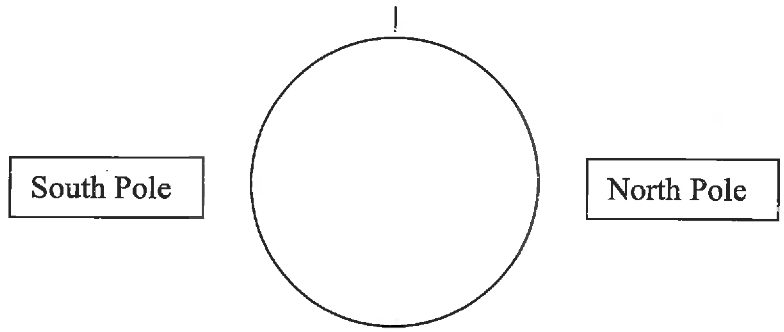
Newton's Proposed Proof of the Rotation of the Earth



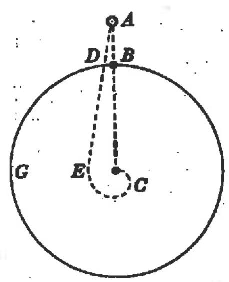
If the Earth rotates, then the translational velocity at the top of the tower is greater than the translational velocity at the bottom by an amount Δr times the angular speed of the Earth.

But then, an object dropped from the top of the tower, instead of landing to the west at the bottom (as the defenders of the motionless Earth would have it) or at the base of the tower (as Galileo and Gassendi would have it), must land to the east of the tower, by an amount equal to $\omega\Delta r$ times the time of descent.

Indeed, if the falling object is heavy enough to minimize the effects of air resistance, the displacement to the east can yield a value for the rotational speed of the Earth that must agree with its known value (15° per hour).

(The phenomenon became known in the 19th century as an instance of Coriolis forces.)

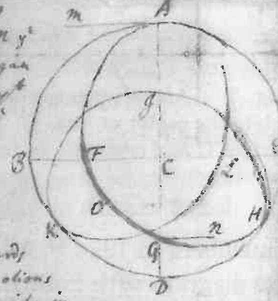
London. I am glad to heare that so considerable a discovery as you made of ye earth's annual parallax is seconded by Mr Flamstead's Observations. In requital of this advertisement I shall communicate to you a fansy of my own about discovering the earth's diurnal motion. In order thereto I will consider ye Earth's diurnal motion alone without ye annual, that having little influence on ye experimt I shall here propound. Suppose then *BDG* represents the Globe of ye Earth carried round once a day about its center *C* from west to east according to ye order of ye letters *BDG*; & let *A* be a heavy body suspended in the Air & moving round with the earth so as perpetually to hang over ye same point thereof *B*. Then imagin this body *B* let fall & it's gravity will give it a new motion towards ye center of ye Earth without diminishing ye old one from west to east. Whence the motion of this body from west to east, by reason that before it fell it was more distant from ye center of ye earth then the parts of ye earth at wch it arrives in its fall, will be greater then the motion from west to east of ye parts of ye earth at wch ye body arrives in it's fall: & therefore it will not descend in ye perpendicular *AC*, but outrunning ye parts of ye earth will shoot forward to ye east side of the perpendicular describing[5] in it's fall a spiral line *ADEC*, quite contrary to ye opinion of ye vulgar who think that if ye earth moved, heavy bodies in falling would be outrun by its parts & fall on the west side of ye perpendicular. The advance of ye body from ye perpendicular east-



ward will in a descent of but 20 or 30 yards be very small & yet I am apt to think it may be enough to determin the matter of fact. Suppose then in a very calm day a Pistol Bullet were let down by a silk line from the top of a high Building or Well, the line going through a small hole made in a plate of Brass or Tinn fastened to ye top of ye Building or Well & yt ye bullet when let down almost to ye bottom were setled in water so as to cease from swinging & then let down further on an edge of steel lying north & south to try if ye bullet in setling thereon will almost stand *in æquilibrio* but yet with some small propensity (the smaller ye better) decline to ye west side of ye steel as often as it is so let down thereon. The steel being so placed underneath, suppose the bullet be then drawn up to ye top & let fall by cutting clipping or burning the line of silk, & if it fall constantly on ye east side of ye steel it will argue ye diurnall motion of ye earth. But what ye event will be I know not having never attempted to try it. If any body may think this worth their triall the best way in my opinion would be to try it in a high church or wide steeple the windows being first well stopt. For in a narrow well ye bullet possibly may be apt to receive a ply[6] from ye straitned Air neare ye sides of ye Well, if in its fall it come nearer to one side then to another. It would be convenient also that ye water into wch ye bullet falls be a yard or two deep or more partly that ye bullet may fall more gently on ye steel, partly that ye motion wch it has from west to east at its entring into ye water by meanes of ye longer time of descent through ye water, carry it on further eastward & so make ye experiment more manifest.

Sir

I agree wth you yt ye body in our latitude will fall more to ye south then east if ye height it falls from be any thing great. And also that if its gravity be supposed uniform it will not descend in a spiral to ye very center but circulate wth an alternate ascent & descent made by it's vis centrifuga & gravity alternately overballancing one another. Yet I imagin ye body will not describe an Ellipsoeid but rather such a figure as is represented by AFOGHIKL &c. Suppose A ye body, C ye center of ye earth ABDE quartered wth perpendicular diameters AD, BE, wch cut ye said curve in F & G; AM ye tangt in wch ye body moved before it began to fall & GN a line drawn parallel to yt tangt. When ye body descending through ye earth (supposed pervious) arrives at G, the determination of its motion shall not be towards N but towards ye coast between N & D. For ye motion of ye body at G is compounded of ye motion it had at A towards M & of all ye innumerable converging motions successively generated by ye impresses of gravity in every moment of its passage from A to G: The motion from A to M being in a parallel to GN inclines not ye body to verge from ye line GN. The innumerable & infinitly little motions (for I here consider motion according to ye method of indivisibles) continually generated by gravity in its passage from A to F incline it to verge from GN towards D, & ye like motions generated in its passage from F to G incline it to verge from GN towards C. But these motions are proportional to ye time they are generated in, & the time of passing from A to F (by reason of ye longer journey & slower motion) is greater then ye time of passing from F to G. And therefore ye motions generated in AF shall exceed those generated in FG & so make ye body verge from GN to some coast between N & D. The nearest approach therefore of ye body to ye center is not at G but somewhere between G & F as at O. And indeed the point O according to ye various proportions of gravity to the impetus of ye body at A towards M, may fall any where in ye angle BCD in a certain curve wch touches ye line BC at C & passes thence to D. Thus I conceive it would be if gravity were ye same at all distances from ye center. But if it be supposed greater nearer ye center ye point O may fall in ye line CD or in ye angle DCE or in other angles yt follow, or even no where. For the increase of gravity in ye descent may be supposed such yt ye body shall by an infinite number



Hooke's Challenge to Newton

... particularly if you would let me know your thoughts of that [hypothesis of mine] of compounding the celestial motions of the planets of a direct motion by the tangent and an attractive motion towards the central body.

24 November 1679

But as to the curve Line which you seem to suppose it to Descend by (though that was not then at all Discoursed of) Vizt a kind of spirall which after sume few revolutions Leave it in the Center of the Earth my theory of circular motion makes me suppose it would be very differing and nothing att all akin to a spiral but rather a kind Elleptueid....

9 December 1679

Your Calculation of the Curve by a body attracted by an aequall power at all Distances from the center Such as that of a ball Rouling in an inverted Concave Cone is right and the two auges will not unite by about a third of a Revolution. But my supposition is that the Attraction always is in a duplicate proportion to the Distance from the Center Reciprocall, and Consequently that the Velocity will be in a subduplicate proportion to the Attraction and Consequently as Kepler Supposes Reciprocall to the Distance. And that with Such an attraction the auges will unite in the same part of the Circle and that the nearest point of accesse to the center will be opposite to the furthest Distance. ... What I mentioned in my last concerning the Descent within ye body of the Earth was But upon the Supposal of such an attraction, not that I believe there really is an attraction to the very Center of the Earth, but on the Contrary I rather Conceive that the more the body approaches the Center, the lesse will it be Urged by the attraction.... But in the Celestiall Motions the Sun Earth or Centrall body are the cause of the Attraction, and though they cannot be supposed mathematicall points yet they may be Conceived as physicall and the attraction at a Considerable Distance may be computed according to the former proportion as from the very Center. This Curve truly Calculated will shew the error of those many lame shifts made use of by astronomers to approach the true motions of the planets with their tables.

6 January 1680

It now remaines to know the proprietys of a curve line (not circular not concentricall) made by a centrall attractive power which makes the velocitys of Descent from the tangent Line or equall straight motion at all Distances in a Duplicate proportion to the Distances Reciprocally taken. I doubt not but that by your excellent method you will easily find out what that Curve must be, and its proprietys, and suggest a physicall Reason of this proportion.

17 January 1680

Demonstration.

[2.][2] If $ef = fg = gh = he = 2fa = 2fb = 2gc = 2ed$. And the globe *b* move from *a* to *b* then $2fa : ab :: ab : fa ::$ force or pression of *b* upon *fg* at its reflecting : force of *b*'s motion. therefore $4ab = ab + bc + cd + da : fa ::$ force of the reflection in one round (viz: in *b*, *c*, *d*, and *a*) : force of

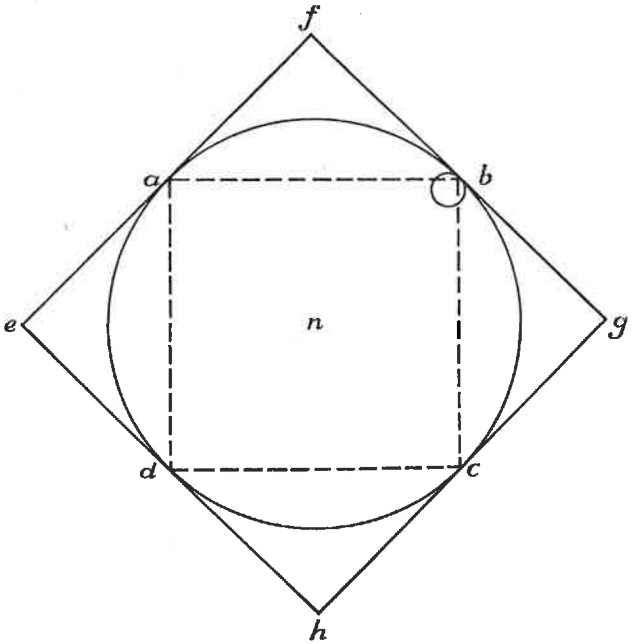


Figure 1.

b's motion. by the same proceeding if the Globe *b* were reflected by each side of a circumscribed polygon of 6, 8, 12, 100, 1000 sides etc. the force of all the reflections is to the force of the bodys motion as the sume of those sides to the radius of the circle about which they are circumscribed. And so if [the] body were reflected by the sides of an equilaterall circumscribed polygon of an infinite number of sides (i.e. by the circle it selfe) the force of all the reflections are to the force of the bodys motion as all those sides (*id est* the perimiter) to the radius.

[3.] If the body *b* moved in an Ellipsis[3] that its force in each point (if its motion in that point bee given) [will?] bee found by a tangent circle of Equall crookednesse with that point of the Ellipsis.

[4.] If a body undulate in the circle *bd* all its undulations of any altitude are performed in the same time with the same radius. Galileus.[4]

[5.] As radius *ab* to radius *ac* : : so are the squares of there times in which they undulate.[5]

[6.] If *c* circulate in the circle *cgef* [Fig. 2], to whose diameter *ce*, $ad = ab$ being perpendicular then will the body *b* undulate in the same time that *c* circulates.[6]

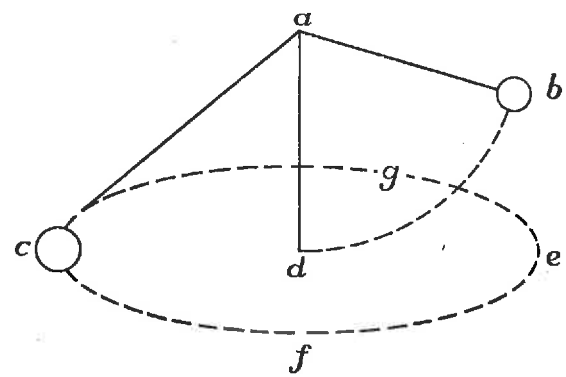


Figure 2.

[7.] And those bodies circulate in the same time whose lines drawne from the center *a* to the center *d* are equall.[7]

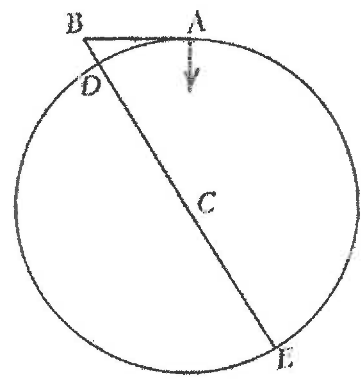
[8.] And $ad : dc ::$ force of gravity to the force of *c* from its center *d*.[8]

[9.] Coroll : hence may the force of gravity of the motion of things falling were they not hindered by the aire may very exactly [be] found[9] (viz. [?] $cd : ad ::$ force from *d* : force from *a*.

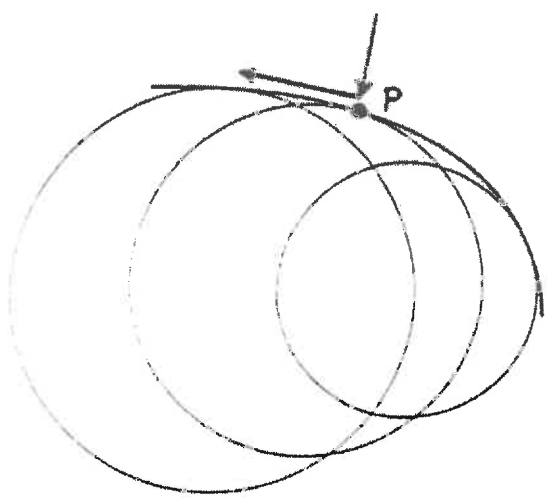
1. For an interpretation of this and the following subsection see above, Part I, Chapter 1.2, pp. 7–11. An equivalent result is derived by an entirely different method in MS. IVa. It seems probable that Newton used this result to derive the peculiar '$\frac{1}{2}R$' formula employed in the calculations of MS. III. See § 2 of the 'Commentary and Interpretation' to that manuscript.

