i,j), i,j \in V(G)$ and type indicator $t \in \{s, w\}$, where $t = s$ indicates the edge is *strong*, otherwise it is considered *weak*. In the cases where the type of an edge does not matter, however, I assume the notation ij for an edge. The exact meaning of strong vs weak ties will be evident in the next section. The *complete graph* $K_{|V(G)|}$ is the graph such that $\forall i,j \in V(G), ij \in E(G)$, where $|V(G)|$ is the carnality of the vertex set. Let $\mathcal{C} = \{G|G \subseteq K_{|V(G)|}\}$ denote the set of all possible graphs that share the same vertex set. I only consider *simple graph* in the discussion, which are graphs with no loops (an edge that has the same endpoints) nor multiple edges between vertexes.

Two vertices $i,j \in V(G)$ are said to be *adjacent* if $ij \in E(G)$. The *neighbour* $N(i)(N(H))$ of $i \in V(G)(H \subseteq G)$ is defined as $\{j \in V(G)|ij \in E(G)\}(\{j \in V(G)|ij \in E(G) \setminus E(H)\})$. A *path* is a simple graph with distinct vertices such that the vertices can be permuted so that i,j are consecutive on the list if and only if i,j are adjacent. The *internal vertices* of an i,j-path are the vertices on this path excluding i and j. The *distance* between two vertices $i,j \in V(G)$ is the number of edges of the shortest i,j-path, denoted by $d_G(i,j)$. If there is no such path, i,j are said to be *disconnected* and the distance is ∞. A graph is *connected* if for $\forall i,j \in V(G)$, there is a i,j-path. A graph G is *empty* if $E(G) = \emptyset$. In the following discussion I will only consider either connected graphs [2] or empty graphs. A *tree* [3] is a connected graph with $|V(G)| - 1$ edges.

Two graphs G and G' are *isomorphic* if there exists a bijection $f : V(G) \to V(G')$, called *isomorphism*, such that $ij \in E(G)$ if and only if $f(i)f(j) \in E(G')$.

[1]Douglas Brent West is a Professor of graph theory at University of Illinois at Urbana-Champaign.

[2]Technically speaking, it is not entirely rigorous to consider only connected graph. However, once the conclusion of a component (a connected subgraph) has been reached, it is a matter of ”direct calculation” (Jackson and Wolinsky, 1996) to determine whether it should be applied to the whole graph. For this reason, I consider only connected graphs.

[3]There are several equivalent characterizations for tree. See p.68 of West’s textbook.

2.2. Value, Efficiency, and Stability

The *value of a graph* G is a function $u = u(G) : \mathcal{C} \to \mathbb{R}$ and the set of all such functions is denoted as $\mathcal{U}$. In this essay I only consider value functions that are *aggregate additive*, i.e., $u = \sum_{i \in V(G)} u_i$, where $u_i = u_i(G) : \mathcal{C} \to \mathbb{R}$.

For $\forall G, G' \in \mathcal{C}$, G *weakly dominates* G' if $u(G) \geq u(G')$. A graph G is *efficient* if G weakly dominates G' for $\forall G' \in \mathcal{C}$.

An *allocation rule* $A : \mathcal{C} \times \mathcal{U} \to \mathbb{R}^n$ specifies the distribution of values among individual vertexes. A graph G is *stable*, or *pairwise stable*[4], if (i) for $\forall e = e(t,(i,j)) \in E(G)$, $A_i(G) \geq A_i(G-e)$ and $A_j(G) \geq A_j(G-e)$, (ii) for $\forall e = e(t,(i,j)) \in E(G)$, $A_i(G) \geq A_i((G-e)+e')$ and $A_j(G) \geq A_j((G-e)+e')$, where $e' = e(t',(i,j)), t \neq t'$, and (iii) for $\forall ij \notin E(G)$, if $A_i(G+ij) > A_i(G)$, then $A_j(G+ij) < A_j(G)$. Intuitively, in a pairwise stable network i) no single player has incentive to delete or change edge type unilaterally, and ii) no pair of players have incentive to add an edge that does not previously exist, regardless of its type.

3. Model

3.1. Jackson and Wolinsky (1996)

In the original (symmetric) model, a graph is used to study how individuals benefit from social communications. Those individuals that communicate directly enjoy mutual benefit of this relationship. However, any two individuals can have such mutual benefits with attenuation so far as they are connected, which only depends on the distance between them. In other words, direction connections generate positive externalities for those who maintain them (Bala and Goyal, 2000). However, establishing direct communication inflicts a cost.

Specifically, given a graph G with $|V(G)| = n$, for $\forall i \in V(G)$, i gets benefit $\delta^{d_G(i,j)}$ from $j \neq i$, where $0 < \delta < 1$, and pays $c > 0$ for each ij such that $j \in N(i)$. The individual utility value of i is

$$u_i(G) = \textstyle\sum_{j \neq i} \delta^{d_G(i,j)} - \sum_{j \in N(i)} c$$

and the value of the graph is therefore $u(G) = \sum_{i \in V(G)} u_i(G)$.

[4]This is slightly different from the usual definition of pairwise stability as there are two types of edges here. Specifically, if any player has incentive to change the type of an edge (not necessarily to dissolve it), the graph is considered to be unstable.

3.2. The Modified Model

Assume the above setting with one modification. Introduce a second type of "strong" edges versus the original "weak" edges. The new type of edge still generate δ for direct connection and costs $lc, l \geq 1$, which makes it more costly than the weak edge, but has the advantage of "neighbourhood sharing": individuals who commit to strong edges now have direct access to their friends' direct friends that are in their neighbourhoods, thus potentially suffering less from the attenuation cost.

Specifically, let $ij = e((i,j), s)$, then if the shortest i, k-path for $\forall j \neq k \in V(G)$ uses ij, the benefit that i gets from k will be $\delta^{d_G(i,k)-1}$. The direct benefit that i gets from j is still δ.

3.3. Interpretation of l

The motivation for introducing this "strong" edge should not be surprising. The mutual benefit gained from a direct relationship, albeit varying in degree, is certainly limited, due to the restriction of personal knowledge and experience. In a symmetric setting, this is modeled as follows: strong and weak edge both generate δ for direct connections, even though strong edges cost more. However, the power of maintaining mutual communication lies mostly in indirect connection: being connected to well-connected individuals allows mutual benefits to out-weight the cost of maintaining the edge. With that being said, people tend not to open fully to their direct friends, and their friend's friends, etc., and this rationalizes the setting that there is decay of mutual benefits with distance. Committing to strong ties remedy this problem to some degree in the sense that people now show more trust to their direct friends and do not act as conservative as before. On the other hand, as Granovetter (1973) put it, "the strength of a tie is a (probably linear) combination of the amount of time, the emotional intensity, the intimacy (mutual confiding), and the reciprocal services which characterize the tie". This would certainly require both parties put more of their resources to maintain the edge, which rationalizes the assumption in the model that strong edges are more costly than weak edges.

