Characterization of Domains of Holomorphy in Several Complex Variables

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Contents

Chapter 1

Introduction

The study of Several Complex Variables began with Weierstrass but did not become significantly important until the work of Hartogs. The Hartogs theorem proved to be one of the most important results in the study of Several Complex Variables. This introductory chapter will provide a look at $\mathbb{C}^n$ for $n \geq 2$ and talk about the major similarities and differences between it and $\mathbb{C}$. Whenever we refer to $\mathbb{C}^n$ we will always have $n \geq 2$. Another important definition which will occur several times is the notion of a polydisc. In $\mathbb{C}^n$, a *polydisc* is the Cartesian product of one-dimensional discs with possibly different radii.

For additional references, see [2] and [5].

1.1 Extension for One Variable to Several

When going from $\mathbb{C}$ to $\mathbb{C}$ several aspects change and definitions have to be altered when working with $n \geq 2$. This section will look at new definitions that occur in Several Complex Variables.

When working over $\mathbb{C}^n$, we will use the standard multi index notation. Therefore, we will frequently write z when we mean $z = (z_1, z_2, ..., z_n)$. Additionally, we will have z^α be $(z_1^{\alpha_1}, z_2^{\alpha_2}, ..., z_n^{\alpha_n})$. Additionally, $\alpha! = \{\alpha_1!, ..., \alpha_n!\}$.

1.1.1 Similarities from One Variable to Several Variables

This section focuses on the things that are the same between $\mathbb{C}$ and $\mathbb{C}^n$.

The Maximum Modulus Principle is the same for both $\mathbb{C}$ and $\mathbb{C}^n$ and can be taken by taking one-dimensional slices of domains in $\mathbb{C}^n$.

Theorem 1.1.1. If f is a holomorphic function on an open simply connected region U of $\mathbb{C}^n$ $n \geq 1$, and $\overline{D}(z_0) \subset U$ is a closed polydisc centered at z_0 then $|f(w)|_{w \in D} \leq \max_{z \in bdD} |f(z)|$. If equality holds, the f is constant.

Another property that is identical between $\mathbb{C}^n$ and $\mathbb{C}$ is the following:

Theorem 1.1.2. If $\{f_k\}$ is a sequence of holomorphic functions on an open set Ω that converges uniformly on compact sets in Ω to f on Ω then f is holomorphic on Ω.

This is a result of Hartogs' theorem which will be proved later.

Both Several Complex Variables and Single Complex Variable are interested in the study of convergence of sequences and series. As with a single variable, we study convergence through the use of absolute convergence.

Another similarity, in some regards, is the Cauchy integral formula for a holomorphic function f on a simply closed curve. In $\mathbb{C}$, we had the formula be:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \tag{1.1}$$

for $z \in C$.

In $\mathbb{C}^n$, the formula is as follows:

$$f(z_1,...,z_n) = \frac{1}{(2\pi i)^n}\int_{C_1}...\int_{C_n} \frac{f(w_1,...,w_n)}{(w_1-z_1)...(w_n-z_n)}dw_1...dw_n \tag{1.2}$$

Another important concept that is the same in Several Complex Variables is the idea of the Cauchy-Riemann equations. If u is the real part of a holomorphic function of 2 complex variables, then we have that u must satisfying the following system of 2nd order PDE's:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} = 0 \tag{1.3}$$

$$\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_2^2} = 0 \tag{1.4}$$

The two equations above are the same as for one complex variable but we have other new differential equations, such as the ones below:

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial y_1 \partial y_2} = 0 \tag{1.5}$$

$$\frac{\partial^2 u}{\partial x_1 \partial y_2} - \frac{\partial^2 u}{\partial y_1 \partial x_2} = 0 \tag{1.6}$$

Equation 1.5 come from the fact that

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 v}{\partial x_1 \partial y_2} \tag{1.7}$$

$$= \frac{\partial^2 v}{\partial y_2 \partial x_1} \tag{1.8}$$

$$= -\frac{\partial^2 u}{\partial x_1 \partial y_2} \tag{1.9}$$

Equation 1.6 can be justified using a similar method.

Also a similar idea between $\mathbb{C}^n$ and $\mathbb{C}$ is Cauchy's estimates. In the case of $\mathbb{C}^n$, we have the theorem appear in the following manner.

Theorem 1.1.3. If f is holomorphic on a polydisc $D = \{z : |z_j| < r_j, j = 1,...,n\}$ and $|f| \leq M$ on D then it follows that $|\partial^\alpha f(0)| \leq \frac{M\alpha!}{r^\alpha}$

This is the result of the Cauchy Integral formula seen in 1.1.

1.1.2 Differences from One Variable to Several Variables

One of the main differences between $\mathbb{C}^n$ $n \geq 2$ and $\mathbb{C}$ is the treatment of power series. In a single variable, convergence would occur inside a disc and divergence would occur outside the disc. However, when working with several variables, the domains can have many different shapes.

Example 1.1.4. If we have the power series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z_1^n z_2^m$ then the open set on which it will converge absolutely is the unit bidisc $\{(z_1, z_2) : |z_1| < 1 \text{ and } |z_2| < 1\}$.

However, the power series $\sum_{n=0}^{\infty} (\frac{z_1 z_2}{2})^n$ converges absolutely on the unbounded region $|z_1 z_2| < 2$, which is clearly not a bidisc.

Another conceptual difference in $\mathbb{C}^n$ has to do with the treatment of an annulus. In $\mathbb{C}$ you could have functions that were holomorphic strictly on an annulus. However, in $\mathbb{C}^n$, if f is holomorphic on a spherical shell, then it is holomorphic inside the shell as well. This idea will be proved as a result of Hartogs' theorem (Theorem 2.5.3). Therefore, the holomorphic functions can be extended to a larger domain.

We will provide a strict definition after the completion of Hartogs' theorem, but for now, we shall define a function to be *holomorphic on an open set* if the function admits a local power series expansion.

Another result of Hartogs' theorem, unlike in $\mathbb{C}$, holomorphic functions in $\mathbb{C}^n$ do not have isolated singularities or zeroes. A holomorphic function cannot have an isolated singularity because an isolated singularity is the same thing as having a shell with inner radius 0 around the isolated singularity, which we have already seen to be impossible. Similarly, a function cannot have an isolated 0 because $1/f$ is still a holomorphic function which would then have an isolated singularity and we have already shown why that is impossible.

Example 1.1.5. A slightly convoluted example to highlight the fact that the zeroes and singularities are not isolated is as follows. If we are working in $\mathbb{C}^3$ and we have $f = \frac{z_3 - 4i}{(2 - z_1 z_2^2) z_3}$. In this case, the zeroes is the set $\{(z_1, z_2, z_3) : z_3 = 4i \text{ and } z_1 z_2^2 \neq 2\}$. The singularities in this case are the set of points $\{(z_1, z_2, z_3) : z_1 z_2^2 = 2 \text{ or } z_3 = 0\}$.

Another interesting difference is the following theorem:

Theorem 1.1.6. If $p(z_1, ..., z_n)$ is a nonconstant polynomial in $\mathbb{C}^n$, then the set of zeroes of p is not a compact subset of $\mathbb{C}^n$.

Proof. Let $p(z_1, ..., z_n)$ be a nonconstant polynomial in $\mathbb{C}^n$. Let us fix $n - 1$ of the z_j's. Therefore, the resulting polynomial is a polynomial of 1 variable and will have a set of zeroes corresponding to the fixed z_j's. Since we can do this for any combination of the z_j's for any complex numbers, the set of zeroes is not compact.

□

The last important difference between $\mathbb{C}$ and $\mathbb{C}^n$ which we will discuss in the introduction is the geometric differences. One major theorem in $\mathbb{C}$ is the Riemann Mapping Theorem which is stated as follows:

Theorem 1.1.7. In $\mathbb{C}$, every simply connected region Ω in the plane (other than $\mathbb{C}$) is conformally equivalent to the open unit disc U.

In the case of $\mathbb{C}^n$, such a conformal mapping may not exist. Although we will not prove this, it can be shown that in $\mathbb{C}^2$ the unit ball is not biholomorphic to the unit bidisc ([1], Section 3.2).

Remark One of the major reasons why there is no Riemann Mapping Theorem for $\mathbb{C}^n$ is because of the fact that biholomorphic maps are not always angle preserving.

Example 1.1.8. Let f be a function from $\mathbb{C}^2$ to $\mathbb{C}^2$ defined as $f(z_1, z_2) = (z_1 + z_2, z_2)$. In this example, f is clearly not angle preserving highlighting the fact that even simple biholomorphic maps may not be angle preserving.

Due to the fact that not even linear maps are always distance preserving, it is impossible to have a Riemann Mapping Theorem for $\mathbb{C}^n$.

Having discussed the similarities and differences between $\mathbb{C}$ and $\mathbb{C}^n$, we can now start talking about domains of convergence and start building the information we need to prove the major theorems.

Chapter 2

Domains of Convergence of Power Series

As mentioned in the last section, if we have a power series, the domain of convergence need not be a ball. This section looks to flesh out the theory related to these domains as well as prove some incredible results about $\mathbb{C}^n$. Particularly, Hartogs' theorem about separate holomorphicity implying joint holomorphicity was one of the biggest results of the study of Several Complex Variables.

Firstly, let us define what it means to be a domain of convergence.

Definition 2.0.1. Given some power series P in $\mathbb{C}^n$, we define the the *domain of convergence* to be the interior of the set E such that P converges absolutely on E.

Example 2.0.1. The power series $\sum_{n=1}^{\infty} z_1^n z_2^{n!}$ converges on the union of three sets $\{(z_1, z_2) : |z_2| < 1\}$, $\{(0, z_2)\}$, and $\{(z_1, z_2) : |z_2| = 1 \text{ and } |z_1| < 1\}$. Therefore, the domain of convergence is the interior of the union of the 3 sets. It is not hard to prove that the domain of convergence is the interior of the first set.

The use of absolute convergence is important, and almost required, in $\mathbb{C}^n$ because it allows terms to be reordered when calculating sums.

In $\mathbb{C}$, domains of convergence were discs. In $\mathbb{C}^n$,domains of convergence can vary. One type of domain of convergence is a polydisc as can be seen from Example 2.0.1.

2.1 Power Series in Several Complex Variables

Since we are using the standard multi index notation, and we are working with power series, we will adapt some standard notation. Therefore, when discussing Power series in Several Complex Variables, we can write $\sum_{\alpha} c_{\alpha} z^{\alpha}$ to be $\sum_{a_1=0}^{\infty} \dots \sum_{a_n=0}^{\infty} c_{a_1,\dots,a_n} z_1^{a_1} \dots z_n^{a_n}$.

2.2 Characterization of Domains of Convergence

When working in $\mathbb{C}^n$ we will derives some properties about domains of convergence of power series which we will provide below.

Remark Without loss of generality, we will only be considering power series centered at 0.

Definition 2.2.1. Domains of Convergence are *multicircular*. This means that if $(z_1, z_2, ...z_n)$ is in the domain of convergence so is $(\lambda_1 z_1, \lambda_2 z_2, ...\lambda_n z_n)$ such that $|\lambda_1| = |\lambda_2| = ... = |\lambda_n| = 1$.

We shall refer to multicircular domains as *Reinhardt domains*.

Moreover, due to the fact that convergence is based on absolute convergence, the above definition works if $|\lambda_j| \leq 1$ for all j. Therefore, we have that a domain of convergence is the union of polydiscs centered at the origin.

After the introduction of two additional definitions, we can start to prove some of the major theorems regarding Domains of Convergence.

Definition 2.2.2. Given some domain $U \in \mathbb{C}^n$, we say D is *complete* if whenever z is in D, the whole polydisc, $\{w : |w_1| \leq |z_1|, ..., |w_n| \leq |z_n|\}$ is in U.

It is clear that the domain of convergence is complete.