2. This demonstration must have followed Newton's first estimate of the *force of the body's endeavour from the centre in half a revolution* given in Ax.-Prop. 22. Particularly interesting in this connexion is the cancellation of the figure 4+ in Ax.-Prop. 24 and its replacement by 6+ corresponding to the 2π of the present section. For a similar 'polygonal' treatment of circular motion see the demonstration of the law of centrifugal force at the end of the *Scholium* to Prop. 4, Theor. 4, Book, I, *Principia*. Ball ([1], p. 13) suggested that this latter demonstration was the one employed by Newton to calculate 'the force with which a ball revolving within a sphere presses the surface of the sphere' prior to his

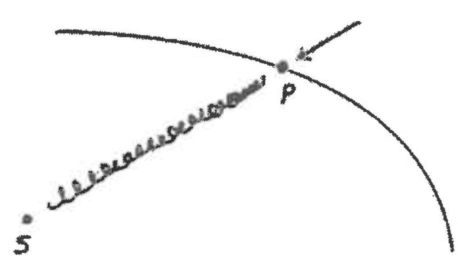
What Newton Gained from Hooke on Curvilinear Motion



The Natural Generalization



Centripetal Force



'The president [Sir Christopher Wren] acquainted the Society with the undertaking of Mr JOHN ADAMS to survey all England, by measuring, taking angles, and also the latitudes of places; and in order to this running three several meridians clear through England: that Mr NEWTON of Cambridge had promised to assist him; and that he designed the next week to wait on the Society, in order to desire their directions and assistance' (Birch, IV, 65; see also pp. 66, 67, 87).

(3) Lady Newton was Mary, daughter of Sir Gervase Eyre of Rempstone, Notts.

249 FLAMSTEED TO CROMPTON

12 FEBRUARY 1680/1

From an extract,(1) made by Newton, in the University Library, Cambridge.
For answer see Letter 251

The appearance of ye Head of ye Comet I could never better compare then to those obscurer spots in ye moon wch wee esteem ye aqueous part of it. Whence it may seem that ye exteriour coat of ye Comet may be composed of a liquid wch reflects but little light. It was never well defined nor shewed any perfect limb but like a wisp of hay. May not this intimate that the globe of wch 'twas once compounded was broke & ye humid part spread over the rest, yet so as here & there some little pieces of ye more solid parts exert themselves above it & reflect to us ye light of ye Sun shining on it? For I noted that at the first I saw some little points of light scattered here & there through the haze. The tayle of ye Comet was nearly but not exactly in a streight line being a little curved backward, towards ye west.

Thus far Mr Flamsteed in a letter from Greenwich Feb. 12. 1680/81.

NOTE

(1) Newton wrote this extract upon a blank sheet of Letter 251.

250 FLAMSTEED TO HALLEY

17 FEBRUARY 1680/1

Extract from the original(1) in the Bodleian Library, Oxford

The Observatory Feb: 17 1681
27 $\overline{0}$

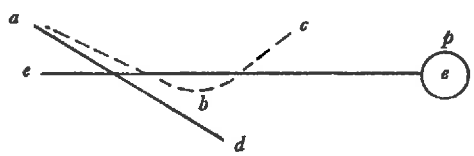
... ...

You tell me you have meditated upon Comets & come to a result yet desire my thoughts as to the Philosophicall part of them. if so you have resolved I doubt not but it is on such good grounds & consideration that my thoughts will be needlesse. might I not also on this intimation from you have expected yours first? You seeme too close but you shall not accuse me of that fault; I shall willingly answer your desires & reckon myselfe a gainer, for betwixt freinds the agreement of opinions confirm them, the difference helpes to correct the faults of either.

I must first thanke you for ye Account you sent me of Gallets[2] course observations at Rome: from them I draw my first argument. I conceave the Comet which appeared in November to be the same wee lately observed.[3] you may remember that I tould you before you went hence when I had onely heard of it that wee should see it againe when it had passed ye Sun. Since, you have seene that prediction verified but the reason I must acknowledge of its late appearance is much different from my conceptions at that time

It appeares by such observations as were made here before Gallets, tho as course, that the Comet had North latitude first, then peirced the plane of ye ecliptick twice[4] & so passed on towards ye Sun.

I conceave therefore that the Sun attracts all the planets and all like bodys that come within our Vortex, more or lesse according to the different substance of theire bodyes and nearenesse or remotenesse from him. that it drew the Comet by its northerne pole, the line of whose motion was at first really inclined from the North into the South part of the heavens but was by this attraction as



it drew nearer to him bent the Contrary way, as if *ee* were the plane of the Ecliptick, *p* ye North pole of ye Sun, *ad* ye line of its first inclination which by ye attraction of ye pole *p* is bent into ye curve *abc* in which Gallet observed it to move. The Comet was then to ye North of ye Solar Æquator & perhaps here the contrary motion of ye Vortex might helpe to beare it up from ye plaine of ye Ecliptick into ye Northerne latitude.

When it came within ye Compasse of ye *orbis annuus* this attraction of ye Sun would have drawne it neare him in a streight line, had not the laterall resistance of ye Matter of ye Vortex moved against it bent it into a Curve. as if in the figure on ye next Page [here p. 339] *αγ* were ye line in which ye motion of ye Comet were directed. this by ye attraction of ye Sun is bent into *β*, *B*. & when ye Comet comes to *B* it would be carried on to ye Sun in ye streight line *BC*, were it not for ye motion of the Vortex beareing it out of yt line from *e* to *g*. the body of ye Comet I conceave to have a'lwayes the same part carried foremost in ye line of its motion so yt when it comes to *C* it moves contrary & crosse ye motion of ye Vortex till haveing ye contrary End[5] opposite to ye Sun hee repells it as ye North pole of ye loadstone attracts ye one end of ye Magnetick needle but repells ye other.[6]

This act of repulsion would carry ye Comet from ye Sun in a streight line, were it not that the crosse motion of ye Vortex bendes it back. which yet the acquired velocity & strenght [*sic*] of its motion may compensate & restore. so that the one countervaileing the other I see no reason why wee may not admitte it to have run from Sun nearely in a Streight line

My theory of the motion differs very little from youres[7] but to tell you the truth I have made no triall by calculation yet but by a large draught on paper. my other affaires not permitting me to bestow so much time as is necessary for this businesse on it.

As for the body of the Comet nothing better occurs to my thoughts at present then that it may have beene some planet belonging formerly to another Vortex now ruined: for Worlds may die as well as men:[8] that its naturall motion being destroyed its body is broke & the humid parts swim over ye rest yet so as some small peeces of ye solid part of ye Masse here & there lie out above them. this its ill defined figure & dusky light persuades me: which In my opinion was not much different from yt of ye obscure large spots in the Moone which are accounted the aqueous part of it; onely the greater distance of the Comet caused its colour to appeare lesse bright but with here & there some very small pointes of light which might be reflected from ye prominent tops of ye broken parts of solid matter.

Onely on ye 12th of December when it was not above 11 degrees distant from ye Sun it looked like a very dul obscure star, its light far lesse & more dusk then that of ♄, which I attribute to its vicinity to the Sun whose rayes reflected from the unequall superficies of it might very well cause it to appeare more bright then, then ever it did after.

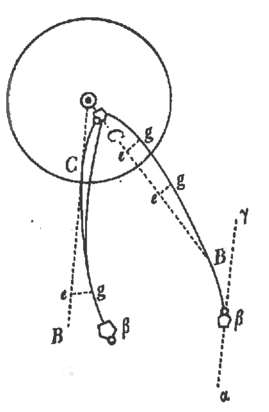
I have given you my opinion fully of it & I thinke answerable to your desires. I shall now expect yours at your leasure.

... ...

I did your Comands & presented your French observations[9] of the Comet to Sr Christopher Wren who is now president of ye Society.[10] I heare of others from Strasburge Dantzick & other places which you sent to Mr Colson.[11] I seldom see him. you may therefore doe well if they were Made in November to send them in your next to:

Sr Your most obliged & reall freind &c

JOHN FLAMSTEED



(as some have thought) *excursions in certain periods into our Systeme or Solar World.* I should rather think them to be made by ye *collection of Exhalations or Effluvia from ye Æther or Ætherial Bodies* & (as much as any) from ye solar. To which I find Mr Horrocks heretofore, & Dr Marsh of late, do incline allso. *The tail I take to be little more, if more, than the tinging of ye sunbeams passing through the Head.* As to what you say, of Worlds dying as well as men, (which to Aristotle, his followers, as to the Incorruptibility of or in the Cœlestial bodies, would be a great Paradox) it doth not seem so strange to me, since yt, divers of the stars noted by ye Ancients, are now wanting: And some others do, now, appear & disappear uncertainly. (which yet argues, yt *disappearing of a Star, is no certain argument that such Star is perished,* since sometimes it appears again). But I can hardly think such dissolutions so frequent as our sight of Comets. And yet, if they were much more frequent, it's yet a great chance *if any of them, or of their Satellites, do stray into our world*: which is so inconsiderable a part of ye vast Regions they have to make such excursions into; that perhaps one, of a thousand would be more than would come to our shore.'

(9) See Letter 252. Halley had visited Paris in December 1680 at the beginning of a few months' stay in France: see p. 349 and note (6), p. 356.

(10) He was president for two years since the previous November.

(11) Possibly Lancelot Colson (fl. 1668), a physician and astrologer, of London, who published almanacks. See *D.N.B.*

There was too Nathaniel Colson (fl. 1674) who produced the *Mariner's New Kalendar.*

(12) This further postscript, like the marginal note (7), was probably added later by Flamsteed.

251 NEWTON TO CROMPTON FOR FLAMSTEED

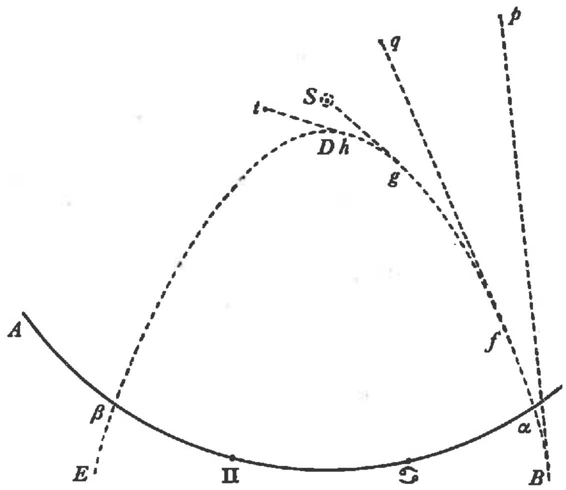
28 FEBRUARY 1680/1

From the original in the Bodleian Library.
In reply to Letters 242, 245, 249; for answer see Letter 252

For Mr Crompton to be sent to Mr Flamstead.

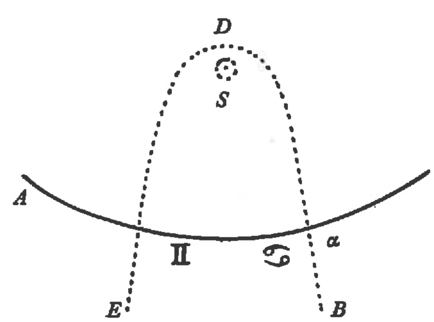
I thank Mr Flamstead for his kind mention of me in his letters(1) to Mr Crompton. And as I commend his wisdom in deferring to publish his hypothetical notions till they have been well considered both by his friends & himself, so I shall act ye part of a friend in this paper(2) not in objecting against it by way of opposition but in describing what I imagin might be objected by others & so leaving it to his consideration. If hereafter he shal please to publish his Theory & think any of ye objections I propound need an answer to prevent their being objected by others, he may describe ye objections as raised by himself or his friends in general wthout taking any notice of me.

First then whereas in ye annexed figure(3) he supposes that when ye earth is in ♊, the comet moved from *B* to *f* & so to *D* & then to *E*, so as to be in its perige twice, the first time at *α* when the earth was in ♊ the next time at *β* when ye Earth was in ♋ & in its perihelion at *D* to be between ye ⊙ & earth: draw *Bp*, *fq*, *gS*, *ht* tangents to ye line of ye Comets motion at *B*, *f*, *g* & *h*, & I can easily allow that ye attractive power of ye Sun as ye Comet approaches ye Sun passing from *B* to *f* & then to *g*, will make ye comet verge more & more from its former line of direction towards ye Sun, so that its line of direction wch at *B* was *Bp*, at *f* shal become *fq*, but I do not understand how it can make ye Comet ever move directly



towards ye ⊙ as at *g* where ye line of direction *gS* passes through ye center of ye Sun *S*, much less can it make ye line of direction verge to ye other side ye sun as at *h* where ye line of direction is *ht*. For if ye Comet at *g* moved directly towards ye ⊙ & the ⊙ also attracted it directly towards himself it would continue to go towards ye ⊙ in ye line *gS* till it fell upon ye ⊙, there being no cause to turn it out of ye line of its direction *gS* towards *h*. The case is as if a bullet were shot from west to east. The attraction of ye earth by its gravity will make ye bullet tend more & more downwards, but it can never make it tend directly downwards much less verge from east to west. Nor will ye motion of ye Vortex[4] releive ye difficulty but rather increase it. For that being according to ye order of ye letters & marks *Aβ* ♊ ♋ *α* would make the Comet verge from ye line *gS* rather towards ye line *fq* then towards *h*. The only way to releive this difficulty in my judgmt is to suppose ye Comet to have gone not between ye ⊙ & Earth but to have fetched a compass about ye ⊙ as in this figure.