In fact, one commonly used measure of tie strength is the frequency of interactions (Granovetter (1973); Gee et al. (2017)). D'Angelo and Lilla (2011) categorized social interactions not only by their intensity but also by their types; serious social interaction such as attending a prestigious club requires more attention and qualification than casual interactions.

Last but not least, the relative magnitude of l can be interpreted as an inverse indicator of players' social skills, as the more skilled people are, the more they can get out of maintaining strong relationship without having too much trouble. This is similar to the interpretation of "non-cognitive skill" in the endogenous social structure model in Ioannides (2016).

4. Efficiency

4.1. Goals and Method

The main goal of this section is to investigate what network structure with $|V(G)| = n \geq 3$ achieves maximum efficiency based on the modified model. Uniqueness of the solution is also examined. It is easy to see that, for a graph of single vertex, the value is zero; for a graph of two vertices, the optimal structure is an empty graph if $\delta \leq c$, otherwise a pair connected by an weak edge.

To achieve this goal, I employ a value-comparison approach that resembles Jackson and Wolinsky's in proposition 1 and 3. My approach in proposition 2 is similar to the dynamic link formation process in Bala and Goyal (2000) in the sense that value-comparisons proceed sequentially. The proof also relies on the main result of social efficiency from Jackson and Wolinsky(1996), which states that (in the current context)

(a)If $c < \delta - \delta^2$, the efficient network structure with only weak edges is a complete graph;

(b) If $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$, the efficient network structure with only weak edges is a star;

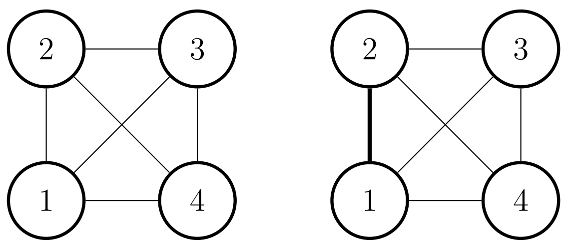
(c)If $c > \delta + \frac{n-2}{2}\delta^2$, the efficient network structure with only weak edges is the empty graph.

4.2. Characterization of Efficient Graphs

It is worth mentioning that the maximum efficiency can certainly be achieved without referring to a specific structure, for $\mathcal{C}$ is compact (finite) and so is $u(\mathcal{C})$, which must assumes maximum in $\mathbb{R}$. However, it is not obvious why simple structures like stars turn out to be the most efficient structure for society and why such specifications are unique. The propositions and proofs are provided in a case-wise manner.

Proposition 1. *If $0 < c < \delta - \delta^2$, the unique efficient graph of the modified model is a complete graph of weak edges if $l > \frac{n}{2}$ and a star of strong edges if $1 \leq l \leq \frac{n}{2}$.*

Proof. I first prove the case when $0 < c < \frac{\delta-\delta^2}{2}$. By (a), if only weak edges are allowed, the efficient graph is a complete graph K_n. If some edge in this graph is to be replaced by a strong edge, the total cost increases while the benefits stay the same, thus it cannot be efficient. This implies that an efficient graph with strong edges must have fewer edges than K_n, which further implies that there exist non-adjacent vertices. Let $i, j \in V(G)$ such that $d_G(i, j) \geq 2$ and $G' = G + e(w, (i, j))$. If it is not the case that $d_G(i, j) = 2$ and ik, kj are both strong, where k is the internal vertex of this shortest i, j-path, G' weakly dominates G since adding $e = e(w, (i, j))$ inflicts a cost of $2c$ but gains at least $\delta - \delta^2$, and possibly more since shortest paths between vertices might be shortened by this extra edge. This suggests that in an efficient graph G with $\forall i, j$ not adjacent, it must be the case that $d_G(i, j) = 2$ and ik, kj are strong, where k is the internal vertex. Furthermore, it is necessary that all of these pairwise non-adjacent vertices share a single internal vertex, for otherwise additional edges are needed to connect these vertices and their internal vertices.



(a) K_4 with weak edges

(b) K_4 with one strong edge

Figure 1. Replacing Weak Edges in Complete Graphs
(thin line = weak edge, thick line = strong edge, same below) Since in a complete graph every pair of vertices are already adjacent, replacing weak edges with strong ones is not efficient.

Let $m (m = 0, 2, 3, \ldots, n-1)$ be the number of pairwise non-adjacent vertices in such a graph G. Observe that, given n and l, the total cost of edges is

$$\begin{aligned} f(m) &= 2[\binom{n-m}{2} + m(n-m-1)]c + 2mlc \\ &= -m^2c + (2l-1)mc + (n^2-n)c. \end{aligned}$$

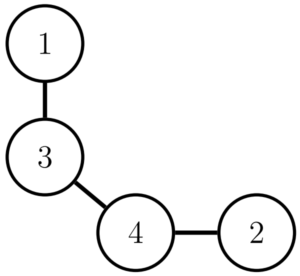
Differentiate with respect to m:

$$\frac{df}{dm} = -2mc + (2l-1)c.$$

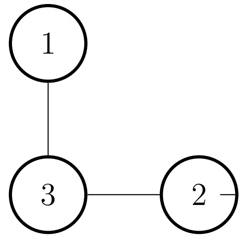
Equating it to zero yields

$$m^* = l - \frac{1}{2}.$$

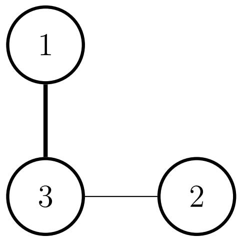
This says the maximum of cost occurs when $m = 0, 2, \ldots, n-1$ is closest to m^*, which



