Literature Review

Children's Understanding of Area of Rectangular Regions and Volumes of Rectangular Shapes and the Relationship of These Measures to Their Linear Dimensions

Qualifying Paper

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Introduction

> …a dramatic transformation has occurred in the operational definition of geometry. For most practicing geometers, geometry has become the study of insights one is led to by mathematical training when one studies visual phenomena. The range of what can be studied is as varied as what the visual world has to offer. (Malkevitch, 1998, p. 21)

Our technological reliance on geometry is rising. Geometry is used in data compression, medical imagery, sophisticated machinery, and space travel. How best should we educate children for the ever changing, highly technical world in which they must be prepared to live and work? What aspects of geometry should be emphasized? Which geometry concepts should be taught, and when should these concepts be introduced? Are there particular sequences or an order in which these concepts should be taught? What materials and tools should be used in the teaching and learning of geometry? Mathematics researchers, mathematicians, and mathematics educators have all offered suggestions and guidance (Mammana & Villani, 1998; National Council of Teachers of Mathematics (NCTM), 1989, 2000, 2006).

On a national level, there is deep concern about our students' lackluster mathematics performance on the Trends in International Mathematics and Science Study (TIMSS) (Mullis, Martin, Gonzalez, & Chrostowski, 2004). Our "Nation's Report Card," more formally referred to as National Assessment of Educational Progress (NAEP), indicates that the majority of our students perform poorly on mathematics questions that require them to write out their thinking and process of solution (Martin & Strutchens, 2000).

In 1992, due to the international concern about the role of geometry in school mathematics, the Executive Committee of the International Commission on Mathematics

Instruction (ICMI) proposed the start of an ICMI Study on the teaching of geometry. Major trends in geometry, implications and perspectives concerning the teaching of geometry now and in the future, and various aspects from social and didactical issues to curriculum design and teacher preparation were among the topics discussed (Mammana & Villani, 1998).

Lappen (1999) referred to geometry as the “forgotten” strand in pre-K through grade 12 mathematics education, and this concern was voiced even earlier by Freudenthal (1981) who stated, “The mathematised spatial environment is geometry, the most neglected subject of mathematics teaching today” (p. 145). And Freudenthal (1973) has stated that “…geometry is grasping space…it is grasping that space in which the child lives, breathes and moves” (p. 403). Van den Heuvel-Panhuizen (2005) continued the plea for “…more space for geometry in primary school” (p. 1); she elaborated on specific aspects of geometry that could be included in a “learning-teaching trajectory” for elementary students (Ibid.).

Measurement is an extremely broad subject involving both spatial and non-spatial quantities. Geometry, particularly in the elementary grades, while certainly not synonymous with measurement, “…has everything to do with measuring” (van den Heuvel-Panhuizen, 2005, p. 7). It is in geometrical measuring that the bonding of space and number comes alive. Measuring is one of the three basic tasks[1] for which numbers are used (Steen, 1990). Answering questions involving “how much,” “how many,” and “how long” are just a few instances of when measurement displays its usefulness (Paulos, 1988; Schwartz, 1996). “Measuring is a process by which a number is assigned to an attribute of an object or event” (Reys, Lindquist, Lambdin, Smith, & Suydam, 2003, p.

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[1] The other two tasks are ordering and coding.

322). The practicality and concreteness of measurement cannot be underestimated. Its connection to spatial sense, particularly in the elementary grades, lays the foundation for more sophisticated applications associated with other topics of study, such as science, art, and social studies (Clements & Bright, 2003; NCTM, 2000; Van de Walle, 2001). Our students in the elementary grades, though, often have difficulty with measurement. There is evidence that they do not possess a strong understanding of such attributes of space as perimeter, area, and volume (Martin & Strutchens, 2000) .

What is needed to strengthen our students' understanding of the attributes of space, to prepare our students for a world that requires increasingly advanced levels of technical sophistications? Again the questions posed are: which concepts should be taught and when should these concepts be introduced? Are there particular sequences or an order in which these concepts should be taught? And what materials and tools should be used in the teaching and learning of geometry?

Goals and Structure of This Paper

In seeking to answer the above questions, this paper focuses on a narrower band of geometric measurement, aimed at harvesting the findings of the reviewed research and suggesting fruitful possibilities for further study. The main goal is to review the literature on studies that have been conducted specifically in the treatment of area and volume, as well as length, with elementary students.

It is important to note that the tools considered in this paper are manipulative 3-D tools. The scope of this paper is restricted to hands-on experiences relative to measurement concepts of length, area, and volume. Senechal states, "Hands-on experimentation is essential. For example, when we make a cube with our own hands, we gain much more insight into its metric, combinatorial, and stability properties than if we just look at one" (1990, p. 178).

This literature review consists of six main sections. In the first section, the theoretical framework of Piaget and his work on children's cognitive development in the conceptions of space and geometry will be presented. In the second section, the constructivist theory of van Hiele and his levels of geometrical thinking will be laid out. In the third section, the key concepts of spatial measurement as delineated by Lehrer (2003) and fundamental notions of shapes as suggested by Senechal (1990, 1991) will be discussed. In the fourth section, research studies that investigate how children come to understand area and volume measurement, particularly of rectangular shapes, will be presented. Included in this section are the following subheadings: unitizing and covering the region; tiling and drawing the array; spatial structuring and enumeration in area (local and global structuring); the importance of linear measurement in relation to spatial structuring and area; and spatial structuring and enumeration in volume studies. In the fifth section, studies that investigated the use of manipulatives and tools in learning measurement concepts are discussed. These studies address: moving square tiles vs. drawing square tiles; when manipulatives mask or distract rather than help or clarify; and the use of rulers. The sixth and concluding section offers suggestions for future research into children's learning of area and volume concepts.

I. Piaget's Theory and the Development of Space and Geometry Concepts

In his own words, Piaget stated; "…my most central concern has always been to determine the contributions of person's activities and the limiting aspect of the object in the process of acquiring knowledge" (Flavell, 1963, p. vii). While respected for his contribution to child psychology and the study of cognitive development, Piaget firmly contended that his work was "…only a link between two dominant preoccupations: the search for the mechanisms of biological adaptation and the analysis of that higher form of adaptation which is scientific thought, the epistemological interpretation of which has always been my central aim" (Gruber & Voneche, 1995, p. xi). In the most general sense, Elkind spoke of Piaget's theory as "…that of subject-object equilibration[2], the view that mental growth is governed by a continual activity aimed at balancing the intrusions of the social and physical environment with the organism's need to conserve its structural systems" (Piaget, 1968, p. v).

"Piaget's work stands as a major landmark in the history of knowledge, because within the framework of the unexamined concept he has given brilliant description one after another of the changing intellect of the child" (Gruber & Voneche, 1995, p. xxix). In particular, Piaget's studies on children's conception of geometry (mainly measurement) and his studies on children's conception of spatial concepts impacted later empirical studies which investigated children's understanding of area and volume; it is these later studies in the literature that are reviewed in this paper.

To appreciate the comprehensive connection of Piaget's work to these studies, Piaget's four stages of cognitive development—sensori-motor, preoperational, concrete operational, and formal operational—are briefly explained, and their corresponding behaviors[3] are described. Four concepts—schemata, assimilation, accommodation, and equilibrium—which are integral to Piaget's theory of cognitive development, are defined. Four overarching factors—physical experience, maturation, social interaction, and

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[2] This is Piaget's generic term for the elimination of perturbations. This is elaborated on p. 11 of this paper.
[3] Piaget was interested in children's behavior as it related to what was going on in the child's mind as opposed to behaviorists whose main interest was only the action that can be observed.

equilibration—which Piaget considered to be significant in cognitive development, are explained. Also, Piaget's findings from his studies on children's conception of geometry and of space are included.

Piaget's Four Stages of Cognitive Development

Piaget viewed cognitive development in a child as an ongoing, dynamic process by which the child organizes experiences and adapts to the external environment in order to make sense of his or her world. The four major distinguishable stages of cognitive development that Piaget proposed are: sensori-motor; preoperational; concrete operational; and formal operational. While there is no set age at which a child should be in a specific stage, every child must advance through these stages in the same order. No stage is bypassed. Due to experiential or genetic factors, some children advance more quickly, while others move at a slower rate of advancement. Apparently not all individuals reach or complete the formal operations stage. Acceleration through the stages cannot be forced, but experiences can afford children the opportunity to gain deeper understanding of a concept. Each of the stages is matched with corresponding, typical children's reasoning at that stage as shown in Figure 1.

Piaget's Stages of Cognitive Development		
Stages of Development		Characteristics of Children's Reasoning
I	Sensori-motor stage (0-2 years)	The child's behavior is primarily motor; knowledge of his/her world is developing, although limited, and is based on physical interactions and experiences.
II	Preoperational stage (2-7 years)	The child's thinking is egocentric; illogical intuitions are formed based upon perceptions; and rapid cognitive development occurs.
III	Concrete operational stage (7-11 years)	The child develops conservation of number, length, liquid, mass, weight, area and volume. Intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Reversibility in thinking is demonstrated, thus the ability to apply logical thought to concrete problems appears.
IV	Formal operations stage (11-15 years)	The child's cognitive structures reach their greatest level of development. The child becomes able to apply logic to all classes of problems. The child demonstrates his/her intelligence through the logical use of symbols related to abstract concepts. Some individuals will not reach this stage at any age.

Piaget's Stages of Cognitive Development
Figure 1.

Pertinent Terms Integral to Piaget's Theory of Cognitive Development

Four pertinent theoretical concepts, which are integral to Piaget's theory of cognitive development and relate to intellectual structures, are: schemata; assimilation; accommodation; and equilibrium. Schemata are the mental structures by which individuals intellectually adapt to and organize their environment. They dictate a child's pattern of reasoning. Schemata of children may not coincide with that of adults, but each child's organization will always be internally consistent.

Assimilation is the cognitive process by which the individual integrates new stimuli into existing schemata. When a child assimilates he or she "…remains unaware of, or disregards, whatever does not fit into the conceptual structure it possesses" (Glasersfeld, 1995, p. 63). If the child is unable to assimilate new stimuli or experiences into his or her existing schemata, then perturbation (agitation or tension) results. Assimilation theoretically does not change schemata, but it does affect their growth.

Accommodation is the creation of new schemata or the modification of old schemata. Assimilation and accommodation go hand-in-hand, and their actions, induced by the environment, account for intellectual adaptation and result in a change in cognitive structures—the schemata. Piaget stressed that interaction with others, especially linguistic interactions, were the most frequent cause of accommodation, and yet detractors of Piaget's theory often contest that Piaget did not consider a social component in development of his theory of cognitive development (Glasersfeld, 1995). When a balance between assimilation and accommodation is met, there is equilibrium, a state of cognitive balance. When there is an imbalance or disequilibrium, there is cognitive conflict and a state ripe for change.

Piaget used the term *décalage* to describe a time lag or minor disparity of action by the child (Gruber & Voneche, 1995). An example of vertical décalage would be when a child is able to act physically on a problem or dilemma but is unable to explain in words what he or she did. Another example would be when a child who is able to classify concrete objects but is unable to do so when the physical objects are not available. Horizontal décalage, or "lack of immediate transfer" (Ginsburg & Opper, 1969, p. 165), occurs when a child displays different levels of achievement when the problem situations involve similar mental operations. An example would be when a child understands the conservation of mass but not the conservation of volume. It takes time, practice, and varied experiences before a conscious realization of an abstract concept takes hold.

