Nonuniform hyperbolicity in Hilbert geometries

A dissertation submitted by

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Abstract

This thesis is a comprehensive case study of the topological dynamics, asymptotic geometry, and ergodic theory for the geodesic flow of a class of 3-manifolds which have a non-Riemannian and nonuniformly hyperbolic geometric structure. These 3-manifolds arise as Hilbert geometries, and they were discovered by Benoist. The geometric structure forces irregularity of the geodesic flow. In particular, there are four major features of the geometry and dynamics which place this dynamical system outside the scope of any existing theory to date. First, the 3-manifolds are non-Riemannian and the geodesic flow is nonuniformly hyperbolic. Geodesic flows in each of those contexts have been studied independently but not simultaneously. Moreover, the manifolds are not CAT(0), and the geodesic flow is not differentiable. In this thesis we are able to extend the long developed framework of smooth ergodic theory to this class of geodesic flows far from the classical setting of Riemannian negative curvature. The main result is ergodicity and mixing of the Bowen–Margulis measure, which is a measure of maximal entropy for the geodesic flow. We conjecture uniqueness of the Bowen–Margulis measure and propose natural extensions of this work to equilibrium states and construction of a natural volume measure.

To my mother.

Acknowlegements

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I would like to take some space to thank the many women in mathematics who have paved a path for my generation and who have collectively built an inclusive and vibrant community. Unsurprisingly, many such women in my life have already been named. To that list I add my undergraduate mentors Michelle LeMasurier and Sally Cockburn, who embody the ideals of the liberal arts educator and are mainly responsible for my graduate exploits. Quickly thereafter, Yulan Qing became my first mathematical companion, and has been a role model for me ever since. Beth Romano has been pleasantly refreshing company over these past few months, and gave very insightful feedback on my application materials. Though not a woman, I am very grateful that Michael Ben-Zvi thinks inexplicably highly of me, and unassumingly sets a feminist ideal that many should follow. But this paragraph culminates in thanking the group of admirable young women who have stuck out this program with me since we first met at that epic open house five years ago this month. Lise Chlebak, Melody Takeuchi, and Meghan O'Connell, I could not be more grateful for your friendship.

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And at last, to my dear Elena, my partner in crime. You make me better every day.

This thesis is submitted in memory of Joseph Bray.

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Chapter 0

Introduction

This thesis is an exploration of the dynamics of the geodesic flow for a class of manifolds with geometric properties that create dynamical obstructions to the direct application of any existing theory.

Hilbert geometries

The manifolds of interest arise as Hilbert geometries. Any bounded, convex domain in real projective space admits a projectively invariant metric, attributed to Hilbert, which realizes such a domain as a Finsler geometry. For the special case when the domain is an ellipsoid, the geometry is Riemannian and uniformly hyperbolic, and it is natural to represent manifold groups as groups of projective transformations which preserve the ellipsoid. Such domains admitting a discrete, cocompact action by a subgroup of projective transformations are of particular dynamical interest and will be called *divisible*. The existence of nontrivial examples of divisible Hilbert geometries is originally due to Kac and Vinberg's constructions of quasi-homogeneous cones.

Benzecri was one of the first to observe that the existence of a nontrivial compact quotient imposes enormous geometric rigidity on the domain Ω, and his work on $\mathrm{PSL}(n,\mathbb{R})$-orbits of marked domains set the stage for Benoist's Dichtomy decades later. The dichotomy, simply put, states that for a divisible Hilbert geometry, a C^1-boundary, strict convexity, and uniform hyperbolicity are all equivalent. In dimension two, the dichotomy reduces to a simpler statement: that a divisible two-dimensional Ω will be either strictly convex with C^1-boundary and quasi-isometric to the hyperbolic plane, or Ω must be a projective triangle, quasi-isometric to $\mathbb{R}^2$.

In dimension three the panorama of divisible Hilbert geometries is slightly more exciting but still quite rigid. Benoist proves that any nonstrictly convex, nonsimplicial divisible Hilbert geometry must have countably many projective triangles properly embedded in Ω, meaning the interior of the triangle is contained in Ω and the boundary of the triangle is contained in $\partial\Omega$. These properly embedded triangles have dense vertices in $\partial\Omega$ but are isolated in the interior, and the closures of properly embedded triangles are pairwise disjoint. Up to index 2, they project to finitely many π_1-embedded tori or Klein bottles in the quotient manifold, and the atoroidal 3-manifold components have the geometry of a finite volume hyperbolic 3-manifold, i.e., the quotient has a nontrivial JSJ decomposition (see Figure 1.3.1). Benoist also proved existence of such examples. Given his comprehensive work on these examples, we will call Hilbert geometries of this class Benoist 3-manifolds.

Dynamics

The dynamics for the geodesic flow of compact quotients of strictly convex Hilbert geometries has been very well studied, pioneered by Benoist's work following his proof of the dichotomy. Given uniform hyperbolicity of the geometry, Benoist then proved uniform hyperbolicity of the geodesic flow of compact quotients, formalized as the Anosov property. Crampon then used the Anosov property in his thesis to compute Lyapunov exponents for the geodesic flow, construct explicitly a measure of maximal entropy which is unique by Benoist's work, and prove an entropy rigidity theorem for strictly convex, divisible Hilbert geometries of dimension n:

$$h_{top}(\varphi) \leq n-1$$

with equality if and only if Ω with the Hilbert metric is isometric to the Riemannian hyperbolic space.

None of this work applies to the Benoist 3-manifolds. A first obstruction is the presence of isometrically embedded flats, realized as properly embedded triangles in the universal cover and immersed tori or Klein bottles in the quotient. The flats prevent uniform hyperbolicity of the manifold, hence of the geodesic flow, and are also an obstruction to the CAT(0) property. The Benoist 3-manifolds might benefit from techniques similar to rank one manifolds, but we must avoid or modify tools dependent on curvature or CAT(0).

A second obstruction is that the regularity of the geodesic flow is given by the regularity of the boundary of the universal cover. Since the boundary of a Benoist 3-manifold is not C^1, neither is the geodesic flow on the unit tangent bundle to the quotient. A direct computation of Lyapunov exponents would be impossible, and Pesin theory for nonuniformly hyperbolic flows does not apply because it relies on the flow being $C^{1+\alpha}$.