This definition along with the previous one implies that every domain of convergence is a complete Reinhardt domain.

Example 2.2.1. Let E be a polydisc centered at the origin with polyradius $r = \{r_1, ..., r_n\}$. Then the domain of convergence of the power series

$$\sum_\alpha \frac{z^\alpha}{r^\alpha} \tag{2.1}$$

is cleary E. Moreover, it is clear that E is a complete Reinhardt domain.

Let $\sum_\alpha c_\alpha z^\alpha$ be a power series with domain of convergence D. Suppose that $z, w \in D$, then we have that $\sum_\alpha |c_\alpha z^\alpha|$ and $\sum_\alpha |c_\alpha w^\alpha|$ both converge in D. We claim that

$$\sum_\alpha |c_\alpha||z^\alpha|^t |w^\alpha|^{1-t} \tag{2.2}$$

will also converge if $0 < t < 1$. This is because we can apply Hölder's Inequality with $p = 1/t$ and $q = 1/(1-t)$. This is because

$$\sum_\alpha |c_\alpha||z^\alpha|^t |w^\alpha|^{1-t} = \sum_\alpha (|c_\alpha||z^\alpha|)^t (|c_\alpha||w^\alpha|)^{1-t} \tag{2.3}$$

By Hölder's inequality, this is less than

$$(((\sum_\alpha |c_\alpha||z^\alpha|)^t)(((\sum_\alpha |c_\alpha||w^\alpha|)^{1-t}) \tag{2.4}$$

which we know to be finite which proves the convergence of

$$\sum_\alpha |c_\alpha||z^\alpha|^t |w^\alpha|^{1-t} \tag{2.5}$$

Also, it is trivial to see that Equation 2.2 will hold when $t = 0$ or $t = 1$.

Definition 2.2.3. A domain D is said to be *logarithmically convex* if whenever z and w are in D then so is any point z' where $|z'_j| \leq |z_j|^t |w_j|^{1-t}$ for all $0 \leq t \leq 1$.

Another way of stating this definition is if z and w are in a domain of convergence, we say the domain is logarithmically convex if the points obtained by forming the geometric average of each component of the moduli also lies in the domain.

Let D be the domain of convergence of a power series. We have seen from above that D is a complete logarithimically convex Reinhardt Domain. Hence, D is determined by the points in it by nonnegative real coordinates and $\log D$ is the subset of $\mathbb{R}^n$ that is replacing the coordinates of each point in D by their logarithm. Moreover, $\log D$ is convex in $\mathbb{R}^n$.

Now we can prove our first major theorem of this section.

Theorem 2.2.2. A complete Reinhardt domain in $\mathbb{C}^n$ is the domain of convergence of some power series $\sum_\alpha c_\alpha z^\alpha$ iff it is logarithmically convex.

Proof. The 'Only if' has already been established. The last major component we need to show is that for every logarithmically convex, complete, Reinhardt domain D there exists some power series which has a domain of convergence of exactly D. We wish to show a method to construct such a power series. We will first prove it when D is bounded and then when D is unbounded.

Assume D is bounded. For each multi index α let $N_\alpha(D) = \sup\{|z^\alpha| : z \in D\}$. Because D is bounded, $N_\alpha(D) < \infty$. We will prove that the series

$$\sum_\alpha \frac{z^\alpha}{N_\alpha(D)} \tag{2.6}$$

is a series which converges on D and diverges outside of D. We prove this is the desired series by showing the series converges for all $w \in D$ and diverges for $w \notin D$.

Since D is open, given $w \in D$ $\exists \epsilon > 0$ such that $(1+\epsilon)w \in D$. Therefore, $(1+\epsilon)^{|\alpha|}|w^\alpha| \leq N_\alpha(D)$ by construction. Therefore, we have that

$$\sum_\alpha \frac{w^\alpha}{N_\alpha(D)} \tag{2.7}$$

must converge absolutely on D because it is dominated by the convergent series

$$\sum_\alpha (1+\epsilon)^{-|\alpha|} \tag{2.8}$$

.

Now, let us assume w is in the exterior of D and the real coordinates of w are positive. If one of the coordinates of w is 0, we can perturb that coordinate slightly to still lie in the exterior of D. We will show that the series $\sum_\alpha \frac{w^\alpha}{N_\alpha(D)}$ diverges. Due to the fact that D is multicircular, having w have positive real coordinates is sufficient. Since $w \notin D$, we know that $\log w \notin \log D$ and because $\log D$ is convex there is a hyperplane that can separate the two in $\mathbb{R}^n$. The hyperplane l is defined by a real linear functional in $\mathbb{R}^n$ defined by $f(x) = \sum_{j=1}^n \beta_j x_j = M$. Therefore $f(\log w_1, ..., \log w_n) > \max_{z \in G} f(\log z_1, ..., \log z_n)$. Because D is a complete Reinhardt domain, we know that the β_j's must be non negative otherwise l could assume arbitrarily large values on D. Because D is bounded, we know that for all $z \in D$, $\log|z_1|, \ldots \log|z_n|$ is bounded by some constant M. When each β_j, is increased by some small amount, ϵ, then the supremum of $\log D$ increases by at most $nM\epsilon$. This fact means we can slightly alter the β_j's and still have f separate $\log w$ from $\log D$. Therefore, we can assume that the β_j's are all rational. We can also multiply the β_j's by a common denominator to have all the β_j's to be positive integers. Let us call this set of β_j's β.

Therefore, $w^\beta > N_\beta(D)$. Additionally, if k is a positive integer, multiplying β by k does not change the fact that the equality holds. Therefore $\sum_\alpha \frac{w^\alpha}{N_\alpha(D)}$ diverges since infinitely many terms are greater than 1.

Now assume that D is unbounded and not all of $\mathbb{C}^n$. Let $\{z_j\}$ be the countable set of points outside of $\overline{D}$ with positive rational coordinates. Now construct the list $\{w(j)\}_{j=1}^{\infty}$ such that each z_j appears in the list infinitely many times.

Let us define D_r to be the set D intersected with a ball of radius r centered at the origin. Now, we know that each D_j is bounded. Using the boundedness of each D_j we can apply the first part of the proof to have a positive multi-index $\beta(j)$ such that $w(j)^{\beta(j)} > N_{\beta(j)}(D_j)$. Since we can multiply the multi-index by any positive number and still have the inequality hold, we can assume that $|\beta(j+1)| > |\beta(j)|$. We now claim that

$$\sum_{j=1}^{\infty} \frac{z^{\beta(j)}}{N_{\beta(j)}(D)j)} \tag{2.9}$$

is the desired power series with domain of convergence D.

First, let us show that $\forall\, z \in D$ that the series converges. So let $z \in D$. Therefore, we know there is some k such that $z \in D_k$ which means that $N_\alpha(D_j) > N_\alpha(D_k)$ whenever $j > k$. Therefore, we have that the absolute value of the sum is dominated by $\sum_\alpha |z^\alpha|/N_\alpha(D_k)$ which converges for any $z \in D_k$ meaning that the domain of convergence is at least D.

Now let us assume that the series were to absolutely converge in some neighborhood outside of D. Therefore, there would be some ζ outside of D such that the series converges for ζ and ζ has rational coordinates. Therefore, we have infinitely many values in our list such that $w(j) = \zeta$. Therefore, we have that

$$\sum_{j=1}^{\infty} \frac{\zeta^{\beta(j)}}{N_{\beta(j)}(D_j)} \tag{2.10}$$

would have infinitely many terms greater than 1. Therefore, the domain of convergence cannot be greater than D.

Therefore, any logarithmically convex, complete Reinhardt domain is the domain of convergence for some power series.

□

2.3 Holomorphic Functions in Several Complex Variables

Having discussed power series, and their domains of convergence, the study of holomorphic functions is the next logical step because convergent power series on a domain D are just local representations of holomorphic functions.

There are several different definitions for a holomorphic function in several complex variables and we shall use the following definition:

Definition 2.3.1. Given an open set G and a function f, we say that f is *holomorphic* on G if it is holomorphic in each variable separately and continuous in all variables jointly.

2.4 Boundaries of Domains of Convergence

Before moving to Hartogs' theorem, we first wish to discuss the natural boundaries for holomorphic functions. We open with a famous theorem uses the Baire-Category Theorem and the Open Mapping theorem.

Theorem 2.4.1. Cartan-Thullen

The domain of convergence of a multivariable power series is a domain of holomorphy. Therefore, for every domain of convergence, there is some power series that converges in the domain and is singular at every boundary point.

Remark When we talk about a function f being singular at a boundary point p, we mean that f does not admit a direct analytic continuation on a neighborhood of p.We provide a stricter definition of singularity in Definition 4.1.1.

Now, let us begin the actual proof.

Proof. Let $\sum_\alpha c_\alpha z^\alpha$ be a power series with domain of convergence D. We wish to show that D is also a domain of holomorphy. We know that $\sum_\alpha c_\alpha z^\alpha$ and $\sum_\alpha |c_\alpha| z^\alpha$ have the same domain of convergence, so we can assume that all coefficients of c_α are nonnegative real numbers.

To continue with the proof, we need to introduce the following lemma:

Lemma 2.4.2. If a power series in $\mathbb{C}^n$ has nonnegative real coefficients, then the series is singular at every boundary point of the domain of convergence at which all the coordinates are nonnegative real numbers.

Suppose that $f(z) = \sum_\alpha c_\alpha z^\alpha$ is a power series in $\mathbb{C}^n$ with nonnegative real coefficients and D be its domain of convergence. Then f is singular at every boundary point p of D.

Proof. We will prove this lemma by contradiction. Assume that f extends holomorphically to a neighborhood of p. Since we are working in a neighborhood of p, we can assume that the coordinates of p are positive. Moreover, dilating the coordinates modifies the coefficients of the power series by positive factors so we can assume that $||p|| = 1$ where $|| \bullet ||$ denotes the standard Euclidean norm in $\mathbb{C}^n$.

Since we are working in a neighborhood of p, we know there exists $\epsilon < 1$ such that the closed ball $\overline{b}$ centered at p with radius 3ϵ lies inside the neighborhood of p on which f extends holomorphically.

We know that the closed ball centered at $(1-\epsilon)p$ with radius 2ϵ is inside the same neighborhood of p because if we have

$$||z - (1-\epsilon)p|| \leq 2\epsilon \tag{2.11}$$

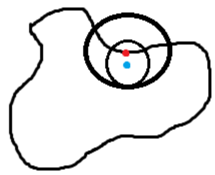

Figure 2.1: Let the noncircular domain be D and p be the red point on the boundary of D with the larger circle being the neighborhood of p. If the blue point is $(1-\epsilon)p$, we can see that the smaller neighborhood with radius 2ϵ centered at $(1-\epsilon)p$ lies inside the neighborhood of p.

then we have

$$||z-p|| = ||z-(1-\epsilon)p-\epsilon p|| \leq ||z-(1-\epsilon)p+++\epsilon||p|| \leq 2\epsilon+\epsilon = 3\epsilon \tag{2.12}$$

by the triangle inequality. An example of this can be seen in Fig. 2.1.