Piaget contends that there are two avenues of abstraction that together contribute to understanding abstract concepts: simple abstraction and reflective abstraction.[4] Simple abstraction relates to the abstraction a child would get from handling an object in the physical world, such as a ball of clay or a wooden cube. The abstraction would come from the object itself; knowledge would be extracted from the physical properties of the object. Reflective abstraction relates to the results of actions carried out on the objects, such as rolling, spinning, and pulling the ball of clay or tumbling, turning, and spinning the cube. Knowledge is extracted from the actions the child takes on the objects. Simple abstraction is referred to as "physical experience," while reflective abstraction is called "logico-mathematical experience" (Gruber & Voneche, 1995, p. 727).

Overarching Factors That Piaget Considered Significant in Cognitive Development

Three overarching factors integral to Piaget's theory of cognitive development are physical experience, maturation, and social interaction. Physical experience indicates the influences of the physical environment. Maturation deals with what one genetically inherits, one's innateness, one's neurological growth. Social interaction includes social transmission, the effects of social influences, and the interchange of ideas between individuals. The interchange could be between peers, parents, or with other adults.

Later, Piaget added equilibration as a fourth factor. Equilibration acts as the coordination of physical experience, maturation, and social interaction and can be considered a "progressive autoregulation" (Piaget, 1971, p. 18). This quest for a consistent equilibrium does not result in a static condition, but rather by expanding equilibration, cognitive development is characterized. This expanding equilibrium means

[4] Taken from Piaget's first lecture of the series, *Genetic Epistemology*, at Columbia University, translated by Eleanor Duckworth.

"…an increase in the range of perturbations the organism is able to eliminate" (Glasersfeld, 1995, p. 67). Equilibration is the individual's internal self-regulating system. Glasersfeld (1995) elaborates on Piaget's equilibration:

> There is a further aspect of equilibration which, although not explicitly stated, is implicit in Piaget's repeated observation that the most frequent occasions for accommodation are provided by interactions with others. Insofar as these accommodations eliminate perturbations, they generate equilibrium not only among the conceptual structures of the individual, but also in the domain of social interaction. Had Piaget emphasized this implicit corollary a little more, the superficial criticism that his mode disregards the social element would have been largely avoided. (p. 67)

Piaget and Children's Conception of Geometry

Piaget painstakingly studied children engaged in what would appear to many adults to be simple measurement tasks. Conservation is often associated with these activities. Conservation is the ability to realize that attributes, such as length, area, and volume of an object, remain constant even though the object may be changed in appearance; for example, a piece of rope has the same length whether it is coiled or if it is laid out straight. Piaget, Inhelder, and Szeminska (1960) in their studies of children's conception of geometry and Piaget and Inhelder (1967) in their studies of children's conception of space questioned whether conservation antecedes measurement, or if in fact conservation was the outcome of measure. Also, Piaget and fellow researchers pondered the relations between the operations of conservation and those of spatial measuring.

The measurement tasks required children, ranging in age from three to nine years of age, to use manipulative materials to solve story-embedded problems that involved length, area, and volume. Piaget contends that conservation of length is fundamental and must be attained in order for a child to be able to find linear measure. This conservation theory, though, has been contested by subsequent researchers such as Nunes, Light, and Mason (1993) and their study will be addressed further on p. 60 of this paper. From his studies with children on length and area, Piaget concludes:

> …the development of conservation and measurement runs exactly parallel whether the objects are lengths or whether they are areas and the level at which they are finally grasped is the same for both. Conservation is always the outcome of complementarity between the two groupings: that of additive subdivision and that of ordered positions and changes of position (Piaget et al., 1960, p. 300).

Referring to volume, Piaget relates:

> …the conservation of a quantity of matter appears as early as level IIIA[5], but the conservation of volume as a physical concept is not elaborated until state IV, being the level of formal operations. (Piaget et al., 1960)

Piaget's findings concluded that children conserve length around the age of six, area around eight years of age, and volume around eleven years of age. In discussing volume, Piaget refers to "occupied space" as a "moveable 'contained'" and "interior volume" as a "fixed spatial 'container'" (1960, p. 394). Lovell's clarifies these distinctions of volume: he relates occupied space to passing "…our hands around a box, block, or ball, to indicate the amount of space taken up by the object"; he relates interior volume to moving "…our hands about inside, say, a box or cupboard, in order to indicate the amount of space within" (1968, p. 122).

Piaget points out that children who have not yet reached the formal operations stage are able to conserve area based on the primitive conception of area (and volume) as that which is "bounded by lines (or faces)" (1960, p. 355). The intuitions of children in

[5] This is Piaget's concrete operational stage.

the early concrete operational stage were "…still topological[6] in character" (Ibid., p. 368) and "…conservation is still limited to interior volume and does not as yet extend to the spatial relations between the object and its surroundings" (Ibid., p. 374). The first sign of children "making full use of the concept of a unit of measurement or, alternatively, that he equates his measurements of boundary sides with his notion of volume" appears in children at the higher end of the concrete operational stage (Ibid., p. 377).

Piaget and his collaborators determined that in order for a child to successfully complete a conservation task he or she must have an understanding of the following three aspects of reasoning: first, the identity of the object; second, compensation of dimensions; and third, reversibility of actions. With mastery of the concept of identity, the child realizes that an object remains the same if nothing is added to or subtracted from it. It would not matter if the clay were rolled into a ball or stretched out to look like a hot dog, as long as no clay was removed and no additional clay was added. With mastery of compensation, the child realizes that changes in one dimension can be offset by changes in another. In a task that involved structures made from blocks, one building could be taller than the original one, if it had less depth or breadth, in order to have the same volume. With mastery of reversibility, the child realizes that mentally reversing the steps and returning to the original situation of the task may cancel out a change. In tasks involving length, after one stick was pushed physically, the stick could be pushed mentally in the opposite direction and resume its original location.

Piaget and Children's Conception of Space

As stated above, Piaget and Inhelder (1967) contended that space is understood initially by topological relations and later by projective and Euclidean relations. This is

[6] Meaning it conserves the most general properties of space – inside and outside.

referred to as Piaget's topological primacy theory[7], which is the historical reversal of the development of Euclidean geometry, which occurred in antiquity. Centuries later projective geometry was developed, and followed later still by topology. This theory has been supported by some researchers (Lehrer, Jacobson et al., 1998) and contested by others (Clements & Batistta, 1992; Clements, Swaminathan, Hannibal, & Sarama, 1999; Lovell, 1968).

Piaget contended that children initially relate to the objects around them in a topological fashion whereby the object's general properties of inside and outside are the distinguishing features. With maturity and experiences, the child relates to objects by taking other viewpoints into consideration. Straight lines are preserved at this stage. And lastly, the more mature child takes on the Euclidean perspective and is able to conserve distance, angles, and parallel lines as well as straight lines. Thus the development of representational space takes time. The infant and young child make sense of objects by being aware and concerned with their perception through proximity, separation, order, and enclosure. Perspective and metric relationships are ignored and children thus initially understand shapes topologically.

For a child in the preoperational stage a square, a circle, or a rhombus would be considered similar. A child's drawing of a square most likely would be closed, but would lack details of having right angles and congruent sides. It is only with maturity and experience that a child develops a more sophisticated mental structure in which to represent spatial objects. Adults may assume that children perceive geometric shapes in a way similar to their own understanding of a three dimensional coordinate system, but in

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[7] For more in-depth information consult: Gruber and Voneche (1995); Piaget and Inhelder (1967); and Piaget et al. (1960).

fact, vertical-horizontal coordination is not fully developed until the ages 8 or 9 (Piaget et al., 1960, p. 4). Contradictory studies by Lovell, though, have indicated that "…horizontal and vertical, and hence axes of reference, were understood by some 7-year-olds far better than one would expect from the Geneva results" (Lovell, 1968, p. 103).

Motor activity is of enormous importance for the understanding of spatial thinking. "…to recognize geometrical shapes the child has to explore the whole contour" (Piaget & Inhelder, 1967, p. 23). After the child makes sense of the shape of objects topologically, he or she will begin to be confronted with the dilemma of making sense of objects as viewed from different perspectives. It is not a dilemma until there is realization that different perspectives can exist; the child's egocentrism is challenged. Children, between ages four and six, begin to probe shapes by haptic perception—by means of the sense of touch, in the absence of visual stimulation—yet in a haphazard manner. They happen upon cues or pointers that distinguish curved and straight sides—recognized by the presence or absence of angles. Children's drawings show attempts to represent these angles, but the drawings are not a photograph of the shape. It is not until the age of seven or eight that a child, whose perceptual activity becomes more complex, returns systematically to a stable point of reference of a shape and thus achieves "reversible co-ordination" (Piaget & Inhelder, 1967, p. 36). "…drawing like the mental image, is not simply an extension of ordinary perception, but is rather the combination of the movements, anticipations, reconstructions, comparisons, and so on, that accompany perception and which we have called perceptual activity" (Piaget & Inhelder, 1967, p. 33).

While a thorough study of the concept of angle is beyond the scope of this paper, it is mentioned here because of its relationship with the study of the concept of parallel lines, which, as will be shown in the next section of this paper, has significance in the study of area, particularly as shown by Outhred and Mitchelmore (2004). Piaget's (1967) studies indicate that "There is no doubt that it is the analysis of the angle which marks the transition from topological relationships to the perception of Euclidean ones" (Piaget & Inhelder, 1967, p. 30). Children's understanding of a straight line is followed by the understanding of the concept of parallel lines and of the concept of angle. It is with the onset of understanding of parallel lines that children can begin to form a frame of reference as a way to coordinate space.

Up to ages seven and eight, children increasingly coordinate their physical actions, and this coordination is complemented by an internal coordination of their schemata. This happens, though, with some anticipation of what will happen, but without long-range planning. At this stage the "linking together of intuitions, the formation of trains of ideas" is beginning to take place (Piaget & Inhelder, 1967, p. 454). Thus, from this process, the level of coordination of schemata indicates the child is capable of mental reversibility and indicates an "initial equilibrium state reached by internalized actions and thus constitutes the first truly operational system" (Piaget & Inhelder, 1967, p. 455).

Piaget's Findings in Relation to Mathematics Education

Even though his research did not address classroom instruction, Piaget did express how findings from his studies with children could be woven into the fabric of mathematics education. At the Second International Congress on Mathematical Education in 1972, Piaget stressed the importance of and difference between physical experience (simple abstraction) and logico-mathematical experience (reflective abstraction). He warned that it "…would be a great mistake, particularly in mathematical education, to neglect the role of actions and always remain on the level of language" (Gruber & Voneche, 1995, p. 727). He went on to say:

> In fact, it is often particularly difficult for the teacher of mathematics, who, because of his profession, has a very abstract type of thought, to place himself in the concrete perspective that is necessarily that of his young pupils. (Gruber & Voneche, 1995, p. 730)

Piaget proposed two principles that bear upon the role of teachers: 1) children may give the impression that they understand a concept merely by repeating what the teacher has said or by applying a concept by duplicating what the teacher has done; 2) children may be able to complete tasks, but not be fully "aware" of and not be able to articulate what he or she is doing (Gruber & Voneche, 1995, p. 731). From these principles, it is logical that teachers need to be "organizers" of situations that "…will give rise to curiosity and solution-seeking in the child…" and when the child struggles "…to suggest such counterexamples that the child's new exploration will lead him to correct himself" (Gruber & Voneche, 1995, p. 727).