A final point of interest is that, as for the non-Riemannian strictly convex case, Liouville volume is not invariant for the geodesic flow. Liouville volume is the classical measure of maximal entropy for constant curvature hyperbolic surfaces, but for surfaces with nonconstant negative curvature the measure of maximal entropy is singular with respect to Liouville volume.

A classical strategy for the study of geodesic flows is to construct the Bowen–Margulis measure as a measure of maximal entropy using the geometry of the group action and the boundary at infinity of the universal cover. This approach has been carried out for geodesic flows of geometrically finite group actions on hyperbolic spaces, manifolds with pinched or variable negative curvature, compact rank one manifolds, and geometrically finite quotients of strictly convex Hilbert geometries. In this thesis, we push this classical approach to the Finsler, non-C^1, non-CAT(0), and nonuniformly hyperbolic geodesic flow of the compact Benoist 3-manifolds.

Overview of results

In Chapter 1 we introduce Hilbert geometries and background results, and include some preliminary results on the geometry and the group associated to the Benoist 3-manifolds. We prove density of the hyperbolic length spectrum in Section 1.4.

In Chapter 2 we prove topological recurrence behavior for the geodesic flow of the Benoist 3-manifolds, including the Anosov Closing Lemma (Theorem 2.3.1), dense unstable sets for regular periodic points (Proposition 2.5.3), and topological mixing (Theorem 2.5.6).

For uniformly hyperbolic dynamical systems, R. Bowen recognized two powerful topological properties which together are sufficient for existence and uniqueness of equilibrium states, including the measure of maximal entropy [12]. In Chapter 3, we adapt some of Bowen's work to the current setting. We open the chapter by exploring weaker properties which do not need any hyperbolicity control. We then prove entropy-expansiveness for the geodesic flow of any compact Hilbert geometry (Theorem 3.2.5), which for the Benoist 3-manifolds allows us to prove the topological entropy is positive (Proposition

3.3.2). This proposition is essential for our study of the Patterson–Sullivan measures and the Bowen–Margulis measure in Chapters 5 and 6. The chapter closes with a remark that Manning's proof that volume entropy agrees with the topological entropy of the geodesic flow of a compact hyperbolic manifold extends to our context (Proposition 3.4.1).

In Chapter 4 we study the essential tools in asymptotic geometry for constructing the Patterson–Sullivan measures at infinity which induce the Bowen–Margulis measure on the unit tangent bundle to the quotient. Due to the irregularity of the topological boundary of the domain, we study the visual boundary and prove almost-uniqueness (Proposition 4.1.3), and almost-continuity of the Busemann function and the Gromov product (Lemma 4.2.2).

In Chapter 5 we study the critical exponent of the group, which is equal to the topological entropy (Theorem 5.1.4), and construct the Patterson–Sullivan measures at infinity (Proposition 5.2.1). We prove the measures are unique up to a constant (Theorem 5.4.2) and that flats are null sets (Proposition 5.3.9). Consequences of uniqueness include sharp asymptotics for sphere growth (Theorem 5.5.1) and divergence of the group Γ (Corollary 5.5.2).

The thesis culminates in Chapter 6, where we construct the Bowen–Margulis measure and prove it is an ergodic and mixing measure of maximal entropy (Theorem 6.2.4, Theorem 6.3.2).

Future Directions

Continuations of this thesis can arise in various forms. It certainly is reasonable to ask whether the measure of maximal entropy is unique: of this the author is quite confident. More ambitious would be a study of equilibrium states for the system. For example, there is the question of whether a natural volume form can be constructed as an equilibrium state for the unstable potential. Since the geodesic flow is not C^1, one would need to construct a geometric analogue for this potential function. Such a project parallels current work around equilibrium states for the geodesic flow of rank one manifolds.

One might also explore Lyapunov exponents for the system with respect to invariant measures which assign trivial measure to the singular set, where the geodesic flow fails to be differentiable. In the strictly convex case, Crampon has explicit formulas for Lyapunov exponents which he uses to construct the stable and unstable SRB measures, so one could pursue a similar result for the Benoist 3-manifolds.

Chapter 1

Hilbert geometries

In this chapter we introduce Hilbert geometries and group actions by isometries on Hilbert geometries. We present several examples, including the class of examples which are of interest for our dynamical investigation. These examples of interest are called the Benosit 3-manifolds after Yves Benoist, who proved their existence and geometric regidity. We prove some properties of the automorphism group of a Benoist 3-manifold and the hyperbolic subset of a discete cocompact subgroup. Lastly, we introduce the geodesic flow as we transition into the study of the dynamics.

1.1 Definition of the Hilbert metric

Definition 1.1.1 (Convexity)**.** An open subset $\Omega \subset \mathbb{RP}^n$ is *convex* if it is convex in an affine chart. Similarly, Ω is *strictly convex* if in an affine chart, there is no open line segment in the Euclidean topological boundary for Ω, denoted $\partial\Omega$, in that and hence any affine chart. A convex domain Ω is *properly convex* if there is an affine chart in which Ω is bounded.

Example 1.1.2. The open first octant of $\mathbb{R}^3$, $O_1 := \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z > 0\}$, projects to a triangle in $\mathbb{RP}^2$. Choosing an affine chart $z = 1$ produces an unbounded domain to represent O_1. This choice can sometimes be nice for computations, but to verify proper convexity we need another affine representation for O_1. One example would be the affine slice $\{x + y + z = 1\}$. We can conclude the projection of O_1 is properly convex in $\mathbb{RP}^2$. See Figure 1.1.1

If we union two octants together in $\mathbb{R}^3$, the projection to $\mathbb{RP}^2$ is not properly convex because there is no affine chart in which such a domain is bounded.