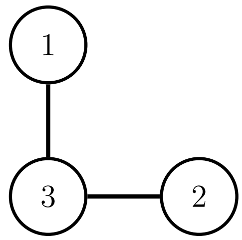
(a) $d_G(1,2) = 3$



(b) $d_G(1,2) = 2$, $13, 23$ are weak

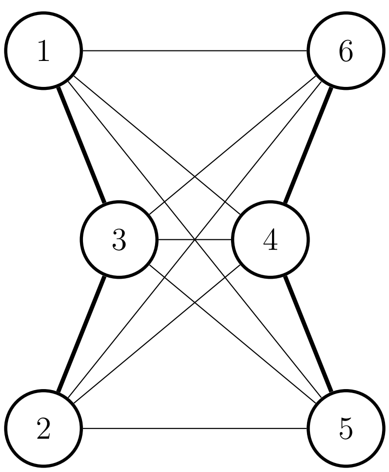


(c) $d_G(1,2) = 2$, 13 weak, 23 is strong

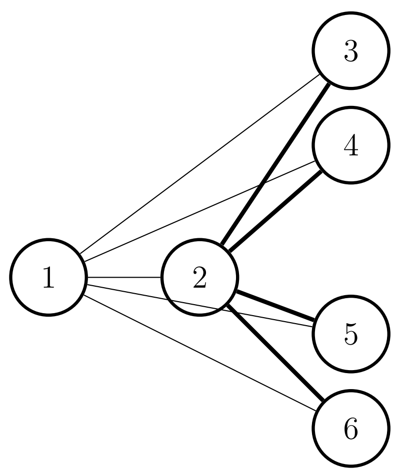


(d) $d_G(1,2) = 2$, $13, 23$ are strong

Figure 2. Non-Adjacent Vertices
In (a)(b)(c), efficiency can be improved by adding $e = e(w,(1,2))$ if $0 < c < \frac{\delta-\delta^2}{2}$, while in (d) it cannot generate any improvement.



(a) multiple internal vertexes



(b) single internal vertex

Figure 3. Multiple Internal Vertices
Both of these two graphs generate benefits 30δ, but (a) costs $8lc + 18c$, whereas (b) costs $8lc + 10c$. The additional $8c$ comes exactly from connecting pairwise non-adjacent vertices to other internal vertices.

indicates that the minimum occurs when m is most distant away from $m*$. Therefore, the minimum occurs at either $m = 0$ or $m = n - 1$. Specifically,

$$\begin{cases} m = n - 1 & \text{if } 1 \leq l \leq |n - l - \frac{1}{2}| + \frac{1}{2} \\ m = 0 & \text{if } l > |n - l - \frac{1}{2}| + \frac{1}{2} \end{cases}$$

or

$$\begin{cases} m = n - 1 & \text{if } 1 \leq l \leq \frac{n}{2} \\ m = 0 & \text{if } l > \frac{n}{2}. \end{cases}$$

However, the first case corresponds to a star with strong edges and the second case corresponds to a complete graph with weak edges and the topological uniqueness comes from the uniqueness of stars and complete graphs.

Now I prove the case when $\frac{\delta-\delta^2}{2} < c < \delta - \delta^2$. The only exception to the above argument is that there could be vertices connecting to the internal vertex using weak edges and such vertices are pairwise non-adjacent to the ones connecting to the internal vertex using strong edges. Suppose there are $k, k = 1, 2, \ldots, m-1$ such vertices and the total number of these vertices plus the vertices that are pairwise non-adjacent to them be $m, m = 2, 3, \ldots, n-1$. Call this set of vertices M. Let $H \subseteq G$ be the graph induced by M and the internal vertex. Notice that since for $\forall i \in V(H), V(G) \setminus V(H) \subseteq V(G)$, changing value of H by adding, deleting, or replacing edges within H will not affect the subgraph induced by $V(G) \setminus V(H) \subseteq V(G)$. Thus, it suffice to consider the value-maximizing structure of H.

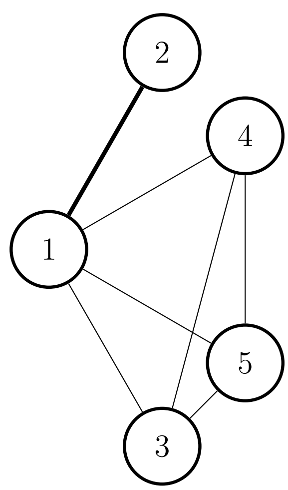


Figure 4. An Example of H

Observe that

$$\begin{aligned} u(H) &= m\delta + m(m-k)\delta + k^2\delta + km\delta^2 - 2(m-k)lc - 2\binom{k+1}{2}c \\ &= m(m-k+1)\delta + k^2\delta + km\delta^2 - 2(m-k)lc - k(k+1)c. \end{aligned} \tag{1}$$

On the other hand, a star with strong edges and same vertex set as $V(H)$ has value

$$m(m+1)\delta - 2mlc \tag{2}$$

and a complete graph with weak edges and the same vertex set has value

$$m(m+1)(\delta - c). \tag{3}$$

Now I compare these three values. Subtracting Eq.(2) from Eq.(1) yields

$$k(m-k)(\delta - \delta^2) - 2klc + k(k+1)c \tag{4}$$

and Subtracting Eq.(3) from Eq.(1) yields

$$k(m-k)(\delta - \delta^2) + 2(m-k)lc - (m-k)(m+k+1)c. \tag{5}$$

Suppose for some feasible m, k, Eq.(4) and Eq.(5) are both negative.This would require

$$\begin{aligned} & k(m-k)(\delta - \delta^2) - 2klc + k(k+1)c < 0 \\ \iff & 2klc > k(m-k)(\delta - \delta^2) + k(k+1)c \\ \iff & lc > \frac{m-k}{2}(\delta - \delta^2) + \frac{k+1}{2}c \\ \iff & l > \frac{m-k}{2}\frac{\delta - \delta^2}{c} + \frac{k+1}{2} \end{aligned} \tag{6}$$

and similarly, from Eq.(5) I deduce that

$$l < \frac{m+k+1}{2}c - \frac{k}{2}\frac{\delta - \delta^2}{c}. \tag{7}$$

However, since $0 < c < \delta - \delta^2$, $\frac{1}{c} > \frac{1}{\delta-\delta^2}$ and $-\frac{1}{c} < -\frac{1}{\delta-\delta^2}$. Then Eq.(6) implies

$$\begin{aligned} l &> \frac{m-k}{2}\frac{\delta-\delta^2}{c} + \frac{k+1}{2} \\ &> \frac{m-k}{2} + \frac{k+1}{2} \\ &= \frac{m+1}{2} \end{aligned} \tag{8}$$

and Eq.(7) implies

$$\begin{aligned} l &< \frac{m+k+1}{2}c - \frac{k}{2}\frac{\delta-\delta^2}{c} \\ &< \frac{m+k+1}{2} - \frac{k}{2} \\ &= \frac{m+1}{2}. \end{aligned} \tag{9}$$