II. Constructivist Theory of Pierre van Hiele

The work of Pierre van Hiele and his late wife Dina van Hiele-Geldof echoes these cautions and principles of cognitive psychologist Piaget. Each of the van Hieles, in contrast, focused on the role of instruction, specifically geometry instruction. Many of van Hiele's tenets, though, are in harmony with Piaget's theory of cognitive development and Piaget's comments on mathematics education as stated above. Van Hiele wrote, "Piaget's point of view, which I support affectionately, was that 'giving no education is better than giving it at the wrong time.' We must provide teaching that is appropriate to the level of children's thinking"[8] (van Hiele, 1999, pp. 310-311).

The research of the van Hieles focused on the thinking of students in secondary geometry classes, but has application to the thinking of younger students as well. Their work began in the mid-late 1950's and their findings indicated a mismatch between the teacher's level of sophistication in instruction (which included deduction and proof) and the students' level of sophistication in understanding what was being taught. They uncovered an existing disparity and showed that learning was thwarted. Van Hiele likened this discord to "the teacher and the students speak[ing] a very different language," and reworded this by saying, "they think on different levels" (van Hiele, 1984, p. 245). Van Hiele-Geldof died soon after completion of her dissertation[9], and the capstone of their related research work is considered "The van Hiele Levels of Geometry Thinking." These levels, which can be considered a structural framework for geometry instruction, will now be presented as well as the phases, which guide children through one level and advance them to a higher level according to the theory. In order to appreciate a broader perspective, the impact that the van Hiele Levels of Geometry Thinking has had on the mathematics education community will be examined; and to view a more specific interest, the relationship of the theory to geometry instruction, Pre-K through Grade 8, will also be included.

The van Hiele Levels of Geometry Thinking

From his analysis of geometry, van Hiele determined five different levels of thought. He formulated The van Hiele Theory of Levels of Geometry Thinking, in which students'

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[8] This was said in reference to his agreement with Piaget on the misguided implementation of the "new math."

[9] The title of her dissertation is *The Didactics of Geometry in the Lowest Class of Secondary School.*

understanding of geometric concepts is described by levels. The theory is applicable from elementary school through high school and beyond (Burger & Shaughnessy, 1986; Fuys, Geddes, & Tischler, 1988; Usiskin, 1982; Wirszup, 1976). The theory contends that if students' levels of thinking are addressed with appropriate levels of instruction, students are invested in the learning process and are more likely to develop insight. The five levels, as shown in Figure 2 on p. 21, describe how a student thinks and the types of geometric ideas about which a student thinks. For van Hiele, age and biological maturation are not the main factors in advancing from one level to a higher level as they are in Piaget's stages of cognitive development. In the advancement of geometrical thinking, the object of a student's thinking changes.

> How do students develop such thinking? I believe that development is more dependent on instruction than on age or biological maturation and that types of instructional experiences can foster, or impede, development… (van Hiele, 1999, p. 311)

In Piaget's stages of development, there is the disparity called décalage, and in the van Hiele Levels of Geometry Thinking, students similarly may be at different levels or oscillate between levels for different geometric concepts or with different tasks (Burger & Shaughnessy, 1986; Mayberrry, 1983). Gutiérrez, Jaime, and Fortuny (1991) found that some students "used several levels at the same time, probably depending on the difficulty of the problem" (p. 250). They do not reject the hierarchical structure of van Hiele's levels, but instead suggest that the van Hieles' theory should be adapted to reflect "the complexity of the human reasoning process" (Gutiérrez et al., 1991). The levels by description appear to be discrete in nature, but according to Gutiérrez, most researchers would "…consider that the movement from one level to the following one is a continuous process, since the acquisition of a thinking level by a student is gradual…" (Gutiérrez et al., 1991, p. 32).

In a longitudinal study of elementary students (started in 1^{st}, 2^{nd}, and 3^{rd} grades and ended in 3^{rd}, 4^{th}, and 5^{th} grades), Lehrer, Jenkins, and Osana (1998) studied the spatial reasoning of students. They found that children defied description by a single level of development. They concluded that "Level mixture was therefore the most typical pattern of response, and children's justifications often 'jumped' across nonadjacent levels of the van Hiele hierarchy" (Lehrer, Jenkins et al., 1998, p. 142).

van Hiele Levels of Geometry Thinking			
Level 0 Base Level	**Objects of Thought**	**Products of Thought**	**Features of Level Activities**
Visualization – Student identifies and operates on shapes (e.g., squares, triangles) and other geometric configurations (e.g., lines, angles, grids) according to their appearance.	Shapes and what they "look like"	Classes or groupings of shapes rather than individual shapes	Students operating at this level judge figures by their shape.
Level 1	**Objects of Thought**	**Products of Thought**	**Features of Level Activities**
Analysis – Student analyzes figures in terms of their components and relationships between components, establishes properties of a class of figures empirically, and uses properties to solve problems.	Classes or groupings of shapes rather than individual shapes	Properties of shape	Students operating at this level are able to generalize about shapes that fit into a class; they are able to list many properties of a shape but may not be able to identify subclasses, e.g., all squares are rectangles, properties are not yet ordered.
Level 2	**Objects of Thought**	**Products of Thought**	**Features of Level Activities**
Informal Deduction – Student formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows and gives deductive arguments.	Properties of shapes	Relationships among properties of geometric objects	Students operating at this level are able to develop relationships between and among properties of geometric objects; they are able to follow and appreciate an informal deductive argument about shapes and their properties.
Level 3	**Objects of Thought**	**Products of Thought**	**Features of Level Activities**
Deduction – Student establishes, within a postulational system, theorems and interrelationships between networks of theorems.	Relationships among properties of geometric objects	Deductive axiomatic systems of geometry	Students operating at this level are able to begin to appreciate the need for a system of logic that rests on a minimum set of assumptions and from which other truths can be derived; they are able to work with abstract statements and make conclusions based more on logic than on intuition. This is the level of the traditional geometry course.
Level 4	**Objects of Thought**	**Products of Thought**	**Features of Level Activities**
Deduction – Student rigorously establishes theorems in different postulational systems and analyzes/compares these systems.	Deductive axiomatic systems of geometry	Comparisons and contrasts among different axiomatic systems of geometry	Students operating at this level are able to appreciate the distinctions and relationships between different axiomatic systems. This is generally the level of college mathematics major who is studying geometry as a branch of mathematical science.

van Hiele Levels of Geometry Thinking

(adapted from van Hiele, 1986, 1999; Fuys et al., 1988; Clements & Battista,1992)

Figure 2.

The major characteristics of van Hiele levels are:

- They are sequential and hierarchical;
- They are not dependant on maturation or age, but rather on instructional experience.

Inherent in the levels and important to appreciate are the facts that:

- What is implicit at one level becomes explicit at the next;
- Material taught to students above their level is subject to "reduction"[10];
- Each level has its own language with use of symbols as a way of thinking;
- Individuals who are thinking at different levels (reasoning at different levels) are unable to follow the thinking process of each other;
- A student goes through various phases before proceeding from one level to the next level.

The original levels of the van Hiele theory and the descriptors of these levels have been questioned. Proposals for an additional non-verbal[11] level prior to level 0 have been suggested (Clements & Batistta, 1992; Clements et al., 1999), and refinement for clarification has been offered (Fuys et al., 1988). Van Hiele himself endorsed the work of Fuys et al. (1988), who, as researchers in geometry education, had translated some of his works; van Hiele agreed that the lowest level in his theory does include non-verbal thinking. It should be noted that in response to a query by van Hiele, Freudenthal had concluded that "…thinking without words is not thinking" and van Hiele acknowledged

[10] When a student resorts to rote learning of a procedure, because the procedure is not truly understood or the student is told a definition rather than be given an opportunity to discover it, it is said that the thinking has been reduced to a lower van Hiele level (Fuys et al., 1988).

[11] This is also referred to as pre-cognition or pre-representational.

this point of view (van Hiele, 1986, 1999). Van Hiele collapsed Levels 3, 4, and 5 into a single level, thus reducing the total number of levels to three (van Hiele, 1986).

Phases in the Learning Process Associated with the van Hiele Levels of Geometry Thinking

Van Hiele proposed five phases that are instrumental in leading students in the teaching and learning process through one level of thinking to the next level. In his own words, van Hiele stated concrete and practical methods:

> Instruction intended to foster development from one level to the next should include sequences of activities, beginning with an exploratory phase, gradually building concepts and related language, and culminating in summary activities that help student integrate what they have learned into what they already know. (van Hiele, 1999, p. 311)

The reference of having students "integrate what they have learned" sounds similar to the interplay of Piaget's assimilation and accommodation. When students are able to access new information, they have the potential to integrate it into what they already understand and the learning process is advanced. Comparably, the processes of assimilation and accommodation, induced by the environment, account for intellectual adaptation and result in a change in children's cognitive structures, the schemata.

Collectively, the five phases make up a period, which is the learning process that leads students from one level to the next level. Van Hiele likened this process to that of "…apprenticeship and not as a ripening of a biological sort" (1984, p. 246). Students should be transitioned through each phase in order to develop understanding of concepts. The five phases are:

- Information[12]
- Guided orientation[13]

[12] This is also referred to as inquiry (van Hiele, 1999).
[13] This is also referred to as bounded orientation (van Hiele, 1986) and direct orientation (van Hiele, 1999).

- Explication
- Free orientation
- Integration

The behaviors that would be manifested by students and teachers in each of these phases are given in Figure 3. These behaviors indicate the interplay between teacher and student and signal the importance placed on the teacher to orchestrate and facilitate opportunities for students to be actively engaged in the learning process.

van Hiele's Phases in the Learning Process		
Phase	Student Behavior	Teacher Behavior
Information	• Becomes acquainted with content or working domain and materials	• Discusses materials • Clarifies content • Holds a discussion • Learns how students interpret the language • Provides information to bring students to purposeful action and perception
Guided Orientation	• Does tasks in which characteristic structures will appear gradually, e.g., measuring, completing a puzzle, finding shapes	• Chooses materials and tasks in which the targeted concepts are striking • Guides students in carefully structured sequenced tasks
Explication	• Becomes conscious of relations • Describes the relations in his/her own words • Learns related technical language	• Brings the objects of study to an explicit level • Leads discussion of the objects of study • Introduces terminology • Encourages students to use the terminology
Free Orientation	• Does more complex tasks that require the synthesis and utilization of concepts and relations • Determines his/her own way in the network of relations • Becomes more proficient	• Presents tasks that are more open-ended, can be completed in more than one way • Introduces concepts and terms as needed • Encourages students to reflect and elaborate on the tasks
Integration	• Reflects on his/her actions • Pulls together what he/she has learned • Builds a coherent summary of what he/she has learned using pertinent language and concepts	• Encourages reflections, explanations, and consolidation of knowledge • Emphasizes use of mathematical structures as a framework for consolidation

van Hiele's Phases in the Learning Process
(adapted from van Hiele, 1984, 1986, 1999; Fuys et al., 1988; Clements & Battista, 1992)
Figure 3.

The Impact of the van Hiele Theory on the Mathematics Education Community

The van Hiele theory was initially well received in Europe, and in particular in the Soviet Union. It is not surprising that psychologists and mathematics educators from Russia, where the learning of geometry traditionally has received much attention, were interested in this theory (Wirszup, 1976). They researched and experimented with the levels and phases of the theory and methods of teaching mathematics. As a result, "between 1960 and 1964 they [Russian psychologists and mathematics educators] verified the validity of his [van Hiele] assertions and principles" (Kilpatrick, Wirszup, Begle, & Wilson, 1975, p. 77). In 1967-68 academic year, a new Soviet mathematics curriculum was unveiled that reflected the Russian psychologists' belief that "…instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages" (Ibid, p. 77). The new curriculum was considered "the most radical change in Russian mathematics education in nearly a century" (Wirszup, 1976, p. 96).

