NUMBER CONCEPT:

THEORETICAL AND EMPIRICAL VIEWS OF NUMBER PROCESSING

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Gabrielle A. Cayton

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Advisor: Prof. Bárbara M. Brizuela

Abstract

Studies of children's numerical understanding over the last decade suggest that there are identifiable progressions in how children develop number concepts. This paper provides a review of the leading research on number concept and how it relates to the research on external representation of number and number processing models. It is evident from this review that, while there are many hypotheses surrounding the processing, representation, and interpretation of numbers, much still has to be learned about this subject in young children. While there are several theories about how children come to understand the number system using various metaphors such as computer programs that learn through experience, currently, the data on young children has yet to provide any conclusive evidence regarding the early childhood number concept and how it gradually develops. There is a strong need for research in which aspects of the leading number processing theories may apply to children in their acquisition of a full-fledged number concept.

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I. Introduction

Studies of children's numerical understanding over the last decade suggest that there are identifiable progressions in how children develop number concepts such as ordinality, cardinality, place value, numerical representations, and others (Cayton, 2008; Cobb, Gravemeijer, Yackel, McClain, & Whitemack, 1997; Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema, 1997). According to Fuson et. al. (1997), children construct meanings for numbers through the various interactions that they have with these numbers both in and out of school. Elementary school mathematics classrooms encourage or facilitate the development of various number concepts through the language that is used by the teachers and students, the type of materials that are used, the types of problems that are solved, and the class activities. These components act in concert with one another to support what is commonly referred to as number concept. However, little is actually known about how children cognitively process numbers, and what relationship, if any, this has to a child's external representations of number[1]. According to Vergnaud (1997), however, the psychological definition of a concept cannot be reduced to simple mathematical definition. Vergnaud goes on to say that "the concept of number cannot be isolated from its functions and properties" (p.15) and he posits that in order to analyze mathematical concepts,

[1] By external representations of number, I mean numbers that are written (be it by numerals, number words, or other markings and drawings), spoken, or any other form of number that is created that can be seen or heard by others. This is taken largely from the definition put forth by Zhang (1997) when he states that external representations are defined as "the knowledge and structure in the environment … and as external rules, constraints, or relations embedded in physical configurations." Zhang continues that "the information in external representations can be picked up, analyzed, and processed by perceptual systems alone" (p.179)

we must consider the invariant properties of the concept, the situations that give meaning to the concept, and the symbols used in the representation.

In addition to simply being symbols, external representations of number are important because they can be seen and heard by others and we can therefore study them directly, as opposed to internal representations of number which cannot be accessed directly by others and are, therefore, difficult to study. This internal-external dichotomy may seem arbitrary (see Nemirovsky, in press) due to the fact that all “internal” representations are inextricably tied to some sort of manifestation or external representation of that concept and that all external representations reflect to some extent the internal representations as well. However, according to Vygotsky (1978), internalization is a central process of development. Vygotsky argued that any higher mental function was external and social before it was internal. While a number of other theorists have a slightly different take on this topic (see Cobb, 2005, Rogoff, 1990; Von Glasersfeld, 1995), it is generally agreed that the interaction between the internal and the external is not a simple one to explain. In dealing with numbers, this is no different. Regardless of whether it is in the external that precedes the internal or vice versa, understanding the interaction between the two is vital to truly understanding. The dichotomy is also in that it helps us isolate different aspects related to number concept and how to study them more thoroughly.

One important and interesting aspect of numbers is that they can be represented through many different formats: written numerals, oral numbers, arrays of dots, tallies, and others. However, these are just the representations that others can see, hear, and reproduce. How does the mind represent numbers? This is an important question as its answer may be able to give us insight into the teaching of numbers and the understanding of numbers, as well as the origin of mathematics itself. It is for this reason that the answer to this question has been sought by both theorists and experimentalists in the fields of education, psychology, cognition, neuroscience, mathematics, and many others (see Dehaene, 1997; Fayol & Seron, 2005; Lakoff & Nunez, 2000). Unfortunately, for those trying to answer this question, we cannot directly access internal representations of number. We can, however, access those external representations mentioned earlier, each of which is, in part, an embodiment of the internal. It is for this reason that we must separate and isolate the internal and the external to better understand each component that comprises our numerical understanding. Before we can claim to have a complete answer to the question of how the mind represents numbers, we must first look beyond each separate field of knowledge and begin by combining the work done by so many. What is agreed upon? What theories and experiments lie in contradiction to each other? What can we see by putting together the big picture?

In this paper, I do not seek to answer all of these questions, but to focus on the smaller component of "number concept." Specifically: How is number concept defined in psychology, education, and mathematics and how can we see a number concept reflected in numerical representation? Since a number must not only be represented internally but also subsequently be represented externally so that we can communicate this to others, I will also explore the noesis and semiosis of numbers. That is, how does the mind encode numbers and how are those numbers then decoded into tangible (external) representations? More specifically, I will be discussing oral and written representations, since those are the two most common and most studied external representations, and their individual relationships with the number concept as well as number processing theories in general.