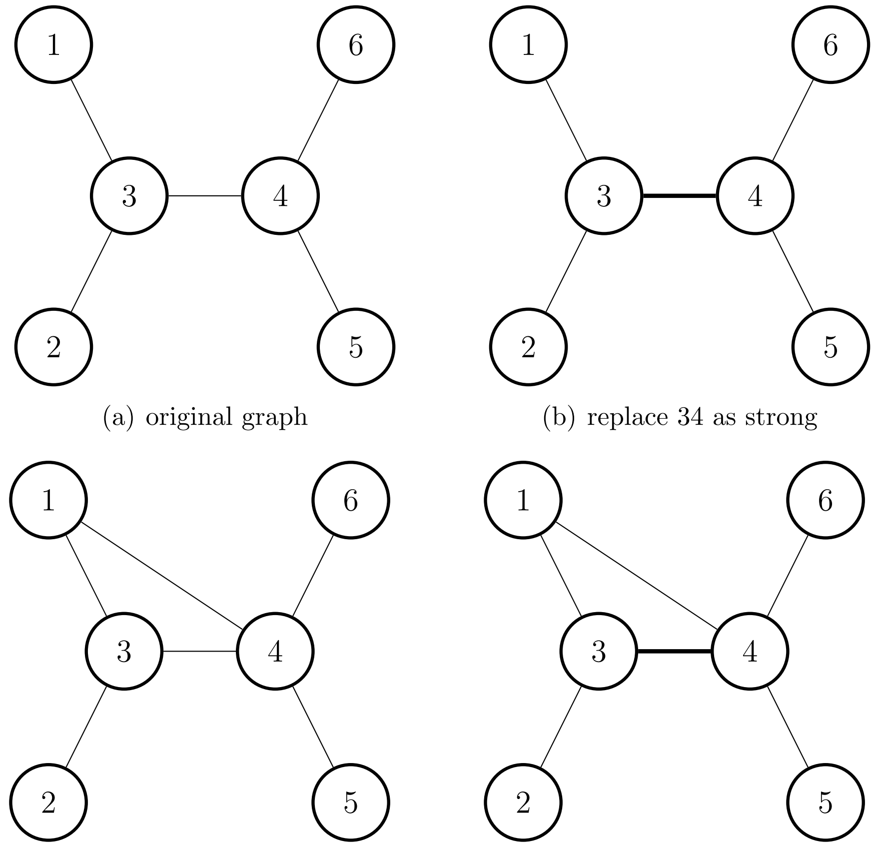
Since it is impossible that (8) and (9) hold at the same time, there is no subgraph H can be more efficient than its boarder cases (stars with strong edges and complete graphs with weak edges) simultaneously. This says that in any efficient graph G there cannot be $i, j \in V(G)$ such that $d_G(i,j) = 2$ and (W.O.L.G) ik is weak and kj is strong, where k is the internal vertex of this i,j-path even when $\frac{\delta-\delta^2}{2} < c < \delta - \delta^2$. Then the proof for the case $0 < c < \frac{\delta-\delta^2}{2}$ still apply here, which completes the proof. □

Proposition 2. *If $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$, the unique efficient graph of the modified model is a star with strong edges if $l \leq \frac{(n-2)(\delta-\delta^2)}{2c} + 1$ and a star of weak edges if $l > \frac{(n-2)(\delta-\delta^2)}{2c} + 1$.*

Proof. First, it is easy to see that (d)in any connected graph G with $e = e(w, (i,j)), i, j \in V(G)$, the maximum increase in benefit (not net gains) by replacing ij with a strong one is $(n-2)(\delta-\delta^2)$, which occurs only when $(N(i) \cap N(j)) \setminus \{i,j\} = \emptyset$ and $|N(i)\setminus\{j\}| + |N(j)\setminus\{i\}| = n-2$.

Second, I show that (e)the increase of benefits in any connected graph G with $|V(G)| = n$ by replacing weak edges with strong ones is at most $(n-1)(n-2)(\delta-\delta^2)$. This is proved by mathematical induction for $n \geq 3$.

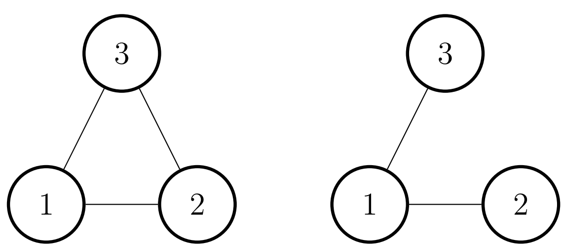
Base Step There are two non-isomorphic connected graphs when $n = 3$: the complete graph K_3 and the star. Replacing any weak edge in K_3 cannot increase benefits because all three vertexes are pairwise adjacent. Replacing weak edges in a star, however, can generate an increase in benefits of $(3-2)(3-1)(\delta-\delta^2) = 2(\delta-\delta^2)$, which happens when both edges in this star are weak. Therefore, the base case is true.



(a) original graph (b) replace 34 as strong

(c) original graph (d) replace 34 as strong

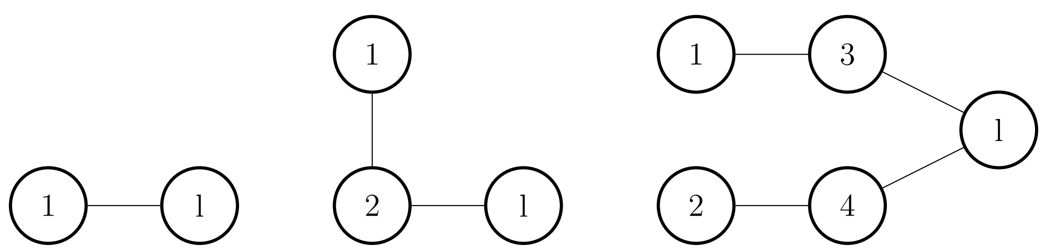
Figure 5. Maximize Increase in Benefits by Replacing a Weak Edge
The transition from (a) to (b) generates $4(\delta - \delta^2)$ increase in benefits. By comparison, the transition from (3) to (4) generates only $3(\delta - \delta^2)$, for 3 and 4 share 1 as their common neighbour.