In addition to the impact that the van Hiele theory had on transforming Russian mathematics education, Freudenthal (1973) expounded on the van Hieles' theory with descriptions of its levels and phases in his own influential book, *Mathematics as an Educational Endeavor*. But it was not until Wirszup (1976) wrote *Breakthroughs in the Psychology of Learning and Teaching Geometry*, which acclaimed the effectiveness of the theory, that van Hiele's work received substantial attention among American mathematics educators. While getting a relatively late start in the American educational arena, "…today, the van Hiele theory has become the most influential factor in the American geometry curriculum" (Van de Walle, 2001, p. 309).

The van Hiele Theory in Relation to Geometry Pre-K Through Grade 8 Instruction

The implementation of van Hiele's theory in school mathematics curriculum could possibly result in a learning trajectory that van Hiele (1984) outlined in *A Child's Thought and Geometry* and in Wirszup's (1976)[14] *Breakthroughs in Psychology of Learning and Teaching of Geometry.* This possible trajectory of the van Hiele Theory, up to Level 3, is found in Figure 4 on p. 27. This trajectory, while lacking specificity, does not conflict with Piaget's findings as to conservation of length, area, and volume and does support Piaget's theories and constructivist philosophy on how children make sense of their world; it also complements the NCTM guidelines of appropriate geometric topics and pedagogy (NCTM, 2000). For the grade band Pre-K-2, NCTM states that teachers "…must provide materials and structure the environment appropriately to encourage students to explore shapes and their attributes" (Ibid., p. 97). For the grade band 3-5, NCTM recommends that teachers "…should emphasize the development of mathematical argument. As students' ideas about shapes evolve, they should formulate conjectures about geometric properties and relationships (Ibid., pp. 166-167). For grade band 6-8, NCTM suggests that "investigations into the properties of, and relationships among, similar shapes can afford students many opportunities to develop and evaluate conjectures inductively and deductively" (Ibid., p. 234).

[14] There is a discrepancy in these two translations involving levels 4 and 5, Wirszup bypasses the 4th level and refers to "discernment in mathematics" at the 5th level, while van Hiele stops at the 4th level and refers to "discernment in mathematics" and does not mention the 5th level. Since both are at the level of proof and beyond the focused area of this paper, it is of no consequence here.

van Hiele's Theory in Relation to Geometry Pre-K Through Grade 8 Instruction				
van Hiele's Levels	Grade Spans	The Aim of Instruction	The Resulting Base on Which the Next Level Will Build	Descriptive Title
Level 1	Start Pre-K	• Geometric figures should become the bearers of their properties. • Students manipulate concrete materials, e.g., a rectangle initially recognized only by its appearance becomes recognized by the fact that it has four right angles.	Manipulations, which the students perform with the materials, would be the base of a new relational system in the process of formation.	*Aspect of Geometry*
Level 2		• Relations, which link properties of figures, are now learned. • Students begin to order properties of figures logically, e.g., in tiling with triangles, other shapes and patterns are recognized.	Components of figures are discerned and relationships among these components are established.	*Essence of Geometry - The Aspect of Mathematics*
Level 3	End Gr. 8	• Understanding what is meant by logical ordering • Students' materials are made up of geometric theorems themselves. • Students can now try to order new domains logically, e.g., understand that it is necessary to know the measure of an angle before it can be said that an equilateral quadrilateral is a square.	Relationships among the properties of a figure and among the figures themselves are established.	*Discernment in Geometry – The Essence of Mathematics*

van Hiele's Theory in Relation to Geometry Pre-K – Grade 8 Instruction
(adapted from van Hiele (1984) and Wirszup (1976))
Figure 4.

III. Key Concepts in Spatial Measurement and Related Visualization

Before introducing the key concepts in measurement and related visualization, it may be advantageous to read a quote from Piaget concerning the complexity of measuring. This realization becomes more obvious when the list of concepts is considered and when findings from the studies are delineated in the following section of this paper. Piaget stated:

> To measure is to take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of a whole: measurement is therefore a synthesis of sub-division and change of position. However, although this way of looking at it seems clear and self-evident, the process is far more intricate in fact. As often happens in psycho-genetic development, a mental operation is deceptively simple when it has reached its final equilibrium, but its genesis is very much more complex. (Piaget et al., 1960, p. 3)

It is important to attend to the concepts and actions necessary in this intricate process of measuring space. Therefore, Lehrer's (2003) comprehensive list of the conceptual foundations of spatial measurement will be listed and terms explained. Also, since the measurement of length, area, and volume are of particular interest in this paper, and they relate to spatial shapes, Senechal's views on visualization that are related and poignant to the study of measurement of spatial objects are also considered of key importance.

Lehrer's Conceptual Foundations for Spatial Measurement

Lehrer (2003) in an inclusive review of spatial measurement studies delineated concepts that constitute a comprehensive network, which broadens the scope of understanding of spatial measure. According to Lehrer (2003), "…some of the most prominent conceptual foundations" for spatial measurement include: unit – attribute relations; iteration; tiling; identical units; standardization; proportionality; additivity; origin (zero-point) (p. 181). For clarity of meaning, each of these terms is defined and examples are given in Figure 5 on p. 29.

Lehrer's Conceptual Foundations for Measurement	
Concept	Meaning
Unit – attribute relations	Appropriate units are needed for measuring an attribute. For example, a square unit is needed to measure area; a cubic unit is needed for volume.
Iteration	The unit used to measure can be used repeatedly. For example, a single meter stick can be used to measure the length of a room. This relates to the understanding that the attribute to be measured can be subdivided.
Tiling	The units fill the space (line, area, volume) without gaps and no overlap. This also relates to the understanding that the attribute to be measured can be subdivided.
Identical units	When the units used are identical, a count of the units will give the measure of the attribute. An example of when units are not identical would be to say that a person is 5' 11" tall.
Standardization	Standard units, such as those in the metric system (meters, square meters, cubic centimeter), or those in the customary system (yards, square inches, cubic inches) as opposed to non-standard units (paper clips, or students' feet) need no explanation for they are conventional and readily understood.
Proportionality	The smaller the unit used to measure, the larger the quantity of the measure. For example, a yard-long board has a measure of 3 feet or 36 inches.
Additivity	In Euclidean space, length, area, and volume can be decomposed and recomposed. For example, the sum of the parts will equal the whole; an area can be broken down into smaller areas, the sum of which will be the same as the original area.
Origin (zero-point)	This relates to a standard unit, such as that found on a ruler, which is used to measure length. This allows for any location on the ruler to act as the origin.

Lehrer's Conceptual Foundations for Measurement
(adapted from Lehrer (2003))
Figure 5.

Children do not understand these concepts in isolation, but through experience, both physical and reflective. It has been considered that the "collective coordination" of these concepts "constitutes an informal theory of measure" (Lehrer, 2003, p. 182). For individual students this coordination occurs over time with varied exposures to problem situations. The studies reviewed in the following section of this paper substantiate this claim. These measurement concepts, as well as the claim that time and varied experiences are necessary for children to understand and to use these concepts proficiently, are in alignment with NCTM's (2000) *Principles and Standards for School Mathematics*. The two overarching measurement standards for pre-kindergarten through grade 12 are that all students should be able to:

1) Understand measurable attributes of objects and the units, systems, and processes of measurement;
2) Apply appropriate techniques, tools, and formulas to determine measurements. (NCTM, 2000)

Developing an understanding of, as well as applying, the measurement concepts listed in Figure 5 "establishes a firm ground for future exploration of the mathematics of measure, and their acquisition also implies coming to understand the (Euclidean) structure of space" (Lehrer, 2003, p. 182).

Senechal's Views on Visualization Related to the Study of Measurement and Geometry

Integral to understanding the structure of space and the measurement of spatial objects is the study of "shape," which Senechal (1990) has equated with being "an undefinable term" (p. 140). She elaborates on this and states that the goals of the study of shape are "…to discover similarities and differences among objects, to analyze the components of form, and to recognize shapes in different representations. Classification, analysis and representation are our three principal tools" (Senechal, 1990, p. 140). These tools correlate to the progressive levels of sophisticated geometrical thinking designated by van Hiele as delineated in the previous section of this paper. For example, topics that relate to analysis would include the construction and deconstruction of polyhedra as well as linear, area, and volume measurement; and topics that relate to representation would include drawing[15]. Each of these topics, classification, analysis, and representation, has relevance to the studies that are reviewed in the next section of this paper.

Senechal (1990) stresses that "…visualization is very important in the study of shape…But it is not true that we instinctively know how to 'see' any more than we instinctively know how to swim. Visualization is a tool that must be cultivated for careful and intelligent use" (p. 168). It is beyond the scope of this paper to delve into studies of visualization, but visualization has a pertinent connection to studies whose findings indicate that students, when measuring area and volume, do not "see" the arrays that structure each of these geometric entities (Batistta & Clements, 1996, 1998; Battista, 1999, 2003; Battista et al., 1998; Outhred & Mitchelmore, 2000, 2004). With this said, Senechal (1990) contends that "it is easy to teach shape as an important first step in developing powers of visualization. The simplest way to teach students to visualize is to provide them with a rich background of hands-on experience with shapes of many kinds…" (p. 171). Banchoff (1990) elaborates on the utility of visualization and contends that "The ability to visualize and interpret multidimensional data sets may be one of the best gifts we can present our students in this modern age" (p. 41).

Similar to, but much broader in scope, is "visual thinking," which Senechal (1991) contends "…is what we are doing when we rapidly recognize and 'automatically' manipulate symbols of any kind" (p. 15). Space perception, which is how Senechal (1991) defines visualization, on the other hand, "…is the mental reconstruction of representations of three-dimensional objects" (p. 15). The idea of visual thinking touches upon knowing without speaking. Senechal (1991) quotes MIT professor Loeb as saying,

[15] For a fuller list of topics related to shape, see Senechal, 1990, p. 172.

"I think that a lot of what is called intuitive is, in fact, nonverbalized knowledge … I think that is very significant in this whole notion of visual thinking" (p. 17). And the connection of intuitive or nonverbalized knowledge relates to Piaget's caution to educators that children often are unable to articulate what they are doing even when they are able to complete a task or solve a problem. Piaget considered that "…the child will be far more capable of 'doing' and 'understanding in actions' than of expressing himself verbally… In fact, it is a general psychological law that the child can do something in action long before he really becomes 'aware' of what is involved—'awareness' occurs long after the action" (Gruber & Voneche, 1995, p. 731).