II. Number Concept

Defining number concept

Many different definitions of "number concept" have been used over the past decades of research. According to Brannon and Van de Walle (2001), a "full-fledged" (p. 54) number concept contains two important and indispensable aspects: cardinality and the ability to represent ordinal relationships between numbers. Under this definition, once a child has acquired both of these aspects, they can be said to have acquired a complete number concept. This is similar to Piaget's (1952) belief that in order to have an operational concept of number, children must think that the actual property (number) remains identical with itself across transformations that do not affect the number (referred to as "conservation"). Munn (1998) adds that an expert's cognitive model of number incorporates number words, abstract number referents, and number symbols to "coordinate, direct, and activate numerical goals" (p. 53).

While the previous definitions all require that a person possess certain distinct components in order to have a complete number concept, other researchers have offered more nonfigurative definitions (i.e., definitions that are not easily defined by distinct and quantifiable components), such as that a number concept is the capability for flexible mental computations, numerical estimation, and quantitative judgment (Greeno, 1991), or that the number concept is the internal representation of a number, removed from either oral counting or written numerals (Fayol & Seron, 2005). This view of the "number concept" is also referred to by many as "nonverbal number representations" or "nonverbal numerical knowledge" (see Gallistel & Gelman, 2000). As Greeno (1991) says "we recognize examples of number sense even though we have no satisfactory definition that distinguishes its features" (p. 170). For the purposes of this paper, I will consider a "number concept" to be the internal representation of number as defined by Fayol and Seron (2005), that "which corresponds to entities that are internal to the subject and refer both to systems of numerical notation and to the numerosity of sets of objects or real or mental events" (p. 3) as it acknowledges that everyone has a number concept, regardless of how advanced or complete this concept is.

Now that we have a working definition of number concept, we can explore how this concept is acquired or created. It has been found numerous times that children learn to count long before they are able to grasp that each number in the counting sequence refers to a specific cardinality (Fluck, Linnel, & Holgate, 2005; Rousselle, Palmers, & Noel, 2004; Wynn, 1990, 1992). Knowing this, it can easily be argued that in order to have a number concept that extends beyond simply subitizing[2], a child must first learn the cognitive model of the natural numbers. That is, the number "one" represents the cardinality of a set containing one item, the number "two" represents the cardinality of a set containing two items, and so on. One theory of how a child learns this concept is by inductive inference (also called the "bootstrap" theory in reference to someone pulling themselves up by their bootstraps; see Carey, 2004). This theory states that a child first learns the concepts of one and two as stated above, then three in the

[2] Subitizing refers to the ability of recognizing the number of briefly presented items without actually counting (see Clements, 1999).

same manner. Next, the child makes the inference that the number after three in the sequence must indicate a cardinality equal to one more than three. Likewise, each number next in the counting sequence indicates a cardinality equal to one more than the number prior in the counting sequence. Once the child makes this leap, we can consider them to have mastered the "count-to-cardinal" transition (Fuson, 1988).

Rips, Asmuth, and Bloomsfield (2006) make a case against the bootstrap theory, however, claiming that the inductive logic Carey (2004) and others claim children use can be easily broken down if we consider other types of counting sequences, such as modular arithmetic where the next number in the sequence may be one that was previously learned. The argument put forth by Rips, Asmuth, and Bloomsfield, however, lacks strength as there is no evidence of any child actually being taught to count in a different system to know if the inductive logic behind bootstrapping would break down or if an attempt to teach a child to count via modular arithmetic would backfire due to the child's bootstrapping. Margolis and Lawrence (2007) provide a very extensive counterargument against the Rips, Asmuth, and Bloomfield's case, emphasizing the weightlessness of their counterclaims. As this debate is not meant to be a major focus in this paper, I will end my review of this dispute by saying that bootstrapping continues to be the leading theory behind the understanding of cardinality (Margolis & Lawrence, 2007) and that there has not yet been a convincible counter argument to this theory.

Oral numerical representations and number concept

Bootstrapping and the acquisition of a cardinality concept are only small pieces of the global number concept. As mentioned earlier, it is hard to discuss the number concept without also discussing external representations of number, as these are the byproducts through which we attempt to make sense of the internal concept. According to Vasco (2007), when you are trying to express something that you are thinking about, you unconsciously choose a semiotic register (such as the number system) and you search for an appropriate type of representation belonging to that register to express your idea (such a numeral or number word). Likewise, in interpreting the external representation, we must recognize what semiotic register was being used in the creation of the external representation.

As mentioned earlier, Brannon and Van de Walle (2001) assert that there are two components to the "number concept": cardinality and ordinality. These researchers go on to point out that both of these aspects are represented in the oral number system[3]. Therefore, we may be led to ask: Does the oral number system reflect the "number concept"? Or, using the terminology of Vasco (2007), is an oral number system a direct representation of the semiotic register that is being activated to represent a number?

This question, however, should not simply be stated as unidirectional. It may be more accurate to ask whether oral representations of number *reflect* a number concept of an individual and/or *impact* the number concept of an individual. As Brannon and Van de Walle point out,

[3] Brannon and Van De Walle refer to this system as the "verbal counting system." I will be adopting the term "oral number system" in this paper to reflect the convention of referring to "nonverbal" numbers as those that are neither oral *nor* written.