Efficient notations for lines and line segments will be very important. Consider x, y in Euclidean space, or an affine subspace of $\mathbb{R}^n$. Let $[xy)$ denote the clopen line segment from x to y, and define $[xy]$ and (xy) to be the closed and open line segments accordingly. Let $\overline{xy}$ denote the line through the points

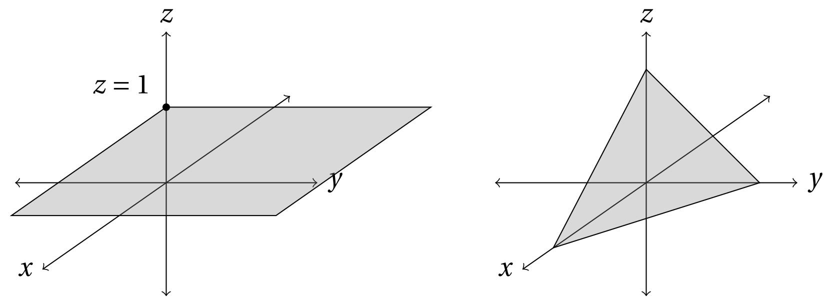


Figure 1.1.1: Two affine slices of the first octant O_1 in $\mathbb{R}^3$, representing the projective triangle in $\mathbb{RP}^2$. On the left is an unbounded slice determined by $z = 1$. On the right is a bounded slice determined by $x + y + z = 1$.

x and y. The Euclidean distance between x and y will be denoted $|xy|$.

Definition 1.1.3 (Hilbert geometry)**.** A *Hilbert geometry* on an open properly convex $\Omega \subset \mathbb{RP}^n$ is determined by the distance function $d_\Omega : \Omega \times \Omega \to \mathbb{R}^+$, defined as follows: Choose an affine chart for Ω in which Ω is bounded. For any $x, y \in \Omega$, there is a unique projective line $\overline{xy}$ passing through x and y. Take a and b to be the intersection points with $\partial\Omega$ as pictured in Figure 1.1.2. Then the Ω-*Hilbert distance* between x and y is

$$d_\Omega(x, y) := \frac{1}{2} \left|\log[a, x, y, b]\right|,$$

where $[a, x, y, b] := \frac{|ay|}{|ax|}\frac{|bx|}{|by|}$.

Remark 1.1.4. Note that our definition of the *cross-ratio* ensures that $[a, x, y, b] \geq 1$, with equality if and only if $x = y$. One can verify the triangle inequality by a cross ratio of four lines argument to conclude that d_Ω is a metric on the affine slice for Ω. By projective invariance of the cross-ratio, this metric is well-defined for any affine representation of Ω.

1.1.1 Properties of the Hilbert metric

The metric d_Ω is complete on the open domain Ω. One can see this by fixing a point x and sending points y_n to $\partial\Omega$ along the same projective line. As the y_n tend to the boundary, the cross-ratio $[a, x, y_n, b]$ goes to infinity and so does the Hilbert distance between x and y_n.

Projective lines are always geodesic in this metric. However, not all geodesics are lines. In the example of the projective triangle, there are infinitely many geodesic paths for $d_\triangle$ between any two points

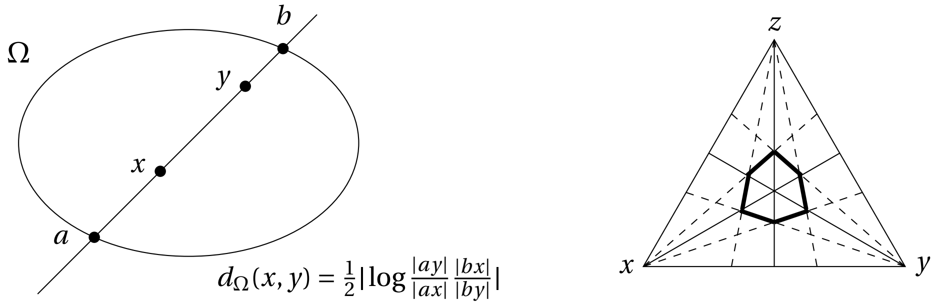


Figure 1.1.2: The ellipsoid with the Hilbert metric is the Beltrami-Klein model for $\mathbb{H}^n$, and the n-simplex with the Hilbert metric is isometric to $\mathbb{R}^n$ with a polygonal norm. On the right we have pictured the hexagonal unit ball in the triangle.

(see Figure 1.1.3). A geometric group theorist may note that this fact implies that the projective triangle endowed with the Hilbert metric is not CAT(0). The absence of the CAT(0) property mandates careful definition of the geodesic flow and becomes significant for the study of the visual boundary in Chapter 4. In contrast, a strictly convex domain is uniquely geodesic for the Hilbert metric, hence CAT(0) (Proposition 1.1.6).

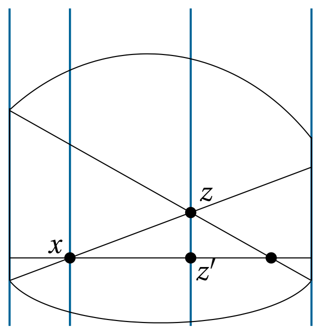
The Hilbert metric is compatible with a Finsler norm on the tangent bundle. A Riemannian norm is a special case of a Finsler norm—this is the case when the norm comes from an inner product. A typical Finsler norm does not have this property, hence angles are undefined in most Finsler spaces. For a Hilbert geometry, the compatible norm is Riemannian only for a very special example: the ellipsoid.

To define the norm, first note that every $(x, v) \in T\Omega$ determines a unique projective line ℓ_v through x in the direction v. Let v^+, v^- be the intersection points of this projective line with $\partial\Omega$ in the forward direction and reverse direction of v, respectively. Then one can check that the d_Ω-length of curves in Ω is determined by the field of norms $F\colon T\Omega \to \mathbb{R}^+$ where

$$F(x, v) := \frac{1}{|v|}\left(\frac{1}{|xv^+|} + \frac{1}{|xv^-|}\right)$$

Fact 1.1.5. *The ellipsoid in $\mathbb{RP}^n$ is isometric to $\mathbb{H}^n$ when endowed with the Hilbert metric. In this metric, angles are defined, though distorted, since the Finsler norm is Riemannian. This model for hyperbolic space is known as the Beltrami-Klein model.*

The following fact is presumably well-known and there are proofs of the forward direction in the



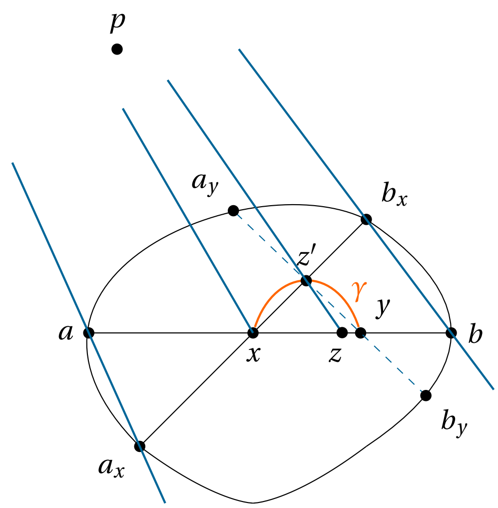


Figure 1.1.3: Left panel: By the cross ratio of four lines, $d_\Omega(x,z) = d_\Omega(x,z')$ and $d_\Omega(y,z) = d_\Omega(y,z')$, so the path from x to z to y is also geodesic for d_Ω. Right panel: for the converse of Proposition 1.1.6. A curve such as γ cannot be geodesic from x to y unless the line segments $(a_x a_y), (b_x b_y)$ are contained in $\partial\Omega$.

literature [18], but not of the converse. Since the arguments in the proof will come up later on, we provide the proof here.