Secondly though I can easily allow an attractive power in ye ⊙ whereby the Planets are kept in their courses about him from going away in tangent lines, yet I am the lesse inclined to beleive this attraction to be of a magnetick nature because ye ⊙ is a vehemently hot body & magnetick bodies when made red hot lose their vertue. A red hot loadstone

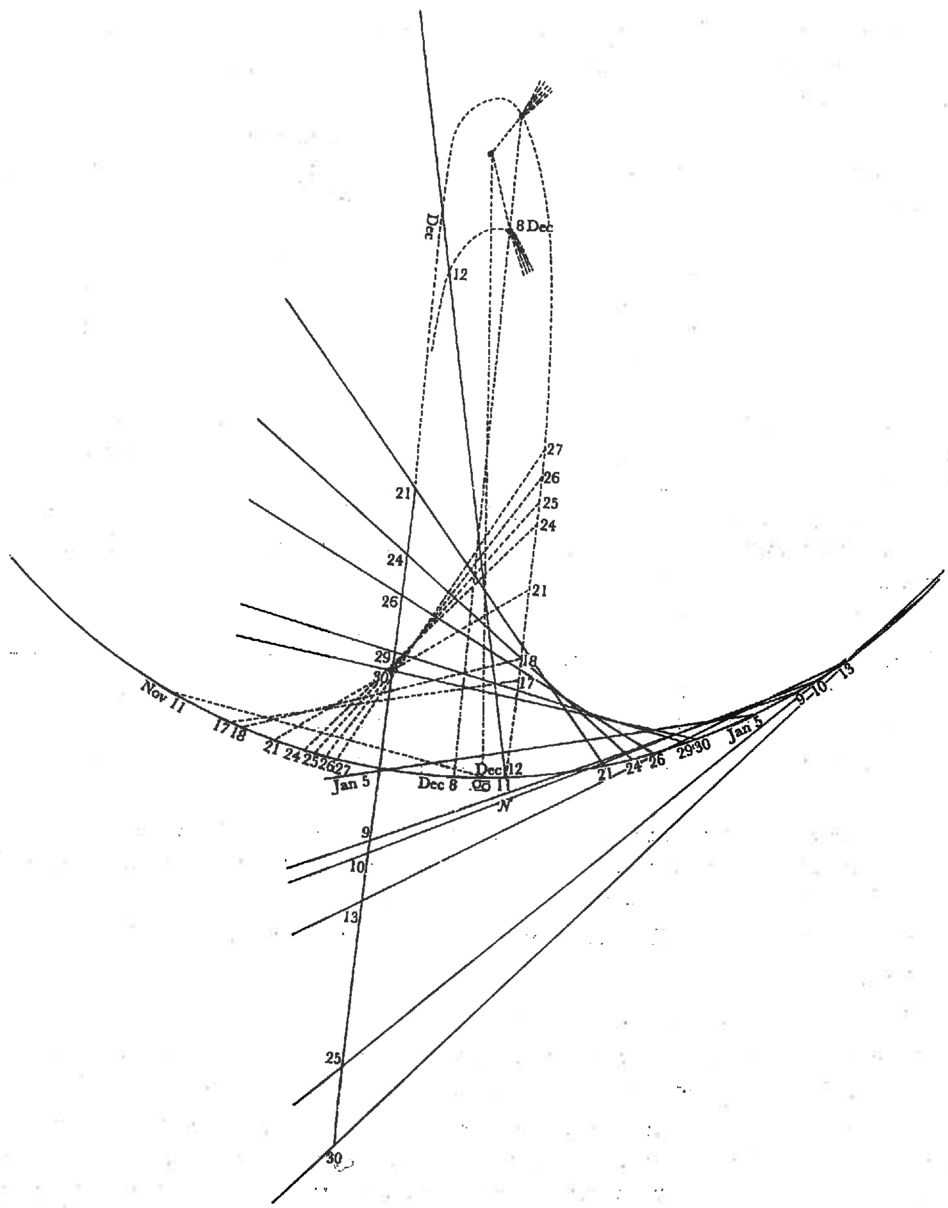


attracts not iron, nor any Loadstone a red hot iron, nor will a loadstone propagate its vertue through a rod of iron made red hot in ye middle. Whence probably ye Earth were it made red hot would lose its magnetism; & ye Sun being more then red hot, must be less capable of it.

But thirdly were ye ⊙ a magnet he would act on ye Comet as a great magnet does on a little one floating in a wallnut shell or other little boat on water. If the little magnet were forcibly turned about by one's hand & put into a wrong or unagreable position the great magnet so long as ye little one was forcibly detained in that wrong position would repell it, but so soon as ye little magnet was set at liberty the great magnet would make it nimbly turn about into an agreeable position & then attract it. And so ye sun, were ye Comet in a wrong position, would make it turn about quickly into a right one & then attract it & keep it constantly in that right position. How then the Comet being at first in a right position so as to be attracted by ye Sun should afterwards get into a wrong one so as be repelled from him I do not conceive. For ye directive vertue of a great magnet is stronger then its attractive vertue. The mariners needle is not sensibly attracted by ye earth, but it's strongly directed by it, so that you cannot make it stand in a wrong position. If then ye Sun be a magnet, the axis of ye Comet ought to be strongly directed by him into such a position as ye laws of magnetism require, & being so directed the Comet will be always attracted by ye Sun & never repelled. And so if ye axes of ye Planets be inclined to ye ecliptick by ye Sun's magnetism, they ought to be so directed as ye laws of magnetism require, wch I feare they are not.

I am further suspicious that ye Comets of November & December wch Mr Flamstead accounts one & ye same Comet were two different ones,(5) & I find Cassini in a Copy of a letter of his wch Mr Ellis(6) shewed me is of my mind. If they were but one Comet, it's motion was thrice accelerated & retarded. From Nov 18 to Nov 21 it moved after ye rate of almost six degrees a day. From Nov 23 to Decemb 5 after the rate of but 36 minutes a day. From Decem 6 to Decemb 12 after the rate of almost 8 degrees a day. From Decemb 12 to Decemb 19 after ye rate of about $3\frac{1}{2}$ degrees a day. From Decemb 24 to Decemb 26 after ye rate of almost $4\frac{1}{2}$ degrees a day. From wch time ye motion decreased continually. This frequent increas & decreas of motion is too paradoxical(7) to be admitted in one & the same Comet without some proof that there was but one. Besides it is very irregular. For after the 20th day of November when ye Comet was in its first perige as Mr Flamstead notes, & moved after ye rate of about six degrees a day, that it's motion should suddenly decrease so much as that from Novemb 23 to Decemb 5 to move but $7\frac{1}{4}$ degrees & consequently in ye middle part of that time (suppose at Novemb 29 or 30) to move after ye rate of less then half a degree a day, & this while ye Comet is going towards ye Sun & so has it's real motion continually accelerated, is very odd, & makes me question Father Gallet's(8) observations on wch ye supposition of but

Figure by Flamsteed, enclosed with Letter 252



[*In Flamsteed's handwriting:*]

hæc figura vestigij vice Cometicæ descripta fuit ab ipso tempore apparitionis et literis communicata cum Domino Crompton quibus responsa Domini Newtoni respiciant. (This map of the comet's path was described at the very moment of its appearance, and communicated in a letter to Mr Crompton, to which Newton's replies relate.)

255 NEWTON TO FLAMSTEED

16 APRIL 1681

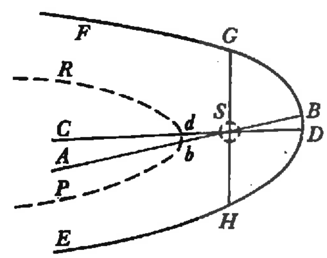
From the original[1] in the Bodleian Library.
In reply to Letter 252

Trin. Coll. Apr. 16. 1681.

Sr

Tis now almost thre weeks since upon my return[2] from a journey I received yours. By some indisposition & other impediments I have deferred answering you longer then I intended. For I was desirous to return you quicker thanks for your kind communications. The complements you are pleased out of extream kindness to give me, might better have suited wth your self from me; nor do I think it suits wth me to judg of an Hypothesis after your thoughts upon it. I only propounded objections for you your self (if you had not thought on 'em before) to judge further of it by; wch therefore being designed only for your use I forbear to urge further & shall only speak to ye question of two Comets. The mistake about ye date of Pere Gallet's observations was in ye Copy I received of them. In ye title 'twas writ *Stylo veteri*, & accordingly the observations all but ye two last were altered from ye new style to ye old one & put November 17, 18, 21, 24, 25. Decem: 6, 7. And ye like alteration from ye new to ye old style had ye French Observations under them. I perceive ye Scholar in his observation of ye transit of ye Comet by Spica was mistaken in ye time. He recollected it only by circumstances & it seems told me Novemb 16 for Novemb. 19. In your argument from ye consent of elongations, if you estimate ye motion of ye Comet not in ye Ecliptick but in ye line of its proper motion (as I think should be done) you will find some difference[3] between November & December. But that is not material, the apparent celerity depending on ye earth's distance from ye Comet at those times, & so whether equal or unequal being but accidental. You[4] seem to incline to think the way of ye Comet wider then in your scheme[5] & so do I for I apprehend ye Comet in Mr Halleys & your first observations Dec 8, 10, 11, 12 to have been remoter from us then ye ☉ & about Jan 2 to have been as far from ye ☉ as ye earth was & its Heliocentrick place yn to have been ♊ 9 degr wth north latitude 19 or 20 degr or there abouts. But you are afraid the long tail will not admitt the Comets passing beyond ye ☉. I apprehend nothing from thence, for I am forced to beleive the tail extended beyond ye sphere of ♂ during ye whole appearance of ye Comet & so was long enough to appear in ye beginning of December as long as it did. Nor was the apparent length of it at that time any way enormous but consonant to ye law it observed all December. For ye tail all that month & (by my observation some days in

January) ended at a great circle wch cut the Ecliptick in ♐ 19½ degrees at an angle of about 52 degr.(6) Now if the December Comet was beyond ye ☉ in ye beginning of December, the comets of November & December could hardly be ye same for this reason. Let *EDF* represent ye line ye Comet moved in, *S* the Sun, *CD* the axis of that line passing through ye Sun, *D* its vertex or Perihelion of ye Comet, and *AB* the plane of ye Ecliptick crossed by the axis in *S*. By contemplating ye figure you will perceive the Vertex or Perihelion *D* was on ye south side ye Ecliptick & consequently since the comet crost ye Ecliptick about ye 3d or at latest ye 4th of December it was in *D* a little before that time vizt above half a day before, for the angle *DSB* was about 6 or 8 degrees at least. But in conjunction wth ye Sun it was Decemb 9. So then in passing from *D* to conjunction there were scarce less then six days spent. But in the Hypothesis the point *D* was opposite to ye earth about December 7 & consequently December 9 was so neare to opposition that is to conjunction wth ye ☉ that the comet in passing thence to conjunction could not have spent many houres. Again drawing *GSH* perpendicular to *CD*, *GS* being equal to *SH*, the Comet should have had as much latitude in *H* as in *G*. In passing from conjunction to *G* it must have spent much more time then in passing from *D* to conjunction & consequently could not be in *G* before ye 12th day if so soon & so in *G* had 8 or 9 degrees(7) north latitude at least, but in *H* by ye Roman observations as they are adapted to ye Hypothesis could not have above a degree south latitude.



If the comet turned short of ye ☉ suppose in ye line *PQR*,(8) the difficulties are thereby something diminished, but I think not taken of. The point *d* or vertex of ye figure described by ye Comet was in conjunction wth ye ☉ Decemb 7. The Comet in conjunction Decemb 9. Therefore, the comets conjunction happened on that side *d* towards *b*. The Comet was in *b* (the point where it crost ye Ecliptick) Decemb 3 or 4. Therefore it spent 5 or six days in passing from *b* to a point between *b* & *d*: wch space by the Hypothesis is yet so little that the comet could not spend many hours in passing it. And I think too the south latitude though it could not be so great as ye north, yet it ought to have been greater then ye Roman observations Novem 26 & 27 make it.

But what ever there be in these difficulties, this sways most with me that to make ye Comets of November & December but one is to make that one paradoxical. Did it go in such a bent line other comets would do ye like & yet no such thing was ever observed in them but rather the contrary. The comets of 1665, 1677 & others which moved towards ye Sun, or some of them at least, had they twisted about ye Sun & not proceeding on forward gone away behind him they would have been seen again coming from him. The many wch have been seen advancing from the ⊙, or some of them at least, would have been seen in the former part of their course advancing towards him, had that former part been performed, not in the line of the latter part shooting on backwards towards the regions beyond ye ⊙ but twisting about him towards any hand. Those which were seen both before & after their perihelium's as the comets of 1472, 1556, 1580 & 1664 would not as they did, have begun in one part of the heavens & ended in ye opposite part, going through almost a semicircle wth motion first slow then swift then slow again as if done in a right line, had it been done in such a line as ye Hypothesis puts. Let but ye Comet of 1664 be considered(9) where the observations were made by accurate men. This was seen long before its Perihelion & long after & all the while moved (by the consent of the best Astronomers) in a line almost straight. So neare was ye line to a straight one that Monsieur Auzout(10) on supposition that 'twas an arch of a great circle about ye dog starr (as Cassini guessed and Auzout was afterward willing should be beleived) or rather a straight one (as ye obviousness of ye Hypothesis, easiness of ye calculation, & number of observations on wch 'twas founded makes me suspect) did from thre observations predict the motion to ye end without very considerable error.

But you ask why the Comet of November staid so long in the same southern latitude if it turned not back? I am not satisfied that it did so. I fear twould be hard to warrant any of ye observations of that Comet to less then a degree & why then might it not have in the time of the Canterbury observation between one & two degrees north latitude, & afterwards crossing ye Ecliptick about ye beginning of ye Roman observations, as Gallet makes it, & from thence advancing continually southward, arrive to between one & two degrees of south latitude at ye end of the Roman.

Your observation of Decemb 12 by your last correction is become much more agreable to ye phænomenon of ye tail then before & yet I fear is not altogether right. I suspect (if you are sure there was no error committed in taking its distance from Venus) that Venus had some minutes more longitude then in your recconning. In your observations last sent(11) ye comet Jan 10 is put in ♈ $20^d\,42'$, in a former copy in ♈ $20^d\,49'\frac{1}{2}$. That of ♈ $20^d\,49'\frac{1}{2}$ seems to correspond best wth your other observations. The Parisian observations compared wth yours seem to have too much longitude.(12) Namely December 29 by about 6', Jan 4 & 6 by about 3 or 4', Jan 8 by 7' or 8', Jan 13 by 12'. The greatest difference being in Jan 13, it may perhaps be worth your while to examin your own observation of that day before you publish it. I made an observation about that time wch though inferior in accurateness to either of yours yet may possibly give some light into ye difference between you. Namely Jan. 11 I observed at