(a) K_3 (b) star with three vertexes

Figure 6. Two Non - Isomorphic Connected Graph When $n = 3$

Induction Step Suppose the statement is true for all connected graphs G with $|V(G)| = k(k \geq 3)$. Now for any connected graph G' with $|V(G')| = k+1$, W.L.O.G, let $V(G') = V(G) \cup \{l\}$ and $E(G') = E(G) \cup \{il | i \in N(l)\}$. By the induction hypothesis, the increase in benefits by replacing weak edges within G is at most $(k-1)(k-2)(\delta - \delta^2)$, plus the increase that comes from $i \in V(G)$ and l. Since G' is connected, $\forall i \in V(G)$, there is a i,l-path. If i, l are adjacent and il is weak, then replacing this edge with a strong one does not generate any increase. If i, l are not adjacent, then the benefits that i gets from l by replacing edges is at most $\delta - \delta^2$. This, however, requires internal vertices that is adjacent to l. For i such that $d_{G'}(i,l) = 2$, making it adjacent to l will lead to a loss in potential increase of at least $\delta - \delta^2$. Thus, it is optimal to have as few as possible vertices adjacent to l. The minimum number is one. Let this internal vertex be j and the maximum increase comes when $d_{G'}(i,l) = 2$, ij is weak for $\forall i \neq j, i, j \in V(g)$. Since $|V(G)| = k$, the possible increase is at most $(k-1)(\delta - \delta^2)$. Now, if $i \in N(l)$ and il is weak, replacing this edge with a strong one cannot increase the benefit that l gets from i. If i, l are not adjacent, the increase that l gets from i is at most $\delta - \delta^2$. The best case scenario is again there is only one internal vertex and all the other vertexes can be reached from l by a distance of two. This leads to a possible increase of $(k-1)(\delta - \delta^2)$. Since $(k-1)(k-2)(\delta - \delta^2) + 2(k-1)(\delta - \delta^2) = k(k-1)(\delta - \delta^2)$, the statement is true when $n = k+1$. Therefore, the statement is true for all $n \geq 3$.



(a) direct connection (b) indirect connection (c) inefficient connection

Figure 7. Maximizing Benefits in the Induction Step

Now I am ready to prove this proposition. For an arbitrary connected graph G, let G_{weak} be the companion graph such that it has the same topological structure as G but all of its edges are weak. Furthermore, $u(G) = u(G_{weak}) + \triangle_G$, where $\triangle_G$ is the changes in the values obtained by replacing weak edges to strong ones in G. Suppose strong edges are indexed and the number of strong edges in G is $s, 1 \leq s \leq \frac{n(n-1)}{2}$. Then $\triangle_G = \sum_i^s \triangle_i$, where $\triangle_i$ is the changes in value by replacing the ith weak edge to a strong one. Notice that $\triangle_i$ might not be unique, but $\triangle_G$ is invariant to indexing, for otherwise same graph will have multiple values, which is impossible.

According to (d),replacing a weak edge with a strong one costs $2(l-1)c$ more, but can increase benefits by at most $(n-2)(\delta-\delta^2)$. In fact, replacing a weak edge in a star with a strong one does just that. If $2(l-1)c > (n-2)(\delta-\delta^2)$ or

$$l > 1 + \frac{(n-2)(\delta-\delta^2)}{2c},$$

then it is not beneficial to replace a weak edge with a strong one in a star, let alone any other G_{weak}. Thus,

$$u(G) = u(G_{weak}) + \triangle_G < u(G_{weak}) \leq u(star_{weak})$$

by (b).

If

$$1 \leq l \leq 1 + \frac{(n-2)(\delta-\delta^2)}{2c},$$

then it is beneficial to replace a weak edge in a star with a strong one. However, since the marginal value changed by taking this action is constant, it is optimal that all weak edges in a star are replaced by strong ones. Now, consider a graph G with $s \leq n-1$. Replacing a weak edge in this graph with a strong one costs the same as in a star with weak edges, but at best achieves the same amount ($(n-2)(\delta-\delta^2)$) in increase in benefits. This implies

$$\begin{aligned} u(G) &= u(G_{weak}) + \triangle_G \\ &= u(G_{weak}) + \sum_1^s \triangle_i \\ &\leq u(star_{weak}) + (s + (n-1) - s)((n-2)(\delta-\delta^2) - 2(l-1)c) \\ &= u(star_{strong}) \end{aligned}$$

by (b). If $s \geq n$, notice that if we start with a star of weak edges, after replacing $n-1$ weak edges with strong ones, the increase in benefits is exactly $(n-1)(n-2)(\delta-\delta^2)$, which is the maximum achievable increase in benefits of such operation by (e). Thus, if a graph uses more than $n-1$ strong edges, either the total benefits continues to increase after replacing the nth strong edge or it stays the maximum value indicated by (e). In either case, since a star with weak edges can achieve this maximum by replacing only $n-1$ edges, it is clear that using more than $n-1$ strong edges is not efficient and we must come to the conclusion that $u(G) \leq u(star_{strong})$ for any such G. The uniqueness comes from the fact that star is the only tree such that the maximum path length is two, which completes the proof.

To summarize, the unique efficient graph is

$$\begin{cases} \text{a star with strong edges} & \text{if } 1 \le l \le \frac{(n-2)(\delta-\delta^2)}{2c} + 1 \\ \text{a star with weak edges} & \text{if } l > \frac{(n-2)(\delta-\delta^2)}{2c} + 1. \end{cases}$$

. □

Proposition 3. *If $\delta + \frac{(n-2)\delta^2}{2} \le c \le \frac{n}{2}\delta$, the unique efficient graph is a star of strong edges if $1 \le l \le \frac{n\delta}{2c}$ and an empty graph if $l > \frac{n\delta}{2c}$. When $c > \frac{n}{2}\delta$, the unique efficient graph is the empty graph.*

Proof. Notice that adding a weak edge costs $2c$, gives direct benefits 2δ and indirect benefits at most $(n-2)\delta^2$, which only occurs when $(N(i) \cap N(j)) \setminus \{i,j\} = \emptyset$ and $|N(i) \setminus \{j\}| + |N(j) \setminus \{i\}| = n-2$. Thus, any graph G with $V(G) = n$ that uses weak edge is not efficient.