Since f is holomorphic on the ball $\overline{B}$, we can express f as a Taylor series about $(1-\epsilon)p$ and we know the series converges absolutely on the closed ball $\overline{B}((1-\epsilon)p, 2\epsilon)$. Therefore, we have that the Taylor series will also converge absolutely at the point $(1+\epsilon)p$. Therefore, we have the Taylor series evaluated at $(1+\epsilon)p$ equals

$$\sum_{\alpha} \frac{1}{\alpha!} f^{(\alpha)}((1-\epsilon)p)(2\epsilon p)^{\alpha} \tag{2.13}$$

Because of the fact that $(1-\epsilon)p$ lies in the domain of the original power series, we can compute the derivative of f at $(1-\epsilon)p$ by differentiating the series $\sum_{\alpha} c_{\alpha} z^{\alpha}$ to get

$$f^{(\alpha)}((1-\epsilon)p) = \sum_{\beta \geq \alpha} \frac{\beta!}{(\beta-\alpha)!} c_{\beta}((1-\epsilon)p)^{\beta-\alpha} \tag{2.14}$$

We can combine the previous two expressions to get the following convergent series on our original domain.

$$\sum_{\alpha} \left(\sum_{\beta \geq \alpha} \binom{\beta}{\alpha} c_{\beta}((1-\epsilon)p)^{\beta-\alpha} \right) (2\epsilon p)^{\alpha} \tag{2.15}$$

We can rearrange the terms in the above series without any problems since all of the terms are nonnegative real numbers. Therefore, after these steps and using the binomial expansion, we have the convergent series

$$\sum_{\beta} c_{\beta}((1+\epsilon)p)^{\beta} \tag{2.16}$$

Note that this new series is the original series of f evaluated at the point $(1+\epsilon)p$. Using the Comparison test, we have that f converges absolutely in the polydisc centered at 0 with boundary point $(1+\epsilon)p$. Therefore, we have that p is also in this polydisc. Therefore, we have that p is not a boundary point of the domain of convergence which is a contradiction. Therefore, f must be singular at p.

□

Using this lemma, we know that the power series with nonnegative coefficients $\sum_\alpha c_\alpha z^\alpha$ must be singular at the boundary points of D. Let us have q be an arbitrary point. We can therefore write $q = (r_1 e^{i\theta_1}, ...r_n e^{i\theta_n})$. Therefore, we can rewrite the series as

$$\sum_\alpha c_\alpha e^{-i(\alpha_1\theta_1+...+\alpha_n\theta_n)} z^\alpha \tag{2.17}$$

and the series will be undefined here. Therefore, for every $q \in bd(D)$ we know there is some power series that converges in D but is singular at q.

Given some domain D, let $\{p_j\}$ be a countable dense subset of $bd(D)$. Given arbitrary natural numbers j, k, we have that $D \cup B(p_j, 1/k)$ is an F-space (See Appendix) and that the set of holomorphic functions on $D \cup B(p_j, 1/k)$ embeds continuously into the space of holomorphic functions on D because of the restriction map. Since the preceding discussion produces a power series that does not extend to the ball $B(p_j, 1/k)$, we know that the embedding is not the whole space. Therefore, we can apply the corollary of the Baire Category Theorem (theorem A.0.2) to have that the image of the embedding must be of first category since it is not the whole space. Since the set of power series on D which extend across any of the the sets $D \cup B(p_j, 1/k)$ is a countable union of sets of first category, and therefore first category itself, we must have that most power series that converge in D must have $bd(D)$ as a boundary. $\square$

2.5 Hartogs' Theorem

This section is dedicated to proving Hartogs' theorem that separate holomorphicity implies joint holomorphicity. All proofs done in this section will be done in $\mathbb{C}^2$ however, the extension to $\mathbb{C}^n$ is straightforward and will be omitted.

An important point of note, this concept does not work in the case of $\mathbb{R}^{2n}$. For example, in the case of $\mathbb{R}^2$ and the function $f(x, y) = xy/(x^2 + y^2)$ for $x, y \neq 0$ and $f(0, 0) = 0$. Even though the function is real analytic in each variable separately, the function itself is not even jointly continuous.

Before proving Hartogs' theorem, we must first use the following theorems by Osgood.

Theorem 2.5.1. (Osgood, 1899). *If $f(z_1, z_2)$ is holomorphic in each variable separately and locally bounded in both variables jointly, then $f(z_1, z_2)$ is holomorphic in both variables jointly.*

Proof. Because the theorem is both local and invariant under both translations and dilations of the coordinates, we may assume the domain of f to be an open set contained in the unit bidisc and the modulus of f to be bounded above by 1.

Fix some z_1 and let g be a function $g : z_2 \to f(z_1, z_2)$ for our fixed z_1. Therefore, g is holomorphic and can be expressed as a power series $\sum_{k=0}^{\infty} c_k(z_1) z_2^k$ which is convergent in the unit disc of z_2. Because of the fact that f is bounded, we know that $|c_k(z_1)| \leq 1$ for all k by the Cauchy's estimate for derivatives. Moreover, by the Weierstrass M test, we have that the series $\sum_{k=0}^{\infty} c_k(z_1) z_2^k$ converges uniformly in both variables jointly for arbitrary compact subsets of the open unit bidisc. To show this fact, let $K \subset D \times D$. Therefore, we have $|z_2| \leq 1$ for all $z_1, z_2 \in K$. This means that we have $|c_k(z_1) z_k^2| \leq |z_2|^k \leq c^k$ where $c \leq 1$.

Therefore, since the series is less than 1 for all points in K, the series converges uniformly by the Weierstrass M test.

All we must now show is that the coefficient function $c_k(z_1)$ is a holomorphic function of z_1 in the unit disc for all k. We do this by having $c_0(z_1) = f(z_1, 0)$. We know this from $k = 0$ where $c_0(z_1)$ is a holomorphic function of z_1 in the unit disc due to the hypothesis of separate holomorphicity. By induction, let us assume that for $k \in \mathbb{N}$, we have that $c_j(z_1)$ is also holomorphic for $j < k$. We also have that

$$\frac{f(z_1, z_2) - \sum_{j=0}^{k-1} c_j(z_1) z_2^j}{z_2^k} = c_k(z_1) + \sum_{m=1}^{\infty} c_{k+m}(z_1) z_2^m \tag{2.18}$$

whenever $z_2 \neq 0$. However, as z_2 approaches 0, we have that the right hand side converges uniformly with respect to z_1 to $c_k(z_1)$ by the fact that $|c_j(z_1)| \leq 1$ for all j which means that the left side converges as well. Now if $z_2 \neq 0$ the left side must be a holomorphic function of z_1 using both the induction hypothesis and the assumption of separate holomorphicity. Therefore, by Morera's theorem and the uniform convergence of $c_k(z_1)$ we see that $c_k(z_1)$ converges for z_1 and we see that we have that the function $c_k(z_1)$ is the limit of holomorphic functions, and holomorphic which completes the induction argument and the proof.

□

The next major result was also achieved by Osgood with the following theorem.

Theorem 2.5.2. (Osgood, 1900). If $f(z_1, z_2)$ is holomorphic in each variable separately, then there is a dense open subset of the domain of f on which f is holomorphic in both variables jointly.

Proof. We need to show that given $D_1 \times D_2$ is an arbitrary closed bidisc contained in the domain of f that there is an open subset of $D_1 \times D_2$ on which f is jointly holomorphic. We do this by constructing sets $E_k = \{z_1 : |f(z_1, z_2) \leq k \, \forall z_2\}$. Since we know that $|f(z_1, z_2)|$ is continuous in z_1 if we fix z_2 we know that E_k must be a closed subset of D_1 because the set for a fixed z_2 is closed and $E_k = \cap_{z_2 \in D_2} \{z_1 : |f(z_1, z_2)| \leq k\}$. Moreover, given $w \in D_1$, there is some E_k such that $w \in E_k$. Now, we can apply the Baire Category theorem to have that there is some k such that E_k has a nonvoid interior. Therefore, there is some open subset of $D_1 \times D_2$ on which f is bounded. Since the function is bounded on a dense subset, we can apply Theorem 2.5.1 to complete the proof.

□

Now, we can solve Hartogs' theorem:

Theorem 2.5.3. If $f(z_1, z_2)$ is holomorphic in z_1 for each fixed z_2 and holomorphic in z_2 for each fixed z_1, then $f(z_1, z_2)$ is holomorphic jointly and $f(z_1, z_2)$ can be represented locally by a convergent power series in two variables.

Proof. Using the previous theorem, and the local nature of the conclusion, we wish to prove that if $f(z_1, z_2)$ is separately holomorphic on a neighborhood of the closed unit bidisc and we have $0 < \delta < 1$ such that f is jointly holomorphic in a neighborhood of the smaller bidisc with $|z_2| \leq \delta$ and $|z_1| \leq 1$ then f is jointly holomorphic on the unit bidisc.

Let us write $f(z_1, z_2) = \sum_{k=0}^{\infty} c_k(z_1) z_2^k$. Using the residue theorem, we can write each coefficient can be written as the integral

$$c_k(z_1) = \frac{1}{2\pi i} \int_{|z_2|=\delta} \frac{f(z_1, z_2)}{z_2^{k+1}} dz_2 \tag{2.19}$$

where the numerator is jointly holormorphic on the smaller bidisc. We know that the joint holomorphicity of f on the smaller bidisc implies that $c_k(z_1)$ is a holomorphic function of z_1 on the unit disc. Let us assume M is an upper bound of $|f(z_1, z_2)|$ on the strip $|z_2| \leq \delta$ and $|z_1| \leq 1$. Then, from Eq. 2.19, for all k, we have that $|c_k(z_1)| \leq M/\delta^k$. This shows that there is a large constant B depending on δ such that for all k, $|c_k(z_1)|^{1/k} < B$. For a fixed z_1, by separate holomorphicity, we have that the series $\sum_{k=0}^{\infty} c_k(z_1) z_2^k$ converges for z_2 when $|z_2| \leq 1$. Therefore $\limsup_{k\to\infty} |c_k(z_1)|^{1/k} \leq 1$ for each z_1 using the formula for radius of convergence.

Now, given $\epsilon > 0$ and a radius r slightly less than 1, we wish to show that there exists $N \in \mathbb{N}$ such that, for a fixed z_1 such that $|z_1| \leq r$, $|c_k(z_1)|^{1/k} < 1 + \epsilon$ whenever $k \geq N$. This property means that $\sum_{k=0}^{\infty} c_k(z_1) z_2^k$ converges uniformly on the set with $|z_1| \leq r$ and $|z_2| \leq 1/(1 + 2\epsilon)$ by the M test. Therefore, we have that $f(z_1, z_2)$ is jointly holomorphic on the interior of this set. Because both r and ϵ were arbitrary, the function is jointly holomorphic on the open unit bidisc. Therefore, we can let $u_k(z_1)$ denote the subharmonic function $|c_k(z_1)|^{1/k}$ and we will get the completion of the proof using the following lemma:

Lemma 2.5.4. Suppose $\{u_k\}_{k=1}^{\infty}$ is a sequence of subharmonic functions on the open unit disc that are uniformly bounded above by a constant B and suppose that $\limsup_{k\to\infty} u_k(z) \leq 1$ for every z in the unit disc. Then for every $\epsilon > 0$ and $0 < r < 1$, there exists a natural number N such that $u_k(z) \leq 1 + \epsilon$ when $|z| \leq r$ and $k \geq N$.

The lemme requires some hard analysis which we will skip ([1], Lemma 3). □

Chapter 3

Convexity Notions

One of the most important notions in Several Complex Variables is the notion of convexity. In this section we will first define the basic idea of convexity as it applies to $\mathbb{R}^n$ and work our way to explore how it relates to $\mathbb{C}^n$. From there, we will discuss several different notions of convexity and build our way up to the idea of holomorphic convexity which will play an important part in solving the main proofs in the next chapter.

The biggest motivation for discussing notions of convexity lies in the fact that it has to do with the notion of separation.

Let us begin by restating the definition for basic convexity:

Definition 3.0.1. Given some arbitrary set $U \subset \mathbb{R}^n$, we say U is convex iff $\forall x, y \in U$, the line segment joining x and y lies entirely in U.

One other important concept to consider when working with convexity is the concept of separation. The notion of separation is the same for both the real case and the complex case and is defined as the following:

Definition 3.0.2. Given a convex set $A \subset \mathbb{C}^n$ and a point $b \notin A$. We say that A can be separated from b if there exists a linear functional, Λ, such that $\text{Re}\,\Lambda\,(z) < \text{Re}\,\Lambda\,(b)\ \forall z \in A$. If we are working over $\mathbb{R}^n$, then we have that $\text{Re}\,\Lambda = \Lambda$.