Proposition 1.1.6. *A properly convex domain Ω in $\mathbb{RP}^2$ is uniquely geodesic if and only if there is at most one line embedded in $\partial\Omega$.*

Proof. We can make all arguments for $\Omega \subset \mathbb{RP}^2$. If $\partial\Omega$ contains two lines, they intersect at a point in projective space which we can take to be infinity. Then the set-up is as in Figure 1.1.3. By the cross-ratio of four lines,

$$d_\Omega(x,z) = d_\Omega(x,z'),\ d_\Omega(y,z) = d_\Omega(y,z')$$

implying the path $(xz'y)$ has the same Hilbert length as the path $(xzy) = (xy)$.

Conversely, suppose there is a geodesic path γ from x to y which is not a projective line segment. There exists a point z on (xy) such that some $z' \neq z$ on γ is equidistant with z to x (and hence y). Now, let a, b be the intersection points of $\overline{xy}$ with $\partial\Omega$, let a_x, b_x be the intersection points of $\overline{xz'}$ with Ω, and let a_y, b_y be the intersection points of $\overline{z'y}$ with Ω.

Let $\{p\} = \overline{a_x a} \cap \overline{b_x b}$. Since $d_\Omega(x,z) = d_\Omega(x,z')$, the cross ratios $[a_x, x, z', b_x]$ and $[a,x,z,b]$ are equal, hence the line $\overline{zz'}$ must also contain $p \in \overline{a_x a} \cap \overline{b_x b} \cap \overline{xp}$. Without loss of generality, we can take $p = \infty$.

By convexity of Ω, a_y must be to the right of $\overline{a_x a}$ and b_y must be to the left of $\overline{b_x b}$. But then $\overline{a_y a} \cap \overline{zz'} \neq \overline{b_y b} \cap \overline{zz'}$, and it would be impossible for $[a_y, z', y, b_y]$ to equal $[a, z, y, b]$. Thus because $d_\Omega(z, y) = d_\Omega(z', y)$, the points b_y, b, and b_x must be collinear and the points a_y, a, and a_x must be collinear. Then both $(a_y a_x)$ and $(b_y b_x)$ are contained in $\partial\Omega$ because $\overline{\Omega} := \Omega \cup \partial\Omega$ is convex. □

1.2 The Automorphism Group

For a properly convex open $\Omega \subset \mathbb{RP}^n$, we define the *automorphism group* of Ω to be

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PSL}(n+1, \mathbb{R}) \mid g\Omega = \Omega\}.$$

Clearly $\mathrm{Aut}(\Omega) < \mathrm{Isom}(\Omega)$, the isometry group of (Ω, d_Ω), since projecive transformations preserve the cross-ratio. The full isometry group of (Ω, d_Ω) is, up to index 2, the group of collineations which preserve Ω [40].

Definition 1.2.1 (Translation length and classifying automorphisms)**.** Let $\Omega \subset \mathbb{RP}^n$ be a properly convex domain. Then for any $g \in \mathrm{Aut}(\Omega)$, we can define the *translation length* of g by

$$\tau(g) := \inf_{x \in \Omega} d_\Omega(x, g.x).$$

A projective transformation $g \in \mathrm{Aut}(\Omega)$ is

- *hyperbolic* if $\tau(g) > 0$ and the infimum is attained.
- *quasi-hyperbolic* if $\tau(g) > 0$ and the infimum is not attained.
- *parabolic* if $\tau(g) = 0$ and the infimum is not attained.
- *elliptic* if $\tau(g) = 0$ and the infimum is attained.

Hyperbolic and quasi-hyperbolic automorphisms arise as the projection of proximal matrices with positive real eigenvalues [8]. A matrix g with real eigenvalues is *proximal* if at least one of its extremal eigenvalues has multiplicity one, and is *biproximal* if both extremal eigenvalues have multiplicity one. If

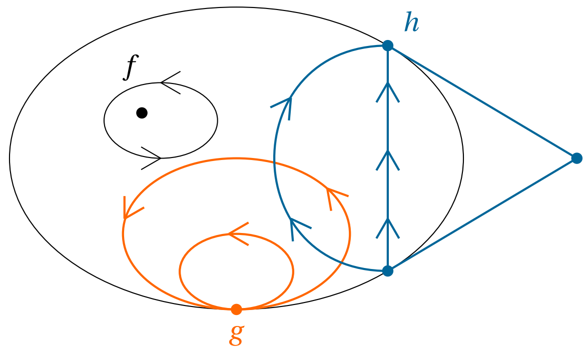


Figure 1.2.1: Some projective transformations which preserve the ellipse. The action of h is hyperbolic, of g is parabolic, and of f is elliptic.

$g \in \mathrm{SL}(n+1,\mathbb{R})$ has eigenspectrum $0 < \lambda_0 \leq \cdots < \cdots \leq \lambda_n$ then[1]

$$\tau(g) = \frac{1}{2}\log\left(\frac{\lambda_n}{\lambda_0}\right).$$

Parabolic automorphisms arise as unipotent matrices (Definition A.2.3). A cocompact group preserving a Hilbert geometry cannot contain unipotent elements (Lemma A.2.4) so we will not discuss them much here.

As for hyperbolic spaces, a quotient of Ω by a subgroup generated by a hyperbolic automorphism h corresponds to a closed geodesic of length $\tau(h)$, and a quotient of Ω by a subgroup generated by a parabolic automorphism corresponds to a cusp of finite volume.

