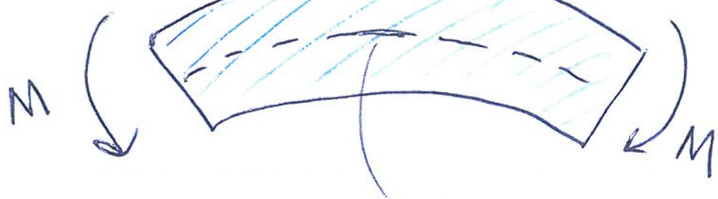
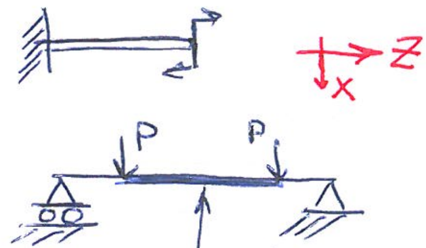
Bending problem

Bending of a beam

(pure bending, no shear force in cross-sections)

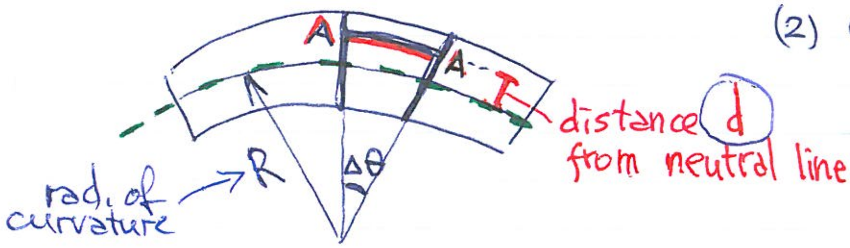




guess: geometry of deformation:

(1) planar cross-sections remain planar

(2) central line is not elongated



Coord system:

$(d = -x)$

Elongation of element AA: $\quad (R+d)\Delta\theta - R\Delta\theta = d \cdot \Delta\theta$

Strain: $\epsilon_{zz} = \dfrac{\Delta S - \Delta S_0}{\Delta S_0} = \dfrac{d \cdot \Delta\theta}{R \cdot \Delta\theta} = -\dfrac{x}{R}$

Hooke's law $\Rightarrow$ $\boxed{\sigma_{zz} = -E\dfrac{x}{R}}$

Additional assumption (guess): material lines do not interact

- do not press on one another $\sigma_{xx} = \sigma_{yy} = 0$
- do not "rub" with friction $\sigma_{zx} = \sigma_{zy} = \sigma_{xy} = 0$

$\Rightarrow$ no stresses other than σ_{zz}

$\therefore$ Our assumptions translate into $\underset{\sim}{\sigma} = \boxed{-E\frac{x}{R}}\, \underset{\sim}{e}_z \underset{\sim}{e}_z$ $\quad \sigma_{zz}$

CHECK

Check eq-ns

(1) Equilibrium: $\partial \sigma_{zz} / \partial z = 0$ (OK)

(2) Compatibility: contain 2^{nd} derivatives of strains only $\Rightarrow$ (OK)

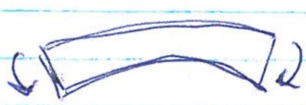
(3) Hooke' law: has been incorporated

Check B.C.

- B.C. on lateral surface: $\underset{\sim}{n} = n_x \underset{\sim}{e}_x + n_y \underset{\sim}{e}_y$ (no z-comp.)

  $\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = 0 \Rightarrow$ traction free as it should be

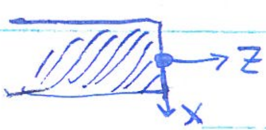
- B.C. at bases



$$\underset{\sim}{n} = \underset{\sim}{e}_z \Rightarrow \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = -E\frac{x}{R}\, \underset{\sim}{e}_z$$

$$\text{Principal vector} = \int_F -E\frac{x}{R}\, dF = -\frac{E}{R}\int_F x\, dF = 0$$

cross-section ↗ F $\qquad$ centroid – (origin of coord)