When considering any arbitrary sets, there is the natural question of how to make that set convex. Therefore, let us also define the idea of a convex hull of an arbitrary set U, and then show the relationship between the convex hull and the idea of separation.

Definition 3.0.3. Given any set $U \subset \mathbb{C}^n$, we define the convex hull of U to be the intersection of all convex sets which contain U. Therefore, the convex hull of U is the smallest convex set containing U. We denote the convex hull of a set U as $\widehat{U}$.

This definition translates to the fact that $\widehat{S} = \mathbb{R}^n \setminus \{x : x \text{ can be separated from } S\}$.

Using this idea, we can introduce the following theorem:

Theorem 3.0.1. If U is any set, then $\widehat{U}$ is also the set of all convex cords of all pairs of points of U.

Another point worth mentioning is the following definition.

Definition 3.0.4. If $U \subset \mathbb{R}^n$ is open and $S \subset U$, then we have the convex hull of S in U to be $\widehat{S} \cap U$.

A quick remark is that $U \setminus \widehat{S}$ is the set of all points in U that can be separated from S.

For example, if we have a set to be the points $\{z_1, z_2\}$, then the convex hull of these two points would be the line segment connecting the two of them together. If we have $U = \{z \in \mathbb{C}^n : |z| = 1\}$ we have that $\widehat{U} = \{z \in \mathbb{C}^n : |z| \leq 1\}$.

Since we can treat $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ we can use some properties regarding convexity for $\mathbb{R}^n$ when talking about convexity for $\mathbb{C}^n$. The first property of interest is the fact that the convex hull of a compact set is compact (The proof can be found in *Functional Analysis* by Rudin and is proof 3.20). The second important fact, which is similar to the first, is that the convex hull of an open set is also open. The third important property is the following theorem.

Theorem 3.0.2. Given U is open in $\mathbb{C}^n$, U is convex iff whenever $K \subset U$ is compact, we have that $\widehat{K} \subset U$.

Proof. $\Rightarrow$

Assume that U is convex. Therefore $U = \widehat{U}$. Given any compact set $K \subset U$. Since we trivially have that $\widehat{K} \subset \widehat{U}$ and $U = \widehat{U}$ we have $\widehat{K} \subset U$.

$\Leftarrow$

Given an open set U such that for all compact sets $K \subset U$ we have $\widehat{K} \subset U$. Given any two points $\{x, y\} \in U$. We have that the set is compact, therefore, $\widehat{\{x, y\}} \in U$ by assumption. Therefore, U is convex and since U is the smallest convex set containing U, we have $\widehat{U} = U$.

$\square$

Now, we can shift our focus to convexity in $\mathbb{C}^n$.

3.1 Classes of Functions

While we just discussed the standard form of convexity, there are other notions of convexity based on classes of functions. Given some class of upper semicontinuous real valued functions, $\mathcal{F}$, on an open set G, let us start by defining a convex hull of a set U, and using this definition to discuss $\mathcal{F}$-convexity. But first, we must begin with a definition of separation in regards to classes of functions.

Definition 3.1.1. Given some set $U \subset \mathbb{C}^n$, a point $p \notin U$ and some class of functions $\mathcal{F}$, we say that p can be separated from U, if there exists some $f \in \mathcal{F}$ such that $|f(p)| > \max_{w \in U} |f(w)|$.

At this point, we can define the $\mathcal{F}$-convex hull of some set U as follows.

Definition 3.1.2. Given any set U, let us define $\widehat{U}_{\mathcal{F}}$ to be the set of all points that cannot be separated from U by some $f \in \mathcal{F}$.

We call $\widehat{U}_{\mathcal{F}}$ the $\mathcal{F}$- convex hull of U.

The requirement that the functions be semicontinuous is important because it ensures that the functions will obtain a maximum value on all compact sets. We can now discuss convexity in relation to classes of functions as follows:

Definition 3.1.3. Given an open set $G \subset \mathbb{C}^n$, we say that G is $\mathcal{F}$ convex iff the $\mathcal{F}$-convex hull of every compact set, $K \subset G$ is also compact.

We must take note of the fact that this notion of $\mathcal{F}$-convexity is different from the notion of logarithmic convexity we saw earlier. In the case of logarithmic convexity, a set D is logarithmically convex if $\log D$ is convex in $\mathbb{R}^n$ while a set D is $\mathcal{F}$-convex if the $\mathcal{F}$-convex hull of every compact set K is also compact. Therefore, logarithmic convexity is based on the properties of D in $\mathbb{R}^n$ while $\mathcal{F}$-convexity is based on the properties of D in $\mathbb{C}^n$.

The remainder of this section will focus on examples and properties of generic classes of functions.

Example 3.1.1. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two classes of functions and $\mathcal{F}_1 \subset \mathcal{F}_2$, then any $\mathcal{F}_1$-convex set is automatically $\mathcal{F}_2$-convex. Moreover, if we have K and U are the same in both cases, we have $\widehat{K}_{\mathcal{F}_2} \subset \widehat{K}_{\mathcal{F}_1}$.

Example 3.1.2. If $\mathcal{F} = \{z^a : a \in \mathbb{N}\}$ and $U = \mathbb{C}$ and $K = \{z : |z| = 1\}$, we have that $\widehat{K}_{\mathcal{F}} = \{z : |z| \leq 1\}$ because there is no function of the form z^a such that $|w^a|_{|w|<1} > \max_{z \in K} |z^a|$. If we took the same U and K and replaced $\mathcal{F}$ with $\mathcal{F} = \{z^a : a \in \mathbb{Z}\}$ then $\widehat{K}_{\mathcal{F}} = K \bigcup 0$ highlighting the fact that the $\mathcal{F}$-convex hull depends on the class of functions rather than the set. Likewise, if $\mathcal{F} = \{z^a : a \in \mathbb{Z}\}$ and $U = \mathbb{C} \setminus 0$ then we have $\widehat{K}_{\mathcal{F}} = K$, which also shows that the $\mathcal{F}$-convex hull also depends on the domain U.

While $\mathcal{F}$ can be any group of upper semicontinuous real valued functions, we will focus strictly on the class of polynomials, linear functions, and holomorphic functions. These classes of functions are of importance because of their relationship to one another, as well as the properties of their corresponding convex hulls.

3.1.1 Polynomial Convexity

In this section, we focus on polynomial convexity. As such, we will always have $\mathcal{F}$ be the set of holomorphic polynomials in complex coordinates $z_1, ..., z_n$.

Before discussing results specifically related to polynomial convexity, we begin with an example over $\mathbb{C}$:

Example 3.1.3. Suppose we are in $\mathbb{C}^n$ and we have $U = \mathbb{C}^n$ and the compact set K to be countably many points $\{z_1, ...z_n\}$. Therefore $\widehat{K}_{\mathcal{F}} = K$ because we can construct a polynomail $f = \prod_{j=1}^{n} (z - z_j)$. Therefore, we have $|f(w)| > 0 \ \forall w \notin K$ and $|f(w)| = 0 \ \forall w \in K$. This example also highlights the fact that a set can be polynomially convex while not being regularly convex. However, regular convexity implies polynomially convex.

One thing we get when working with polynomial convexity is the following theorem:

Theorem 3.1.4. For any set U and compact subset K, $\widehat{K}_{\mathcal{F}} \subset \widehat{K}$.

Proof. Given $\widehat{U}$ and a compact set $\widehat{K} \subset \widehat{U}$. We will show the relation by showing that $\widehat{K}^C \subset \widehat{K}^C_{\mathcal{F}}$. Given some point p that can be separated from $\widehat{K}$, we have that it is separated from $\widehat{K}$ by some real-linear function Re $l(z)$. Because of this, we also have that p can be separated from $\widehat{K}$ by $e^{l(z)}$ and, consequently, by $|e^{l(z)}|$. Since we are working on compact sets, and $e^{l(z)}$ is entire, it can be approximated uniformly by polynomial functions. Therefore, $p \in \widehat{K}^C_{\mathcal{F}}$.

□

This proof also establishes the fact that if there is an entire function that separates a point p from a compact set K, then that point can also be separated from K by a polynomial.

When working with polynomial convexity over $\mathbb{C}^n$, there are many questions to consider.

One fact we get is if K is a compact subspace of the real part of $\mathbb{C}^n$, then we have that K is polynomially convex.

Another interesting fact is the following theorem:

Theorem 3.1.5. If K_1 and K_2 are disjoint, compact, convex sets in $\mathbb{C}^n$, then their union is polynomially convex.

The proof can be found in (Boas, 2013).

We will end this section with a few examples and theorems:

Example 3.1.6. If we are working in $\mathbb{C}^2$, we have the set $A = \{(\cos\theta, \sin\theta) : 0 \leq \theta \leq 2\pi\}$ is polynomially convex. This is because $A \subset \mathbb{R}^2 \subset \mathbb{C}^2$ meaning that both the complex coordinates happen to be real numbers. Since the polynomially convex hull is a subset of the regular convex hull, we just need to find a polynomial that separates the interior of A from A. We can construct the polynomial $f = 1 - z_1^2 - z_2^2$. therefore, we have $|f(w)| = 0$ if $w \in A$ and $|f(z)| \neq 0$ if z lies on the interior of A.

Example 3.1.7. A slightly more interesting example is as follows. Let K be a polynomially convex compact subset of $\mathbb{C}^n$ and let p be some polynomial. We therefore have the graph $A = \{(z, p(z)) \in \mathbb{C}^{n+1} : z \in K\}$ is polynomially convex.

Suppose $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}$, and that (α, β) is not in A. Let us first assume that $\alpha \notin K$. Therefore, by our assumption, we have a polynomial in $\mathbb{C}^n$ that separates K from α. Therefore, if we treat g as a polynomial in $\mathbb{C}^{n+1}$ and have it be independent of z_{n+1}, then we have it separates (α, β) from p. Assume $\alpha \in K$ and $p(\alpha) \neq \beta$. Therefore, let us define $g = z_{n+1} - p(z)$ such that g is identically 0 on K and nonzero at (α, β). Therefore, g separates (α, β) from A.

Definition 3.1.4. One way to construct a polynomially convex set in $\mathbb{C}^n$ by constructing a polynomial polyhedra: $\{z \in \mathbb{C}^n : |p_1(z)| \leq 1, ..., |p_k(z)| \leq 1\}$ where each p_j is a polynomial. The sets are still polynomially convex if $\leq$ is replaced by $<$. The set is polynomially convex because every point in the complement must be separated by some p_j that defines the polyhedron.

Polynomial polyhedra may not always be bounded, so one way to avoid this complication, and still have the result be polynomially convex, is to have the polynomial polyhedra intersect a polydisc.

Although we will not discuss polynomial polyhedra, we do get the following interesting theorem.

Theorem 3.1.8. If G is a polynomially convex set in $\mathbb{C}^n$, then it can be approximated by polynomial polyhedra:

(1). If K is a compact polynomially convex set, and U is an open neighborhood of K, then there is an open polynomial polyhedron P such that $K \subset P \subset U$.

(2). if G if polynomially convex open set, then G can be expressed as the union of an increasing sequence of open polynomial polyhedra.

Proof. For case (1).

Since we know K is bounded, we know that it is contained in some closed polydisc $\overline{D}$. If $\overline{D} \subset U$ then D is our required polyhedron. If we have that $\overline{D} \setminus U \neq \emptyset$, we know that for each $w \in \overline{D} \setminus U$ there exists a polynomial f that separates w from K. Therefore, we can find some constant, λ such that $\max_{z \in K} |\lambda f(z)| < 1 < |\lambda f(w)|$.

Because of the fact that $\overline{D} \setminus U$ is compact, we now there are finitely many polynomials, $f_1, ..., f_k$ such that $\cap_{j=1}^{k} \{z : |f_j(z)| < 1\}$ contains K and does not intersect $\overline{D} \setminus U$. If we take this set and intersect it with D we have a polyhedron containing K that is contained in U.