IV. Research Studies on Length, Area, and Volume

> A number of geometricians believe that instruction in elementary notions of space should begin with the idea of volume, because that idea is less abstract, since all the objects we encounter in our everyday experience are in fact three-dimensional rather than two-dimensional or linear. There is some justification for their point of view where the argument is limited to early topological intuitions of space (although, it should be borne in mind that volumes are always bounded by areas and areas in turn are bounded by lines, so that even at the earliest levels of representation, linear considerations are far more important than they are in perception). (Piaget et al., 1960, p. 360)

While the main focus of this paper is area and volume, linear measurement plays an integral part. The length of a line segment can be found conveniently using a tool, such as a ruler or tape measure, which consists of uniform, linear units. If someone positions zero on the ruler at one end of the object that is to be measured, one can line up the ruler alongside the object and read off the number associated with the marking at the other end of the object. With this direct comparison, the number that is read off is the count of the units of length. When it comes to area, though, there is no tool (unless one considers a transparent grid of unit squares) that provides the built-in units to give the area measure. The same is true for volume (unless one considers individual cubes). If the actual units, unit squares for area or unit cubes for volume, were to be used, then counting these units would in fact give the corresponding measures of area and volume. Yet, this would become an extremely tedious and inefficient method, and this process does not sufficiently address the issue that linear, area, and volume measurements are continuous and never exact. For practical purposes, the efficacy of using formulas is to be appreciated, and these formulas are based on linear measure. Understanding the relationships between linear measure (1-D), area measure (2-D), and volume (3-D) is indeed an abstract concept. And it is no wonder that children's understanding of the concepts involved in measuring 2-D and 3-D objects, as well as the best methods and sequence for teaching these measurement concepts, have been the subject of researchers both in psychology and mathematics education.

In the last decade or so, there has been considerable attention in the research given to investigating children's understanding of the underlying array structures that hopefully expose to students the formulas for area and volume in a concrete way. In light of the van Hiele levels of geometric thinking, Piaget's stages of cognitive development, the aforementioned conceptual foundations of measurement and of shape, these studies will now be presented. Overviews of the studies that focus on unitizing and covering regions in an array as well as tiling and drawing the array will be presented. First, the limitations, concerns and significance of these experiences will also be explored. To further elaborate on the array, its spatial structuring and enumeration processes will be examined. Since linear measure is central to the question of how students understand rectangular area and volume of rectangular prisms, studies that relate linear measurement to spatial structuring will be presented. Lastly, the spatial structuring and enumeration of the array structure in volume studies will be discussed.

Unitizing and Covering the Region; Tiling and Drawing the Array

Wheatley and Reynolds (1996), in a longitudinal study of 2nd through 5th graders constructing abstract units in numeric and geometric settings, contend that tiling is a "rich source for developing the unitizing operation" (p. 82). They suggest that "unitizing seems to be a fundamental mental operation in coming to act mathematically" (Wheatley & Reynolds, 1996, p. 67). Students in the study tiled a variety of shapes in creating patterns on dot paper and counted the shapes. The levels of sophistication varied by the students' methods of counting. Wheatley and Reynolds (1996) stressed the importance of composite units in geometric situations, and they note that students' construction and coordination of units in geometric settings are analogous to their unit construction in numeric settings.

In a related study with four students during individual problem-solving sessions, Reynolds and Wheatley (1996) contend that to understand area a child must construct and coordinate units. In these two related studies, the units included shapes other than the usual square unit because the focus of each study was to understand the coordination of the units in a tiling and a measurement setting, not necessarily culminating in the area formula. In these studies involving shapes and coverage the researchers state that "…determining the area can be thought of essentially as a tiling of the plane with congruent regions that become units of measure" (Reynolds & Wheatley, 1996, p. 567).

Some researchers have uncovered limitations and concerns when tiling is stressed without thoughtful consideration of the resultant coverage of the planar region. For example, Outhred and Mitchelmore (1992) found that most students (ages 6-10) were able to cover rectangular shapes with an array of square tiles and that they were successful counting the tiles to find the area of the rectangle. The researchers, though, determined that young children do not "… automatically interpret arrays of squares in terms of their rows and columns" and concluded that "…this could hinder their learning about area measurement using diagrams" (Outhred & Mitchelmore, 1992, p. 202).

Outhred and Mitchelmore (1992) contend that unless the array of squares is *seen* as groups of rows and columns, students will not understand the significance of the lengths of the sides of the rectangle to find the area by formula nor that of the multiplication principle. Outhred and Mitchelmore (1992) engaged sixty-six students in drawing tasks that tapped skills that would act as transitions from manipulation of square tiles to finding the area of a rectangle pictorially — pictorially representing the area formula. It is interesting to note that many students drew an array by drawing first the rectangular boundary of the array and then struggled to figure out how to partition each side. Choosing as the starting point the edges of the rectangles indicates the significance students placed on them and this relates back to Piaget's finding that young children conceive area as that "which is bound by lines…" (Piaget et al., 1960, p. 355).

To be successful with all the tasks in the Outhred and Mitchelmore study (1992) the students had to apply numerical, spatial, and measurement skills. In the counting tasks, 50% counted by ones, whereas only 12% counted by array multiplication; the rest, 38%, counted by groups of either rows or columns. There was a wide discrepancy in the students' abilities to draw the array, with 30% of the students unable to accurately *copy* the array they had made even with the array in front of them. In some cases, a picture of a correctly formed array was given to students to copy, and all students were able to succeed on this task. The researchers linked the counting, drawing, and measuring aspects of the task and correlated these activities. The results indicate strongly that unless children can represent the array structure correctly with its rows and columns, they are unlikely to employ multiplication or group counting to determine the area of the rectangle. Success on the measurement tasks, which had the students find side lengths of the rectangle, was strongly correlated to the children's' drawing of an array. Outhred and Mitchelmore (1992) concluded that some drawing techniques must be developed, but that the most important skill needed in representing an array is based on understanding fundamentally that a rectangular array is made up of elements that "…are collinear in two directions. It is this property which allows the partitioning of the array into rows and columns" (p. 200).

Similar to Hart and Sinkinson (1988), who stressed the need for interim activities to bridge the gap between concrete activities and the application of a formula, Outhred and Mitchelmore (1992) have indicated that even interim activities, such as depicting pictorially an array that has been formed with concrete materials, may also be challenging. Students whose drawings indicated increased structuring of row-by-column also showed increased use of the multiplicative strategies in enumerating the squares—which relates to the area formula. While these studies have shown the positive effects of drawing, this applied skill is not consistently applied into American elementary mathematics classes.

Stigler and Perry (1990) suggested that the high performance of Japanese elementary students (on tests involving visualization and paper-folding) might be due to the Japanese classroom practice of concepts being visually represented and to the expectations that students gain proficiency in drawing. The lack of drawing has been noted by Senechal (1990), who stated: “Before the camera became available to every one, drawing was widely taught. Today very few people know how to draw accurately and, consequently, they no longer notice things as carefully as they once did” (p. 164).

An unexpected result of the Outhred and Mitchelmore (2000) study was the benefits of drawing as *both* a teaching and a learning tool. They found that children made insightful connections in their attempts to draw a covering of the rectangle, connections that were not made by merely covering a rectangle with square tiles. They contend that “…graphic materials provide little prestructuring, and success depends on an operational understanding of the structure of the rectangular array” (Outhred & Mitchelmore, 2000, p. 146). Outhred and Mitchelmore (2000) note the paucity of related studies in the literature of representation through drawing. The few studies involving drawing that are connected to mathematical learning include: Mitchelmore (1983), who worked with 6th and 7th graders; Outhred and Mitchelmore (1992), who worked with first through sixth grade students; Outhred and Sardelich (1997), who worked with kindergarten students who over an eight-month period showed increased sophistication in drawings of their manipulation of shapes; and Lo and Watanabe (1997), who conducted a teaching experiment with 5th graders working on proportion problems in which drawing was a focus. Outhred and McPhail (2000) stressed the value of drawing and contend that students do not appreciate the significance of the side lengths of the rectangle, as mentioned above in this paper, unless they can visualize and *draw* arrays as an iteration of rows and columns. Also, Vergnaud (1990) contends that one of the avenues through which students identify "relevant objects and relationship," is through drawings of some kind (p. 28).

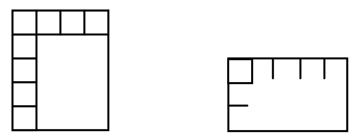
Outhred and Mitchelmore (2004) further pursued the study of structural development of students' drawing of arrays; in this study with children ages six to nine, they particularly investigated the significance of using lines instead of drawing individual squares. Their findings indicated that "…representing an array of units using two-perpendicular sets of parallel lines is more difficult than might be expected, indicating that the structure of a square tessellation is not obvious to students but must be learned" (Outhred & Mitchelmore, 2004, p. 470). The researchers contend that while some students may only see the lines in an array as a "visual" feature unrelated to numerical structure, "…drawing lines in one dimension appeared to be a precursor to recognizing rows as composite units" (Outhred & Mitchelmore, 2004, p. 471).

Spatial Structuring and Enumerating: Local and Global

Battista et al. (1998) analyzed the spatial structuring[16] and enumeration of two-dimensional rectangular arrays of squares; this study extends the work of Battista and Clements (1996) who investigated the spatial structuring that students applied while enumerating three-dimensional arrays configured in rectangular prisms. Battista, Clements, Arnoff, Battista, & Van Auken Borrow (1998) contend that spatial structuring "…is an essential mental process underlying students' quantitative dealings with spatial situations" (p. 503). In agreeing with Outhred's and Mitchelmore's (1992) findings in two-dimensions, and confirming analogous findings of Battista's and Clements' (1996) three-dimensional study, Battista et al. (1998) also determined that many students do not

[16] This is defined as "the mental operation of constructing an organization or form for an object or set of objects" (Battista et al., 1998, p. 503).

see the row-by-column structure in rectangular arrays. Their study examined the work of twelve second-graders (7-8 year olds) in separate videotaped interviews over the course of the school year. Students knowing the size of one square were asked to make an *original prediction* of the number of squares it would take to cover the inside of a rectangle; a sample of two such tasks is shown in Figure 6.



Rectangle Interview Tasks
How many squares does it take to completely
cover the inside of the rectangle?
(Battista et al., 1998, p. 507)
Figure 6.

The students were then asked to draw where they thought the squares would be located on the rectangle and to predict again how many squares would be needed to cover the rectangle; this was called the *drawing prediction*. Lastly, the students determined the number of tiles needed by covering the rectangles with square tiles.

The Battista et al. study (1998) categorized the students' structuring of rectangular arrays into the following levels of sophistication:

- Level 1 work indicated total lack of row- or column-structuring. Tiles when drawn were done so without attention to equivalency between rows or between columns. The structuring was considered *local* (attention given mainly to individual squares) as opposed to *global* (consideration of equal number of squares in full rows and accurate placement of equivalent rows and columns), and the structuring would be inadequate to enumerate the total number of tiles. Inability to appropriately structure often causes double counting of perimeter tiles.

There was evidence that students seem "…to structure the rectangular array in terms of its perimeter and interior" (Battista et al., 1998, p. 510). This relates to Piaget's contention that children of this age still think of areas "…in terms of their boundaries, the linear factor is still uppermost in children's minds" (Piaget et al. 1960, p. 345).

- Level 2 work indicated partial row- or column-structuring. There is some use of row or column as a composite unit, but not used to cover the whole array.
- Level 3A work indicated the use of row-or column-composites for structuring an array. The structuring is global; the student can conceptualize the rectangular array as being covered by copies of composites but does not coordinate the composite properly with the orthogonal dimension.
- Level 3B work indicates visual row- or column-iteration with students able to make a composite unit of a row or column, but still requiring concrete material to carry out the enumeration of the tiles in the array. A student may try to estimate and get confused or often may lose count without the concrete material.
- Level 3C work indicates row-by-column structuring whereby the iterative process is used without the dependence on concrete materials.