If $1 \le l \le \frac{n\delta}{2c}$,suppose G uses k edges, $k \ge n-1$, for $n-1$ is the minimum number of edges required for a graph to be connected. Consequently, the direct benefits is $2k\delta$ and the total cost is $2klc$. This leaves $\binom{n}{2} - k = \frac{n(n-1)}{2} - k$ indirect connections and the benefits derived from such connections is at most $(n(n-1) - 2k)\delta$. Thus,

$$u(G) \le k(2\delta - 2lc) + (n(n-1) - 2k)\delta. \tag{10}$$

Meanwhile,

$$u(star_{strong}) = (n-1)(2\delta - 2lc) + (n-1)(n-2)\delta. \tag{11}$$

Therefore, subtracting Eq.(10) from Eq.(11) yields

$$\begin{aligned} u(star_{strong}) - u(G) &= (n-1)(2\delta - 2lc) + (n-1)(n-2)\delta - u(G) \\ &\ge (n-1)(2\delta - 2lc) + (n-1)(n-2)\delta - k(2\delta - 2lc) - (n(n-1) - 2k)\delta \\ &= 2klc - 2(n-1)lc \\ &\ge 0. \end{aligned}$$

which implies that star is efficient. To show uniqueness, first notice that the above lower bound on difference in values is zero if and only if $k = n-1$, which implies G is a tree. A tree has $\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$ indirect connections. To achieve $(n-1)(n-2)\delta$ as indirect benefits, all of these indirect connections must be of length two. It turns out that star is the only tree such that the maximum path length is two.

Adding a strong edge costs $2lc$, gives direct benefits 2δ and indirect benefits at most $(n-2)\delta$. If $2lc > n\delta$ or $l > \frac{n\delta}{2c}$, it is not beneficial to use any strong edge whatsoever.

Similarly, if $c > \frac{n}{2}\delta$, then

$$\frac{n\delta}{2c} < \frac{n\delta}{2\frac{n\delta}{2c}} = 1,$$

suggesting no $l \geq 1$ could make a non-empty efficient graph, which completes the proof. □

5. Pairwise Stability

5.1. Goals

If the allocation rule is $A = u$, the pairwise stability is said to be *without side payments* in the original paper. In the following discussion, I first use this notion of stability and present an example that shows stable networks can be inefficient. Then I characterize the stability of efficient graphs with strong edges. It is interesting to see that they are not stable for any range of parameter values. I also investigate the question of whether there exists allocation rules other than u that can render efficient graphs with strong edges pairwise stable.

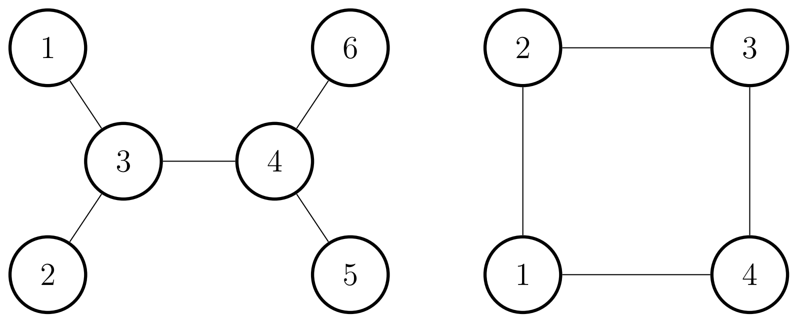
It is worth mentioning the pairwise stability considered here is only one of the many possible definitions and it is amenable to refinements[5]. It should also be noted that network pairwise stability is a relatively weak notion of stability of networks, for it does not depend on any particular network formation procedure and it focuses on equilibrium outcome of a network if the relationship between a pair of vertices is changed, while holding rest of this network constant.

5.2. Examples

There certainly exist stable networks that are not efficient.

Let's first consider Figure 8.(a). We know from the characterizations of efficient graphs that this network is not efficient. However, this network is stable. Vertices $1, 2, 5, 6$ maintain one weak edge, have one direct connection, two indirect connections of length two, and two indirect connections of length three. The individual utility value is therefore $0.5+2\times0.5^2+2\times0.5^3-0.5 = 0.75$. If the weak edge is deleted, these vertices get zero. If this edge is replaced by a strong one, the individual utility value becomes $0.5 + 2 \times 0.5 + 2 \times 0.25 - 10 \times 0.5 = -3$. If any pair of these vertices form weak edges among themselves, the individual utility value does not change. If they form strong edges among themselves, the individual utility values decrease. Thus, these vertices do not have incentive to deviate from the current

[5]See section 5 of the original paper.



(a) a double star, $c = 0.5, \delta = 0.5, l = 10$

(b) a cycle, $c = 0.35, \delta = 0.5, l = 1.5$

Figure 8. Stable Networks That are Not Efficient

configuration. Now consider vertices 3 and 4. They are indifferent between maintaining or deleting edges with the peripheral vertices. If such edges are replaced by strong ones, the individual utility values of 3 and 4 decrease because they get the same benefits but strong edges cost more. If $e = e(w, (3, 4))$ gets replaced by a strong one, the individual utility value changes by $2 \times (0.5 - 0.25) - 4.5 = -4$ and if it gets deleted, the individual value changes by $-(0.5 + 2 \times 0.25) + 0.5 = -0.5$. Furthermore, 3 and 4 are indifferent between maintaining the current structure and forming weak edges with peripheral vertices that are not directly connected to them and if they form strong edges, the individual value decreases. Therefore, I conclude that this graph is stable.

Now let's consider Figure 8.(b). This graph is clearly not efficient. W.L.O.G, I test whether vertex 1 has incentive to deviate from the current structure. The individual value of 1 is $2 \times 0.5 + 0.25 - 2 \times 0.35 = 0.55$. If 14 gets deleted, its individual value becomes $0.5 + 0.25 + 0.125 - 0.35 = 0.525$. If 14 is replaced by a strong one, its individual value becomes $3 \times 0.5 - 0.5 - 2 \times 0.5 = 0$. If $e = e(w, (1, 3))$ is formed, its individual value is $3 \times 0.5 - 3 \times 0.35 = 0.45$ and is less if 13 is strong. Therefore, no pair of vertexes have incentive to deviate and we conclude that this graph is pairwise stable.

5.3. Characterization of Stable Graphs that are Efficient

Jackson and Wolinsky (1996) [6] characterizes stable networks in the generic connection model, including the stability of efficient graphs with weak edges. This section, however, instead of characterizing all the stable graphs in the modified model, focuses only on the stability of stars with strong edges. Up to this point, I can only posit that *graphs that use both types of edges are unstable.* This is an interesting conjecture because it might help

[6] See proposition 2, section 3.1.2

understand how inequality generates instability within society. I will update this proposition if I am lucky enough to be able to prove it in the near future.