For case (2).

Since G is open, we know that G can be expressed as an increasing union of compact sets. Moreover, the polynomial convex hulls of these sets form another increasing sequence of compact sets that equal G. Let us now remove certain sets and renumber them such that G is exhausted by a sequence $\{K_j\}_{j=1}^{\infty}$ of polynomially convex compact sets such that K_j is contained in the interior of K_{j+1}. By the first part of the proof, we know that there is a sequence of open polynomial polyhedra, $\{P_j\}_{j=1}^{\infty}$ such that $K_j \subset P_j \subset K_{j+1}$ for all j. $\square$

Theorem 3.1.9. (Oka-Weil) If K is a compact, polynomially convex set in $\mathbb{C}^n$, then every function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials.

3.1.2 Linear and Rational Convexity

In this section, our focus is on linear function and rational functions. As such, we will always have $\mathcal{F} = \{f : f = \frac{a_0 + a_1 z_1 + ... a_n z_m}{b_0 + b_1 z_1 + ... b_n z_n}\}$. If we want to focus on specifically linear functions, we will take the denominator to strictly be 1. When working with rational functions, instead of polynomial functions, one of the main differences is that the domain, G now matters because we need to have that every $f \in \mathcal{F}$ does not have a 0 in the denominator for all $z \in G$.

As was the case with polynomial convexity, we can have a set that is $\mathcal{F}-$convex that is not regularly convex.

Example 3.1.10. The open set $A = \mathbb{C}^2 \setminus \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$ is $\mathcal{F}$-convex but not regularly convex. This is because if $K \subset A$ is a compact set, then the function $\frac{1}{z_2}$ is bounded on K making it so that $\widehat{K}_{\mathcal{F}}$ stays away from the boundary of A.

Example 3.1.11. If we have K to be an arbitrary compact set, then $\widehat{K}_{\mathcal{F}} = \widehat{K}$, where $\widehat{S}$ refers to the standard convex hull. The proof of this example will be done later.

When working with $\mathcal{F}$ instead of polynomials, we gain the following theorem:

Theorem 3.1.12. Given an open set G, we say G is $\mathcal{F}$-convex iff for every boundary point of G there is a complex hyperplane that intersects the boundary point but does not intersect G.

We can now provide the following definition:

Definition 3.1.5. A domain is called linearly convex if its complement can be written as the union of complex hyperplanes.

When we choose to work with the class of rational functions, instead of linear functions, we can define rational convexity as follows:

Definition 3.1.6. Given a compact set $K \subset \mathbb{C}^n$, we say K is rationally convex if for every point $w \notin K$, we can separate w from K by some rational function that is holomorphic on $K \cup w$. This definition translates to the standard definition, where we have some rational function f such that $|f(w)| > \max_{z \in K} |f(z)|$.

A remark about the definition is the fact that if $f(w)$ is undefined we can slightly alter the coefficients of f such that $|f(w)|$ would become an arbitrarily large number and f on K would not change too much.

Example 3.1.13. If K is any compact subset of $\mathbb{C}$ we have that K is rationally convex because, for all $w \notin K$, we can have the rational function $\frac{1}{z-w}$ which blows up at w. We can also take a small enough ϵ such that $\frac{1}{z-w-\epsilon}$ will be large at w and mostly unchanged for all values in K.

We can also introduce the following two theorems related to rational convexity.

Theorem 3.1.14. If K is a compact subset of $\mathbb{C}^n$ and $w \notin K$ and there is a polynomial p such that the image of w under p is not contained in the image of K under p, then this is equivalent to rational convexity.

Proof. $\Rightarrow$

Assume $p(w) \notin p(K)$. Therefore, we can find an ϵ such that $\frac{1}{1-p(z)-p(w)-\epsilon}$ is a rational function of z that is holomorphic in a neighborhood of K and we have that the modulus at w is larger than the modulus anywhere on K.

$\Leftarrow$

If f is holomorphic on $K \cup w$ then we have that $\frac{1}{f(z)-f(w)}$ is a rational function of z that is holomorphic on K and singular at w. Therefore, we can rewrite this function as a quotient of polynomials with the denominator being a polynomial equal to 0 at w and nonzero on K. $\square$

Theorem 3.1.15. The rationally convex hull of a compact subset of $\mathbb{C}^n$ is a compact subset of $\mathbb{C}^n$.

3.1.3 Holomorphic Convexity

The final category of convexity we wish to consider is Holomorphic convexity. Therefore, given a domain G, we have $\mathcal{F} = \{f : f \text{ is holomorphic on } G\}$. We have the idea of $\mathcal{F}$-convexity be the same as before.

If our domain is all of $\mathbb{C}^n$, then we always have that holomorphic convexity is the same as polynomial convexity, as we saw earlier in the section on polynomial convexity.

Given the fact that $\mathcal{F}$ consists of holomorphic functions on some domain G, if we have $G_1 \subset G_2$, and a compact $K \subset G_1$, we have $\widehat{K}_{\mathcal{F}_1} \subset \widehat{K}_{\mathcal{F}_2}$ because of the fact that $\mathcal{F}_2 \subset \mathcal{F}_1$ since there are more holomorphic functions on G_1 than there are on G_2. Moreover, we have the following theorem:

Theorem 3.1.16. Given a compact set K and a domain G such that $\widehat{K} \subset G$, we have that $\widehat{K}_{\mathcal{F}} \subset \widehat{K}$.

Proof. We solve this proof by showing $\widehat{K}^C \subset \widehat{K}^C_{\mathcal{F}}$. Let $w \in \widehat{K}^C$. Let f be the function that separates K from w and let us take the real part of f. Therefore, we have $|\text{ Re } f(z)| < |\text{ Re } f(w)|$. Therefore, we have that $|e^{Ref(z)}| < |e^{Ref(w)}|$ therefore, e^{Ref} is our desired holomorphic function and $w \in \widehat{K}^C_{\mathcal{F}}$. $\square$

We will end this section with some examples and a theorem which we shall prove later.

Example 3.1.17. If we are working in $\mathbb{C}$ and have K to be the unit circle, if $G = \mathcal{C}$ and $\mathcal{F} = \{f : f \text{ is entire }\}$, then $\widehat{K}_{\mathcal{F}} = \{z : |z| \leq 1\}$.

Example 3.1.18. Let K be the unit circle in $\mathbb{C}$. If $G = \mathbb{C}$ and $\mathcal{F}$ is the class of entire functions we have that $\widehat{K}_{\mathcal{F}}$ is the unit circle by the Maximum Modulus Principle.

If $G = \mathbb{C} \setminus 0$ and $\mathcal{F}$ is the set of functions holomorphic on G then $\widehat{K}_{\mathcal{F}} = K$ because $f = \frac{1}{z}$ is holomorphic on G and separates K from its interior.

Theorem 3.1.19. Given K is a holomorphically convex compact subset of G and $p \in G$ such that $p \notin K$, then $K \bigcup p$ is a holomorphically convex compact subset of G.

These major classes of functions are all related and play a role in solving the official theorems of this paper.

Holomorphically convex sets are particularly important because any open holomorphically convex set is also a domain of holomorphy.

Chapter 4

Official Theorems

4.1 Statement of the First Theorem

Before stating the official theorem, we first wish to introduce some additional definitions.
Let us assume G is a domain in $\mathbb{C}^n$.

Definition 4.1.1. We shall say a point p on the domain of G is *completely singular* for a holomorphic function f, if for all open connected neighborhoods, U, of p, there is no holomorphic function g with $f = g$ on some nonempty open connected subset of $U \cap G$.

For example, if we work over $\mathbb{C}$ and take G to be the unit disc, we have that $p = 0$ is completely singular for $f(z) = \frac{1}{z}$. If we have a domain G and have a holomorphic function $h = \frac{f}{g}$ where $f(z) \neq 0 \ \forall z \in G$, then we have the set of completely singular points is the set of zeroes of g in G. A picture example of completely singular can be seen in Figure 4.1.

If G is completely singular at each boundary point p, we say that G is a *weak domain of holomorphy.* We have that any convex domain is a weak domain of holomorphy. Given some point p on the boundary of some convex domain, there is a holomorphic function f such that $f(p) = 0$ and $f(z) \neq 0 \ \forall z \in G$. Therefore, $h = \frac{1}{f}$ is our desired holomorphic function.

If there exists a holomorphic function f that is completely singular at every boundary point p of G, we say that G is a *domain of holomorphy.* In $\mathbb{C}^n$, we have the unit ball is a

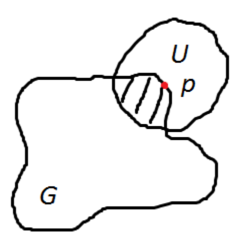

Figure 4.1: Given our domain G, we say p is completely singular for a holomorphic function f, if for any open connected neighborhood of p, we have no function g such that $g = f$ on any open connected subset of $U \cap G$. In the case of this example, there is no holomorphic function g, such that $f = g$ on the shaded region of the photo.

domain of holomorphy. Let us have $f = f_1 f_2 ... f_n$ where f_j is a function of a single complex variable that is 0 on the boundary of the unit disc in $\mathbb{C}_j$. If we have $h = \frac{1}{f}$, we have that h is a function that completely singular at every point on the boundary of the unit ball.

Given $z \in G$, let $\delta(z) = \inf_{w \in \mathbb{C}^n \setminus G} ||z - w||$, or the distance from z to the boundary of G. We can expand this definition to a set $S \subset G$ and have $\delta(S) = \inf\{\delta(z) : z \in S\}$ where $\delta(S)$ is equivalent to the minimum distance from a point in S to the boundary of G.

We can also expand these definitions to include a unit vector v. Let us have $\delta_v(z) = \sup\{r > 0 : z + \lambda v \in G, \lambda \in \mathbb{C}^n, |\lambda| < r\}$. We therefore have $\delta_v(z)$ is a disc of the largest radius, centered at z, such that it is contained in G. In the case of $\delta_v(S)$, we have $\delta_v(S) = \inf\{\delta_v(z) : z \in S\}$.

The first theorem we wish to prove is the following theorem:

Theorem 4.1.1. Given a domain G of $\mathbb{C}^n$, the following are equivalent:

1. G is holomorphically convex.

2. Given any sequence of unique points $\{p_j\} \in G$ with no accumulation point in G, there exists a holomorphic function f on G such that $\lim_{j \to \infty} |f(p_j)| = \infty$.

3. Given any sequence $\{p_j\} \in G$ with no accumulation point in G, there exists a holomorphic function f on G such that $\sup_j |f(p_j)| = \infty$.

4. For all compact sets K in G, and for all unit vectors v in $\mathbb{C}^n$, the distance from K to the boundary of G in direction v is equal to the distance from $\widehat{K}$ to the boundary of G in direction v. Alternatively, $\delta_v(K) = \delta_v(\widehat{K})$.

5. For all compact sets K in G, the distance from K to the boundary of G is equal to the distance from $\widehat{K}$ to the boundary of G. Alternatively, $\delta(K) = \delta(\widehat{K})$.

6. G is a weak domain of holomorphy.

7. G is a domain of holomorphy.

4.2 Proof of the First Theorem

We already have (2) $\Rightarrow$ (3) and (7) $\Rightarrow$ (6) trivially. We will begin with the proof of (4) $\Rightarrow$ (5):

Proof. Let us first define $f(z) = \inf\{\delta_v(z) : ||v|| = 1\}$. Now, for any $r < \delta(z)$, we must have $B(z, r) \subset G$ by our definition of $\delta(z)$. Therefore, $\forall u$ such that $||u|| = 1$, we must have that $z + \lambda u \subset G$ if we have that $|\lambda| < r$. Therefore, we have $r \leq \delta_u(z)$ and this gives us, $\delta(z) \leq \delta_u(z)$.