> the ability to correctly employ the verbal counting system enables one to encode and compare numerosities with a high degree of accuracy. The fact that oral counting is such a powerful, accurate tool for dealing with number and the fact that the use of such systems is culturally widespread raise the question of how it is related to nonverbal numerical knowledge. (p. 55)

Understanding this relationship is key to a better awareness of how children come to learn about numbers in general and to give researchers and educators a better grasp on assessing a child's knowledge as once we better understand the processes, we will be better able to teach to those processes.

Gelman and Gallistel (1978) suggest that non-verbal counting principles guide children's acquisition of oral counting:

> One could argue that skill in reciting count-word sequences precedes and forms a basis for the induction of counting principles. We, however, advance the opposite thesis: A knowledge of counting principles forms the basis for the acquisition of counting skills. (p. 204)

One observation Gelman and Gallistel made in support of this statement is that children generate their own counting strategies that differ from the left-to-right sequence that adults typically use in counting activities, suggesting that children are not simply mimicking a routine. While adults tend to count in a specific (left-to-right) order so as to count every item, children do not, often counting in a seemingly-random order

Supporting the notion that the oral number system and the number concept work in interaction is the evidence that the mastery of the cardinality concept appears to improve dramatically as children master oral counting (e.g., Brannon and Van de Walle, 2001; Lipton & Spelke, 2005; Sophian & Adams, 1987; Wynn, 1990, 1992). Others, however (e.g., Rouselle, Palmers, & Noel, 2004) disagree with this position. For instance, Brannon and Van de Walle (2001) claim that this phenomenon is simply a measure of two spontaneously developing concepts (cardinality and counting) that appear at approximately the same age (also termed the "language-irrelevant hypothesis").

To tease apart these two conflicting views, Brannon and Van de Walle (2001) conducted an experiment to assess ordinal numerical knowledge in 2- and 3-year-old children. One hypothesis under which they operated was that children may come to realize ordinal relations only after they have learned how the words in the oral counting sequence relate to numerosity. Termed the "strong language hypothesis," Brannon and Van de Walle hypothesize that children may become aware of the ordered positions of the words in the counting sequence and then come to understand that these ordered words represent specific ordered quantities in the real world. According to the strong language hypothesis, children should be entirely unable to make ordinal number comparisons until after they have attained a high degree of oral counting proficiency. The strong language hypothesis also predicts that the more number words children grasp, the greater range of ordinal comparisons they should be able to formulate.

An alternative hypothesis that Brannon and Van de Walle also put forth is the "weak language hypothesis." According to this alternative model, it is not that children *must* realize how the words in the oral counting sequence relate to numerosity before grasping the notion of ordinal relations, but that acquiring verbal quantification allows children to extend any existing preverbal representations they already possessed for small sets onto larger sets.

In conducting an experiment to tease apart the strong language hypothesis, weak language hypothesis, and language-irrelevant hypothesis, Brannon and Van de Walle found that children who completely lacked oral numerical knowledge were also unable to make any ordinal judgments about numbers. However, children with even minimal oral numerical competence were able to make ordinal judgments for numbers larger than that to which they could count. This implies that number words may become a relevant facet of number only as children learn to count. It is possible that an analog magnitude representation of number may underlie success on the ordinal task and that learning the oral counting sequence is a prerequisite for displaying knowledge of some numerical concepts. This most closely supports the weak language hypothesis.

However, if we look at other data, it may not be a language hypothesis that was tested in Brannon and Van de Walle's study at all. Multiple studies (e.g., Fayol & Seron, 2005; Rourke, 1993) have found that the transition from preverbal number knowledge to verbal knowledge is greatly dependent on the use of fingers. In fact, tactile skills of children at age five are better predictors of arithmetic performance at ages six and eight than cognitive development measures at the same age (Fayol & Seron, 2005).

While this may initially seem an outlandish comparison, fingers can be used to preserve the one-to-one matching relationship between the set and the (finger) representation. They may also aid in counting with the order of the fingers constituting a conventional sequence which children might associate with the ordinal numbers (Fuson, 1988). With larger numbers, they can also aid in the learning of base-ten as children start with the first finger again to represent eleven, second to represent twelve, and so on. Occasionally, finger number representations are actually referred to as being their own semiotic register (Vasco, 2007). Thus, in Brannon and Van de Walle's study, it is possible that certain children were better able to use their fingers to aid them in the task and thus it was not truly language-relevance that was being tested.

According to Rouselle and colleagues (Rousselle et al, 2004), however, the results of Brannon and Van de Walle argue in favor of their hypothesis as well: that numbers begin to be a salient dimension of the situation for children when they start to understand how oral numbers map onto numerosity independent of perceptual variables. They go on to state that,

> ... minimal counting knowledge is necessary in situations in which numerosity is the only relevant dimension to discriminate or compare sets (i.e., quantifying heterogeneous objects or collections that are very well controlled perceptually), but not in conditions in which perceptual features and number covary. In the latter case, children would rely on perceptual

variations and not need to abstract numerosity to resolve the problem. (p. 64)

One method used to test Rouselle et al's (2004) hypothesis was with a comparison task. Children were shown two sets of sticks and asked to indicate which set has more. If children possess a numerical mechanism for representing numerosity, their model would predict both a ratio and size effect. In fact, no size effect was observed at all. Thus, children did not rely on verbal counting to make the comparisons.