Much like in the study of hyperbolic geometries, the dynamics of automorphisms of Ω is classified by translation length for sufficiently regular Ω, and there are no quasi-hyperbolic automorphisms.

Theorem 1.2.2 ([20, Theorem 3.3])**.** *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex open subset with C^1-boundary. Then any $g \in \mathrm{Aut}(\Omega)$ is one of the following types:*

- *elliptic: g acts by rotation about a fixed point in Ω.*
- *parabolic: g fixes exactly one point in $\partial\Omega$, which is an attractor for all other points in $\partial\Omega$ under iterates of g.*
- *hyperbolic: g fixes two points in $\partial\Omega$ and acts on $\overline{\Omega}$ with North-South dynamics.*

[1] For the proof, see Appendix A.

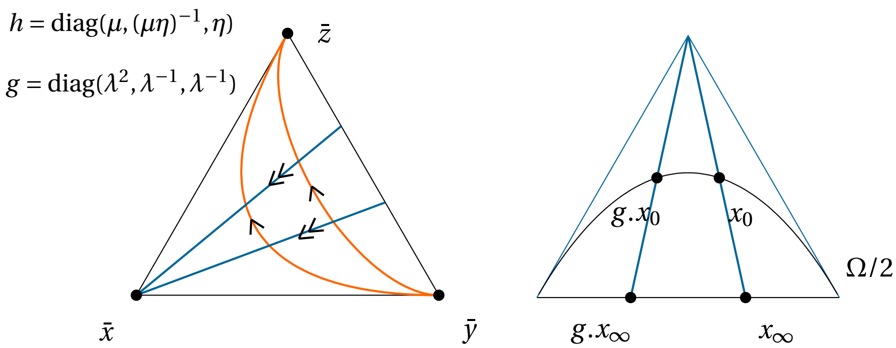


Figure 1.2.2: Left panel: Let $\triangle$ be the projection of the first octant in $\mathbb{R}^3$, with vertices $\overline{x}, \overline{y}, \overline{z}$ the projection of the axes into $\mathbb{RP}^2$. Then we have pictured the induced action of $g, h \in \text{SL}(3,\mathbb{R})$ on $\triangle$. Right panel: A half-ellipse $\Omega/2$ admits two supporting hyperplanes which intersect in $\mathbb{RP}^2$. Any $g \in \text{Aut}(\Omega/2)$ must stabilize this supporting triangle and leave $\partial\Omega/2$ invariant.

1.2.1 Automorphisms of nonstrictly convex Hilbert geometries

In this subsection, we illustrate the necessity of the hypotheses for Theorem 1.2.2.

Example 1.2.3. The projective triangle admits hyperbolic automorphisms which fix three points at infinity, with sink-source-saddle dynamics (see Figure 1.2.2). The projective triangle also admits hyperbolic automorphisms which fix a point and fix pointwise a line in $\partial\Omega$, with North-South dynamics between the line and the point.

Note that in both cases, $\tau(g)$ is positive and realized at infinity, since that is where the axis of g occurs. However, $\tau(g)$ is also realized in the interior of Ω by the cross-ratio of four lines argument presented in Figure 1.2.2. In some sense, g fits the definition of a hyperbolic projective transformation by coincidence rather than intention. And the dynamics of g do not agree with the North-South dynamics of a hyperbolic action on strictly convex domains, as in Theorem 1.2.2.

Example 1.2.4. By a cross-ratio of four lines argument, any projective transformation $g \in \text{Aut}(\triangle)$ with $\tau(g) > 0$ must realize that infimum, hence there are no quasi-hyperbolic isometries. However, a modified nonstrictly convex Ω does admit a quasi-hyperbolic action from the same projective transformation that preserved the triangle. See Figure 1.2.2.

1.3 Quotients of Hilbert geometries

Definition 1.3.1 (Divisibility)**.** A properly convex domain $\Omega \subset \mathbb{RP}^n$ is *divisible* if it admits a cocompact action by a discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{R})$. We say Γ *divides* Ω.

1.3.1 Examples

1. **Triangles**. Consider the $\mathbb{Z}\times\mathbb{Z}$ subgroup generated by the diagonal matrices $g, h \in \mathrm{Aut}(\triangle)$ in Figure 1.2.2. The action is geometric and the quotient manifold $\triangle/\langle g, h\rangle$ is a torus.

2. **Hyperbolic space**. Let Ω be the ellipse. There is a representation of any Fuchsian group $F <$ $\mathrm{PSL}(2,\mathbb{R})$ in the subgroup of $\mathrm{Aut}(\Omega)$ containing all conjugates of isometries with real eigenvalues $\lambda, 1, \frac{1}{\lambda}$ with $\lambda > 1$. Then Ω/F is the point in Teichmüller space associated to F.

3. **Coxeter groups**. These are the original Kac–Vinberg examples, constructed via quasi-homogeneous cones as a modification of the Tits cone construction for hyperbolic triangle groups [39]. Consider any matrix B in $\mathrm{GL}(3,\mathbb{R})$ satisfying the following:

   (a) The determinant of B is negative.

   (b) All entries are negative except the diagonal, on which $B_{ii} = 2$.

   (c) Off the diagonal, $B_{ij}B_{ji} < 4$ for $i \neq j$.

   (d) B satisfies an asymmetry condition: $B_{12}B_{23}B_{31} \neq B_{21}B_{32}B_{13}$.

   We can think of B as a bilinear form on $\mathbb{R}^3$, where $\langle e_i, e_j\rangle_B = B_{ij}$ with e_i the standard basis vectors of $\mathbb{R}^3$. Asymmetry of the matrix (3d) is equivalent to asymmetry of the norm.

   This matrix B is associated to a reflection group R_B in $\mathrm{GL}(3,\mathbb{R})$ with an associated triangular cone T_B. The action of R_B on T_B tiles a nonhomogeneous cone C_B which is properly convex as a cone in $\mathbb{R}^3$. Then C_B projects to some properly convex domain $\Omega \subset \mathbb{RP}^2$, with $T \subset \Omega$ the projection of T_B and $R < \mathrm{Aut}(\Omega) < \mathrm{PSL}(3,\mathbb{R})$ a representation of R_B. Condition (3c) guarantees that the tiling of Ω by $R.T$ is a hyperbolic tiling (valency of the vertices of the tiling satifies a hyperbolicity condition). Then there is a torsion-free finite-index subgroup $\Sigma < R$ such that the quotient surface Ω/Σ is a finite cover of T. Hence Ω is divisible by Σ, but is not Riemannian by the work of Kac and Vinberg. For more details on this construction, see [32].