Now, let us assume that $\delta(z) \leq \inf\{\delta_v(z) : ||v|| = 1\}$. Let us assume $\delta(z)$ is strictly less than $\inf\{\delta_v(z) : ||v|| = 1\}$. Therefore, let us choose an s such that $\delta(z) < s < \inf\{\delta_v(z) : ||v|| = 1\}$.

By our definition of $\delta(z)$, we must have that $B(z, s) \cap G^c \neq \emptyset$. Let $w \in B(z, s) \cap G^c$. Therefore, $||w - z|| < s$, moreover, we have that $||w - z|| = \lambda$. We can now define $u = \frac{w-z}{||w-z||}$.

This gives us the fact that $w = z + \lambda u \in G^c$. But, this means that $\delta_v(z) < s$ which is impossible from our assumption. Therefore, $\delta(z) = \delta_v(z)$.

Using this fact, and our initial assumption, given any compact set $K \subset G$, we have that $\delta(\widehat{K}) = \inf\{\delta_v(\widehat{K}) : ||v|| = 1\}$ because of the fact that $\delta(\widehat{K}) = \inf\{\delta(z) : z \in \widehat{K}\}$ and $\delta(z) = \delta_v(z)$. Moreover, we have that $\delta(\widehat{K}) = \inf\{\delta_v(\widehat{K}) : ||v|| = 1\} = \inf\{\delta_v(K) : ||v|| = 1\} = \delta(K)$.

□

The next implication we wish to prove is that (5) ⇒ (1).

Proof. Given a compact set K in G, we have that (5) implies K is a closed subset of G with positive distance from the boundary. Thus, given a point $p \notin \widehat{K}$, we must have a holomorphic function f, on G, such that $|f(p)| > \sup_{w\in K} |f(w)|$. Similarly, we can multiply f by some λ with $|\lambda| = 1$ such that $|f(p)| = \lambda f(p)$ and we shall call λf simply f for convenience sake. Now let us have $\delta = f(p) - \sup_{w\in K} |f(w)|$ and construct the set $A = \{z \in G : |f(z)| > f(p) - \frac{\delta}{2}\}$. Given the fact that A is open, and is the set of points that can be separated from K by f, we have that $\widehat{K}$ is relatively closed since its compliment is open.

By (5), we have that $\delta(\widehat{K}) > 0$. Therefore, if we can show that $\mathbb{C}^n \setminus \widehat{K}$ is open, we must have that $\widehat{K}$ is closed. For $p \in G$, we have already found an open set, A, such that $p \in A$ and A is open. Given $p \notin G$, let us have $r = \delta(\widehat{K})$. Therefore, $d(p, r)$ is a disc centered at p with radius r that is open and $d(p, r) \cap \widehat{R} = \emptyset$. This gives us the fact that $\widehat{K}$ is closed.

Now that we have the fact that $\widehat{K}$ is closed, we can complete the proof by showing that the holomorphically convex hull $\widehat{K}$ is a subset of the ordinary hull of K, since we know that the ordinary hull of K is bounded. We will prove this with the following lemma:

Lemma 4.2.1. Let $\mathcal{F}$ be the set of all affine functions of the form $\{|a_0 + a_1 z_1 + ... + a_n z_n|\}$, we have that $\widehat{K}_{\mathcal{F}} = \widehat{K}$.

Proof. In order to show $\widehat{K}_{\mathcal{F}} \subset \widehat{K}$ we will show that $\widehat{K}^C \subset \widehat{K}^C_{\mathcal{F}}$. Therefore, given some $p \in \widehat{K}^C$ we need to show that $p \in \widehat{K}^C_{\mathcal{F}}$. We know that $\exists f$ such that f is of the form $b_1 z_1 + b_2 z_2 + ... b_n z_n$, and $|f(p)| > \sup_{w\in K} |f(w)|$. Let us now choose a b_0 such that $b_0 + b_1 p_1 + b_2 p_2 + ... b_n p_n > 0$.

If $\sup_{w\in K}(b_0 + b_1 w_1 + b_2 w_2 + ... b_n w_n) > 0$ then we are done since this function is our desired affine function. Let us suppose that $\sup_{w\in K}(b_0 + b_1 w_1 + b_2 w_2 + ... b_n w_n) \not> 0$. Since K is compact, we know that the function is bounded. Therefore, there is some c_0 such that $\sup_{w\in K}(c_0 + b_0 + b_1 w_1 + b_2 w_2 + ... b_n w_n) > 0$.

Calling this affine function h, we have that $|h(p)| > \sup_{w\in K} |h(w)|$. Therefore $p \notin \widehat{K}_{\mathcal{F}}$ and $p \in \widehat{K}^C_{\mathcal{F}}$ giving us $\widehat{K}_{\mathcal{F}} \subset \widehat{K}$.

To show that $\widehat{K} \subset \widehat{K}_{\mathcal{F}}$ we will show that $\widehat{K}^c_{\mathcal{F}} \subset \widehat{K}^c$. Given $p \in \widehat{K}^c_{\mathcal{F}}$, we know there exists an affine function f such that $|f(p)| > \sup_{w\in K} |f(w)|$. However, any affine function is a hyperplane that separates K from p, therefore $p \in \widehat{K}^c$ which completes the proof.

□

This lemma functions for our purposes using the fact that $\forall p \in G$ such that $p \notin K$, $p \bigcup K$ is a compact set, so thus the lemma holds in this case which gives us the fact that the holomorphically convex hull of K is a subset of $\widehat{K}$ which gives us the fact that it is closed and bounded and thus compact. □

Next, we show that (1) $\Rightarrow$ (2).

Proof. Given some holomorphically convex domain G and some sequence $\{p_i\}$ of unique points with no accumulation point in G, we wish to show that there exists a function f such that $\lim_{i\to\infty}|f(p_i)| = \infty$. We will begin by first constructing a sequence of increasingly large holomorphically convex compact sets $\{K_i\}$ such that their interiors exhaust G and we will construct a sequence $\{q_i\}$ by rearranging terms in $\{p_i\}$ such that $q_i \in K_{i+1} \setminus K_i$.

We construct the sets as follows. Firstly, we have $K_1 = \emptyset$. For all $m \in \mathbb{N}$, we form the set $\{z \in G : ||z|| \leq m \text{ and } \delta(z) \geq \frac{1}{m}\}$. We shall also have L_m be the corresponding holomorphically convex hull of this set. Let L_{m_1} be the first set that intersects G and contains points of $\{p_1\}$. Note that it must contain fintely many points of $\{p_i\}$ otherwise the sequence would have a limit point in G, which would be a contradiction. We can take this subsequence of $\{p_i\}$ and arrange it to form the sequence $\{q_1, ..., q_k\}$. Let us define $K_{j+1} = \{q_1, ..., q_j\}$ for $1 \leq j \leq k$.

Next, find L_{m_2} such that $L_{m_2} \setminus L_{m_1}$ contains some finite subsequence of $\{p_i\}$. If we label these points $\{q_{k+1}, ..., q_{k+k_1}\}$, for $k \leq j \leq k + k_1$ we can define $K_{j+1} = L_{m_1} \bigcup \{q_{k_1}, ..., q_j\}$. We will prove that the K_{j+1}'s are holomorphically convex compact subsets with the following lemma:

Lemma 4.2.2. Given K is a holomorphically convex compact subset of G, and $p \in G$, we have that $K \bigcup \{p\}$ is a holomorphically convex compact subset of G.

Proof. Given a holomorphically convex compact subset K of G, and a point $\{p\} \in G$, we wish to show that $K \bigcup \{p\}$ is holomorphically convex compact. We have compact by construction, so we just need to show that the new set is holomorphically convex. Therefore, given some $\{q\} \in G$ such that $\{q\} \notin K$ and $q \neq p$, we wish to find some holomorphic function h such that $|h(q)| > \max_{w \in K \bigcup \{p\}} |h(w)|$.

First, let f' be a holomorphic function on G such that $|f'(q)| > |f'(p)|$. Let us now define $f = f' - f'(p)$. Therefore we have $|f(q)| > |f(p)$ and $f(p)| = 0$.

Since K is holomorphically convex, there is some g' such that $|g'(q)| > \max_{w \in K} |g'(w)|$. Let us define $M = \max_{w \in K} |f(w)|$. If $\frac{|g'(q)}{\max_{w \in K} |g'(w)|} \leq M$ let us redefine g' as $g' = e^{g'}$. We will continue to exponentiate until we have some $g = e^{g'}$ such that $\frac{|g(q)|}{\max_{w \in K} |g(w)|} > M$.

Therefore, our desired holomorphic function is $h = fg$.

□

We continue this process until we exhaust G. Therefore, we have a series of increasingly large holomorphically convex compact subsets and a sequence $\{q_i\}$, such that the interiors of these sets exhaust G as well because $\forall K \subset G$ where K is compact, we know that it is contained in the union of open subsets. Therefore, there must be some K_j° such that K_j° contains the desired open subsets, and since $K_j^\circ \subset K_j$, we have that K_j° and K_j exhaust G.

Now that we have constructed our desired holomorphically convex compact sets and our rearranged sequence, we can proceed with the proof. For each j, we wish to find a holomorphic function on G such that $|f_j(z)| < 2^{-j}$ for $z \in K_j$ and $|f_j(q_j)| > j + \sum_{k=1}^{j-1} |f_k(q_j)|$.

Since we know that K_i are holomorphically convex, and $q_i \notin K_i$, there exists a function f_i such that $|f_i(q_i)| > \max_{w \in K_i} |f_i(K_i)|$. If f_i does not satisfy the properties above, we can make it do so in the following manner. The first step is to make sure the difference between

$|f_i(q_i)|$ and $\max_{w\in K_i}|f_i(w)|$ is large enough. If the gap is not sufficiently large, we can replace f_i with $f_i = e^{f_i}$. We can continue this process until the gap is large enough. From there, we can divide the resulting f_i by a large enough constant such that $|f_j(z)| < 2^{-j}$ for $z \in K_j$ and $|f_j(q_j)| > j + \sum_{k=1}^{j-1}|f_k(q_j)|$.

Therefore, the infinite sum $\sum_{j=1}^{\infty} f_i$ will converge uniformly on all compact subsets of G to a function f. Moreover, we have that $|f(q_j)| > j-1$. That is because

$$|f(q_j)| = |\sum_{l=1}^{\infty} f_l(q_j)| = |\sum_{l=1}^{j-1} f_l(q_j) + f_j(q_j) + \sum_{l=j+1}^{\infty} f_l(q_j) \tag{4.1}$$

$$\geq |f_j(q_j)| - |\sum_{l=1}^{j-1} f_l(q_j)| - |\sum_{l=j+1}^{\infty} f_l(q_j)| \tag{4.2}$$

$$\geq |f_j(q_j)| - \sum_{l=1}^{j-1}|f_l(q_j)| - \sum_{l=j+1}^{\infty}|f_l(q_j)| \tag{4.3}$$

.

Since $|f_j(q_j)| - \sum_{k=1}^{j-1}|f_k(q_j)| > j$ by our assumption, we therefore have:

$$|f_j(q_j)| - \sum_{l=1}^{j-1}|f_l(q_j)| - \sum_{l=j+1}^{\infty}|f_l(q_j)| \tag{4.4}$$

$$\geq j - \sum_{l=1}^{j-1}|f_l(q_j)| + \sum_{l=1}^{j-1}|f_l(q_j)| - \sum_{l=j+1}^{\infty}|f_l(q_j)| \tag{4.5}$$

$$= j - \sum_{l=j+1}^{\infty}|f_l(q_j)| \tag{4.6}$$

Since $q_j \in K_l$, we have that $\sum_{l=j+1}^{\infty}|f_l(q_j)| \leq \sum_{l=1}^{\infty} 2^{-l} < 1$.

Therefore, $|f(q_j)| > j-1$ and $\lim_{j\to\infty}|f(q_j)| = \infty$.