We cannot ignore, however, that oral numbers exist far beyond counting. When an individual who has mastered oral numbers hears an oral number, he/she need not count from one up to the number heard to extract its meaning. Once the system is understood, it exists on its own, without counting being necessary. Some theories propose that oral numbers exert an influence on nonverbal representations only at encoding stages while others propose that the oral representations play a role at more central processing stages[4].

In my opinion, based on the review of the literature and my own research (see Cayton 2007; Cayton & Brizuela, 2007, 2008), it appears that oral numbers do play a role in central processing. For example, people who speak languages that have very long number names tend to take longer to process numerical problems, while the opposite is true for speakers of languages with shorter number names (Dehaene, 1997). In addition, in our research we have found that at particular ages, oral number forms are more salient representations

[4] I discuss the various number processing models in depth below in Section III.

for children than other representations (e.g., Cayton 2007; Cayton & Brizuela, 2007, 2008; Cayton, Brizuela, & Gravel, 2008).

Written numerical representations and number concept

According to Vasco (2007), one important step in understanding numbers and numerosity is to distinguish concepts as mental artifacts from signs as amphibious artifacts. By amphibious he means those artifacts that are both sometimes mental and therefore not directly observable and those that are sometimes materialized and therefore observable directly. Vasco also comments that a parallel step is to distinguish numbers as concepts from numerals as signs. He states, "are the concepts themselves just a particular type of mental sign internalized from the language of adults?" (p. 14).

This is extremely interesting and important because it deals directly with Piaget's (1962, 1965, 1971; Piaget & Inhelder, 1971, 1973) discussions of figurative and operative aspects of thought. In these writings, Piaget insists that there is a necessary interaction between figurative aspects and operative aspects of thought. In the use of numerical representations, the reproduction of the numbers themselves could superficially be thought of as figurative; and the number system as a conceptual object is certainly operative. However, we cannot understand the first without the second, just as Piaget thought of both aspects of thought as wholly interwoven and interconnected.

Though numerous attempts to introduce the distinction between numerals as signs and numbers as concepts in elementary mathematics curricula have been thwarted (Vasco, 2007), the distinction is still of the utmost importance when we look at children's representations of numbers. In the words of Bialystok and Codd (2000), "what do children believe that written representations of quantity mean?" (p. 117). This question is extremely important in the field of education, as it is possible that the words of teacher and student are incommesurable (see also Carey, 1991). That is, does the educator know what his/her words mean to the student?

One window through which to look at this topic is the concept of "zero" and the notation for "zero." Zero is a fitting window for this topic since the "zero" as a distinct number with a distinct representation was created by humankind much later than other quantities. It was not automatically obvious that zero is a quantity that could be represented (Bialystok & Codd, 2000; Dehaene, 1997). Children are also reluctant to accept that a set without elements can be a set, seeming to infer that zero is not an intuitive number of objects at all (Vasco, 2007). In fact, zero was used as a place holder long before it was treated as a number in its own right[5]. This may lead us to wonder if zero would exist as a number at all if a place-value number system were not also used. Thus, we may think of zero as a cultural construct just as much as a numerical concept. Therefore, its use and understanding may help us to understand the bridge between concept and notation since we can actually trace a number, notation, and numerical idea from its conception.

As stated in the previous paragraph, zero would potentially not exist at all if it were not for the use of a place-value number system such as our Arabic

[5] The use of zero as a place holder was probably first developed around 1500 BCE by the Babylonians; however, the use of zero as a number occurred much later, probably not until at least 800 CE (Bialystok & Codd, 2000; Dehaene, 1997; Ifrah, 1998).

number system[6]. The Arabic number system creates an ideal balance between memory usage, ease of manipulation, and simplicity of understanding. This is likely why, though many numerical systems have existed throughout time, the world has converged on one system in a manner unprecedented by any other form of communication. However, when expecting children to converge upon the same understanding, it is important to keep in mind the thousands of years that it took for this system to develop and become the world's norm.

Place-value coding is also a necessity if one wants to perform calculations using simple algorithms (Dehaene, 1997). Consider calculations in the Greek/Roman numerical system. They are inconvenient for a variety of reasons; one apparent reason being that nothing indicates that N/L (50) is greater than E/V (5). For this reason, Greeks and Romans could never perform calculations without the use of the abacus. Place value allows, for us, 5, 50, 500, etc. to be of transparent magnitude. We must only memorize 10 digits and products 2x2 through 9x9.

Looking at the history of all numerical notations, we must consider Roman numerals. As Dehaene (1997) points out, the first three digits in the Roman number system are self-explanatory as one, two, and three lines certainly fit into the category of a symbol. However, after this, the numerals suddenly become signs[7], they add V as a notation for "five" and, moreover, utilize subtraction in IV standing for "four." The mystery only deepens when we look at *all* other known

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[6] Also referred to as the Hindu-Arabic number system

[7] Throughout this paper, I will be using the terms "sign" and "symbol" as distinguished by Saussure (1931) and Piaget (1965). Symbols are idiosyncratic and bear a resemblance to the objects represented, such as tally marks and pictures, while signs are arbitrary and do not resemble the object represented and have their source in convention, such as letters and most numerals.

numerical notation systems. Each system utilizes dots or lines for "one," "two," or "three" (Arabic notation has this history, too, as 2 and 3 were originally horizontal bars but became connected through handwritten shortcuts). This may not seem mysterious; however, most civilizations including Chinese, Ancient Indian, and Arabic chose to use a drastically different sign for "four" and beyond. This is likely due to our ability to easily subitize quantities below five while after five this becomes increasingly difficult, leading us to seek another method for representing quantities (Dehaene, 1997).