1.3.2 Rigidity of divisible Hilbert geometries

The following landmark theorem of Benoist for the study of divisible Hilbert geometries is equivalence of the regularity of the boundary, convexity of the boundary, and hyperbolicity of the flow based on abstract properties of the group.

Theorem 1.3.2 ([8, Theorem 1.1])**.** *Suppose* Γ *is a discrete torsion-free subgroup of* $\mathrm{PSL}(n+1,\mathbb{R})$ *dividing an open properly convex domain* $\Omega \subset \mathbb{RP}^n$*. Then the following are equivalent:*

(i) The domain Ω *is strictly convex.*

(ii) The boundary $\partial\Omega$ *is of class* C^1*.*

(iii) The group Γ *is* δ*-hyperbolic.*

Essential to Benoist's theorem is Benzecri's thesis work on the PGL-orbits of marked properly convex sets in projective space [10]. In fact, an application of the work of Benzecri shows that in dimension two, a divisible Ω is either strictly convex with C^1-boundary or a projective triangle. For more, see Appendix A.

1.3.3 Benoist's nonstrictly convex, divisible 3-manifolds

It was once hypothesized that the dichotomy in dimension two might hold for all dimensions: that is, a divisible Hilbert geometry is either strictly convex with C^1-boundary or a simplex. Benoist disproved this hypothesis by constructing nontrivial Hilbert geometries in dimension three which are nonstrictly convex and nonsimplicial via a modification of the Kac–Vinberg Coxeter construction [9, Proposition 1.3]. Moreover, Benoist proved strong geometric rigidity for such Hilbert geometries.

Definition 1.3.3 (Properly embedded triangles)**.** A *properly embedded triangle* in Ω is a triangle $\triangle \subset \Omega$ such that $\partial\triangle \subset \partial\Omega$. Let $\mathcal{T}$ denote the collection of triangles $\triangle$ properly embedded in Ω, and $\Gamma_\triangle := \mathrm{Stab}_\Gamma(\triangle) = \{\gamma \in \Gamma \mid \gamma\triangle = \triangle\}$ be the subgroup of Γ stabilizing $\triangle \in \mathcal{T}$.

Theorem 1.3.4 ([9, Theorem 1.1])**.** *Let* $\Gamma < \mathrm{SL}(4,\mathbb{R})$ *be a discrete torsion-free subgroup which divides an open, properly convex, nonsimplicial* $\Omega \subset \mathbb{RP}^3$*, and* $M = \Omega/\Gamma$*. Then*

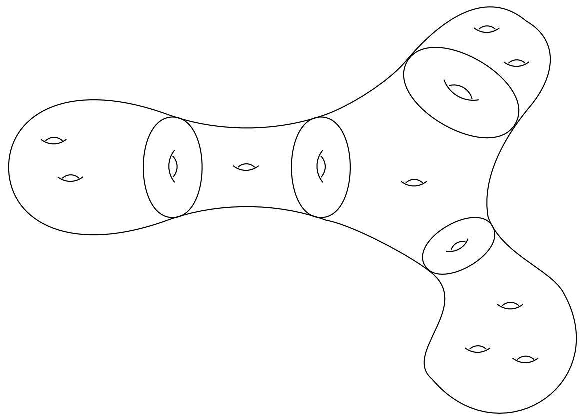


Figure 1.3.1: A 3-manifold which admits a JSJ decomposition. When cut along the finitely many boundary tori, the remaining components are atoroidal 3-manifolds with boundary.

(a) Every subgroup in Γ *isomorphic to* $\mathbb{Z}^2$ *stabilizes a unique triangle* $\triangle \in \mathcal{T}$.

(b) If $\triangle_1, \triangle_2 \in \mathcal{T}$ *are distinct, then* $\overline{\triangle_1} \cap \overline{\triangle_2} = \varnothing$.

(c) For every $\triangle \in \mathcal{T}$*, the group* $\Gamma_\triangle$ *contains an index-two* $\mathbb{Z}^2$ *subgroup.*

(d) The group Γ *has only finitely many orbits in* $\mathcal{T}$.

(e) The image in M *of triangles in* $\mathcal{T}$ *is a finite collection* Σ *of disjoint tori and Klein bottles, denoted by* T*. If one cuts* M *along each* $T \in \Sigma$*, each of the resulting connected components is atoroidal.*

(f) Every nontrivial line segment is included in the boundary of some $\triangle \in \mathcal{T}$.

(g) If Ω *is not strictly convex, then the set of vertices of triangles in* $\mathcal{T}$ *is dense in* $\partial\Omega$.

We will call nonstrictly convex, nonsimplicial divisible Hilbert geometries in dimension three *Benoist 3-manifolds*, and compact quotients are *compact Benoist 3-manifolds*. The topological decomposition as in 1.3.4(e) is called a Jaco–Shalen–Johansson (JSJ) decomposition, pictured in Figure 1.3.1.

Regularity of fixed points at infinity for hyperbolic automorphisms

In the case of Benoist's 3-manifolds, isometries which stabilize properly embedded triangles do not rightly fit into any of the previous classifications for isometries. An isometry could meet the definition of hyperbolic, but still stabilize a triangle and behave like a Euclidean isometry. As such, we were motivated to formulate a new classification of the isometries of a Benoist 3-manifold.

Lemma 1.3.5. *Let Ω be a Benoist 3-manifold. If $g \in \mathrm{Aut}(\Omega)$ stabilizes a side or vertex of a properly embedded triangle $\triangle$ in Ω then $g \in \mathrm{Stab}(\triangle)$.*

Proof. It suffices to prove $g(\partial\triangle) = \partial\triangle$. Let $\{v_i\}_{i=1,2,3}$ be the vertices of $\triangle$ with s_i the open sides opposite to each vertex. Since $g \in \mathrm{Stab}(\triangle)$ if and only if $g^t \in \mathrm{Stab}(\triangle^*)$ and vertices are dual to sides, we consider the case where g fixes a vertex of $\triangle$. If $g.v_1 = v_1$, then g must preserve $\overline{s_2} \cup \overline{s_3}$. Else, since g is a projective transformation which preserves Ω, g would be moving s_2 or s_3 to a distinct side embedded in $\partial\Omega$ which, by Benoist's Theorem 1.3.4(f), bounds a distinct properly embedded triangle whose closure intersects $\partial\triangle$ at v_1, contradicting Theorem 1.3.4(b).