We also must have $\lim_{i\to\infty}|f(p_i)| = \infty$. This is because of the fact that $\forall\, M\, \exists\, m$ such that $|f(p_n)| > |f(q_M)|\ \forall n > m$ because of the fact that the set $\{q_1, ..., q_M\}$ must be mapped to some set $\{p_{m_1}, .., p_{m_M}\}$. Therefore, we can have $m = \max\{m_1, ..., m_M\}$.

□

The next implication we wish to show is that (3) ⇒ (1).

Proof. We first assume K to be a compact subset of some domain G. Due to the fact that K is compact, we have any holomorphic functions on K will be bounded, and hence they must also be bounded on $\widehat{K}$ because it is also compact. Therefore, if $\{p_i\}$ is a sequence in $\widehat{K}$, it must have an accumulation point in G. Because of (3), if we had the opposite case, then there is a holomorphic function on $\widehat{K}$ that is bounded on $\widehat{K}$ but approaches infinity at the boundary. This property also gives us the fact that any accumulation point of $\{p_i\}$ must be in $\widehat{K}$ because it is relatively closed by definition. Therefore, we have that $\widehat{K}$ is sequentially compact, completing the proof.

□

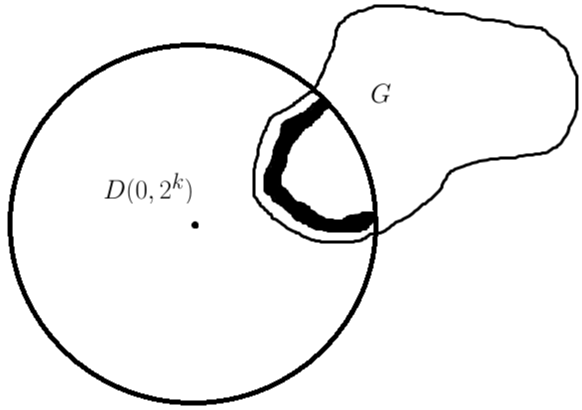

Figure 4.2: The following is an example of some set A_k. The shaded region is the intersection of the ball of radius 2^k, and the set $\{z \in G : 2^{-(k+1)} \leq \delta(z) \leq 2^{-1}\}$ and is our desired A_k.

The proof of (2) $\Rightarrow$ (7) is as follows:

Proof. We begin by constructing a series of compact sets as follows: for each k, we have $A_k = \{z \in G : 2^{-(k+1)} \leq \delta(z) \leq 2^{-k}$ and $||z|| \leq 2^k\}$. Due to the fact that G could potentially be unbounded, the second condition is required for A_k to be compact. An example of A_k can be seen in Figure 4.2.

Because of the fact that A_k is compact, it can be covered with finitely many balls of radius $2^{-(k+2)}$. These balls with have their centers in G and will not reach the boundary of G. For all the A_k's, we can form a sequence $\{p_j\}$ where each p_i is the center of some ball covering an A_k.

If we take some arbitrary compact set K, we know that that $\delta(K) > 0$. Therefore, we must have that $K \cap \{p_j\}$ will have finitely many points of the sequence meaning $\{p_j\}$ has no accumulation point in G. Moreover, we have that every point on the boundary of G must be an accumulation point of $\{p_j\}$. This is because of the fact that any ball centered at a boundary point of G must intersect infinitely many A_k.

Now, let us take some open connected set U, and have it intersect G at the boundary. Moreover, let V be some connected component of $U \cap G$ as seen in Figure 4.3. We must have that any of the p_j's in V must accumulate to every boundary point of V that is contained in U. This means that bd $V \cap U \neq \emptyset$ because if bd $V \cap U = \emptyset$ then $U = (U\cap$ int $V) \bigcup (U\cap$ ext $V)$ but this would mean U is the union of 2 open sets, contradicting the fact that U is connected. We must also have that any boundary point of V contained in U is a boundary point of G. Assume that $z \in G$ is a boundary point of V that is not a boundary point of G. Then we have that either $z \in G$ or $z \notin G$. The first case is impossible because we could form a closed ball around z contained in G which would contradict our earlier fact that $\{p_j\}$ has no accumulation point in G. The latter case is also impossible based on our construction of V. Also, since $z \in G \cap G$, we have that z must be contained in a connected component of the intersection. Therefore, $z \in U$.

Now let us have some $q \in$ bd $V \cap U$. We can now find some large integer n such that

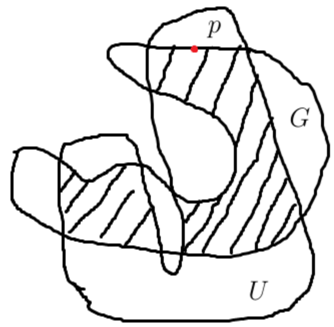

Figure 4.3: The following is an example of taking some open connected set U and having it intersect G. The shaded regions are the connected components of $U \cap G$. V is the set containing the point p.

$B(q, 2^{-n}) \subset U$ and $||q|| < 2^{-n}$. Given some $m > n$, we can have q' be in a $V \cap B(q, 2^{-m})$. For some k that is at least as large as m, we have that $2^{-(k+1)} \leq \delta(q') \leq 2^{-(k)}$. This also gives us that $B(q', 2^{-(k+2)}) \subset B(q', 2^{-(m+2)}) \subset B(q, 2^{-n}) \subset U$.

Since q' is in some A_k, we know that some $\{p_l\}$ has distance less than $2^{-(k+2)}$ from q'. Because we know that $B(q', 2^{-(k+2)}) \subset V$, because of the fact that the ball is connected and must lie in some connected component of $U \cap G$ so therefore it lies in V, we have that $\{p_l\} \in V$. Therefore, we have we have some subsequence of $\{p_j\} \subset V$ that converges to q.

Since q was arbitrary, and by (2), we have some holomorphic function f, such that f will be singular at every point on along the boundary of G.

□

The remaining two proofs of (6) ⇒ (5) and (3) ⇒ (4) follow a similar pattern. The proof of the former is as follows:

Proof. Given some compact set K, due to the fact that $K \subset \widehat{K}$, we have $\delta(\widehat{K}) \leq \delta(K)$. We will assume that that $\delta(\widehat{K}) < \delta(K)$ and arrive at a contradiction.

By the construction of δ, we have some $w \in \widehat{K} \setminus K$ and $q \in \text{bd } G$, such that $||w - q|| < \delta(K)$. Let us construct the n-tuple $(r_1, ..., r_n)$ such that $r_j = \min\{r : q_j \in D(w_j, r\} \forall j$, or that r_j is the smallest radius such that an open disc centered at w_j with radius r_j will contain q_j. This is equivalent to stating $r_j = |w_j - q_j|$. Therefore, if $r = (r_1, ..., r_n)$, we have that $q \in D(w, r)$, where $D(z_0, s)$ is a polydisc centered at z_0 with polyradius s. We also have $\forall z \in K$, $D(z, r) \subset G$ and $B = \bigcup_{z \in K} \overline{D}(z, r)$ is a compact subset of G.

By (6), we have some holomorphic function f that is completely singular at q. Moreover, we know that f is bounded by some constant M on B. If we take α to be any multi-index, we must have that $|f^{(\alpha)}(z)| \leq \frac{M\alpha!}{r^\alpha}$ for $z \in K$ by Cauchy's estimate for derivatives.

Given any holomorphic function ϕ and compact set K, if we have $|\phi(z)| \leq M \; \forall z \in K$, we must have $|\phi(w)| \leq M \; \forall w \in \widehat{K}$ otherwise ϕ would separate w from K which would contradict w being in $\widehat{K}$. We have proved that $|f^{(\alpha)}(w)| \leq \frac{M\alpha!}{r^\alpha}$ must hold $\forall w \in \widehat{K}$. Therefore, the Taylor series of f centered at w is of the form $f(w) = \sum_\alpha \frac{f^\alpha(w)}{\alpha!}(\zeta - w)^\alpha$. Moreover, by the M test, we have that this Taylor series must converge in $B(w, r)$ which means that f cannot be singular at q. Therefore, $\delta(\widehat{K}) = \delta(K)$. □

The final proof of (3) $\Rightarrow$ (4) follows a similar format:

Proof. Assume for the purpose of contradiction that (4) does not hold. Then there is a unit vector $v \in \mathbb{C}^n$, some compact set $K \subset G$, and some $w \in \widehat{K} \setminus K$ such that $\delta_v(w) < \delta_v(K)$.

Given some $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| = \delta_v(w)$ such that the complex polydisc $w + \lambda_0 v$ is in G, we can have (3) allow us to produce a holomorphic function f such that $\lim_{j\to\infty} |f(w+\frac{j}{j+1}\lambda_0 v)| = \infty$.

By restricting ourselves to a function of one complex variable, we can have a function $g : \lambda \to f(w + \lambda v)$. Since f is holomorphic on G, we know that g has a Maclaurin series that has a radius of convergence equal to $\delta_v(w)$.

Given r such that $\delta_v(w) < r < \delta_v(K)$, we can form a compact set: $R = \{z + \lambda v : z \in K$ and $|\lambda| \leq r\}$. We have that R is compact by a proof similar to the proof in the last section. By construction we have that $R \subset G$ and that $|f|$ restricted to R is bounded by M. Given $z \in K$, we have that the kth coefficient of g is bounded by $\frac{M}{r^k}$ because of Cauchy's estimate for derivatives. Also, because of the chain rule, the Maclaurin coefficient at z is $g^{(k)}(\lambda) = \sum_{m_1+\ldots+m_n=k} \frac{\partial^k f(w+\lambda v)}{\partial z_1^{m_1}\ldots\partial z_n^{m_n}} v_1^{m_1}\ldots v_n^{m_n}$, or a linear combination of the partial derivatives of of f. Since g is analytic, we also have that $g(\lambda) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}\lambda^k$. Therefore, we have the resulting coefficient is a holomorphic function on λ with a radius of convergence greater than or equal to r.

Using what we just have, and following a similar approach as the last proof, we can therefore construct a one variable holomorphic function $\psi : \lambda \to f(w + \lambda v)$. We also have that the Maclaurin coefficients of ψ are also bounded by $\frac{M}{r^k}$. Because we can do this for any k, we have that the radius of convergence of ψ must be at least r. However, from our assumption, we had that the radius of convergence should be $\delta_v(w) < r$. Thus we arrive at our desired contradiction.

□

Thus, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (7) $\to$ (6) $\Rightarrow$ (5) $\Rightarrow$ (1), (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1), and (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) which completes the proof.

4.3 Statement of the Second Theorem

The following Theorem is based off of the previous theorem and utilizes the Baire Category theorem in the fact that the existence of one singular function implies that most functions are singular.

Before we begin the actual statement and proof of the theorem, we must first introduce some terminology and discuss some important points.

If have a dmain G and the space of holomorphic functions on G we can induce a metrizable topology by the uniform convergence on compact subsets. Therefore, if we have that $\{K_j\}$ s a sequence of compact subsets which exhaust G, we can define

$$d_j(f, g) = \max_{z\in K_j} |f(z) - g(z)| \tag{4.7}$$

and the metric to be defined as

$$d(f,g) = \sum_j 2^{-j} d_j(f,g)/(1 + d_j(f,g)) \tag{4.8}$$

. Since the limit of holomorphic functions on a compact set is a holomorphic functon, we have that the set of holomorphic functions on G is a complete metric space with the metric d.

We will also refer to the "generic" holomorphic function which we mean to be "belonging to the residual set" or in the complement of a set of first category.

The theorem is as follows:

Theorem 4.3.1. Given G is a domain in $\mathbb{C}^n$, the following properties are equivalent:

1 For every boundary point p, there is a holomorphic funcion, f, on G that is completely singular at p. Therefore, G is a weak domain of holomorphy.

2 For every boundary point p, the generic, in the sense of the Baire Category Theorem, holomorphic function on G is completely singular at p.