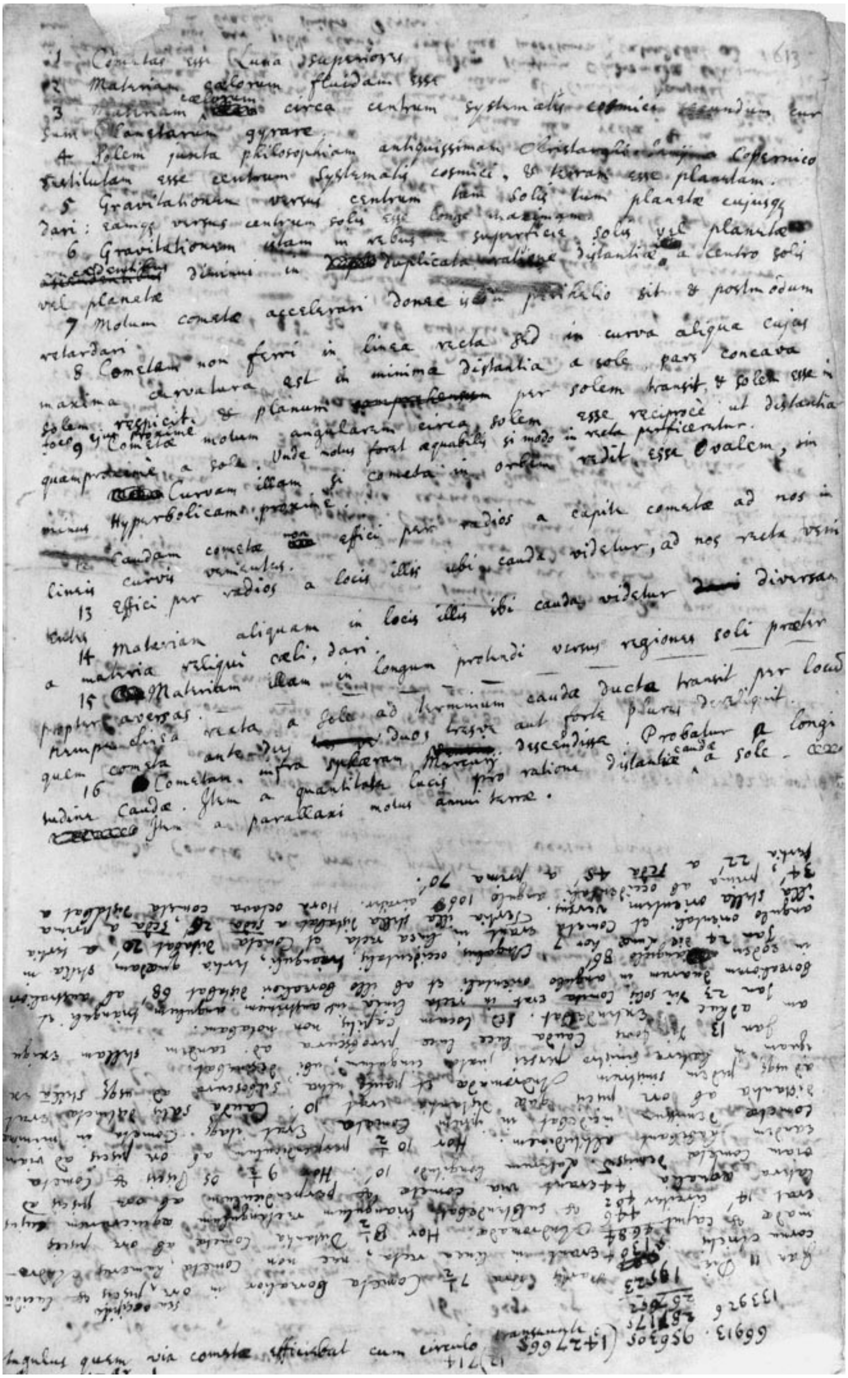

1 Cometas esse Luna superiores
2 Materiam caelorum fluidam esse
3 Materiam caelorum circa centrum systematis cosmici secundum cursum Planetarum gyrare
4 Solem juxta philosophiam antiquissimam [illegible] Copernico [illegible] esse centrum systematis cosmici, & Terram esse planetam.
5 Gravitationem versus centrum tam solis tum planetae cujusque dari: eamque versus centrum solis esse longe maximam
6 Gravitationem illam in rebus [illegible] superficie solis vel planetae [illegible] diminui in duplicata ratione distantiae a centro solis vel planetae
7 Motum cometae accelerari donec in perihelio sit & postmodum retardari
8 Cometam non ferri in linea recta sed in curva aliqua cujus maxima curvatura est in minima distantia a sole pars concava solem respicit & planum [illegible] per solem transit [illegible]
9 [illegible] Cometae motum angularem circa solem esse reciproce ut distantia [illegible] a sole. Unde motus foret aequabilis si modo in recta perficeretur.
[illegible]

Plate 1. *Propositiones de Cometis*, U.L.C. manuscript Add 3965.14 fol. 613r, published by permission of the Syndics of Cambridge University Library

16. Cometam infra sphaeram (18) Mercurij descendisse. Probatur a longitudine caudae. Item a quantitate lucis pro ratione distantiae caudae (19) a sole. Item a parallaxi motus annui terrae.

Translation [Propositions on comets]

1. Comets are higher than the moon.
2. The matter of the heavens is fluid.
3. The matter of the heavens revolves around the center of the cosmic system in the direction of the courses of the planets.
4. According to the most ancient philosophy of Aristarchus of Samos, restored by Copernicus, the sun is the center of the cosmic system, and the earth is a planet.
5. There is gravitation toward the centers of the sun and each of the planets, and that toward the center of the sun is far greater.
6. That gravitation in things diminishes in duplicate ratio to the distance from the center of the sun or a planet as they recede from the surface of the sun or planet.
7. The motion of a comet is accelerated until it is in perhelion and retarded afterwards.
8. A comet does not travel in a straight line but in some curve the maximum curvature of which is at the minimum distance from the sun, the concave part faces the sun, and the plane passes through the sun, and the sun is in its near focus.
9. The angular motion of a comet around the sun is very nearly reciprocal to the distance from the sun. Whence the motion would be uniform only if performed in a straight line.

[X.] That curve is an oval if the comet returns in an orbit, if not [the curve] is nearly a hyperbola.

12. The tail of a comet is not produced by rays coming in curved lines to us from the head of a comet.
13. [The tail is] produced by rays coming in straight lines to us from those places where the tail is seen.
14. There is some matter different from the rest of the matter in the heavens, in those places where the tail is seen.
15. That matter stretches far out into the regions more or less opposite to the sun. - certainly a straight line drawn from the sun to the end of the tail passes through the place the comet left behind two or three or perhaps more days before.
16. The comet descended below the sphere of Mercury. This is proved from the longitude of the tail. The same [point is proved] from the quantity of light by reason of the distance of the tail from the sun. The same from the parallax of the earth's annual motion.

Tell tale tails

The topics are examined in reverse order to approximate the probable sequence of discovery or fomulation. Proposition 16 can refer only to the comet of 1680/1. The

(13) *Phil. Trans.* 16 (1686), 3–21.

(14) Here Newton wrote and then crossed out the word 'way'.

(15) See Letter 146.

(16) The passage here quoted stands at the end of the long first section of the 'Hypothesis' (Letter 146, vol. I, at p. 366). The 'foregoing words' contain the following passage: 'the vast body of the Earth, wch may be every where to the very center in perpetuall working, may continually condense so much of this Spirit as to cause it from above to descend with great celerity for a supply. In wch descent it may beare downe with it the bodyes it pervades with force, proportionall to the superficies of all their parts it acts upon....'

(17) The 'Hypothesis' in fact contains no more precise formulation of the law of the inverse square than is suggested in the final phrase quoted in note (16) above.

(18) In Hooke's *Diary*, for Friday, 15 February 1688/9, is noted: 'At Hallys met Newton—vainly pretended claim yet acknowledged my information. Interest has noe conscience; *a posse ad esse non valet Consequentia*' (MacPike, *Halley*, p. 184).

289 HALLEY TO NEWTON

29 JUNE 1686

From the original in King's College Library, Cambridge

Sr

I am heartily sorry, that in this matter, wherin all mankind ought to acknowledg their obligations to you, you should meet with any thing that should give you disquiet, or that any disgust should make you think of desisting in your pretensions to a Lady, whose favours you have so much reason to boast of. Tis not shee but your Rivalls enviing your happiness that endeavour to disturb your quiet enjoyment, which when you consider, I hope you will see cause to alter your former Resolution of suppressing your third Book, there being nothing which you can have compiled therin, which the learned world will not be concerned to have concealed; These Gentlemen of the Society to whom I have communicated it, are very much troubled at it, and that this unlucky business should have hapned to give you trouble, having a just sentiment of the Author therof.

According to your desire in your former,[1] I waited upon Sr Christopher Wren, to inquire of him, if he had the first notion of the reciprocall duplicate proportion from Mr Hook, his answer was, that he himself very many years since had had his thoughts upon making out the Planets motions by a composition of a Descent towards the sun, & an imprest motion; but that at length he gave over, not finding the means of doing it. Since which time Mr Hook had frequently told him that he had done it, and attempted to make it out to him, but that he never satisfied him, that his demonstrations were cogent. and this I know to be true, that in January 83/4,[2] I, having from the consideration of the sesquialter proportion of Kepler, concluded that the centripetall force decreased in the proportion of the squares of the distances reciprocally, came one Wednesday to town, where I met with Sr Christ. Wrenn and Mr Hook, and falling in discourse about it, Mr Hook affirmed that upon that principle all the Laws of the celestiall motions were to be demonstrated,[3] and that he himself had done it; I declared the ill success of my attempts; and Sr Christopher to encourage the Inquiry sd, that he would give Mr Hook or me 2 months time to bring him a convincing demonstration therof, and besides the honour, he of us that did it, should have from him a present of a book of 40s. Mr Hook then sd that he had it, but that he would conceale it for some time that others triing and failing, might know how to value it, when he should make it publick; however I remember Sr Christopher was little satisfied that he could do it, and tho Mr Hook then promised to show it him, I do not yet find that in that particular he has been as good as his word. The August following when I did my self the honour to visit you, I then learnt the good news that you had brought this demonstration to perfection, and you were pleased, to promise me a copy therof, which the November following I received with a great deal of satisfaction from Mr Paget; and therupon took another Journey down to Cambridge, on purpose to conferr with you about it, since which time it has been enterd upon the Register[4] books of the Society as all this past Mr Hook was acquainted with it; and according to the philosophically ambitious temper he is of, he would, had he been master of a like demonstration, no longer have conceald it, the reason he told Sr Christopher & I now ceasing. But now he sais that this is but one small part of an excellent System of Nature, which he has conceived, but has not yet compleatly made out, so that he thinks not fit to publish one part without the other. But I have plainly told him, that unless he produce another differing demonstration, and let the world judge of it, neither I nor any one else can belive it.

As to the manner of Mr Hooks claiming this discovery, I fear it has been represented in worse colours than it ought; for he neither made publick application to the Society for Justice, nor pretended you had all from him. The truth is this: Sr John Hoskins[5] his particular friend being in the Chair, when Dr Vincent presented your Book, the Dr gave it its just Encomium, both as to the Novelty and dignity of the subject. it was replied by another Gentleman, that you had carried the thing so far that there was no more to be added. to which the Vicepresident replied, that it was so much the more to be prized, for that it was both Invented and perfected at the same time. this gave Mr Hook

NEWTON'S PROBLEM: TO INFER FORCES FROM MOTIONS

The ***centripetal force*** retaining a body in uniform circular motion varies as

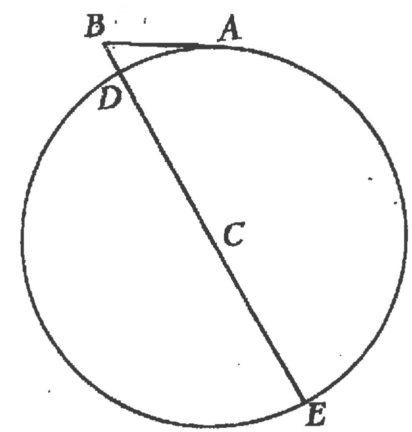
$$\mathbf{BD}/\delta t^2$$

which – by Euclid 3,36 – becomes

$$\propto (\mathbf{AB}^2/\mathbf{BE})/\delta t^2$$

which, as **B** approaches **A,**

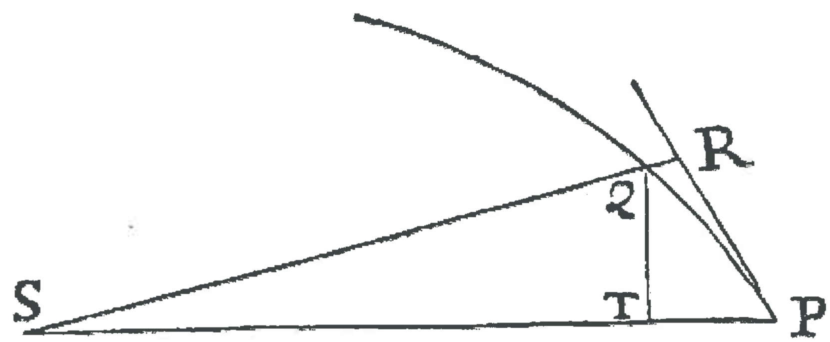
$$\propto v^2/r \propto r/P^2$$



***Problem*: How to generalize from uniform circular to arbitrary curvilinear motions – e.g. Kepler's ellipse?**

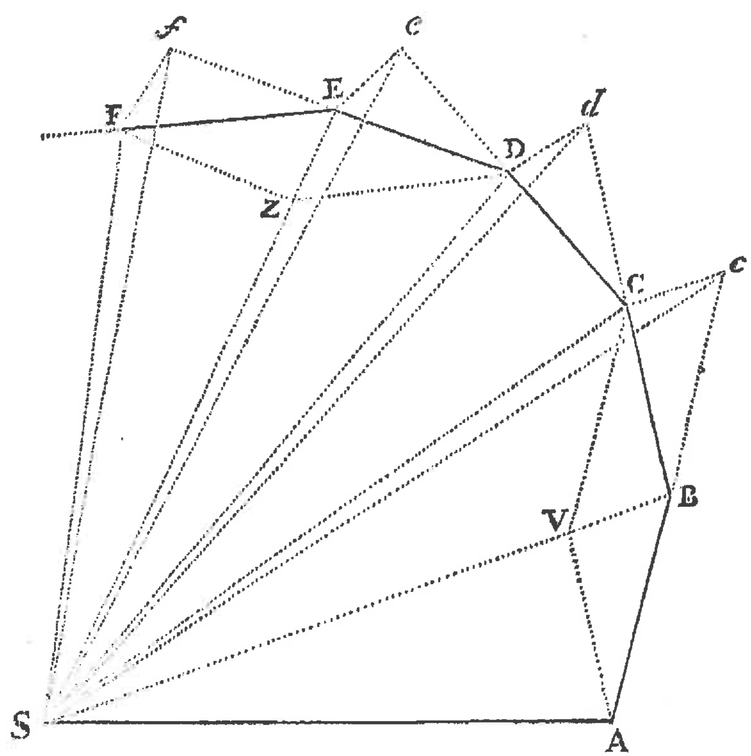
Hooke's Proposal: Consider Only Forces Directed Toward a Single Point in Space

Prop. 6: To infer the "centripetal" force, toward S, from the curvilinear motion:



force $\propto \lim$ **QR/δt^2**

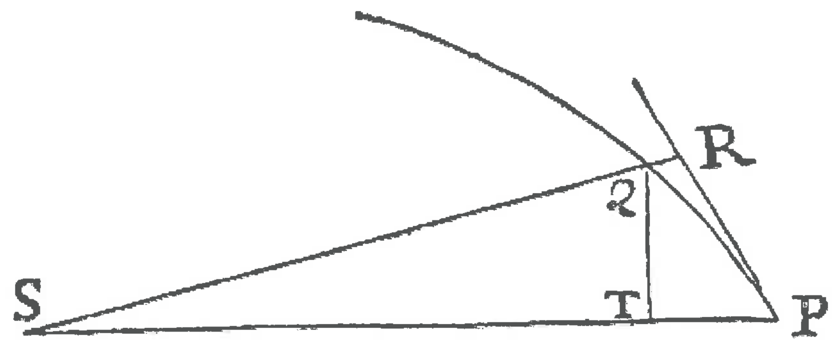
The Area Rule: the First Key



Theorem 1: If all departures of a body from uniform motion in a straight line are directed toward a single point in space S – i.e. the external force on the body is centripetal – then the body sweeps out equal areas in equal times with respect to S.

Newton's Basic Solution for Motion Under Centripetal Forces: the Second Key

Prop. 6: To infer the centripetal force, toward S, from the curvilinear motion:



force $\propto$ lim **QR/δt²**

$\propto$ lim **QR/(QT²xSP²)**

$$\frac{1}{2r^2}\left(\frac{1}{r}+\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right)$$

De Motu Corporum in Gyrum

Definition 1. A 'centripetal' force I name that by which a body is impelled or attracted towards some point regarded as its center.

Definition 2. And the force of — that is, innate in — a body I call that by which it endeavours to persist in its motion following a straight line.