Dehaene (1997) discusses one possible link from our numerical cognition to our number system: it has been shown in numerous studies that human infants distinguish between one, two, and three objects, yet not much beyond this point (Starkey & Cooper, 1980; Strauss & Curtis, 1981). While our number system was not created by infants, this innate numerical intuition in childhood does have an extension to number discrimination in adults. Bourdon (1908, as cited in Dehaene, 1997) found that the time required to name the number of dots in an array grew slightly from 1 to 3, then increased sharply beyond this point. Thus, the decision to move from bars or dots to signs after "three" was likely influenced by number discrimination abilities. Distinguishing III from IIII at a glance is difficult (Dehaene, 1997)[8].

While four different civilizations—Babylonian, Chinese, Mayan, and Arabian—seem to have created place-value notation, the first three never

[8] While little has been written about the possible link between subitizing and bootstrapping, the inferences are obvious: perhaps we must learn to infer that each number in sequence represents a quantity of one more than the previous number since we have no other method of deduction; we cannot subitize larger numbers, so we must know how to count them.

achieved the simplicity of the last due to three other inventions that Arabic notation included: “zero,” a unique base number, and the discarding of the additive principle for the digits 1 through 9 (Dehaene, 1997). For example, the Babylonian system, the oldest known place-value system, had a base number of sixty. In principle, this would have needed sixty distinct symbols for 0-59. Since this is obviously impractical, the Babylonians wrote numbers using additive base-ten. The mixture of additive and place-value coding rendered the system understandable to only the elite few (Dehaene, 1997; Vasco, 2007).

In addition to the complex base system, since the Babylonian system was also lacking in a zero, the Babylonians would simply leave blank any unused places. This made some numbers only understandable through context (for example, it would even be unclear if a 1 was a 1 followed by a blank, or two blanks, and so on). Finally, in about 300 B.C. the Babylonians invented a place-holder symbol, although this symbol did not also have the same meaning of “zero” or “null quantity” that our “0” has today.

We can still see the concept/notation of zero as being problematic today. Recently, it has been found in numerous studies that transcoding zeros within numbers proves particularly problematic amongst young children (see Cayton, 2007; Cayton & Brizuela, 2007; Scheuer, Sinclair, Merlo de Rivas, & Tieche-Christinat, 2000; Seron & Fayol, 1994). These errors with zeros almost always involve the addition of zeros, such as 100502 to represent one hundred and fifty-two. Seron and Fayol (1994) noted that these errors can be either lexical or syntactical in nature. That is, they may be derived from semantics or from misunderstood production rules, respectively. We rarely, if ever, see these same errors in adults. One noted exception is described by Grana, Lochy, Girelli, Seron, and Semenza (2003) in a case study of a patient who suffered a left cerebrovascular accident. As a result, the patient was inclined to delete zeros from their number representations; this was in opposition to insertions of zeros within numerals, which are seen in young children. Thus, we may conclude that this representational error was of quite a different nature than those shown by children. Thus, it appears that zero-insertions likely have to do with the appropriation of this particular aspect of the number system and that, once the rules of zero have been entirely appropriated, we no longer exhibit difficulties of this kind.

To further understand this distinction, we turn to Gardner and Wolf (1983) who state that the main stepping stones of notational symbolization are the abilities to reduce a large corpus of information, to use elements and codes which are consistent across the system, and to produce notations themselves which are legible for other readers. However, according to Strauss and Stavey (1982), in the course of mastering notational systems children often lose, at least temporarily, some of the basic intuitions underlying a domain because they are so overpowered by the implicit demands of the system itself. This could also be explained in part by Goldin (1998) who claims that the interaction between internal and external representations is fundamental to effective teaching and learning. Therefore, as the internal and external systems interact, children may lose certain aspects of the domain until they realize that the particular aspect is necessary and reacquire it.

This was also shown in our own work, in which first graders who were conventionally able to represent four- and five-digit numbers erred in representing the same numbers when in the second grade (Cayton & Brizuela, 2008). This demonstrates that even by the second grade, children are still both losing and acquiring fundamental aspects of the number system. This is additionally problematic as by this age, children have already "learned" basic mathematical algorithms that rely on the understanding of the number system in its entirety. If these students have not yet fully appropriated all of the rues and elements of the system, they could not have possibly understood the algorithm as it was meant to be understood.