In moving from local to global structuring students in the study often coordinated small parts locally and in putting the parts together saw a more efficient way to make composites of the rows or columns. During the interviews, the researchers commented on a variety of physical movements of the students while counting and drawing and while answering specific questions. These movements and gestures included: nodding, motioning, pointing, moving (a finger, end of pencil, whole pencil), sweeping, and tracing. One student visually slid pre-drawn squares down to form a bottom row and only gradually did he create the array. These non-verbal expressions attest to Piaget's contention "…the child will be far more capable of 'doing' and 'understanding in actions' than of expressing himself verbally…" (Gruber & Voneche, 1995, p. 731).

To get to the more sophisticated levels of global structuring, which is a paradigm shift, a student must first abstract a row (or column) as a composite and be able to compose the whole array out of these composites. And finally, to reach the pinnacle of sophistication a student must coordinate these row-composite units with the elements of the orthogonal column so that each row can be distributed over the elements of the column. This "seeing" of the global structure of the array has connection to Hiebert (1986), who contends that a student must be able to "anticipate the result" of partitioning a region into equal parts. Battista et al. (1998) emphasize that students' spatial structuring of arrays come as a result of their "…organizing *actions* (motor and perceptual) on the sets of squares… they do not 'read off' these structures from the objects, but instead, employ a process of 'constructive structualization'[17] that enriches objects with non-perceptual content" (Battista et al., 1998, p. 530).

This structuring process is a form of abstraction that Battista et al. (1998) took "…to be the process by which the mind selects, coordinates, unifies, and registers in working memory a set of mental items or actions that appear in the attentional field" (p. 504). The levels of abstraction of an item or action include: isolating it and grasping it as a unit; internalizing it, which holds it in working memory so that it can be 'replayed'

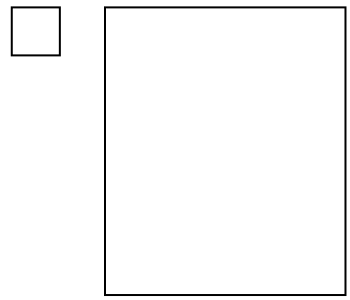
[17] Battista et al. (1998) referenced (J. Piaget & Inhelder, 1969).

without it; and interiorizing it, which strips it of its original sensory construct so that it can be applied in a different situation.

In another study that involved covering a rectangular shape by drawing units, Outhred and Mitchelmore (2000) studied the strategies of one hundred fifteen students in grades 1 – 4. Outhred and Mitchelmore (2000) focused on the relationship between linear dimensions of the rectangle and the size of the array that it encloses. They did not reference area measurement or its formula in the analysis of the students' drawings, yet the researchers stated their assumption that "…children will at some stage be able to interpret covering as a means of measuring area and that their understanding of rectangular covering will then play a vital role in their understanding of area measurement" (Outhred & Mitchelmore, 2000, p. 147).

Although the focus of Outhred's and Mitchelmore's (2000) study was different from Battista et al. (1998), their findings on spatial structuring were in agreement. They agree that the structure of the rectangular array is not obvious and that the formation of the iterable row is a foundation for an understanding of array structure. This was again reinforced in a study by Outhred and Mitchelmore (2004): understanding that the number of units in each row is constant follows after the rows are determined to be geometrically equivalent; determination of the number of rows in the array comes next; and lastly, enumeration of the total number of units concludes the structuring process.

Outhred and Mitchelmore (2000) observed students completing three different array-based tasks: Task M1 used a manipulative square; Task M2 provided the students with only pictures; and Task M3 provided the students with only instructions without pictures, diagrams, or manipulatives. For Task M2, students were asked how many small squares would be needed to cover the rectangle as shown in Figure 7.



Task M2
Students were asked to work out how many 1-cm squares would be needed to cover the drawn 6-cm x 5-cm rectangle.
(Outhred & Mitchelmore, 2000).
Figure 7.

While Outhred and Mitchelmore (2000) saw many "differences in details" in students' drawings, they also saw "many similarities in their development" (p. 157). They then classified the strategies into levels that represent a developmental sophistication. These levels complement those of Battista et al. (1998) as delineated on pp. 39-40 of this paper. The five levels designated by Outhred and Mitchelmore (2000) are as follows:

- Level 0 was considered an incomplete covering of the rectangle with gaps and/or overlaps. Some drawings suggest that students focused on the edges of the rectangle rather than on the internal structure.
- Level 1 was considered a primitive covering of the rectangle. While the units did completely cover the rectangle with no overlaps, there was no systematic organization. Units may have different size and shape of units with poor alignment of the units. Focus on the edge may still be evident.

- Level 2 was considered an array covering, constructed from units. While there is a correct array structure, the array is not constructed by iterating rows and the significance of row congruence is not fully grasped.
- Level 3 was considered an array covering, constructed by measurement. There is complete row iteration, but the array may not be fully drawn. The students' counting procedures indicate that they "see" the full array. Younger students tend to count individual units, while the older students tend to use multiplication or repeated addition.
- Level 4 was considered an array implied, with solution by calculation. While there is no array drawing, the student is able to calculate from the size of the unit and the dimensions of the rectangle. Multiplication is usually used, but there were some students who used repeated addition. This level "indicates an operational sophistication equivalent to knowledge of the area formula" (Outhred & Mitchelmore, 2000, p. 158), but the researchers are careful to claim this could indicate that the student "either visualized the array or over-learned a procedure that is based on the array structure" (Ibid.)

An interesting study by Bonotto (2003) had fourth grade students investigating strategies to determine the area of a rectangle surface. Appealing to the students' interest to solve an out-of-school practical problem, Bonotto asked students what was the surface area of each sheet of paper in a typical school binder of 90 sheets. The students were given the information provided on the manufacturer's label that each sheet was 21 x 29.7.[18] The students were also asked that if they could lay out sheets in any way they

[18] This is a standard centimeter size of paper in Europe.

wished, would the sheets cover a surface of 1 m^2. Bonotto (2003) also asked students to solve additional area tasks, such as, "How many squares whose sides measure 5mm can you draw on one sheet of paper?" and "Please fill in this sheet with 1cm x 1cm squares." The students engaged in whole-group discussions and were asked to describe their methods of solutions to the problems. Bonotto's (2003) study used an out-of-school situation in which students mathematized a problem situation in the sense that Freudenthal highly encouraged—"mathematics … taught within contexts" (Freudenthal, 1981, p. 144). The students were not given directions to perform an activity involving area, but instead were given a situation and asked to strategize a solution. To the students the surface area of the figure was not just a number, but rather it represented the magnitude of something, the square. In much the same way (but not necessarily involving area) Nunes, Schliemann, & Carraher (1993) found that street-market arithmetic involved an "oral practice" which preserves meaning to a mathematical-related activity. In a similar fashion, the Brazilian street vendors in Nunes's et al. (1993) study determined values without school-taught computational algorithms, just as the students in Bonotto's (2003) study determined area without specific instructions.

The Importance of Linear Measurement in Relation to Spatial Structuring and Area

Outhred's and Mitchelmore's (2000) investigation went into another aspect of spatial structuring that further documents the complexity of measurement. In their study they considered "…the significance of the relation between the size of the unit and the dimensions of the rectangle" (Outhred & Mitchelmore, 2000, p. 165). They emphasized the prevailing importance of linear measure as the foundation on which sophisticated measurement in 2-D and 3-D is dependent in order to develop. The relational understanding of linear measurement, as well as the linking of area measurement to both linear measurement and multiplicative concepts, must occur *before* the area formula can be meaningfully learned. Linear measurement thus becomes a basic concept, although by its very nature it may in fact be a more abstract concept than its 2-D and 3-D counterparts (Wilson & Rowland, 1993).

Seemingly contradictory, linear measurement has also been considered somewhat overused and often misused by early elementary students (Lehrer, 2003). Children's introduction to linear measurement customarily entails the use of non-standard, informal units. The emphasis is on counting which "may obscure the linear nature of the unit of measure if not made explicit when informal units are introduced" (Bragg & Outhred, 2000, p. 118). In their study of 120 students (ages 6-10), Bragg and Outhred (2000) investigated children's performance on a variety of length measurement tasks; they concluded that many students did not make the connection between linear units and a formal scale, even though most students were able to measure with rulers and count informal units. Many errors involved the alignment of the ruler when used to measure. Bragg and Outhred (2000) contend that this type of error may be due to emphasis on counting objects starting with 'one.'

Realization of the distinction between counting discrete objects and subdividing a continuous property into units underpins the role 'zero' plays as an indicator of no length. The ability to rename the 'zero' is a necessary skill when what is to be measured cannot be aligned with zero. In conclusion, the researchers contend that that the common components of linear measurement "…in this case *length* as represented by a line and *linear sub-units* of that line had not been made explicit when informal units had been introduced" (Bragg & Outhred, 2000, p. 117). And they found that the more able students "seem to have extracted this information for themselves" (Ibid). Emphasis on ruler technique, though, does not help students acquire understanding of the concepts of unit size and the structure of unit iteration. The researchers strongly contend that "…if students do not understand how scales are constructed, they will not have the basic knowledge to relate measurement of length and number lines, nor have the foundation to develop area, volume and other higher order mathematical applications" (Ibid, p. 118). There is also a substantial body of evidence that shows many secondary students do not a firm grasp of linear, area and volume concepts (Hart, 1981, 1989).

Kamii (1996, October 12-15), in a study with 177 children in grades 4, 5, 7, and 9, questioned which students' difficulties in using area formula were due to confusion between area and perimeter or to their unidimensional linear thinking. Kamii (1996) concluded that "…children are not just confusing area and perimeter" but that "…children have to make sense of the idea that one can get an area (which is two-dimensional) out of two lines (which are both unidimensional)" (p.225).

There is also the caution that linear measurement predominates early measurement experiences (Lehrer et al., 1998). This is not to be interpreted to mean that linear measurement is merely overemphasized, but rather it is a criticism that instruction of linear measure has failed to promote the development of essential ideas in measurement. And due to this failing Lehrer suggests that young students may erroneously assume that units of length act as "…a universal, all-purpose measure…" (Lehrer et al., 1998, p. 164).

In their longitudinal study, Lehrer et al. (1998) examined the development of children's conceptions of measurement of length of thirty-seven children (initially 6-8 years old) randomly selected. They found that the modal response of 78% used the manipulatives provided (squares, right triangles, circles, and rectangles) to cover the square, indicating that the majority of the students equate area measure with coverage. There was a minority of 22% who relied on linear notions to find the area; 16% actually indicated area as an iterative length. Some students matched the edge of a manipulative to one or more of the sides, while others used the ruler to measure the side of the square and to slide the ruler across the square adding the value of the lengths each time. Longitudinally, the growth of children's conception of area indicated that "a significant proportion of the children persisted in thinking about length units (i.e., iterative length strategy) as appropriate units of area measure" (Lehrer et al., 1998, p. 158). This argument is supported by Curry and Outhred (2005) as well as by Curry, Mitchelmore, and Outhred (2006).

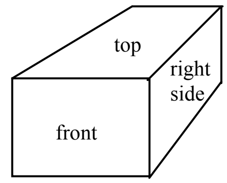
Linear measures are needed in order to apply the area formula, for example, for the rectangle—the length of the sides of the rectangle must be found and then those numbers are multiplied. Keep in mind that linear measure is not needed to find area measurement, *but* it is essential for the area formula (Outhred & Mitchelmore, 2000). Outhred and Mitchelmore (2000) determined that in the exercises of covering a rectangle with squares, students saw the count of the squares as generally one-dimensional and additive whereas the area formula is two-dimensional and multiplicative. Outhred and McPhail (2000), in a study of 16 teachers of elementary school students, learned that most teachers placed little importance on structuring in the teaching of measurement.