Definition 3. While 'resistance' is that which is the property of a regularly impeding medium.

Hypothesis 1. In the ensuing nine propositions the resistance is nil; thereafter it is proportional jointly to the speed of the body and to the density of the medium.

Hypothesis 2. Every body by its innate force alone proceeds uniformly into infinity following a straight line, unless it is impeded by something from without.

Hypothesis 3. A body is carried in a given time by a combination of forces to the place where it is borne by the separate forces acting successively in equal times.

Hypotheses 4. The space which a body, urged by any centripetal force, describes at the very beginning of its motion is in the doubled ratio of the time.

From Huygens's *Horologium Oscillatorium*

PART II

The Falling of Heavy Bodies and Their Motion in a Cycloid

HYPOTHESES

I

If there were no gravity, and if the air did not impede the motion of bodies, then any body will continue its given motion with uniform velocity in a straight line.

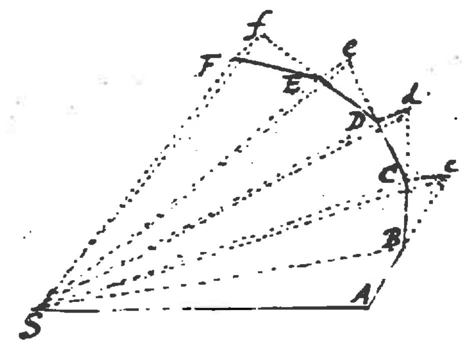
II

By the action of gravity, whatever its sources,[1] *it happens that bodies are moved by a motion composed both of a uniform motion in one direction or another and of a motion downward due to gravity.*

III

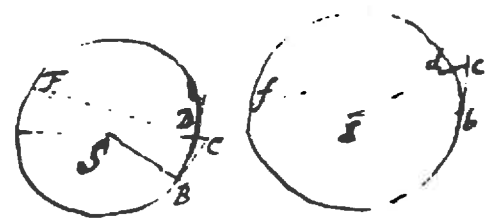
These two motions can be considered separately, with neither being impeded by the other.

De Motu Corporum in Gyrum

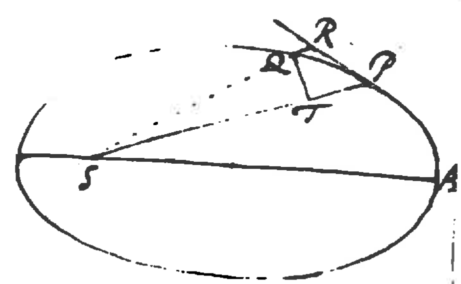


Theorem 1. All orbiting bodies describe, by radii drawn to their center, areas proportional to the times.

Theorem 2. Where bodies orbit uniformly in the circumferences of circles, the centripetal forces are as the squares of arcs simultaneously described, divided by the radii of their circles.



Corollary 5. If the squares of the periodic times are as the cubes of the radii, the centripetal forces are reciprocally as the squares of the radii. And conversely so.



Theorem 3. If a body P in orbiting around the center S shall describe any curved line APQ, and if the straight line PR touches that curve in any point P and to this tangent from any other point Q of the curve there be drawn QR parallel to the distance SP, and if QT be let fall perpendicular to this distance SP: I assert that the centripetal force is reciprocally as the "solid" $SP^2 \times QT^2/QR$, provided that the ultimate quantity of that solid when the points P and Q come to coincide is always taken.

Scholium to Theorem 2. **The case of the fifth corollary holds true in the heavenly bodies: the squares of the periodic times are as the cubes of their distances from the common center round which they revolve. That it does obtain in the major planets revolving round the Sun and also in the minor ones orbiting round Jupiter and Saturn astronomers are agreed.**

A worry. **The orbits of the major planets are indeed nearly circular in shape: the largest deviation, for Mercury, has only a 2 percent variation from the largest diameter to the smallest.**

Nevertheless, the motions are far from uniform. The ratio of the maximum to minimum velocity is (1+e)/(1-e), which for Mercury is around 1.2/0.8, so that the maximum velocity is 50 percent greater than the minimum, and correspondingly for Mars, the maximum velocity is around 20 percent greater than the minimum.

What sort of conclusion can then be drawn on the basis of the fifth corollary about an inverse-square ratio of the centripetal force holding the planets in orbit around the Sun?

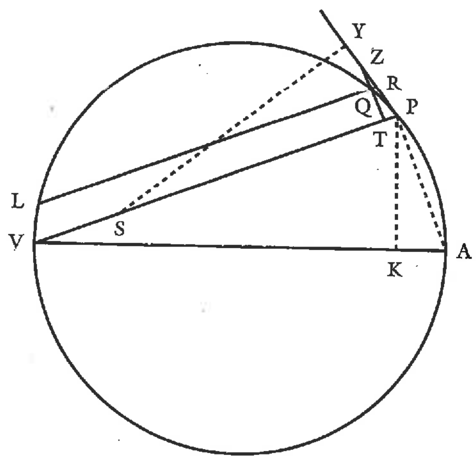


FIGURE 4
Excentric Circle Diagram from Book 1 Proposition 7, Added in the Second Edition

only *quam proxime* a Keplerian ellipse. No matter. For Newton had a much better way of showing this.

In the second edition of the *Principia* he inserted a new proposition (book 1, proposition 5), which appears in his papers from the early 1690s, that gives the rule of centripetal force for an excentric circle—that is, a circle in which a body sweeps out equal areas in equal times relative to a point S off the center. The rule is that the centripetal acceleration and force vary inversely as the product of SP^2 and PV^3. A corollary to this result then provides the basis for the CG^3/RP^2 rule I used above. Now, PV can readily be expressed in terms of SP in the case of a circle. I have instead chosen to apply the CG^3/RP^2 rule to the excentric circle. Either way, the upshot is that the centripetal force and hence acceleration vary inversely as a combination of four terms in which SP occurs respectively in powers of 5, 3, 1, and –1:

$$\left(\frac{SP}{a}\right)^5 + 3(1-\varepsilon^2)\left(\frac{SP}{a}\right)^3 + 3(1-\varepsilon^2)^2\left(\frac{SP}{a}\right) + (1-\varepsilon^2)^3\left(\frac{SP}{a}\right)^{-1}$$

Notice that SP to a power of 2 is nowhere to be found in this expression.[6]

The evidence at the time that the planetary orbits are ellipses was confined to Mercury and Mars; and even in the case of Mercury, the most elliptical of the orbits then known, the minor axis is only 2 percent shorter than the major axis. The orbits really are *nearly* circular—so much so that an excentric circular orbit together with Kepler's area rule gives results for Venus, Jupiter, and Saturn of the same level of accuracy as Kepler's ellipses gave. In other words, to the level of precision of the data at the time, the orbits of Venus, Jupiter, and Saturn were not observationally distinguishable from excentric circles in which the planets sweep out equal areas in equal times with respect to the Sun. But then, postulating that the orbits are Keplerian ellipses and inferring a force rule exponent of –2 from these ellipses was a risky move.[7] If some of these orbits are Keplerian ellipses only *quam proxime* and excentric circles exactly, then an exponent of –2 will not hold even remotely *quam proxime*.

Did Newton know this? I don't know if he took the trouble to derive the full formula, but his published result for the excentric circle is enough to make clear that the inverse-square need not hold *quam proxime*. The more interesting question is not whether he knew it, but when he knew it. The excentric circle proposition first appears in Newton's surviving papers from the early 1690s. But the corollary to it—the force varies inversely with the fifth power when the center of force is on the circumference—appears in the version of *De Motu* registered by the Royal Society in December 1684. Newton scholars have long found this proposition strange, for why would Newton or anyone else have asked what the rule of force is when the center of force is on the circumference? The proof of the full excentric circle proposition is easy, and the diagram for it is virtually the same as the one for the special case that appears in *De Motu*. Perhaps Newton had the full result early on, before writing *De Motu*, and then included only the simple limiting case in the tract. If so, he had already explored the excentric circle before the first edition of the *Principia* and would have seen that the inverse-square need not hold *quam proxime* when the orbit approximating the ellipse too closely approximates an excentric circle as well.

This then is what I regard as the best answer to my "why not" question: even though the Keplerian ellipse entails the inverse-square, one cannot always infer that the inverse-square holds *quam proxime* when the Keplerian ellipse holds only *quam proxime*. In particular, observations at the time were unable to distinguish clearly between the Keplerian ellipse and the excentric circle for Venus, Jupiter, and Saturn. The centripetal acceleration rule for a body in a circular orbit sweeping out equal areas in equal times about any point off center is far removed from inverse-square. What Newton did instead was to infer the inverse-square for the planets from their nearly circular, nonprecessing excentric orbits. This licensed the further inference that the orbits must be exact ellipses in the absence of any further components of acceleration. To quote more of the entry in Huygens's notebook from which I quoted earlier:

Problem 2. Proof for "Central" Ellipse

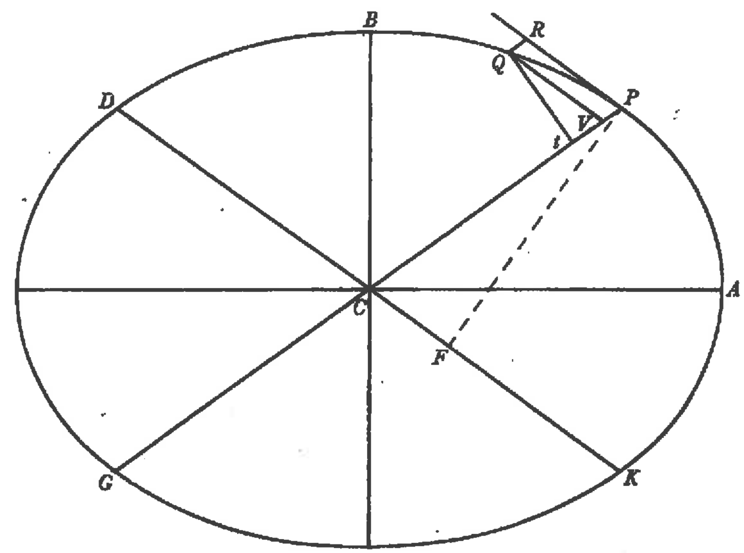
Background (from Apollonius)

- $PV \times VG : QV^2$ as $PC^2 : CD^2$
- $QV^2 : Qt^2$ as $PC^2 : PF^2$ (from similar triangles, QVt and PFC)
- All circumscribed parallelograms equal in area: $4CD \times PF = 4CB \times CA$

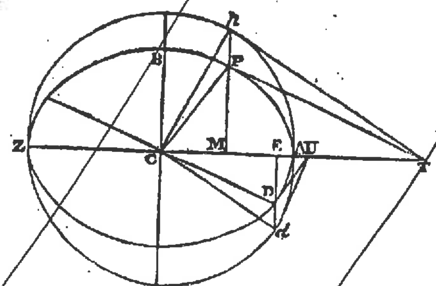
Proof

- $PV \times VG/Qt^2 = (PC^2 \times PC^2)/(CD^2 \times PF^2)$
- $QR \times VG/Qt^2 = PC^4/(BC \times CA)^2$
- But $VG \Rightarrow 2PC$
- $\mathbf{QR/PC^2 \times Qt^2 = PC/2(BC \times CA)^2}$

  $\mathbf{\propto PC}$



P, D; and let PT, pT, DU, dU be tangents to the ellipse and to the circle. (Prop. 4.)



Since Mp, Ed are perpendicular to CA, by similar triangles $CM \cdot MT = Mp^2$, hence (B. 1. Art. 17)

$$CM \cdot MT : MP^2 :: Ed^2 : ED^2,$$

and $CM \cdot MT : Ed^2 :: MP^2 : ED^2$;

whence $CM \cdot MT : Ed^2 :: MT^2 : CE^2$, by parallels PT, CD,

whence $CM \cdot MT : MT^2 :: Ed^2 : CE^2$,

or $CM^2 : CM \cdot MT :: Ed^2 : CE^2$;

therefore $CM^2 : Mp^2 :: Ed^2 : CE^2$,

and $CM : Mp :: Ed : EC$.

Hence the triangles CMp, dEC are similar, the angle dCE is the complement of MCp, and pCd is a right angle. Q. E. D.

Cor. 1. If CD be conjugate to CP, CP is conjugate to CD.

Cor. 2. $CM = Ed$, and $CE = Mp$.

Cor. 3. $CP^2 + CD^2 = AC^2 + BC^2$.

For $CP^2 + CD^2 = CM^2 + MP^2 + CN^2 + ED^2$

$$= CM^2 + MP^2 + Mp^2 + ED^2$$

$$= Cp^2 + MP^2 + ED^2.$$

But $\dfrac{MP^2 + ED^2}{BC^2} = \dfrac{MP^2}{BC^2} + \dfrac{ED^2}{BC^2} = \dfrac{Mp^2}{AC^2} + \dfrac{Ed^2}{AC^2}$

$$= \frac{Mp^2 + CM^2}{AC^2} = 1.$$

Hence $MP^2 + ED^2 = BC^2$, and $CP^2 + CD^2 = AC^2 + BC^2$.

Cor. 4. If $CM = x$, $Ed = x$, $ED^2 = \dfrac{b^2}{a^2} x^2$,

and $CE^2 = Mp^2 = a^2 - x^2$.

Hence $CD^2 = a^2 - x^2 + \dfrac{b^2}{a^2} x^2 = a^2 - \dfrac{a^2 - b^2}{a^2} x^2 = a^2 - e^2 x^2$.

Cor. 5. In like manner $CP^2 = b^2 + e^2 x^2$.

Cor. 6. $SP \cdot HP = CD^2$. For, as in Art. 14, Book 1, $SP = a + ex$, $HP = a - ex$:

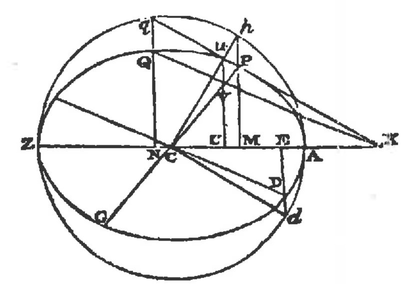
hence $SP \cdot HP = a^2 - e^2 x^2 = CD^2$, by Cor. 4.

Prop. VI.

In an ellipse, if QV be an ordinate to a diameter CP, CD the semi-conjugate diameter to CP;

$$PV \cdot VG : QV^2 :: CP^2 : CD^2.$$

Let $APZD$ be an ellipse, GP a diameter, QV an ordinate to the diameter, CD the semi-diameter conjugate to CP. Let $ApZd$ be the circular projection of the ellipse; p, q, d the circular projections of points P, Q, D. Let UV, perpendicular to AZ, meet Cp in v; and join qv.



By Prop. 5, Cd is a perpendicular to Cp. Also

$$Uv : UV :: Mp : MP, \text{ which is } :: Nq : NQ.$$

Hence qv, QV will meet NU in the same point X. And since VX is parallel to CD (because QV is an ordinate), and that $UV : Uv :: ED : Ed$, it is easily seen that Xv is parallel to Cd; and therefore qv is perpendicular to Cp: and hence $Cp^2 - Cv^2 = Cq^2 - Cv^2 = qv^2$.