There is a reason why I want to emphasize the stability of efficient graphs, however. Surely, efficiency should always be on the top of the list of priorities for people who have the power to design social or institutional structures. At the same time, it is imperative that feasibility should also be examined fully before any plan is implemented. In the current context, we desire an efficient structure that can sustain its value production without side payment, i.e., an equilibrium under the default allocation rule. Therefore, such analysis is necessary and important, especially in the case where there is no evaluation metric available that incorporates these two goals simultaneously.

The following proposition states that some efficient graphs can never be stable.

Proposition 4. *When $l > 1$, stars with strong edges are not pairwise stable. Furthermore, efficient graph that use strong edges cannot be stable.*

Proof. Consider the central vertex within a star with strong edges. W.L.O.G, let this vertex be i and let j be one of the peripheral vertices. i gets δ from j and spends lc on ij. If this edge is replaced by a weak one, i gets the same payoff from j but only has to pay c. Thus, $u_i(G) < u_i((G - e) + e')$, where $e' = e(w, (i, j)), e = e(s, (i, j))$ and we conclude that stars with strong edges are not pairwise stable. We know from the characterization of efficient graphs that the only class of efficient graphs that use strong edges are stars with strong edges. □

5.4. Existence of Stabilizing Allocation Rules

We see from proposition 4 that stars with strong edges are not stable under the default allocation rule. Since the value of a graph is solely determined by its structure, one question that naturally occurs is therefore how such networks can be made stable by allowing different allocation rules. Furthermore, I want to investigate whether pairwise stability can be achieved without outside transfers, i.e., the allocation rule achieves stability only by redistributing the value generated within the graph itself. This restricted version of allocation rule has important implications in the real world situations because it differentiates economic systems that can remedy themselves and those can't. In time of crisis, if the major players (i.e., government) can successfully identify those institutions (banks, companies, etc.) that are capable of self-remedy, it might just take a internal rule change (personnel change, etc.) and very little resources to keep them from liquidation. Consequently, limited resources could be directed to helping out entities that are intrinsically unable to stabilize themselves.

To formally formulate this question in terms of the symmetric setting of the modified connection model, I need to introduce the following restrictions on allocation rule.

Definition An allocation rule A is *anonymous* if for any two isomorphic graphs G and G' with isomporphism $f : V(G) \to V(G')$, we have $A_i(G) = A_{f(i)}(G')$ under all possible u.

Definition An allocation rule A is *balanced* if $\sum_i A_i(G) = u(G)$ for all possible G and u.

Intuitively, anonymity states that players in the same position within a network should be paid the same regardless of their names. This property, of course, is required by the symmetry setting of the model, but in some sense it introduces equality into the allocation rule. In the following discussion, I will denote vertices of degree one in a star as *peripherals* and the vertex of degree $n-1$ as *center*[7]. Under an anonymous allocation rule, all peripherals are distributed exactly the same value, which suggests that if they were to be taxed (which they will), each vertex is going to be taxed the same amount. On the other hand, under a balanced allocation rule one can only make transfers within a graph and there is no value exchanged with outside environment, which mimics self-sufficiency of an economic system.

Theorem 1 of the original paper states that if $n \geq 3$, there there is no A that is anonymous and component balanced (which is a weaker version of A being balanced) such that for each u there exists a efficient graph that is pairwise stable. It cannot apply to the current settings because the original model uses only one type of link, but it suggests that it is necessary to contain our attention when searching for such rules. Specifically, I relax the requirement that such rules must hold for all possible graphs and value functions. The question is therefore to investigate whether stars with strong edges can be made stable by allowing anonymous and balanced allocation rules with the relaxation on only such graphs.

The next three propositions characterize the existence of such rules. In the following discussion, I assume that allocation rules are anonymous and balanced. Furthermore, such rules are restricted on stars with strong edges and companion value function is the default one.

Proposition 5. *There exists stabilizing allocation rules with restrictions when* $\delta + \frac{(n-2)\delta^2}{2} \leq c \leq \frac{n}{2}\delta$ *and* $1 \leq l \leq \frac{n\delta}{2c}$.

Proof. In this case, we know from proposition 3 that stars with strong edges are uniquely efficient. Notice that any pair of peripherals i and j within a star with strong edges do not

[7] In graph theory, the *center* of a graph G is the set of vertices $i \in V(G) | minmax_{i \neq j}\{d_G(i,j)\} = diam(G)$. In the case of a star, these two terms coincide.

have incentive to add an edge between them, regardless of the type because adding ij cannot generate any increase in benefit but inflicts a cost of at least c on both i and j. This is true even when value is diverted from or added to the peripherals by some allocation rule A. Now consider the center-peripheral pair k and i. Since $c = \delta + \frac{(n-2)\delta^2}{2}$, the center gets negative value maintaining ki, regardless of types. Thus, if the center does not get any positive transfer from the peripheral, it will strictly prefer deleting this edge and getting a value of zero. For peripherals, since the individual value of maintaining a weak edge is negative, the choice is always between maintaining a strong edge and deleting this edge, even with the presence of a different allocation rule A.

The peripheral gets value $(n-1)\delta - lc$ by maintaining a strong edge. Since $1 \leq l \leq \frac{n\delta}{2c}$, $(n-1) - lc > (n-1)\delta - \frac{n\delta}{2c}c = \frac{n-2}{2}\delta > 0$. Let t be the amount of value that is transferred from the peripheral to the center. The maximum of such transfer is $(n-1)\delta - lc$ if we want the peripheral not to deviate. The center now gets $\delta + (n-1)\delta - 2lc$, which is positive. This says it is possible make the center prefer the strong edge rather than deleting this edge. This completes the proof. □

Proposition 6. *There might or might not exist stabilizing allocation rules with restrictions when* $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$ *and* $l \leq \frac{(n-2)(\delta-\delta^2)}{2c} + 1$.

Proof. In this case, we know from proposition 2 that stars with strong edges are uniquely efficient. Similar to the previous analysis, no pair of peripherals have incentive to add an edge between them, regardless of the type. In terms of the center-peripheral pair, first we need to figure out the maximum taxable value of the peripheral. The current individual value is $(n-1)\delta - lc$. If it instead kept a weak edge, its individual value would be $\delta + (n-2)\delta^2 - c$. Since $\delta - \delta^2 < c < \delta + (n-2)\delta^2$ and $l \leq \frac{(n-2)(\delta-\delta^2)}{2c} + 1$,

$$\begin{aligned}
(n-1)\delta - lc - (\delta + (n-2)\delta^2 - c) &= (n-2)(\delta - \delta^2) - (l-1)c \\
&\geq (n-2)(\delta - \delta^2) - \frac{(n-2)(\delta - \delta^2)}{2} \\
&= \frac{(n-2)(\delta - \delta^2)}{2} \\
&> 0,
\end{aligned}$$

which implies the maximum taxable value from the peripherals is $(n-2)(\delta - \delta^2) - (l-1)c$ if we want them not to deviate from keeping strong edges.