Recent research has also shown that written representations of number are related to number concept. Children who have not yet fully appropriated the written number system and the notion that a digit changes value depending on its place in a number attempt to represent these differing quantities in physical ways such as rotating the digits (see Alvarado, 2002; Alvarado and Ferreiro, 2002; Brizuela, 2004). In our own research (see Cayton 2007; Cayton & Brizuela, 2007, 2008), we have found that certain unconventional strategies for representing numbers appear to be related to differing number concepts as demonstrated by their relation to unconventional or conventional representations in other external systems. In addition, we have found that at particular ages, children that have mastered other forms of external representation are more likely to also write numbers conventionally than children that have not mastered these systems (e.g., Cayton 2007; Cayton & Brizuela, 2007, 2008; Cayton, Brizuela, & Gravel, 2008), pointing to a link between written numbers and some numerical concepts.

III. Models of number processing

Since it appears from the previous sections that there is, in fact, a relationship between external representations (both oral and written) and the number concept, the next step is to deduce where these connections take place. This movement from the external to the internal and the reverse can be considered "number processing." Better understanding of these processes may help us to further understand the internal/external connection. According to Zhang and Norman (1995):

> In complex numerical tasks, as well as in many other tasks, people need to process the information perceived from external representations and the information retrieved from internal representations in an interwoven, integrative, and dynamic manner. External representations are the representations in the environment, as physical symbols or objects (e.g. written symbols, beads of abacuses) and external rules, constraints, or reflections embedded in physical configurations. (Zhang & Norman, 1995, p. 279)

Zhang and Norman, in the quote above, have summed up the task of number processing, interpretation, and production. While mathematical cognition, as a field, has been in existence for a long time, models of number processing have received relatively little attention in comparison to language processing (Zorzi, Stoianov, & Umilta, 2005). In this section, I will lay out the leading theories of number processing and experimental evidence that favors and rejects some of these theories. To date, no theory exists without some of its components being called into question.

Some theorists believe that the integers constitute the foundation by which all other numerical concepts arise (see Wynn, 1998) while others believe that we can conceptualize real numbers as being on a continuous scale and integers being a discrete set of that scale (see DeCruz, 2005; Gallistel & Gelman, 2000). In 1994, Zhang and Norman developed a "theory of distributed representations" to account for behavior in *distributed cognitive tasks* (tasks that involve both internal and external representations) and concluded that the representation of a cognitive task is not solely internal or external but distributed over these two indispensable parts. Thus, external representations do not necessarily need to be re-represented internally: they directly activate processes. This theory most certainly applies to number, which, along with language, accounts for a majority of our distributed cognitive tasks.

Any model of number processing must account for the fact that educated adults are able to recognize and produce numbers in the written number system and the oral number system (Fayol & Seron, 2005). While there had been previous suggestions as to how this may occur, a formal theory behind the perception, production, and learning of numerical notation was put forth by Power and Longuet-Higgins in 1978. This theory describes the learning of the number system as analogous to a computer program that learns through example. According to this model, when people translate between oral numbers and Arabic numerals they construct an intermediate semantic representation. A complex numeral such as one thousand four hundred and thirty-two has the structure ((one thousand) ((four hundred) and (thirty-two))) which is articulated when read aloud. The writer must then transfer the representation created by the oral structure into a numeral using the known set of rules (or vice versa). Complicating this writing is also the mastery of an "over writing" (Power & Dal Martello, 1990; Seron & Fayol, 1994) operation that children must learn in order to avoid the addition of zeros. This overwriting comes hand in hand with the comprehension of a concept of place-value as the child must learn the manner in which to assure that each numeral fits into the correct "place" and when, as well as when not, a position in a number requires the inclusion of a zero. The Power and Longuet-Higgins model, however, does not provide much description regarding what happens with this intermediate semantic representation once the system of over-writing takes place. Does it adapt? One should expect that once the written number system is fully acquired, the internal representation should adapt to this instead of requiring over-writing each time a number is represented.

What is the extent to which different numerical skills involve independent cognitive systems? One prominent theory on this topic is the modular model developed by McCloskey and colleagues (McCloskey, 1992). McCloskey's model proposes three functionally distinct number processing systems: a number-processing system that recodes stimulus numbers into an abstract semantic code, a calculation system that includes memory for number facts, and a number production system that receives output from the comprehension or calculation systems and converts it to written or spoken responses. The three systems and related subsystems are assumed to be functionally and neurologically independent.

In this model, simple number repetition engages phonological input and output as well as syntactic processing mechanisms. Individual number words or elements gain lexical access. Syntactic processing then allows the determination of the relations between elements. In other words, the comprehension of an external number requires that it be translated into an internal, abstract representation, whereas the production of a number (in any external format) requires that its internal, semantic representation be translated into the proper external representation, or "output format." For instance, the comprehension of the numeral 47 entails the selection of the separate elements 4 and 7 and syntactic ordering into correct sequence.

Although McCloskey's model is both simple and comprehensive, the assumptions on which it has been based have been called into question on a number of accounts and some alternative architectures have been proposed (e.g., Campbell, 1994; Campbell & Clark, 1992; Dehaene, 1992). Dehaene (1992) introduced the "triple code theory" which proposes that number processing operates on three types of codes: a visual-Arabic form, an auditory-verbal code, and an analog magnitude representation. The Arabic form mediates digital input, output, and some multi-digit operations, the analog magnitude code provides the basis for numerical size comparisons and estimation, and the auditory-verbal code mediates verbal input and output as well as counting operations and memorized multiplication and addition facts. In this model, number processing proceeds independently of the initial notation after input. This differs from McCloskey's hypothesis since in the Dehaene model differences in mathematical performance can be attributable to different pathways invoked by various notations. Thus, Arabic numerals and number words could differ in their capacity to activate certain calculation subsystems. This implies that the type of representation that is used can significantly impact understanding and use of the number.