Moment (about y-axis): must equal the applied mom

$$\int_F \sigma_{zz} \cdot x\, dF = -\frac{E}{R}\int_F x^2 dF = -\frac{E}{R} I_y$$

corresponds to:

$\oplus$ – sign for M

our analysis based on sketch

$$\Rightarrow \cancel{-} M = \frac{E I_y}{R} \Rightarrow \boxed{\frac{1}{R} = \frac{1}{E I_y} |M|}$$

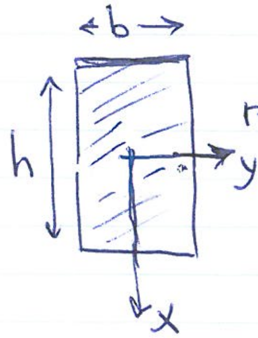
bending stiffness $\qquad$ phys. $\qquad$ geom.

$$\Rightarrow \boxed{\sigma_{zz} = -\frac{M}{I_y} x}$$

$\Rightarrow$: most economical shape of cross-section (smallest σ_{zz} ~~for~~ a given cross-section)

$$\sigma_{zz} = -\frac{M}{I_y}x \qquad \Rightarrow \qquad \sigma_{zz}^{max} = -M \underbrace{\frac{x_{max}}{I_y}}_{\text{determined by geometry of cross-section}}$$

$$I_y = \int_F x^2 \, dF$$



rectangle :

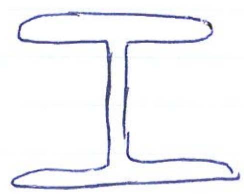
$$I_y = \int_F x^2 \underbrace{dA}_{bdx} = 2\int_0^{h/2} x^2 b \, dx = \frac{bh^3}{12}$$

$$x_{max} = h/2$$

$$\frac{x_{max}}{I_y} = \frac{6}{bh^2} = \frac{6}{area} \cdot \frac{1}{h}$$

– the more elongated – the better

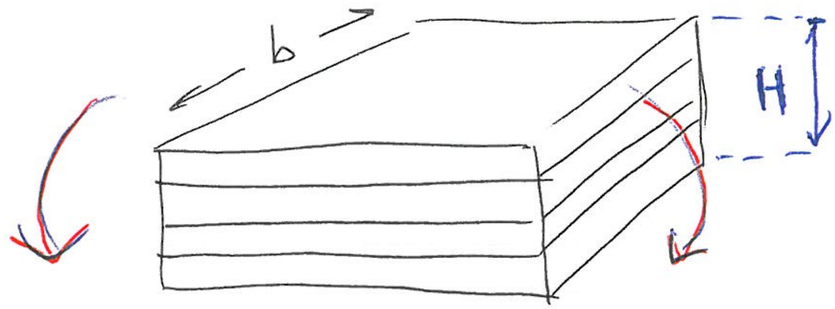
From $I_y = \int x^2 dA$: it is clear that most economical shapes are:



(rail)

After sol'n obtained: Try to extend to more complex cases

Bending of a layered beam



n sheets

Lubricated layers : no friction between them

$\Rightarrow$ each layer: bends as a separate beam; for it:

$$\sigma_{zz}^{max} = \underbrace{M}_{\sim \frac{1}{n} \text{ (shared)}} \cdot \frac{x_{max}}{I_y} \qquad x_{max} = x^{max}_{\text{(entire beam)}} \cdot \frac{1}{n}, \quad I_y = \frac{1}{12} b \left(\frac{H}{n}\right)^3 \qquad \sim n \cdot \sigma_{zz}^{max,\text{ glued}}$$

– Slicing the beam increases stress n times –

Curvature :

$$\frac{1}{R} = \frac{M \; (\sim \frac{1}{n})}{E I_y \; (\sim \frac{1}{n^3})} \sim n^2 \cdot \frac{1}{R_{glued}} \quad \text{(solid) beam}$$

– slicing makes the beam much more "compliant"

$\Rightarrow$ for max. flexibility : slice the beam

But : higher stress $\Rightarrow$ higher reqs to material strength

$\Rightarrow$ importance of lubrication of suspensions :

(friction : start to resemble glued beam)