Now $CP^2 : CV^2 :: Cp^2 : Cv^2$;

$$\therefore\ CP^2 - CV^2 : Cp^2 - Cv^2 :: CP^2 : Cp^2;$$

$$\therefore\ CP^2 - CV^2 : \quad qv^2 \quad :: CP^2 : Cp^2.$$

Also $qv^2 : QV^2 :: Cd^2\ (Cp^2) : CD^2$;

$\therefore$ compounding the two last propositions,

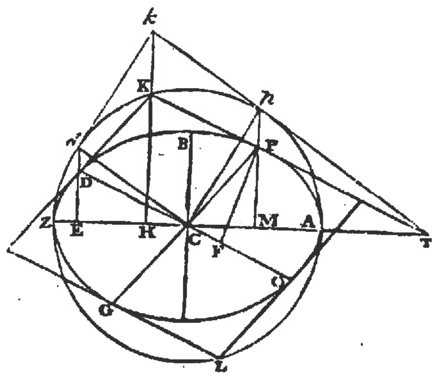
$$CP^2 - CV^2 : QV^2 :: CP^2 : CD^2,$$

or $$PV.VG : QV^2 :: CP^2 : CD^2.$$

Prop. VII.

The parallelograms made by drawing tangents at the extremities of two conjugate diameters of an ellipse are all equal in area.

Let $APDZ$ be an ellipse, $ApdZ$ its circular projection,



CP, CD two semi-diameters conjugate to each other: KL a parallelogram made by drawing tangents at the extremities of the diameters PC, DC.

Draw HK perpendicular to AZ, meeting in k the tangent of the circular projection at p. Therefore since the tangent of the ellipse and of its circular projection meet AZ in the same point T, we have $HK : Hk :: MP : Mp$, that is, $HK : Hk :: BC : AC$. For the same reason the tangent at D will meet Hk in a point determined by the same proportion. Therefore the two tangents at p and d meet HK in the same point k. And Cp is at right angles to Cd and equal to it; therefore $Cpkd$ is a square.

Now the triangles THK, THk are as their bases $HK : Hk$; that is,

$$THK : THk :: BC : AC.$$

Also $TMP : TMp :: BC : AC$; hence the differences are in the same proportion; that is,

$$\text{trapezium } MPKH : MpkH :: BC : AC.$$

In like manner, trapezium $EDKH : EdkH :: BC : AC$.

Also triangle $CPM : CpM :: BC : AC$,

and triangle $CDE : CdE :: BC : AC$.

Add together the two former of these four sets of proportionals, and subtract the two latter, and we have

$$CPKD : Cpkd :: BC : AC.$$

Whence, $CPKD : Cpkd :: AC.BC : AC^2$.

But $Cpkd$ is equal to Cp^2 or AC^2. Therefore $CPKD$ is equal to $AC.BC$.

The parallelogram KL is four times the parallelogram $CPKD$. Therefore the parallelogram $KL = 4AC.BC$, and is constant.

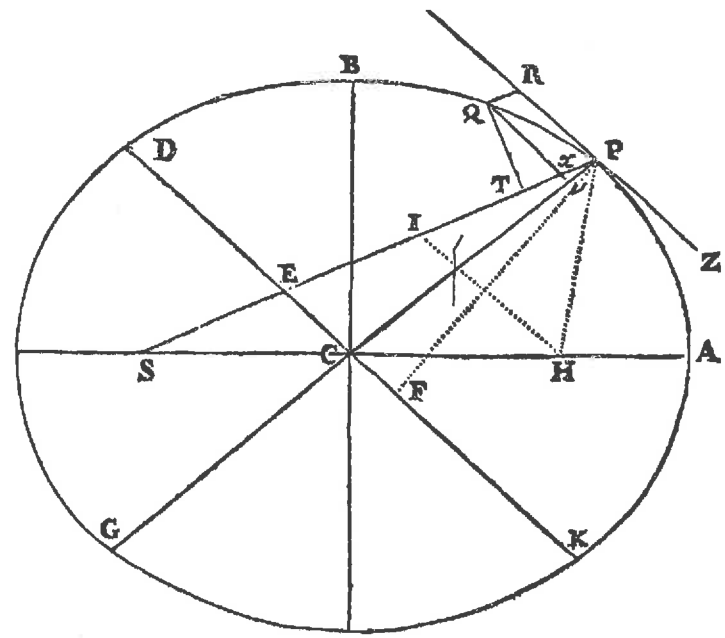
Cor. If PF be drawn perpendicular on DC, the parallelogram $CPKD$ is equal to $CD.PF$. Therefore

$$CD.PF = AC.BC.$$

Problem 3. Proof for "Keplerian" Ellipse

Background (from Apollonius)

- $PV \times GV : QV^2$ as $PC^2 : CD^2$
- $QX : QT$ as $PE : PF$
- All circumscribed parallelograms equal in area: $4CD \times PF = 4CB \times CA$
- Latus rectum $L = 2BC^2/AC$



Outline of proof

- $EP = AC$ since $EP = (PS + PI)/2$
- $QR/PV = EP/PC$
- $QX^2/QT^2 = EP^2/PF^2$
- $\mathbf{L \times QR/QT^2 =}$

  $\mathbf{\{(AC \times L \times PC \times CD^2)/(GV \times CD^2 \times CB^2)\} \times (QV^2/QX^2)}$

  $\mathbf{= (2PC/GV) \times (QV^2/QX^2)} \Rightarrow$ unity
- So, **force $\propto$ $\mathbf{1/(L \times SP^2)}$**

De motu corporum in gyrum.

Def. 1. Vim centripetam appello qua corpus impellitur vel attrahitur versus aliquod punctum quod ut centrum spectatur.

Def 2. Et vim corporis seu corpori insitam qua id conatur perseverare in motu suo secundum lineam rectam.

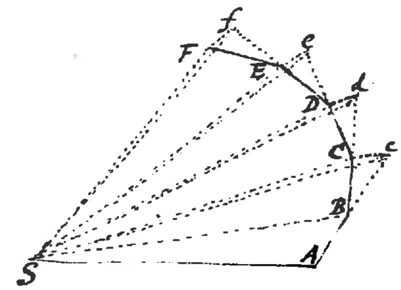
Def 3 Et resistentiam quae est medij regulariter impedientis.

Hypoth. 1. ~~Corpora nec medio impediri nec alijs causis externis~~ Resistentiam ~~esse ut corporis celeritas et medij densitas conjunctim~~ in proximis novem ~~quae minus omnibus insita et centripeta aequalia eadem~~ propositionibus nullam esse, in sequentibus esse ut corporis celeritas et medij densitas conjunctim.

Hypoth. 2. Corpus omne sola vi insita uniformiter secundum rectam lineam in infinitum progredi nisi aliquid extrinsecus impediat.

~~Hypoth. 3. Resistentiam in proximis novem propositionibus nullam esse in sequentibus esse ut corporis celeritas et~~

Theorema 1. Gyrantia omnia radijs ad centrum ductis areas temporibus proportionales describere.

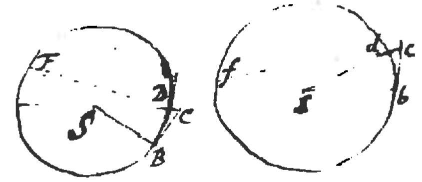


Dividatur tempus in partes aequales, et prima temporis parte describat corpus vi insita rectam AB. Idem secunda temporis parte si nil impediret[a] recta pergeret ad c describens lineam Bc aequalem ipsi AB adeo ut radijs AS, BS, cS ad centrum actis confectae forent aequales areae ASB, BSc. Verum ubi corpus venit ad B agat vis centripeta impulsu unico sed magno, faciatque corpus a recta Bc deflectere et pergere in recta BC. Ipsi BS parallela agatur cC occurrens BC in C et completa secunda temporis parte[b] corpus reperietur in C. Junge SC et triangulum SBC ob parallelas SB, Cc aequale erit triangulo SBc atque adeo etiam triangulo SAB. Simili argumento si vis centripeta successive agat in C, D, E &c, faciens corpus singulis temporis momentis singulas describere rectas CD, DE, EF &c triangulum SCD triangulo SBC et SDE ipsi SCD et SEF ipsi SDE aequale erit. Aequalibus igitur temporibus aequales areae describuntur. Sunto jam haec triangula numero infinita et infinite parva, sic, ut singulis temporis momentis singula respondeant triangula, agente vi centripeta sine intermissione, & constabit propositio.

a Hyp. 1.

b Lem. 1.

Theorem. 2. Corporibus in circumferentijs circulorum uniformiter gyrantibus vires centripetas esse ut arcuum simul descriptorum quadrata applicata ad radios circulorum.



Corpora B, b in circumferentijs circulorum BD, bd gyrantia simul describant arcus BD, bd. Sola vi insita describerent tangentes BC, bc his arcubus aequales. Vires centripetae sunt quae perpetuo retrahunt corpora de tangentibus ad circumferentias, atque adeo hae sunt ad invicem ut spatia ipsis superata CD, cd, id est productis CD, cd ad F et f ut $\frac{BC\,quad}{CF}$ ad $\frac{bc\,quad}{cf}$ sive ut $\frac{BD\,quad}{\frac{1}{2}CF}$ ad $\frac{bd\,quad}{\frac{1}{2}cf}$. Loquor de spatijs BD, bd minutissimis inque infinitum diminuendis sic ut pro $\frac{1}{2}CF$, $\frac{1}{2}cf$ scribere liceat circulorum radios SB, sb. Quo facto constat Propositio.

Hyp. 3. Corpus in dato tempore viribus conjunctis eo ferri quo viribus divisis in temporibus aequalibus successive.

Hyp 4.

Cor 2 Et reciprocè ut … temporum periodicorum applicata ad radios.

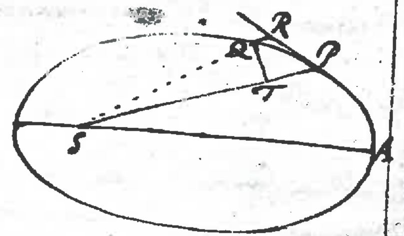
Cor 3 Unde si quadrata temporum periodicorum sunt ut radij circulorum vires centripetæ sunt æquales. Et vice versa.

Cor 4 Si quadrata temporum periodicorum sunt ut quadrata radiorum vires centripetæ sunt reciprocè ut radij. Et vice versa.

Cor 5 Si quadrata temporum periodicorum sunt ut cubi radiorum vires centripetæ sunt reciprocè ut quadrata radiorum. Et ~~si denique~~ vice versa.

Schol. Casus Corollarij quinti obtinet in corporibus cælestibus. Quadrata temporum periodicorum sunt ut cubi distantiarum a communi centro circum quod volvuntur. Id obtinere in Planetis majoribus circa Solem gyrantibus inque minoribus circa Jovem et Saturnum jam statuunt Astronomi.

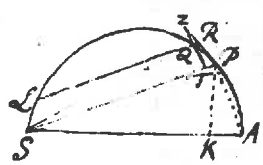
Theor. 3. Si corpus P circa centrum S gyrando, describat lineam quamvis curvam APQ, et si tangat recta PR curvam illam in puncto quovis P et ad tangentem ab alio quovis curvæ puncto Q agatur QR distantiæ SP parallela ac demittatur QT perpendicularis ad distantiam SP: dico quod ~~pars~~ vis centripeta sit reciprocè ut solidum $\frac{SP\,quad. \times QT\,quad.}{QR}$, si modò solidi illius ea semper sumatur quantitas quæ ultimò fit ubi coeunt puncta P et Q.



Namque in figura indefinitè parva QRPT lineola QR dato tempore est ut vis centripeta et data vi ut quadratum temporis atque adeo neutro dato ut ~~quadratum~~ vis centripeta et quadratum temporis conjunctim, id est ut vis centripeta semel et area SQP tempori proportionalis (vel duplum ejus $SP \times QT$) bis. Applicetur hujus proportionalitatis pars utraque ad lineolam QR et fiet unitas ut vis centripeta et $\frac{SPq \times QTq}{QR}$ conjunctim, hoc est vis centripeta reciprocè ut $\frac{SPq \times QTq}{QR}$. Q.E.D. ex Hyp. 4. & Lem. 2

Corol. Hinc si detur figura quævis et in ea punctum ad quod vis centripeta dirigitur, inveniri potest lex vis centripetæ quæ corpus in figuræ illius perimetro gyrare faciet. Nimirum computandum est solidum $\frac{SPq \times QTq}{QR}$ huic vi reciprocè proportionale. Ejus rei dabimus exempla in problematis sequentibus.

Prob. 1. Gyrat corpus in circumferentia circuli requiritur lex vis centripetæ ~~gravitatis~~ tendentis ad punctum aliquod in circumferentia.



Esto circuli circumferentia SQPA, centrum vis centripetæ ~~gravitatis~~ S, corpus in circumferentia latum P locus proximus in quem movebitur Q. Ad SA diametrum et SP demitte perpendicula PK QT et per Q ipsi SP ~~age~~ parallelam age LR occurrentem circulo in L et tangenti PR in R, et coeant TR, PR in Z. Ob similitudinem triangulorum ZQR, ZTP, SPA erit RPq (hoc est QRL) ad QTq ut SAq ad SPq. Ergo $\frac{QRL \times SPq}{SAq} = QTq$. Ducantur hæc æqualia in $\frac{SPq}{QR}$ et punctis P et Q coeuntibus scribatur SP pro RL. Sic fiet $\frac{SPqc}{SAq} = \frac{QTq \times SPq}{QR}$. Ergo vis centripeta ~~gravitas~~ reciprocè est ut $\frac{SPqc}{SAq}$, id est (ob datum SAq) ut quadrato-cubus distantiæ SP. Quod erat inveniendum.

Schol. Cæterum in hoc casu et similibus concipiendum est quod postquam

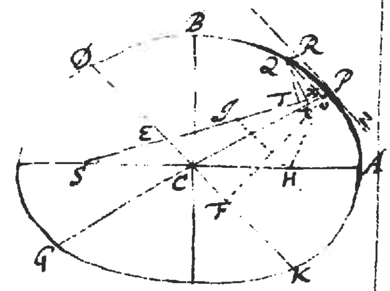
note: gravitatis

gravitas

postquam corpus pervenit ad centrum S, id non amplius redibit in orbem sed abibit in tangente. In spirali quae secat radios omnes in dato angulo vis centripeta tendens ad spiralis principium est in ratione triplicata distantiae reciprocè, sed in principio illo recta nulla positione determinata spiralem tangit.

gravitas

Prob. 2. Gyrat corpus in Ellipsi veterum: requiritur lex ~~gravitatis~~ vis centripetae tendentis ad centrum Ellipseos.