In fact, we know that Arabic numerals are more efficient than Roman and many other types of numerals for calculation, even though they all represent the same entities (see Cayton, 2007 for a complete review). Zhang and Norman (1995) point out that this "representational effect" (that different representations of the same abstract notion can lead to dramatically different cognitive behaviors) has had profound influence throughout history in the development of arithmetic, algebra, and mathematics in general.

Campbell and Clark (1992) also have a multiple-code theory. In this theory, there are verbal codes and nonverbal codes. All of these codes can directly activate each other without intervention by a central code. Each of these codes can also be used for any type of numerical processing. This model explicitly rejects the abstract form of number representation and the form on modularity in the McCloskey model (Noel & Seron, 1997). While Campbell and Clark (1992) do cite evidence from adult studies in favor of their model, it is hard to explain the evidence from children according to this model. If each of these codes can be used for any type of numerical processing and there was no mediating internal system, why would the children from Cayton's study (2007) as well as various other studies (see Brizuela, 2004; Scheuer et al., 2000; Seron & Fayol, 1994) demonstrate an appropriation of *different* external representations at *different* times?

DeLoache and Seron (1987) have a slightly different approach than Dehaene (1992). According to their model, the mechanisms that process the elements of the number system are conceptualized as psycholinguistic procedures operating in a stack-structured environment. That is, each lexical primitive is characterized by two sets of categorizing information: lexical class (i.e., unit, teen, or decade) and position value class (e.g., one through nine). This model, however, has no basis in experimentation. There is no data, for example, that shows a long processing time to distinguish between sixteen and seventy, which should be the case according to this model, as they only differ by one spot in each class of categorizing information. In fact, Seron himself, along with a collegue (Noel & Seron, 1993) found evidence of a patient whose number reading deficit conflicted with his own model when he was unable to decode arabic numerals but could produce them properly. He would then compare values on the basis of his (incorrect) method of decoding.

In contrast to the models above, Kamiloff-Smith's (1979) proposal of the use of representational systems is based directly on studies of children. In observing children learning both mathematical procedures and language skills, Karmiloff-Smith (1979) noted that "each time a procedure in a representational system is functioning adequately and automatically, the child steps up to a metaprocedural level and considers the procedure as a unit in its own right" (p.

91). Further, "each time children develop an adequate tool for representing their knowledge, and once the tool functions well procedurally, then the tool is considered metaprocedurally as a problem-space in its own right" (p.92). For example, in the case of place-value, once a child has fully appropriated the place-value system, the child is then able to use the system for arithmetic algorithms and for representing numbers he/she has never previously seen.

We can think of this experience as being related to the process/object distinction that Sfard (1991) and Dubinsky (1991) both describe in algebra. According to both Sfard and Dubinsky, there are two approaches to concept development, one operational focusing on processes, the other structural, focusing on objects. Once one has mastered the process of algebra, he/she can then use algebraic functions as objects in and of themselves. This process is also known as "reification." In this context, we can think of place value at first being a tool for numerical processes, but as performance with it evolves, it gradually (or perhaps suddenly) becomes an object.

While most of the models described above, as well as other models (see Cornet, Seron, & Deloache, 1988) treat number processing and arithmetical processing as one and the same, evidence exists to suggest that perhaps the two are quite dissociated. DeLoache and Seron (1987) note that there are cases of digit alexia without calculation difficulties and calculation difficulties without digit alexia. In addition, there is also evidence of oral acalculia without number alexia and the reverse and written acalculia without oral acalculia and the reverse (see also Pesenti, Depoorter, & Seron, 2000). It has also been found that incongruence between the physical size of numbers and their values affects numerical value comparisons for numerals but not alphabetical English forms of numbers (Besner & Coltheart, 1979, as cited by DeLoache & Seron, 1987).

DeLoache and Seron also describe a patient who was able to read numbers of one, two, and three digits, but neither three digit numbers containing zeros nor numbers with more than three digits. Number writing was also possible for this patient with one and two digit numbers, but not numbers of three digits or more. These difficulties imply that perhaps the children's appropriation of the number system is not a matter of knowledge building, but the combining of separate systems to work in interaction with one another, for if they were one and the same, it would be impossible to lose one part and keep another. This would help explain the kind of evidence provided by Cayton (2007), Brizuela (2004), Scheuer et al. (2000), and Seron and Fayol, 1994 which demonstrates an appropriation of *different* components at *different* times.

In an attempt to provide a number processing model that accounts for some of the above difficulties with each model, Noel and Seron (1997) propose that intermediate representations related to lexical primitives and the arithmetical relationships specific to the particular numeral presented are activated with all external numerical representations. These lexical and syntactic encoding mechanisms lead to an intermediate representation. They also leave open the possibility that there are other internal codes activated as well, aside from this one intermediate form. While Noel and Seron do provide evidence supporting this hypothesis, it is also a rather difficult hypothesis to dispute, as the "other internal codes" could produce almost any result possible.