Even though the sample size of the study was small, no teacher mentioned the structure of unit iteration, and all considered area as a process of covering and counting. Simon and Blume (1994), in a study with 26 pre-service teachers, determined that they did not fully understand the connections between the linear measures of the rectangle's sides and quantification of its area. If teachers are unaware and/or do not place significance on the structure underlying area, it is no wonder that dependence on rote formulas unfortunately could become the only option for students.

Spatial Structuring and Enumeration in Volume Studies

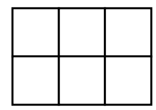
Battista and Clements (1996) studied how 3rd, 4th, and 5th grade students conceptualize and enumerate cubes in three-dimensional arrays of rectangular prisms using both interviews and quantitative analysis of the students' strategies. Their findings, which complement those on studies involving structuring of square arrays for rectangular regions (Battista et al., 1998; Outhred & Mitchelmore, 2000, 2004), show that the spatial structuring of layering arrays of cubes to determine the volume is also not intuitive, but rather, the process is created by mental actions (cognitive operations). Successful enumeration strategies of the cubes require coordination of the views of the rectangular prism faces; integration of the information gained by the views of the rectangular prism faces "to form a coherent conception of the whole" (Battista & Clements, 1996, p. 271), and a global structuring of the cubes. Students were asked to determine the number of unit cubes it would take to construct rectangular prisms. For some tasks, the students were given an isometric[19] picture of a prism of interlocking centimeter cubes and for other tasks, orthogonal[20] views were given, as shown in Figure 8.

Suppose we completely fill the rectangular box below with a rectangular cube building. The box is transparent, so you can see the building through the box's sides.

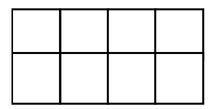


After we fill the box, we look straight at the building from its front, top, and right side. [Indicate orthogonal viewing lines with a pencil.]

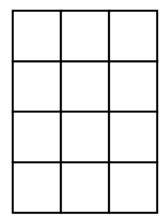
From the *front*, it looks like this. [Indicate figure at right.]



From the *right side*, it looks like this. [Indicate figure at right.]



From the *top*, the building looks like this. [Indicate figure at right.]



A. How many cubes does it take to make the building?

B. Can you make the building with cubes?

Isometric and Orthogonal Views Used in Study
(Battista & Clements, 1996, p. 261)
Figure 8.

The students' strategies, which were classified into five categories, and the outcomes of each strategy, are shown in the following table in Figure 9 on p. 51.

[19] This view represented the rectangular prism as it had been constructed on isometric dot paper.
[20] This view showed the top, side, and front rectangular faces that in 3D would be at right angles to each other

Classification of Students' Strategies for Enumerating the Cubes in 3D Array		
	Strategy Used	Outcome of the Used Strategy
A	Concept of layers used	Most effective
B	Concept of space filling considered	Inconsistent organizing of the cubes into layers Some unsuccessful grouping attempted
C	Concept of cubes was confused with the faces of the cubes	Least effective
D	Formula of L x W x H was used	No indication of understanding in terms of layers
E	Other	Mixed results

Classification of Students' Strategies for Enumerating the Cubes in 3D Array
(condensed from Battista & Clements, 1996, p. 263)
Figure 9.

There was a fairly consistent distribution of strategies across problems. Therefore, the researchers concluded that the use of non-layering strategies could not be attributed to any misinterpretation of diagrams, as had been suggested in some studies (Ben-Haim, Lappen, & Houang, 1985). About 60% of fifth graders, but less than 20% of third graders, used layering strategies. No student used the formula meaningfully. Double counting of cubes was the cause of many errors. Students who applied a local rather than global structuring would lose track of what they were doing, and thus were unable to make a correct enumeration. They often would group some cubes that made up a portion of a side, column or row, but no scheme for organizing these groups. Battista and Clements (1996) propose that students must be able to spatially structure a three-dimensional array of cubes in order to enumerate the cubes in a meaningful way. This structuring involves the recognition of a unit, the iteration of the units, and the proper placement of the iterated units on the local level (i.e., layer) such that they will generate the composite (i.e., layering the layers), resulting in new unit (i.e., the 3-D rectangular array). This relates to Piaget's findings that

> …the reconstruction of shapes is not just a matter of isolating various perceptual qualities, nor is it a question of extracting shapes from the objects without more ado. The reconstruction of shapes rests upon an active process of *putting in relation*, and it therefore implies that the abstraction is based on the child's own actions and comes about through their gradual co-ordination. (Piaget & Inhelder, 1967, pp. 78-79)

Battista and Clements (1996) identify some cognitive milestones in structuring 3D arrays as shown in the table in Figure 10.

Cognitive Milestones in Structuring 3-D Arrays In Order of Sophistication	
Medley of views	Student is able to organize cubes or sets on cubes seen in only one face of the prism at a time (local structuring)
Composite units	Student is able to conceptualize one or more faces of the prism as a composite of 2-D arrays of cubes
Coordination	Student recognizes the interrelatedness of the different views of the 3-D array and coordinates accordingly
Integration	Student constructs a coherent mental model and coordinates the orthogonal views of the prism

(condensed from (Battista & Clements, 1996, pp. 283-287))

Figure 10.

While there are differences in how students structure an entire 3-D array, *the most effective way* is by layering the composite 2-D array of one face. Included in this cognitive milestone lies the "maintaining the faces-as-composites." In doing this, a student has to be mindful that the 2-D array of one face is a representation of "composite units of cubes." To advance from a "medley of views" perspective, a student must apply mental coordination, the coordination of the orthogonal views. Shared cubes are often double counted when enumeration of the cubes in the 3-D array is being made. Students must consider that different views of the prism would be showing different faces of particular cubes. And finally, there is integration. Integration of the views of the 3-D prism requires the student to construct “a coherent mental model” of the object that possesses these views; it requires the student to coordinate the orthogonal views. This coordination relates back to Piaget’s contention that 2-D shapes bound 3-D objects.

Battista and Clements (1996) hypothesize two processes by which a mental model of spatial structuring could be formed by an integration operation: the first process depends on “recall” coordination; the second process requires performing transformations on and coordination of images of objects, for example, visualizing an iterative translation of a single layer through a distance determined by the third dimension. It is interesting to note that the researchers suggest that this “…spatial structuring provides the input and organization for enumeration. However, …sometimes it seems that attempts at enumeration engender spatial structuring or restructuring” (Battista & Clements, 1996, p. 288). This two-way synergy of enumeration-supporting-structuring and vice versa was also suggested by Ben-Haim et al. (1985). This study indicated that middle school students, after an instructional intervention, showed improvement in problems involving orthogonal views. Ben-Haim et al. (1985) contend that spatial visualization can be improved by “training.”

Battista and Clements (1996) are in agreement with Cobb, Yackel, and Wood (1992) in their belief that “…neither spatial structure nor mathematical meaning is inherent in objects …which are intended to embody certain concepts. Such meaning must be constructed by the individual” (Battista & Clements, 1996, p. 289). Thus, Battista and

Clements (1996), as well as Cobb et al. (1992) agree with Piaget and Inhelder (1967) that the visual reconstruction of shapes occurs when children are able to abstract the structure of the object as a result of coordination of their own actions. More will be said about the issues related to the use of concrete materials and the development of measurement concepts in the section on tools and manipulatives, pp. 55-61 of this paper.

Battista and Clements (1996) propose a "…rudimentary, and by no means complete, theory of the development of students understanding of 3-D cube arrays" (p. 291). The three components are: first, the notion of spatial structuring; second, the forming of composites; and third, the coordination and integration operations. Battista (1999) verified the findings of Battista and Clements (1996) and also concluded that efficiently enumerating the cubes in a 3D array, which required coordination and integration, was complex. The Battista study (1999), though, went beyond the scope of the Battista and Clements study (1996) in that it investigated the sociocultural aspects of a constructivist paradigm in an inquiry-based, problem-centered classroom. Battista used "emergent" perspective to integrate psychological and sociological aspects of constructivism in this detailed analysis.

Similar to the Battista's et al. (1998) two-dimensional study, which involved students making predictions prior to manipulation of the square tiles, Battista (1999) required that the students make predictions as to the number of cubes needed to fill various boxes. He analyzed the process that pairs of students worked through to complete various tasks. Battista (1999) fine-tuned and explicated the mental processes involved in the spatial structuring and placed this constructivist learning in the context of the inquiry-based classroom. He elaborated on the psychological mechanisms for spatial structuring cube arrays and further broke down abstraction into levels of: 1) perception, unitizing the item; 2) internalization, being able to re-present the item; 3) interiorization, disembedding the item from its original context so that it can be freely operated on in novel situations. For full understanding of the spatial structuring Battista (1999) contends that reflective abstraction in Piaget's sense is necessary, which relates to van Hiele's levels in which the implicit of one level becomes the explicit in the next. The interchange between the pairs of students had considerable effect on the mental processes of each partner. In discussions that arose between pairs of students and small groups, students were forced to justify their thinking and at the same time extend their own conceptualizations in order to make sense of their partner's justifications.

While all five pairs of students in Battista's (1999) case study initially structured mental models of 3-D arrays as uncoordinated sets of orthogonal views, those who gained the most understanding of how to fill the box with cubes did so by coordinating the views. Those students who did not coordinate the views were not able to correct their structuring errors. Students who were able to interiorize the layer structure did so recursively, "…cycling through sequences of acting (structuring and enumerating), reflecting, and abstracting" (Battista, 1999, p. 442). However, students were often not consistent in using the layering structure. This speaks to the complexity of the task and, as Battista (1999) suggests, it implies that the entire process is not fully interiorized from the first view of the task through to the final layering of arrays.

V. Use of Manipulatives and Tools in Learning Measurement Concepts

Commenting on mathematics education at the Second International Congress on Mathematical Education (ICME) in 1972, Piaget warned, particularly to teachers, that "the pupil will be far more capable of 'doing' and 'understanding in actions' than of expressing himself verbally' " (Piaget, 1975, p. 9). Manipulatives are often employed in classrooms to help children understand measurement concepts by their actions on the objects. Do manipulatives really allow students to act on them in order to gain understanding of a concept? Are plastic tiles, wooden or paper squares, cubes, and rulers necessary, helpful, and valuable in the learning of area and volume tasks? Do they support measurement concepts that teachers often assume, or do they mask, distract, or prove to be too cumbersome to be worthwhile in the understanding of area and volume measurement? These questions will now be addressed.

Moving Square Tiles vs. Drawing Square Tiles

Doig, Cheeseman, and Lindsey(1995), in an "adaptive design" study of 8-year-olds, investigated the use of concrete materials for measuring area of rectangular shapes. The findings from their study indicated that tiles made of wood (plastic would be the same) were better than tiles made from paper when it came to teaching and learning about covering a space to find area. The tiles did not overlap due to their thickness and stiffness and they easily tessellated, creating an array with little difficulty. But, because the wooden tiles have no chance to overlap, Doig et al. (1995) contend that students may create an array, and the teacher may falsely assume that students understand the row-column structure of the array. Since the paper squares can overlap easily, the researchers contend that students must be aware when using the paper manipulative to form arrays in order to represent the covering (and hopefully the partitioning) of the shape indicating the representation of area.

Along these same lines, Outhred and Mitchelmore (2000) go further and suggest that the activity of drawing squares is more valuable than physically moving concrete squares. They contend that the tiles may only support the coverage of the shape, but that drawing of the square tiles may help students realize that they are partitioning the space. And the most sophisticated method of completing the full partitioning of the rectangle, and the most abstract, is by drawing two sets of parallel perpendicular lines.