Sunto CA, CB semi-axes Ellipseos, GP, DK diametri conjugatae, PF, Qt perpendicula ad diametros, Qv ordinatim applicata ad diametrum GP et QvPR parallelogrammum. His constructis erit (ex Conicis) PvG ad Qv^2 ut PC^2 ad CD^2 et Qv^2 ad Qt^2 ut PC^2 ad PF^2 et conjunctis rationibus PvG ad Qt^2 ut PC^2 ad CD^2 et PC^2 ad PF^2, id est vG ad $\frac{Qt^2}{Pv}$ ut PC^2 ad $\frac{CD^2 \times PF^2}{PC^2}$. Scribe QR pro Pv,

a et $BC \times CA$ pro $CD \times PF$, nec non (punctis P et Q coeuntibus) 2PC pro vG et ductis extremis et medijs in se mutuò, fiet $\frac{Qt^2 \times PC^2}{QR} = \frac{2BC^2 \times CA^2}{PC}$. Est ergo vis centripeta reciprocè ut $\frac{2BC^2 \times CA^2}{PC}$ id est (ob datum $2BC^2 \times CA^2$) ut $\frac{1}{PC}$, hoc est directè, ut distantia PC. Q.E.I. — a Per Lem. 4

gravitas

Prob. 3. Gyrat corpus in ellipsi: requiritur lex vis centripetae ~~gravitatis~~ tendentis ad umbilicum Ellipseos.

Esto Ellipseos superioris umbilicus S. Agatur SP secans Ellipseos diametrum DK in E. Patet EP aequalem esse semiaxi majori AC eò, quod actâ ab altero Ellipseos umbilico H linea HI ipsi EC parallela, ob aequales CS, CH aequentur ES, EI adeo ut EP semisumma sit ipsarum PS, PI id est (ob parallelas HI, PR et angulos aequales IPR, HPZ) ipsarum PS, PH quae conjunctim axem totum 2AC adaequant. Ad SP demittatur perpendicularis QT. Et Ellipseos latere recto principali (seu $\frac{2BC^2}{AC}$) dicto L, erit $L \times QR$ ad $L \times Pv$ ut QR ad Pv id est ut PE (seu AC) ad PC. et $L \times Pv$ ad GvP ut L ad Gv et GvP ad Qv^2 ut CP^2 ad CD^2, et Qv^2 ad Qx^2 puta ut M ad N et Qx^2 ad QT^2 ut EP^2 ad PF^2 id est ut CA^2 ad PF^2 sive [a] ut CD^2 ad CB^2. et conjunctis his omnibus rationibus, $L \times QR$ ad QT^2 ut AC ad PC + L ad Gv + CP^2 ad CD^2 + M ad N + CD^2 ad CB^2, id est ut $AC \times L$ (seu $2BC^2$) ad $PC \times Gv$ + CP^2 ad CB^2 + M ad N, sive ut 2PC ad Gv + M ad N. Sed punctis Q et P coeuntibus rationes 2PC ad Gv et M ad N fiunt aequalitatis: Ergo et ex his composita ratio $L \times QR$ ad QT^2. Ducatur pars utraque in $\frac{SP^2}{QR}$ et fiet $L \times SP^2 = \frac{SP^2 \times QT^2}{QR}$. Ergo vis centripeta reciprocè est ut $L \times SP^2$ id est in ratione duplicata distantiae SP. Q.E.I. — a Per Lem. 4

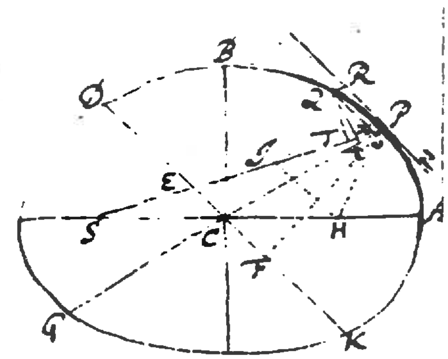
gravitas

Scholium. Gyrant ergo Planetae majores in ellipsibus habentibus umbilicum in centro solis, et radijs ad solem ductis describunt areas temporibus proportionales, omninò ut supposuit Keplerus. Et harum Ellipseon latera recta sunt $\frac{QT^2}{QR}$, ~~existentibus lineis QT, QR~~ punctis P et Q spatio quam minimo et quasi infinitè parvo distantibus.

Theorem. 4.

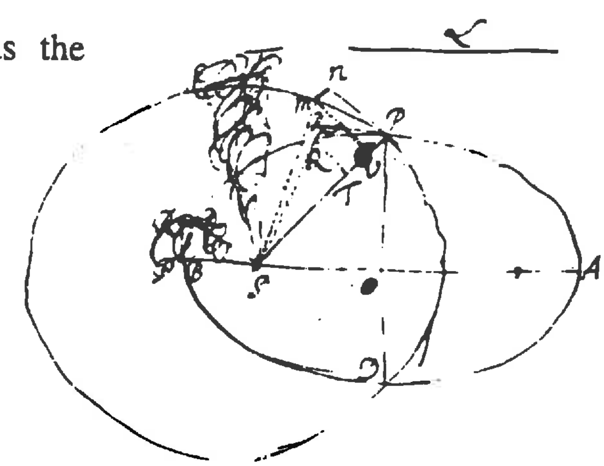
De Motu Corporum in Gyrum

Problem 3. A body orbits in an ellipse: there is required to find the law of centripetal force tending to a focus of the ellipse.



Scholium. The major planets orbit, therefore, in ellipses having a focus at the center of the Sun, and with their *radii* drawn to the Sun describe areas proportional to the times, exactly as Kepler supposed.

Theorem 4. Supposing that the centripetal force be reciprocally proportional to the square of the distance from the center, the squares of the periodic times in ellipses are as the cubes of their transverse axes.



Theorem 4. 3/2 Power Rule Holds for Confocal "Keplerian" Ellipses

From Problem 3 and Stipulation

- $L \times QR = QT^2$; $2SP \times MN = MV^2$
- Force at P is the same, so effect of force in same time the same: $QR = MN$
- Latus rectum $L = PD^2/AB = PD^2/2SP$

Proof

- $L/2SP = QT^2/MV^2$
- So, $QT/MV = PD/2SP$
- Area-SPQ/Area-SPM = PD/2SP =
  = $(\frac{1}{4}\pi AB \times PD)/(\pi SP^2)$ =
  = Area-ellipse/Area-circle
- **But then *incremental times* always in this ratio, so that the *period* for the ellipse = the *period* for the circle, and hence the conclusion follows from Corollary 5 of Theorem 2**

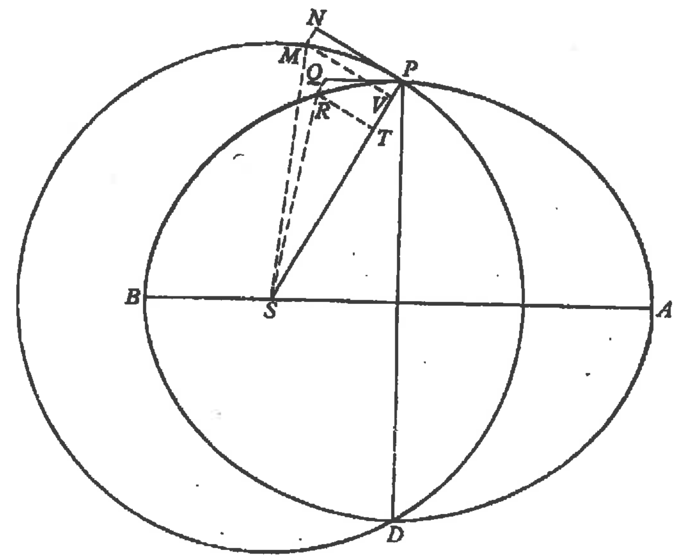
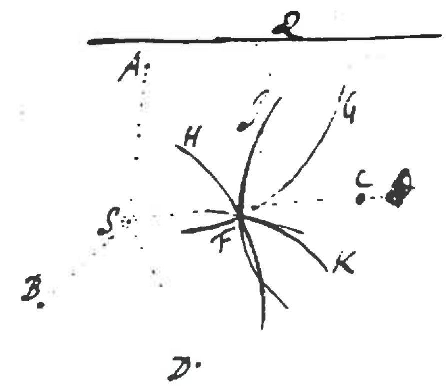


Fig. 63

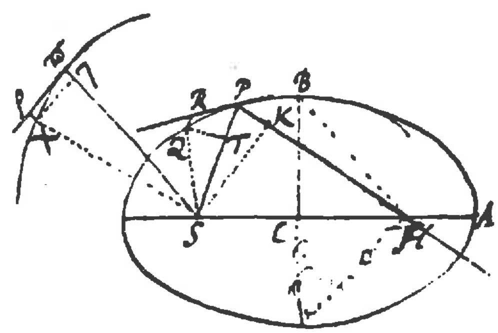
Note: Q and R reversed as above in the original

De Motu Corporum in Gyrum

Scholium. Hereby in the heavenly system from the periodic times of the planets are ascertained the proportions of the transverse axes of their orbits. It will be permissible to assume one axis: from that the rest will be given. Once their axes are given, however, the orbits will be determined in this manner.



Problem 4. Supposing that the centripetal force be reciprocally proportional to the square of the distance from its center, and with the quantity of the force known, there is required the ellipse which a body shall describe when released from a given position with a given speed following a given straight line.



$$F_{cent} \propto \frac{[a^3/P^2]_S}{r_{SP}^2}$$

Scholium. A bonus, indeed, of this problem, once it is solved, is that we are now allowed to define the orbits of comets, and thereby their periods of revolution, and then to ascertain from a comparison of their orbital magnitude, eccentricities, aphelia, inclinations to the ecliptic plane, and their nodes whether the same comet returns with some frequency to us.

Problem 4. Solution for the "Initial-Value" Problem

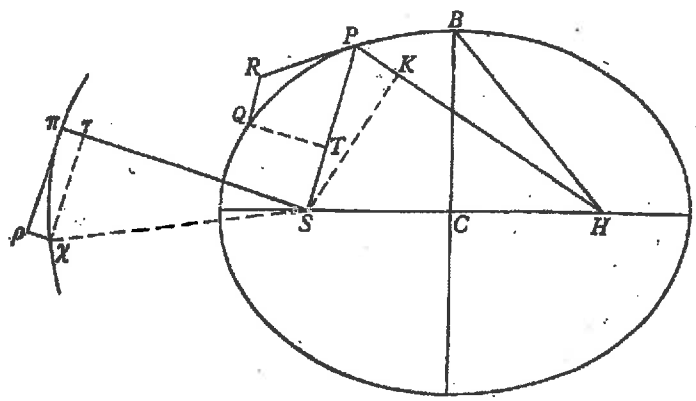


Fig. 66

Given: **The velocity PR, in direction and magnitude, at P; the strength of the inverse-square centripetal tendency toward S: $(a^3/P^2)_S$. That then gives the areal velocity of P about S and the uniform motion in the circle πχ about S.**

From Prop. 3 then: **$QT^2/QR : \chi\pi^2/\chi\rho :: L : 2S\pi$, and so L, the latus rectum of the trajectory, is given**

From the geometry of the ellipse, **∠ RPH = 180° – ∠RPS, and so the line PH, from P toward the other focus H, is given in direction, leaving only the problem of finding its length.**

From a series of steps,

$$(SP + PH)/PH = (2SP + 2KP)/L$$

and so the length PH is given, determining the location of the other focus H, and hence too the length of the major axis = (SP+PH) and its direction relative to S, P, and PR.

If **L = (2SP + 2KP), then the trajectory is a *parabola*;**
L > (2SP + 2KP), then the trajectory is an *hyperbola*.

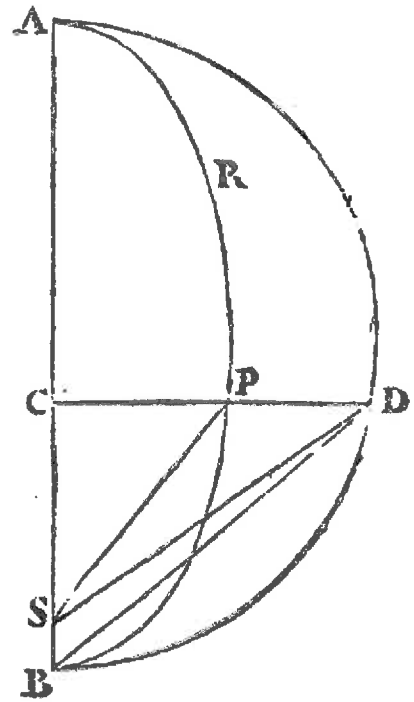
Vertical Descent and Ascent under Inverse-Square Forces

Problem 5

time $\propto$ area **ASP**

$\propto$ area **ASD**

diminish the orbit **ARPB** indefinitely until it coincides with **ACB** and **S** comes to coincide with **B**

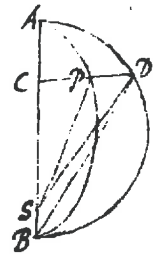


HKM (per tabulam segmentorum vel secus) æquale fiat triangulum SKN. Ad EG demitte perpendiculum NQ, et in eo cape PQ ad NQ ut Ellipseos axis minor ad axem majorem et erit punctum P in Ellipsi atque acta recta SP abscindetur area Ellipseos EPS tempori proportionalis. Namque area HSNM triangulo SNK aucta et huic æquali segmento HKM diminuta fit triangulo HSK id est triangulo HSC æquale. Hæc æqualia adde areæ ESH, fient areæ æquales EHNS et EHC. Cum igitur Sector EHC tempori proportionalis sit et area EPS areæ EHNS, erit etiam area EPS tempori proportionalis.

gravitas

Prob. 5. Posito quod vis centripeta sit reciproce proportionalis quadrato distantiæ a centro, spatia definire quæ corpus recta cadendo datis temporibus describit.

Si corpus non cadit perpendiculariter describet id Ellipsin puta APB cujus umbilicus inferior puta S congruet cum centro terræ. Id ex jam demonstratis constat. Super Ellipseos axe majore AB describatur semicirculus ADB et per corpus decidens transeat recta DPC perpendicularis ad axem, actisque DS, PS, erit area ASD areæ ASP atque adeo etiam tempori proportionalis. Manente axe AB minuatur perpetuo latitudo Ellipseos, et semper manebit area ASD tempori proportionalis. Minuatur latitudo illa in infinitum et Orbita APB jam coincidente cum axe AB et umbilico S cum axis termino B descendet corpus in recta AC et area ABD evadet tempori proportionalis. Definietur itaque spatium AC quod corpus de loco A perpendiculariter cadendo tempore dato describit si modo tempori proportionalis capiatur area ABD et a puncto D ad rectam AB demittatur perpendicularis DC. Q.E.F.