Recently, we have found that many children are able to produce and interpret numbers in one external system (e.g., in writing or orally) without being able to produce or interpret the same number in a different system (Cayton, 2007; Cayton & Brizuela, 2007). The difficulties described by DeLoache and Seron above may be the result of a reemergence of this divide between systems of numerical representation. Alternatively, perhaps there is a separate, intermediary system whose role is transferring between external systems. In this scenario, children may come to slowly acquire this system as they learn about external representations, and it is the breakdown of this intermediary system that is seen in the above difficulties. This could be thought of as closely related to Zhang and Norman's (1994) "theory of distributed representations" discussed earlier. When Zhang and Norman state that representation of a cognitive task is not exclusively internal or external but distributed over these two parts, perhaps we can think of the intermediary system as being the "distributor" of the parts.

Now that we have looked at the most commonly cited models of number processing, perhaps it is time to take a step back and look specifically at number processing in children. The identification of very early abilities to compare and evaluate quantities has led researchers to believe that the first numbers are probably acquired very quickly and easily (Fayol & Seron, 2005). However, beyond this point, much of the field is up for debate.

Surprisingly, while there is a great deal of research on numerical understanding in prelinguistic infants, very little work has been done on numerical processing models in children who are beginning to read and write numbers. There is no logical reason for this gap, as children have repeatedly shown difficulties in transforming between mathematical story problems and symbolic mathematical expressions (Zhou, Wang, Wang, & Wang, 2006). Thus, it is clear that there must be some intermediary processes that occur in the transition of processing numbers as a prelinguistic infant to processing numbers as an adult.

One example of the existence of this intermediary processing system is that it can be very hard for young children to switch back and forth between instantiated and symbolic numerical models (Hughes, 1986) as well as between "count" and "measure" numbers (Munn, 1998). Adults expect children to see the logical similarities between the representation and the object, unaware that the inferences are not directly from one to the other, but through the cognitive model (Munn, 1998).

According to Tolchinsky (2008), the features of writing and written number systems are influential throughout the entire learning process, not only after they have been learned. We, therefore, learn the system by immersing ourselves in it. Sinclair and Tieche-Christinat (1992) found that children begin by having intuitions about the precise meaning of digits in particular positions by setting up part-to-part relations in the spoken number that correspond part-to-part to the notation. This finding is backed by Fuson and Kwon (1992) who propose that children learn spoken patterns and written patterns simultaneously, by relating a particular number-word to a particular numeral. However, understanding that the "2" in "26" corresponds to the word "twenty" still occurs many years before understanding that the "2" doesn't just signify to pronounce "twenty" but that because of its placement in the number the "2" itself actually represents the cardinal number twenty, which, further, is ten times more than 2 because of the position in which it is placed. In fact, Munn (1998) found that preschool children were able to create tally marks or similar number markings and use them to do simple addition problems, yet, once formal schooling began, many more children began using conventional numerals, yet ignored these numerals in their attempt to do the arithmetic. Thus, the school methods lead children to "rote-learn" the numeric forms, without truly relating them to quantity, in the eyes of the children.

In summary, there has not yet been a comprehensive model of number processing that encompasses all aspects of numerical comprehension: notation, calculation, production, and number sense. Or, in the words of Zhang and Norman, that shows how we "process the information perceived from external representations and the information retrieved from internal representations in an interwoven, integrative, and dynamic manner" (1995, p.279). We can conclude from this review that each of the described models are likely complementary, each encompassing components that are not contained in the others. Perhaps the true difficulty in creating a comprehensive number processing model is that we still have much to learn about both internal and external numbers and thus the interplay between the two is still out of our grasp.

IV. Conclusions

While there are many hypotheses surrounding the processing, representation, and interpretation of numbers, much still has to be learned about this subject in young children. While there are several theories about how children come to understand the number system using various metaphors such as computer programs that learn through experience, currently, the data on young children has yet to provide any conclusive evidence regarding the early childhood number concept and how it gradually develops. There is a strong need for research in which aspects of the leading number processing theories may apply to children in their acquisition of the "full-fledged" number concept that Brannon and Van de Walle (2001) define[9].

Though many numerical systems have existed throughout time, the world has converged on one system in a manner unprecedented by any other form of communication. This is likely due to the manner in which the Arabic number system creates an ideal balance between memory usage, ease of manipulation, and simplicity of understanding. However, when expecting children to converge upon the same understanding, it is important to keep in mind the thousands of years that it took for this system to develop and become the world's norm. From what is currently known, we can see that there are clear progressions in how children develop number concepts and that the varying aspects of number can be acquired at different times.

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[9] See page 5 of this document

We can begin to understand these relationships by looking at the connections between number concepts and oral and written representations of number, such as evidence that as children master oral counting, they continue to master the concept of cardinality, that certain strategies for written numbers are related to strategies of representation in other systems; and that once written and oral number systems are fully appropriated, we do not see the same errors that we saw previous to the complete acquisition.

We can also see that external number representations are only transparent to those who have already acquired the system, while for children who must learn the rules, there is much left that seems implicit to adults, which perhaps must been taught and demonstrated explicitly to children. When expecting children to become skilled with a system that took thousands of years to develop, we must stop to understand why this is so.

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