Now consider the center. First notice that

$$\delta - lc + (n-2)(\delta - \delta^2) - (l-1)c - \delta + c = (n-2)(\delta - \delta^2) - 2(l-1)c \geq 0 \tag{12}$$

which implies keeping a strong edge with the peripheral and receiving maximum transfer is always better than maintaining a weak edge with it. However,

$$\begin{aligned} \delta - lc + (n-2)(\delta - \delta^2) - (l-1)c &= \delta + (n-2)(\delta - \delta^2) - 2lc + c \\ &\geq \delta + (n-2)(\delta - \delta^2) - 2c(1 + \frac{(n-2)(\delta - \delta^2)}{2c}) \\ &= \delta - c. \end{aligned} \tag{13}$$

Since it is possible that $\delta < c$, this inequality might have a negative lower bound, which can be reached when $l = 1 + \frac{(n-2)(\delta-\delta^2)}{2c}$. This implies that even though keeping a strong edge and receive transfers is better than keeping a weak edge with no transfer, it might be worse than not having any connection whatsoever. This completes the proof. □

Proposition 7. *There exists stabilizing allocation rules with restrictions when* $0 < c < \delta - \delta^2$ *and* $1 \leq l \leq \frac{n}{2}$.

Proof. We know from proposition 1 that stars with strong edges are uniquely efficient. Similar to the previous analysis, no pair of peripherals have incentive to add an edge between them, regardless of the type. Also, the maximum taxable value from a peripheral without leading to deviation from maintaining a strong edge is still $(n-1)\delta - lc - (\delta + (n-2)\delta^2 - c)$ and is positive. In terms of the center - peripheral pair, notice that since $c < \delta - \delta^2 < \delta$, the lower bound (13) is now positive. (12) and (13) combined indicate that keeping a strong edge with peripherals and receive transfers is now always better than deleting this edge or deviating to a weak edge. This completes the proof. □

5.5. A Brief Discussion on Pairwise Stability

The proof of proposition 7 might be counterintuitive. Wouldn't peripherals deviate from keeping strong edges with the center and add weak edges among themselves if they anticipate their taxes exceed certain amount? In fact, they won't. This is an example illustrating that why pairwise stability is considered to be a relatively weak concept among many notions of stability. The reason is that pairwise stability only concerns with two players and their relationship at a time, while keeping the rest of the network constant. Thus, if we are looking at the pairs from peripherals, since they all have strong edges connecting to the center, adding either type of edge between them will not increase benefits they get from each other but will increase individual cost. Similarly, when considering how transfers between peripherals and the center could affect their relationships, we hold the relationship between pairs from peripherals constant (which is no edge connecting them). One might want to refer to Bala and Goyal (2000)[8] for a comparison in link formation. In the repeated link formation game described in this paper, when it is some player's turn to form a link, he or she gets to observe the whole network and choose the best option (forming link with the most connected player). One advantage of pairwise stability, however, is that its simplicity allows us to study the evolution of networks sequentially [9] (Jackson (2003); Jackson and Watts (2002)).

6. Conclusion

We see from the characterizations of efficient networks that stars with strong edges play a important role in achieving efficiency. Intuitively, if the relative cost l is low, people will be able to exploit the "neighbour sharing property" of strong edges and save from maintaining direct connections. However, the exact mechanism differs across different range of c. When maintaining weak connections is cheap ($0 < c < \delta - \delta^2$), the efficiency of star with strong edges comes solely from reduction of edges due to neighbourhood sharing. If the cost of maintaining weak connections is moderate ($\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$), stars with strong edges are efficient because the attenuation of mutual benefits is avoided, thanks to the neighbourhood sharing property of strong edges. When maintaining weak connections is costly ($\delta + \frac{n-2}{2}\delta^2 < c < \frac{n}{2}\delta$), star of strong edges can still achieve a positive value because the benefits brought by neighbourhood sharing exceed the cost. Of course, if the relative cost l is high or the cost of maintaining weak connections is too costly, the benefit of using strong edges vanishes and the efficiency result coincide with that of only using weak edges.

Stability is another desired goal when we design networks. However, we observe that it is not always compatible with efficiency. Furthermore, stars with strong edges are the least stable in the sense that they are never pairwise stable under the default value function as allocation rule, yet they are the most resilient ones in terms of achieving efficiency across a wide range of parameters values. This dilemma might be resolved by employing anonymous and balanced allocation rules with restrictions. One policy implication is that if self-stabilizing

[8]The equilibrium structures (cycles for one-way flow case and either stars or the empty graph for two-way flow case) are stable in the sense that they are the unique limiting structures of dynamic link formation game staring from any network.

[9]See the section discussing *improving paths*.

capability of institutions can be identified, outside resources should be prioritized for those which are not capable of self - remedy. The stability of general networks with both strong and weak edges, however, is only conjectured and it needs future research.

References

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

D'Angelo, E. and Lilla, M. (2011). Social networking and inequality: The role of clustered networks. *Cambridge Journal of Regions, Economy and Society*, 4(1):63–77.

Gee, L. K., Jones, J., and Burke, M. (2017). Social networks and labor markets: How strong ties relate to job finding on facebook's social network. *Journal of Labor Economics*, 35(2):485–518.

Granovetter, M. S. (1973). The strength of weak ties. *American Journal of Sociology*, 78(6):1360–1380.

Ioannides, Y. (2016). Endogenous social networks and inequality in an intergenerational setting. *Working Paper*.

Jackson, M. O. (2003). *The Stability and Efficiency of Economic and Social Networks*. Studies in Economic Design. Heidelberg and New York: Springer.

Jackson, M. O. and Watts, A. (2002). The evolution of social and economic networks. *Journal of Economic Theory*, 106(2):265–295.

Jackson, M. O. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74.

Kramarz, F. and Skans, O. N. (2014). When strong ties are strong: Networks and youth labour market entry. *Review of Economic Studies*, 81(3):1164–1200.

West, D. B. (2001). *Introduction to graph theory*. Prentice Hall, Upper Saddle River, NJ, 2nd edition.