3 G is a domain of holomorphy.

4 The generic, with respect to the Baire Category Theorem, holomorphic function on G is completely singular at every boundary point.

4.4 Proof of the Second Theorem

We already have (4) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) $\Rightarrow$ (1). We are only interested in proving that (1) $\Rightarrow$ (4) because we have that (1) and (3) are equivalent, which was proved in the previous theorem.

Proof. Assume that (1) holds. Let U be an open connected set which intersects $bd(G)$ and let $V \subset U \cap G$ be a connected component of the intersection. Now let f be a holomorphic function such that f cannot be extended holomorphically from V to U.

We wish to show that most functions satisfy the above requirement. Let us define $H(G)$ to be the set of holomorphic functions on G. We know that the vector space of holomorphic functions on G is an F-space (Def. A.0.3). If we define $H_U(G) \subset H(G)$ to be the set of holomorphic functions on G which can be extended from V to U, we also have that $H_U(G)$ is an F-space such that the metric of it is the sum of the metrics of $H(G)$ and $H(U)$. We also have that this subspace is continuously embedded into $H(G)$. From our assumption, we also have that the image of the embedding is not the whole of $H(G)$ which means that the image is of first category (see Remark in Appendix). Therefore, we have that the functions in the residual set of $H(G)$ cannot be extended holomorphically from V to U.

Now, let $\{p_j\}$ be a dense countable subset of points on the boundary of G. For each p_j, let B_k^j be a countable neighborhood basis of open balls whose radii are reciprocals of positive integers. Therefore, we have that $B_k^j \cap G$ is countably many connected components for all j and k. Let us arrange all of these components to form the countable list $\{V_j\}$. Using the previous part of the proof, we have that the set of holomorphic functions that extend from V_j to its corresponding B_k^j's is the countable union of sets of first category, and of first category itself. This definition is equivalent to saying that the complementary set of holomorphic functions on G that extend from no V_j to the corresponding B_k^j is a residual set.

Now, all we have to show is that every member of the previously mentioned residual set is completely singular at every boundary point. Let U be an arbitrary connected open set such that $U \cap bd(G) \neq \emptyset$. Let V be a component of $U \cap G$. Then, we have some ball in our constructed sequence is contained in U and is centered at a boundary point of V. Therefore, there is a V_j corresponding to this ball that must also be a subset of V so all the functions in the residual set fail to extend holomorphically from V to U. Therefore, every function in the residual set is completely singular at every boundary point of G.

$\square$

Appendix A

Baire Category and Open Mapping Theorem

This section is focused on using the Baire Category Theorem and proving the Open Mapping Theorem. The Open Mapping Theorem has played a large part in multiple proofs and is worth exploring. Due to the fact that the work behind proving the Open Mapping Theorem is irrelevant to the rest of the work, it is proved here in an appendix.

Let us first begin by stating the Baire Category Theorem ([3], Theorem 5.6).

Theorem A.0.1. If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.

Since $\mathbb{C}^n$ and the vector space of all holomorphic functions on a region in $\mathbb{C}^n$ (topologized by uniform convergence on compact sets) are a complete metric space (see equation 4.8), the Baire Category Theorem, and subsequently the Open Mapping Theorem, will apply to them.

Proof. Given X is a complete metric space. Let $V_1, V_2, ...$ be dense open subsets in X and let $V = \cap V_n$. Given some nonempty open set W, we wish to show $W \cap V \neq \emptyset$.

Since V_1 is dense in X, we have that $W \cap V_1 \neq \emptyset$. Therefore, we have some point $x_1 \in W \cap V_1$ and some radius r_1 such that $\overline{B}(x_1, r_1) \subset W \cap V_1$ where $\overline{B}(x_1, r_1)$ is the closed ball centered at x_1 with radius r_1. Moreover, we can have it so that $0 < r_1 < 1$.

For the case of $n \geq 2$, we can assume we have a chosen x_{n-1} and r_{n-1} such that $V_n \cap \overline{B}(x_{n-1}, r_{n-1}) \neq 0$. We can now find some x_n and some $r_n < \frac{1}{n}$ such that $\overline{B}(x_n, r_n) \subset V_n \cap B(x_{n-1}, r_{n-1})$.

Through this process, we have produced a sequence of $\{x_n\}$ such that, if $j, k > m$, we have that $x_j, x_k \in B(x_m, r_m)$ and $\rho(x_j, x_k) < r_n < 2r_n = \frac{2}{n}$. Therefore, $\{x_n\}$ is a Cauchy sequence and we have some x such that $x = \lim_{n\to\infty} x_n$.

Once again, for $j > m$, we have $x_j \in \overline{B}(x_m, r_m)$ which means that $x \in \overline{B}(x_m, r_m)$ which also means that we have $x \in V_n$ by construction. We also have that $x \in \overline{B}(x_1, r_1)$ and since $\overline{B}(x_1, r_1) \subset W \cap V_1$, we have $x \in W$ which completes the proof.

□

Before discussing an alternate version of the Baire Category Theorem, we need to introduce the following definitions.

Definition A.0.1. Given any set $E \subset X$, we say that E is *nowhere dense* if the closure $\overline{E}$ contains no nonempty open subsets of X.

We can use this definition to introduce the following two definitions:

Definition A.0.2. We say a set is of *first category* if it is the countable union of nowhere dense sets. All other sets are sets of *second category*.

Another statement of the Baire Category Theorem, and where the name comes from, is the following corollary:

Theorem A.0.2. No complete metric space is of first category.

This corollary is a result of taking the complement of A.0.1.

This version of the Baire Category Theorem will be the primary version of the theorem in which we are interested.

Another important notion we have to introduce before proceeding to the Open Mapping theorem is the idea of an F-space.

Definition A.0.3. If X is a topological vector space, we say X is an *F-space* if its topology, τ, is induced by a complete translation invariant metric, δ.

Finally, we must talk about what it means to be an open map before talking about the Open Mapping Theorem.

Definition A.0.4. Let S and T be topological spaces and let $f : S \to T$ be a map. We say that f is an *open map* if $f(U)$ is open whenever U is open.

With all the preliminaries, we can now state the Open Mapping Theorem (This is one of the most important theorems in functional analysis).

Theorem A.0.3. (Open Mapping Theorem) ([4] Theorem 2.11)

Suppose:

(A) X is an F-space

(B) Y is a topological vector space

(C) $\Lambda : X \to Y$ is continuous and linear

(D) $\Lambda(X)$ is of second category in Y

then

(1) $\Lambda(X) = Y$

(2) Λ is an open mapping

(3) Y is an F-space

Proof. Due to the fact that Y is the only open subspace of Y, we immediately have that (2) $\rightarrow$ (1).

To prove (2), we assume that V is a neighborhood of 0 in X and show that $\Lambda(V)$ contains a neighborhood of 0 in Y.

Let d be an invariant metric that is compatible with the topology induced on X. We can now construct $V_n\{x : d(x,0) < 2^{-n}r\}$ such that $r > 0$ and $V_0 \subset V$.

Using this construction, we will show there is some $W \subset Y$ such that $0 \in W$ and

$$W \subset \overline{\Lambda(V_1)} \subset \Lambda(V) \tag{A.1}$$

Before we continue, we need the following lemma:

Lemma A.0.4. If X is a topological space and if $A \subset B$ and $B \subset X$, then $\overline{A} + \overline{B} \subset \overline{A + B}$

Proof. Given $a \in \hat{A}$ and $b \in \hat{B}$, let $a + b \in W$. Therefore, we have a neighborhood, W_1 of a and a neighborhood W_2 of b such that $W_1 + W_2 \subset W$. Moreover, we have at least one $x \in A \cap W_1$ and $y \in B \cap W_2$. Therefore, $x + y \in (A + B) \cap W$ meaning that $a + b \in A + B$. $\square$

By construction, we have that $V_1 \supset V_2 - V_2$, we can use the previous lemma to have

$$\overline{\Lambda(V_1)} \supset \overline{\Lambda(V_2) - \Lambda(V_2)} \supset \overline{\Lambda(V_2)} - \overline{\Lambda(V_2)} \tag{A.2}$$

Therefore, we can prove the first relation of Eq A.1 if we show that $\overline{\Lambda(V_2)}$ has a nonempty interior. Since V_2 is a neighborhood of 0, we know that

$$\Lambda(X) = \cup_{k=1}^{\infty} k\Lambda(V_2) \tag{A.3}$$

Since X is of second category, at least one $k\Lambda(V_2)$ must be of second category. However, since $f : y \rightarrow ky$ is a homeomorphism on Y, $\Lambda(V_2)$ is of second category and therefore has nonempty interior.

To prove the second relation of Eq A.1, let us fix $y_1 \in \overline{\Lambda(V_1)}$. We will find $y_n \in \overline{\Lambda(V_n)}$ for $n \geq 1$ as follows. We know that each $\Lambda(V_n)$ is a neighborhood of 0, so we have that

$$(y_n - \overline{\Lambda(V_{n+1})}) \cap \Lambda(V_n) \neq \emptyset \tag{A.4}$$

If we set $y_{n+1} = y_n - \overline{\Lambda(V_{n+1})}$, then we have that $y_{n+1} \in \overline{\Lambda(V_{n+1})}$

Due to construction, we have that $d(x_n, 0) < 2^{-n}r$, we have that the sum $x_1 + ... + x_n + ...$ forms a convergent Cauchy sequence that converges to a point $x \in X$ such that $d(x, 0) < r$. Therefore, since

$$\sum_{n=1}^{m} \Lambda x_n = \sum_{n=1}^{m} (y_n - y_{n-1}) = y_1 - y_{m+1} \tag{A.5}$$

we have that $x \in V$. Due to the continuity of Λ, we have that $y_{m+1} \rightarrow 0$. Therefore, $y_1 = \Lambda(x) \in \Lambda(V)$. Therefore, we just proved the second inclusion of Eq A.1 and proved (2).

To prove (3), we need the following lemma which we will not prove

Lemma A.0.5. Let N be a closed subspace of a topological vector space X. If X is an F-space, then so is X/N.

Proof. Given X is an F-space and N is a closed subspace of X. We wish to show that X/N is also an F-space. To do this, let us assume Let d be an invariant metric on d.

Let us define $\pi(x) = x + N$ to be the coset of N that contains x.

Now, given our invariant metric d, let us define

$$\rho(\pi(x), \pi(y)) = \inf\{d(x - y, z) : z \in N\} \tag{A.6}$$

To complete the proof, we have to show that ρ is complete whenever d is complete.

Let $\{u_n\}$ be a Cauchy sequence in X/N relative to the metric ρ. We can therefore construct the subsequence $\{u_{n_j}\}$ such that $\rho(u_{n_j}, u_{n_{j+1}}) < 2^{-j}$. We can therefore choose a correspondng $x_j \in X$ such that $\pi(x_j) = u_{n_j}$. By the completeness of d, we know that x_j will converge to some x. Therefore, since π is continuous, we have that $\{u_{n_j}\} \to \pi(x)$ meaning that ρ is complete. $\square$

Using this lemma, we can have N be the null space of Λ. Therefore, to prove (3), all we need to do is to find an isomorphism f from X/N to Y such that f is also a homeomorphism. We can do this by defining $f(x + N) = \Lambda x$. If we have π to be the standard quotient map, we have that f is an isomorphism and that $\Lambda(x) = f(\pi(x))$. Therefore, given some open subset V of Y, we have that $f^{-1}(V) = \pi(\Lambda^{-1}(V))$ and $f^{-1}(V)$ is open due to the continuity of Λ and the openness of π.

Therefore, f is continuous and if E is an open subset of X/N we have that $f(E) = \Lambda(\pi^{-1}(E))$ is open, meaning f is our desired homeomorphism.

$\square$

Remark One important point worth noting is that if we did not have property D in the Open Mapping Theorem, then the image of $\Lambda(X)$ is of first category.

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