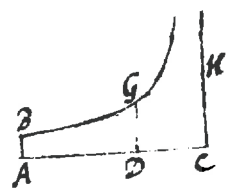


Schol. Priore Problemate definiuntur motus projectilium in aere nostro, hoc motus gravium perpendiculariter cadentium ex Hypothesi quod gravitas reciproce proportionalis sit quadrato distantiæ a centro terræ quodque medium aeris nihil resistat. Nam gravitas est species una vis centripetæ.

gravitas

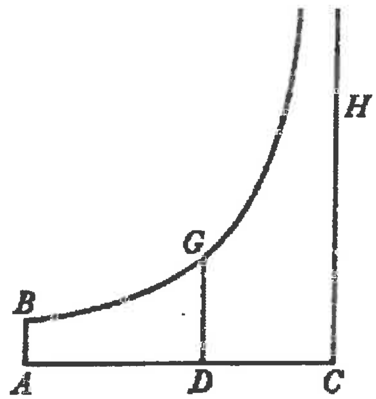
Prob. 6. Corporis sola vi insita per medium similare resistens delati motum definire.

Asymptotis rectangulis ADC, CH describatur Hyperbola secans perpendicula AB, DG in B, G.



Exponatur tum corporis celeritas tum resistentia medii ipso motus initio per lineam AC elapso tempore aliquo per lineam DC et tempus exponi potest per aream ABGD atque spatium eo tempore descriptum per lineam AD. Nam celeritati proportionalis est resistentia medii et resistentiæ proportionale est decrementum celeritatis, hoc est, si tempus in partes æquales dividatur, celeritates ipsarum initiis sunt differentiis suis proportionales. Decrescit ergo celeritas in proportione Geometrica dum tempus crescit in Arithmetica. Sed tale est decrementum lineæ DC et incrementum areæ ABGD, ut notum est. Ergo tempus per aream et celeritas per lineam illam recte exponitur. Q.E.D. Porro celeritati atque adeo decremento celeritatis proportionale est incrementum spatii descripti sed et decremento lineæ DC proportionale est incrementum lineæ AD. Ergo incrementum spatii per incrementum lineæ AD, atque adeo spatium ipsum per lineam [illegible]

Problem 6. Horizontal Motion with Resistance $\propto v$



In the figure, represent

time **by the increasing area under the rectangular hyperbola BG**

distance **by the increasing length AD**

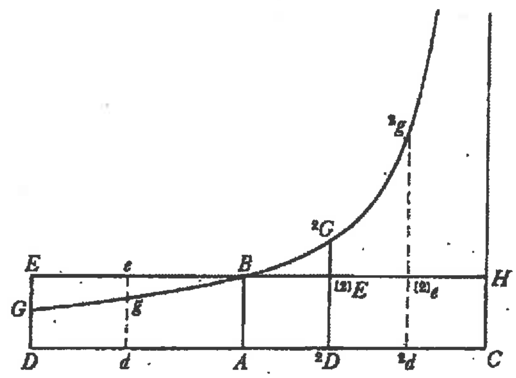
velocity **(and *resistance*) by the decreasing length DC**

i.e.

$$x = \left(\frac{u_0}{k}\right)(1 - e^{-kt})$$

insofar as the *velocity* and hence the *resistance* decrease in a *geometrical progression* as the *time* increases in an *arithmetical progression*

Problem 7. Vertical Motion with Resistance $\propto v$



<u>In the figure</u>, in ascent represent

centripetal force **by the area of the rectangle ABHC**

resistance at the start of ascent **by the area ABED taken in the opposite way**

time **by the increasing area DGgd**

distance **by the increasing area EGge**

velocity **(and *resistance*) by the decreasing area ABed**

<u>In the figure</u>, in descent represent

time **by the increasing area AB^2G^2D**

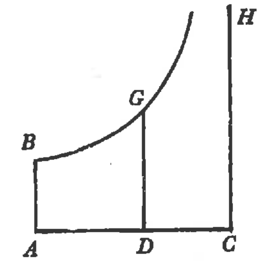
distance **by the increasing area B^2E^2G**

velocity **(and *resistance*) by the increasing area AB^2E^2D**

terminal velocity **by the area BACH**

In the figure, $AC = u/\lambda$, which 'expresses' (is proportional to) both the initial resistance $u\lambda$ and the velocity u: and $DC = a - x = \dot{x}/\lambda$ expresses the velocity at time t. From (i) the time is given by

$$t = \frac{1}{\lambda}\log\frac{u}{u-\lambda x} = \frac{1}{\lambda}\log\frac{a}{a-x}$$



which is the area $ADGB$ of the hyperbola, if $\lambda = 1/ab$.

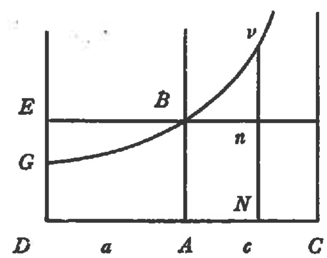
The figure is therefore a graph of the *reciprocal* of the velocity co-ordinated with the space described ($y = ab/\dot{x}$, x = space described), so that the area $\int y\,dx$ of the graph expresses the time t.

(4) This statement is the heart of the argument: the increment dt of the time is constant, so that $-d\dot{x}$, the decrement of the velocity, satisfies

$$-d\dot{x} = \lambda\dot{x}dt,$$

by the Second Law of motion, the operating force being the resistance $\lambda\dot{x}$.

(5) This is known in either of two ways: (i) from Napier's original definition of a logarithm. If D moves from A to C, with a velocity always proportional to DC, and simultaneously if d moves from a in another straight line, but with a constant velocity, then ad is proportional to log AD. Also (ii) from Grégoire de St Vincent's geometrical discovery (1647), which, in effect, replaces the second motion, that of d, by the hyperbolic graph. In fact D moves 'geometrically', and d 'arithmetically' (in Napier's phraseology) and the area $ADGB$ increases 'arithmetically'. Cf. Napier, *Mirifici Logarithmorum Canonis descriptio* (Edinburgh, 1614), or English Translation (1616), p. 2; and Grégoire de Saint Vincent, *Geometricum Quadraturæ Circuli et Sectionum Coni* (Antwerp, 1647).

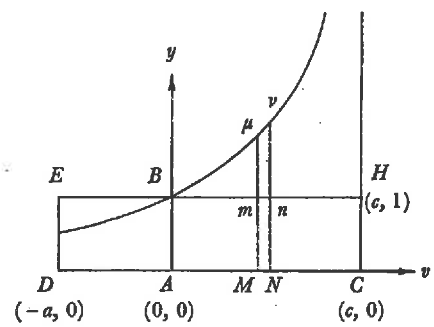


(6) The graph $B\mu v$ is the hyperbola $y(c-v) = c$, with A as origin, $AM = v$, $M\mu = y$, $MN = dv$, $AC = c$, and $AB = 1$. Newton invokes the unusual combination of plotting y, proportional to the reciprocal of the acceleration, against v, the velocity: for then at a time t, proportional to the area $AB\mu M$, the space s traversed by the projectile is proportional to the area $B\mu m$. This follows from the equation of motion (reckoned downward)

$$dv/dt = g - \lambda v, \tag{1}$$

where g is the 'ever-equal centripetal force'—gravity, and λv is the resistance (λ = constant). This integrates as

$$v + \lambda s = gt, \tag{2}$$



where s, t, v are the space traversed, the time, and the velocity, all reckoned from the highest point of flight at which v, s, t vanish. If $c = g/\lambda$ and $dt/dv = y/g$, then (1) becomes the equation $y(c-v) = c$ of the hyperbola. But $g\,dt = y\,dv$; so that $\int y\,dv = gt = v + \lambda s$ by (2): that is the area $AB\mu M = v + \lambda s$. Hence this whole area is gt and the parts, BM and $B\mu m$, are v and λs.

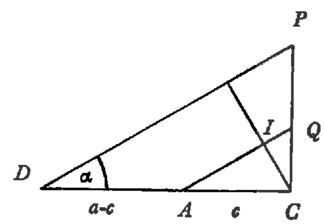
Since $MC = c - v = (g - \lambda v)/\lambda$, MC is proportional to the 'absolute force', the resultant of gravity and resistance at the time t. If $DA = a$, then a is the initial velocity of upward ascent (at a time when t is negative). In Fig. 2, p. 457, Newton marked F as E (ULC. Add. 3965 (7), fo. 62).

(7) The curve $DarFK$ is the trajectory of a body moving, under constant acceleration g vertically downwards, against a resisting force λv directly opposed to the motion. Take D as origin, $DR = x$, $Rr = y$, u = initial velocity of projection at an elevation α, the angle ADP. Then

$$\ddot{x} = -\lambda\dot{x}, \quad \ddot{y} = -g - \lambda\dot{y}, \quad \text{so that} \quad \dot{x} = -\lambda x + u\cos\alpha \quad \text{and} \quad \dot{y} = -gt - \lambda y + u\sin\alpha. \quad \text{(i)}$$

Let $DC = a$, $AC = c$, $CH = 1$, so that $DP = a\sec\alpha$, which 'represents' the initial velocity (in direction and magnitude): say

$$u = \lambda a \sec\alpha. \quad \text{(ii)}$$



Newton defines CI by $DA:CI::\lambda u:g$, giving $CI = (a-c)g/u\lambda$. (There is an error in placing I on DP in the figure: it should be on a line through A parallel to DP, so that $CI = c\sin\alpha$.) Hence

$$(a-c)g/u\lambda = c\sin\alpha. \quad \text{(iii)}$$

He takes the hyperbola

$$y(a-x) = c \quad \text{(iv)}$$

which passes through the point $B(a-c, 1)$. Then $EG = 1 - c/a$. He then takes

$$N = EG.DC/CP = (1 - c/a)\cot\alpha,$$

on using (ii) and (iii).

His equation for the trajectory is

$$y = \frac{1}{N}\left(x - c\log\frac{a}{a-x}\right) \quad \text{(v)}$$

and for the time is

$$t = c\log\{a/(a-x)\}; \quad \text{(vi)}$$

so that $Ny = x - t$. This requires that $c = 1/\lambda$ and $N = \lambda/g$: for then the equations (v) and (vi) agree with the equations (i). Also (iii) is verified so that $CI = c\sin\alpha$ (see second figure).

The maximum of y (when $x = DA = a - c$) follows from $\dot{y} = 0$, so that $\dot{x} = 1$: but $\dot{x} = \lambda(a-x)$ by (i) and (ii). Hence, at A, $x = a - 1/\lambda = a - c$. At F, $y = 0$ and $x = t$, so that the areas $DFsE$, $DFSBG$ are equal. (vii)

The velocity at the point r is represented in magnitude and position by the vector rL, that is $v = \lambda.rL$, which is true since $\dot{x}$, its horizontal component, is $\lambda(a-x) = \lambda.RC$. (viii)

Finally, if $DF = x'$, the gradient of the curve (v) at F is, by differentiation,

$$\frac{1}{N}\left(1 - \frac{c}{a-x'}\right) = -\frac{sS}{N}, \quad \text{since } Fs = 1 \text{ and } FS = \frac{c}{a-x'}$$

by (iv), and at D(where $x = 0$) the gradient is

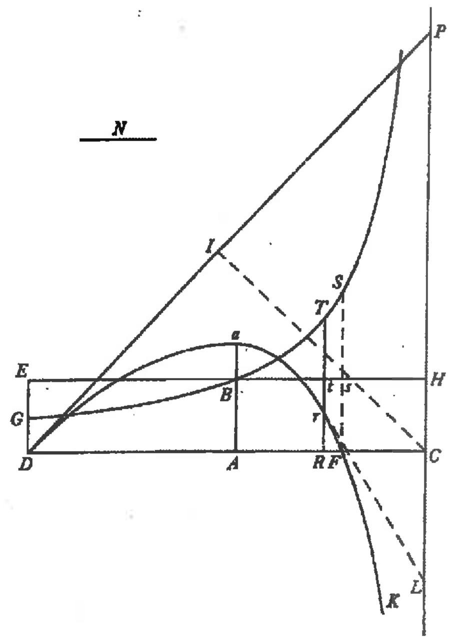
$$\frac{1}{N}\left(1 - \frac{c}{a}\right) = \frac{EG}{N};$$

which verifies Newton's statement concerning the ratio of sS to EG. (ix)

(8) See note (7) equation (ii).

(9) See (iii).

Solution for Projectile Motion with Resistance $\propto v$



<u>In the figure</u>, represent

The initial velocity by DP, and the ratio of the initial resistance to gravity by DA/CI.

Erect the perpendicular AB of any length, complete the rectangle DABE, and construct a rectangular hyperbola through B with asymptotes DC and CP.

Define N such that N/EG = DC/CP.

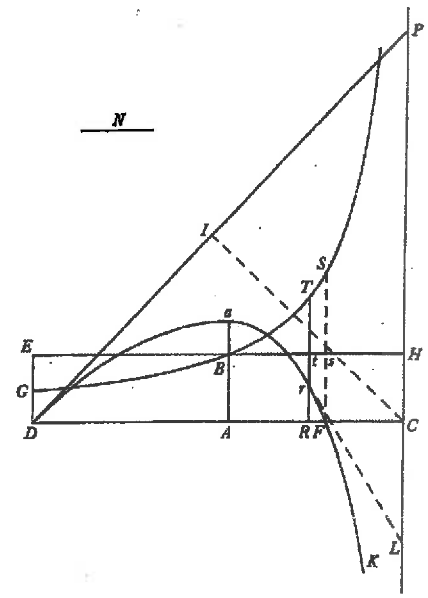
Then at any point R on the horizontal DC, r is given by

Rr = (area-DRtE – area-DRTBG)/N

The projectile will reach point r along the trajectory DarFK at *time* DRTBG, reaching the horizontal at F where the two areas become equal.

The *velocity* at every point is given by the tangent to the curve extended to CP, e.g. rL

To Determine Resistance for a Given Sphere and v



Measure the angles ADP and the angle AF*r* at impact

On DF form the rectangle DFsE of any height.

Construct a rectangular hyperbola with asymptotes DC and CP such that DfsE = DFSBG, sS/EG = tan(AF*r*) / tan(ADP)

From point B where the hyperbola crosses the line Es, draw a line perpendicular to DC intersecting it at A.

Then DA/CI gives the ratio of the initial resistance to gravity.

<u>To extend this result to other spheres</u>, let

The *deceleration from resistance* vary directly as the *density* of the fluid, the *surface area* of the sphere, and its *velocity*, and inversely as the *weight* of the sphere

Principal Results from the Registered Version
De Motu Corporum in Gyrum

- **A sufficient condition for Kepler's area rule to hold exactly**
- **A necessary and sufficient condition for Kepler's 3/2 power rule to hold exactly for multiple bodies moving uniformly in concentric circles**
- **A necessary condition for bodies to be moving exactly in ellipses in which all departures from uniform motion in a straight line are directed toward a focus of the ellipse**
- **A sufficient condition for Kepler's 3/2 power rule to hold exactly for multiple bodies moving in confocal ellipses**
- **A solution for the motion of a projectile along a conic-section trajectory under a $1/r^2$ centripetal force that in principle can be applied even to comets**
- **A solution for vertical fall under a $1/r^2$ centripetal force that allows the difference between this rule and uniform acceleration in free fall to be determined**
- **A solution for Galilean motion under resistance forces that vary linearly with velocity which, in principle, allows the differences between Galileo's solutions for free fall and projectile motion and the corresponding motions in resisting media to be calculated**

So what?

What conclusions about the actual world do these results enable anyone to draw that could not have been drawn before them?