Outhred and Mitchelmore (2000) involved children who had not yet learned the area formula. By working with the visual representation of the array as partitioning the rectangle, the foundations that support the area formula of *length x width* is being laid. The formula, which is based on a multiplication method, can only be mapped onto the array model if the student has made the connection of equivalent number of squares in each row and column respectively. The number of squares in the rows and the number of squares in the columns can be determined by the linear partitioning of the rows and columns. Thus, Outhred and Mitchelmore (2000) recommend that children need clear understanding of linear unit and iteration of that unit in order to apply that understanding to the dimensions of the rectangle. And, they contend that there exists a likelihood that this connection may not be made by mere manipulation of concrete squares.

When Manipulatives Mask or Distract Rather Than Help or Clarify

In a study investigating the concurrent development of children's understanding of length, area, and volume measurement in grades 1- 4, Curry et al. (2006) found that it was impossible to determine if students, when using a ribbon to determine an area, were using the edge of the ribbon to get linear measures of the sides or the width of the ribbon to cover the shape[21]. Outhred and McPhail (2000) also uncovered confusion as to which attribute of a manipulative should be the focus of attention in measurement tasks. They found that children often are distracted by details and that teachers may not be aware of their plight. Outhred and McPhail (2000) contend that unless it is clear to a student "…they may not know whether to focus on an edge, a face or the block itself" (p. 491). Similarly, in a study with 120 students grades 1-5, Bragg and Outhred (2001) found that less than 5% of 1st through 4th graders were able to state which part of a 1cm cube was used when measuring a length, and of the 5th graders only about 16% were able to correctly identify the edge of the cube as that which was used informally to measure a length. Similar findings resulted when Hart (1989) worked with students involved in volume tasks. Because teachers had used blocks to measure length, students were confused as to what was being counted/measured when blocks were used to find volume.

Certain manipulatives may actually mask measurement concepts rather than help a child uncover them. Lehrer, Jenkins, & Osana (1998) and Lehrer, Jaslow, and Curtis (2003) found that many students were apt to use *resemblance* as criterion for area unit selection, thus squares would be chosen for measuring squares and rectangles for rectangles, and kidney-shaped beans for kidney shaped regions. The use of these manipulatives to find the area of a plane region encourages the routine of merely covering and counting to find area and it may or may not address overlap considerations. Area is looked at purely as a discrete count of objects somewhat covering a region rather than a continuous region that has been partitioned into uniform units.

[21] The researchers were unable to derive parallel scores assessing five measurement principles across length, area, and volume due partly to the ambiguous use of the ribbon.

Use of Rulers

Also to be considered in this discussion is the use of the ruler, which has the built in iterated, uniform, standard linear units. Lehrer, Jenkins & Osana (1998) states that "…children often use tools proficiently (e.g., rulers to measure length, grid paper to measure area), the gulf was wide between proficient use of these tools and the understanding of the essential qualities of length and area measure" (Lehrer, Jenkins, & Osana, 1998, p. 159). Bragg and Outhred (2000) also contend that students often can use a ruler with ease, but they concluded that "many students did not understand the relationship between linear units and a formal scale" (p. 117). Prior to having children use rulers, it has been general practice to have children engage in linear measurement activities which involve arbitrary non-standard units such as counters, links, or other units of their fancy. This was to ensure that students would gain insight into the need for a uniform and iterated unit. Boulton-Lewis, Wilss, and Multch (1994) indicated, though, that requiring children to go through a sequence of using arbitrary non-standard units before progressing to standard units may actually " defeat the purpose of what it is intended to achieve" (p. 130). The load that is placed on children's information processing may actually defeat any conceptual gains that were intended to be made. Their findings indicate that children, even if not fully understanding a standard measuring device, will choose to use it. Boulton-Lewis et al. (1994) made the point that to use a ruler is a real-world and less demanding task than it is "…to reconcile the varying lengths and numbers of arbitrary units and reason transitively" in order to determine that arbitrary units are unreliable (p. 130).

On a contradictory note on the early use of rulers, Kamii and Clark (1997), in a study of 383 students in grades 1-5, confirmed Piaget's contention that children do not construct transitive reasoning until age 7 to 8. In their study, 72% of the children were able to reason transitively by second grade. They emphatically state: *"For children who cannot yet reason transitively, rulers are completely useless for comparing two lengths that cannot be placed next to each other"* [italics in original] (Kamii & Clark, 1997, p. 118).

While Kamii's findings supported Piaget's theory on the ages children develop the conception of transitivity in measurement activities and does not encourage early use of rulers, some studies refute Piaget's theory that children must be able to conserve length before they are able to measure it. Nunes, Light, and Mason (1993) investigated the use of tools in finding length and area. In a playful scenario, pairs of students (30 children, 6-8 years of age, and 60 children, 9 –10 years of age) talked by phone and carried out comparisons of magnitude without the benefit of perceptual comparisons. Rulers, broken rulers, and string were used for length tasks; bricks (1cm^2 tiles) and rulers were used for area tasks. The children, each in separate rooms, had a line drawn on a sheet of paper and just one of the measurement tools. They conversed by phone to determine which of them had the longer line. Those children who used the ruler answered correctly 84% of the trials, those who used the broken ruler (a ruler which does not start at zero) answered correctly 63% of the time. Children using the string encountered more challenges due to the need to iterate or to subdivide the string or to merely estimate. In agreement with the findings of Boulton-Lewis et al. (1994), Nunes et al. (1993) concluded that the use of the standard tool did support, rather than hinder, the problem-solving activities of measurement. Both Boulton-Lewis et al. (1994) and Nunes et al. (1993) support Wilson and Rowland (1993) who urged teachers to provide appropriate measurement activities from children even if they were unable to conserve the attribute in question.

In the part of the Nunes et al. (1993) study that involved area, children had rulers and area units (ten bricks glued in a row as well as loose bricks) and paper and pencil at their disposal; they worked cooperatively and had visual access to both their rectangular shape as well as their partner's. They were to determine the relative size of the "walls" painted on their sheets of paper. Children had more success using the bricks than using the ruler. It was first hypothesized that counting bricks was easier than using rulers because the use of rulers would require the children to understand the multiplicative relations involved with area measurement. On closer examination, it was determined that many children using bricks were in fact using a repeated addition or multiplication strategy, rather than single counting of the bricks.

VI. Conclusion and Suggestions for Future Research

This paper has attempted to review the literature on children's understanding of the area of rectangular regions and the volumes of rectangular shapes and the relationship of those measures to their linear dimensions. The culminating evidence from the varied studies concurs that measurement is deceptively more difficult to understand than it may appear. It is also striking that studies have shown manipulative tools, which have been highly touted as beneficial to helping a child uncover the concept of area or volume or to clarify the underpinnings of the area formula, have missed their mark (Bragg & Outhred, 2001; Curry et al., 2006; Doig et al., 1995; Hart, 1989; Lehrer, Jacobson et al., 1998;

Lehrer et al., 2003; Outhred & McPhail, 2000; Outhred & Mitchelmore, 2000). It is not the tools that have mislead the students as much as it is possibly the oversight that teachers have had by focusing on the tool rather than on the *thinking* of the child *acting on* and *acting with* the tool. This supports Piaget's theory that children's understanding can only be constructed cognitively by their mental actions with their physical world.

Studies have substantiated the value of the array structure in comprehending the area of rectangles (Battista et al., 1998; Outhred & Mitchelmore, 1992, 2000, 2004) and the volume of rectangular prisms (Battista & Clements, 1996; Battista, 1999). And the progressive development of using units and composites of units to enumerate and structure the 2-D rectangular regions and the 3-D rectangular prisms seemed to unfold nicely in the description of the various studies. These studies verify that the concepts of unit and identical unit, iteration, and tiling are among the conceptual foundations of spatial measurement as noted by Lehrer (2003). These studies also incorporated components of the van Hiele phases of the learning process, which include information, guided orientation, explication, free orientation, and integration. The research methods in these studies also act as models for interpreting students' work and analyzing students' mistakes. The skills of analyzing and interpreting students' work are considered essential for effectiveness in teaching.

Teaching effectiveness in the mathematics classroom has been reported in the literature from various standpoints using an assortment of methods (e.g., Hill, Rowan, & Ball, 2005; Ma, 1999; Shulman, 1986; Stigler & Hiebert, 1999, 2004) with the majority of them related to number and operations or algebra. This body of research could be further developed by studies focused on teachers' effectiveness as indicated by elementary children's thinking and achievement when specific manipulative tools are employed in their study of geometry and measurement.

The simple tool of a pencil and the activity of drawing also appeared promising in studies (Outhred & McPhail, 2000; Outhred & Mitchelmore, 1992, 2000, 2004; Outhred & Sarelich, 1997; Vergnaud, 1990). Further research into the effect of drawing in the mathematics classroom seems warranted. As children have more opportunities to discuss and draw mathematical and physical objects that they act on, their powers of observation and their awareness of the limitations of 2-D representation may improve. Investigating the effectiveness of drawing in mathematics classroom in terms of van Hiele levels may prove beneficial. And considering Piaget's warning to teachers that children cannot always verbally express what they know, the action of drawing may facilitate bridging the gap between understanding and articulating that understanding.

Making choices as to what should be taught in geometry classes and the order in which it should be taught has been repeatedly discussed for many years, but this imperative debate has been taken anew in recent years (Mammana & Villani, 1998). The related studies of Curry and Outhred (2005) and Curry et al. (2006) have investigated students' parallel development of length, area, and volume. They are unique in the literature in that they investigate children's understanding of *three* measurement attributes within the same study. While such studies may be somewhat overloaded with details to investigate, and questions to answer, they offer much in the way of uncovering children's thinking in related concepts. The conclusions from such studies in turn contribute to the discussion on a better sequence of topics in measurement and geometry in the elementary classroom. Information may be gained as to deficiencies in understanding that hold some children back from "seeing" mathematical structures. In the same vein as the Curry and Outhred (2005) and Curry et al. (2006) studies, it may prove fruitful to investigate children's thinking while investigating the surface area and volume of rectangular prisms. Due to the absence of such a study in the literature, this would be an interesting and valuable contribution to research. This study would contribute to the mathematics educators' call to determine appropriate orders in which pertinent concepts should be taught.

A surface area/volume study could be illuminating considering what the literature has said about the "global vs. local" understanding when children are working with arrays. Children must be able to "see" the array before the array is of value to them – first in finding area and then in finding volume. Investigating surface area and volume of rectangular prisms offers an opportunity for further study. Each face of a rectangular prism constitutes an array. When finding the volume of the prism there is the need to integrate the faces so as not to double count the cubes. When finding the surface area of the rectangular prism there is no need to worry. Would children be more apt to "see" the prism by building the sides and then filling it, or would they be more successful in stacking the arrays of cubes to build the prism and then determining the area of its faces? It would be interesting to study if children, who first focus on parts, are more successful with one approach over the other; the same would be true for the children who prefer the "big picture" first and then prefer to break it down into parts. Such an investigation could also be designed to pay particular attention to children's thinking while using tools to determine surface area and volume.

Clearly, there is much to be studied in the area of elementary geometry and measurement and the different methods to do so. By learning more about *how* elementary school children understand geometric measurement we can: implement better teaching methods; suggest better sequencing of math topics; and engage children in more engaging tasks that invoke thinking and help them to construct meaning.

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