If g preserves $\overline{s_2} \cup \overline{s_3}$ then g permutes v_2 and v_3 and hence also stabilizes s_1, so $g\partial\triangle = \partial\triangle$ implying $g \in \mathrm{Stab}(\triangle)$. □

Proposition 1.3.6. *For Benoist's nonstrictly convex $\Omega \subset \mathbb{RP}^3$, there are no quasi-hyperbolic group elements in $\mathrm{Aut}(\Omega)$.*

Proof. If $\tau(g) > 0$ and g preserves Ω, then the line segment(s) connecting the eigenvectors of g associated to the extremal eigenvalues must be contained in $\partial\Omega$. Choose one such axis and call it the axis of g, denoted ℓ_g. To be quasi-hyperbolic, ℓ_g must be contained in $\partial\Omega$. Then by Theorem 1.3.4(f), $\ell_g \subset \partial\triangle$ for some properly embedded $\triangle$ in Ω. By Lemma 1.3.5, $g \in \mathrm{Stab}(\triangle)$ and thus $\tau(g)$ is realized in $\triangle$ by the cross ratio of four lines, so g must be hyperbolic. □

Definition 1.3.7 (Hyperbolic and flat automorphisms)**.** We will say some $g \in \mathrm{Aut}(\Omega)$ is *hyperbolic* if and only if $\tau(g) > 0$ and the infimum is attained uniquely along a g-invariant projective line in Ω. Any other $f \in \mathrm{Aut}(\Omega)$ for which $\tau(f)$ is realized, but not uniquely along a g-invariant projective line in Ω, will be called *flat*. Recall Definition 1.2.1 for proximal and biproximal matrices. Then an automorphism of Ω is said to be *dynamically hyperbolic* if g is biproximal and there are exactly two fixed points of g in $\overline{\Omega}$.

Definition 1.3.8 (Proper, extremal boundary points)**.** We also classify points in $\partial\Omega$. We say that $p \in \partial\Omega$ is *proper* if there is a unique supporting hyperplane to Ω at p. A hyperplane $H \subset \mathbb{RP}^n \setminus \Omega$ is a supporting hyperplane at p if $p \in H \cap \partial\Omega$. Also, p is *extremal* if there is no open line segment containing p embedded in $\partial\Omega$, i.e., if H is a supporting hyperplane at an extremal point p then $H \cap \partial\Omega = \{p\}$.

Remark 1.3.9. The proper extremal points form the complement of the union of the boundaries of properly embedded triangles: by Benoist's Proposition A.2.7 for the Benoist 3-manifolds, nonproper points are precisely the vertices of properly embedded triangles, and nonextremal points are precisely the sides of properly embedded triangles.

Proposition 1.3.10. *Let Ω be a Benoist 3-manifold with discrete, torsion-free dividing group Γ and compact quotient M. Then for all $g \in \Gamma$,*

- *g is hyperbolic if and only if g is dynamically hyperbolic. Then g acts on $\overline{\Omega}$ with north-south dynamics between the repeller g^- and the attractor g^+, which are proper extremal points in $\partial\Omega$.*
- *g is flat if and only if $g \in \mathrm{Stab}(\triangle)$ for some properly embedded $\triangle$. If such a g is biproximal, it acts on $\triangle$ with sink-source-saddle dynamics. Else, g acts with north-south dynamics between a vertex and its opposite edge in $\partial\triangle$.*

These are the only possible automorphisms of a divisible, indecomposable domain in $\mathbb{RP}^3$.

Proof. Since Γ is discrete and torsion-free, there are no elliptic isometries. Since M is compact, there are no unipotents, hence no parabolic isometries (Lemma A.2.4). By Proposition 1.3.6, there are no quasi-hyperbolic elements.

Thus, it suffices to characterize the group elements with translation length realized in Ω. First, if a group element is only proximal, not biproximal, and preserves Ω, then the subspace generated by the eigenvectors with redundant eigenvalues must be an open line segment in a properly embedded triangle, hence by Lemma 1.3.5, $g \in \mathrm{Stab}(\triangle)$ and g is not hyperbolic. If g does not fix exactly two points then g must fix three noncollinear points[2], and since g preserves Ω then the triangle formed by the eigenvectors must be properly embedded in Ω implying $g \in \mathrm{Stab}(\triangle)$. Then by the cross-ratio of four lines, $\tau(g)$ is not realized uniquely along a single g-invariant projective line, hence g is flat, not hyperbolic.

Also, if g is dynamically hyperbolic then g is hyperbolic because the axis of g is contained in Ω and is the unique g-invariant projective line along which $\tau(g)$ is realized. If g is flat and $\tau(g)$ is realized along a

[2] if g fixed four noncollinear points or more, then Ω would be a simplex or g would be the projective identity.

conic but no g-invariant projective line, then g must be a biproximal element which stabilizes a properly embedded triangle. If g is flat and $\tau(g)$ is realized along more than one g-invariant projective line, then extending the lines to infinity, there must be an open line segment σ between the endpoints embedded in Ω which is preserved by g. Then g stabilizes the maximal open line segment containing σ, which is the edge of a properly embedded triangle $\triangle$ by Benoist's Theorem 1.3.4(f), hence $g \in \mathrm{Stab}(\triangle)$ by Lemma 1.3.5. □

The quotient geometry

As noted after the statement of the theorem, the geometric description in Theorem 1.3.4(e) is better known as a Jaco–Shalen–Johannson (JSJ) decomposition. When Ω is not strictly convex, the JSJ decomposition is nontrivial, meaning the atoroidal components have nonempty boundary tori. We can prove these components are not Seifert fibered using Benoist's geometry and conclude by Thurston's geometrization (cf. [14, 25]) the following:

Fact 1.3.11. *For an indecomposable, nonstrictly convex Ω in $\mathbb{RP}^3$ admitting a discrete, cocompact, torsion-free action by some $\Gamma < \mathrm{PSL}(4,\mathbb{R})$, the quotient M admits a nontrivial JSJ decomposition into atoroidal components glued along boundary tori, and these atoroidal components are diffeomorphic to finite-volume quotients of $\mathbb{H}^3$.*

Proof. By Thurston's hyperbolization theorem (cf. [25]), the 3-manifold components of the JSJ decomposition are either a circle bundle over a surface, known as a Seifert fibered space, or they admit a complete finite-volume hyperbolic structure. Gabai completed the Seifert-fibered conjecture which characterizes Seifert fibered spaces completely by the existence of a cyclic normal subgroup of the fundamental group [22]. In our context, to prove that any component is not a Seifert fibered space, we need only that after lifting the boundary tori to any two properly embedded triangles in Ω, the stabilizers of the properly embedded triangles do not share an infinite center. This fact follows from Benoist's Proposition 3.1(d) [9], that $\mathrm{Stab}(\triangle_1) \cap \mathrm{Stab}(\triangle_2) = \{\mathrm{Id}\}$ for all $\triangle_1 \neq \triangle_2 \in \mathscr{T}$. □

Fact 1.3.12 ([4, Baker–Cooper subgroup])**.** *The fundamental group of a complete, finite volume hyperbolic 3-manifold contains a hyperbolic surface subgroup called the Baker–Cooper subgroup.*

Remark 1.3.13. Let Γ_{hyp} denote the hyperbolic elements of Γ. Since a hyperbolic surface group cannot act discretely on Euclidean space, Σ cannot stabilize any properly embedded triangle. Then by Proposition 1.3.10 and Fact 1.3.11, $\Sigma < \Gamma_{\text{hyp}}$.

Corollary 1.3.14. *There exist infinitely many noncommuting hyperbolic group elements in* Γ.

1.4 The hyperbolic length spectrum is dense

In this section we assume that Ω is a divisible, indecomposable, nonstrictly convex 3-manifold by a discrete torsion-free $\Gamma < \mathrm{PSL}(4,\mathbb{R})$.

The goal of this section is to prove density of the hyperbolic length spectrum of Γ in $\mathbb{R}$ via Zariski density of the Baker–Cooper subgroup Σ (Fact 1.3.12). We will often refer to density of the length spectrum assuming that the ambient topological space is $\mathbb{R}$. Represent $\Gamma < \mathrm{Aut}(\Omega)$ as a subgroup of $\mathrm{SL}(4,\mathbb{R})$ and denote the *length spectrum* of $\Gamma < \mathrm{SL}(4,\mathbb{R})$ by $\mathrm{Spec}(\Gamma) := \langle \tau(g)\rangle_{g\in\Gamma}$, a subgroup of $\mathbb{R}$ with the addition operation. Let $\ell(g) := (\log|\lambda_0|,\ldots,\log|\lambda_3|) \in \mathbb{R}^4$ for $g \in \mathrm{SL}(4,\mathbb{R})$ with eigenvalues λ_i in increasing order. Let $\mathfrak{a} := \{\overline{x} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$. Note that $\ell(\mathrm{SL}(4,\mathbb{R})) = \mathfrak{a}$ and that $\mathfrak{a}$ is closed under the group operation of addition.

Lemma 1.4.1 ([8, Fact 5.5])**.** *If* Γ *is a Zariski dense subgroup of* $\mathrm{SL}(4,\mathbb{R})$*, then* $\langle \ell(g)\rangle_{g\in\Gamma}$ *is dense in* $\mathfrak{a}$.

Corollary 1.4.2 (of Lemma 1.4.1)**.** *If* Γ *is a Zariski dense hyperbolic subgroup of* $\mathrm{SL}(n+1,\mathbb{R})$ *preserving a properly convex domain* $\Omega \subset \mathbb{RP}^n$*, then* $\mathrm{Spec}(\Gamma)$ *is dense in* $\mathbb{R}$.

Proof. We will make the argument in dimension three, but note that the proof is completely general. Recall Definition 1.2.1 for the translation length of an automorphism of $\Omega \subset \mathbb{RP}^3$. In particular, for a hyperbolic automorphism g we can take as our definition of the translation length $\tau(g) := \frac{1}{2}\log|\lambda_3(g)/\lambda_0(g)|$ where λ_3 is the largest eigenvalue of $g \in \mathrm{SL}(4,\mathbb{R})$ in magnitude and λ_0 is the smallest. From Lemma 1.4.1 it follows that for any $r \in \mathbb{R}^+$ and $\epsilon > 0$, there are finitely many $g_i \in \Gamma$ such that $\frac{1}{2}\sum \ell(g_i)$ is $\epsilon/2$ close to $(r/2, 0, 0, -r/2) \in \mathfrak{a}$. Then $\frac{1}{2}\sum\log|\lambda_0(g_i)|$ is within $\epsilon/2$ of $-r/2$ and $\frac{1}{2}\sum\log|\lambda_3(g_i)|$ is within $\epsilon/2$ of $r/2$, and

$$\frac{1}{2}\sum\log|\lambda_3(g_i)| - \frac{1}{2}\sum\log|\lambda_0(g_i)| = \sum\frac{1}{2}\log|\lambda_3(g_i)/\lambda_0(g_i)| = \sum\tau(g_i)$$

is within ϵ of $r/2 - (-r/2) = r$ for any $r > 0$, which suffices after adding inverses. $\square$

If Ω is not an ellipsoid, then the hypotheses of Lemma 1.4.1 hold whenever Γ is acting cocompactly on an indecomposable properly convex Ω in projective space. This result is due to Benoist:

Theorem 1.4.3 ([7, Theorem 1.2]). *If Γ is a discrete subgroup of $\mathrm{SL}(n+1,\mathbb{R})$ which divides an indecomposable, nonhomogenous, properly convex cone in $\mathbb{R}^{n+1}$, then Γ is Zariski dense in $\mathrm{SL}(n+1,\mathbb{R})$.*

If Ω is an ellipsoid, then density of the length spectrum for a cocompact group action on hyperbolic space is well known and has been generalized to many Riemannian contexts (compact rank one manifolds, convex cocompact hyperbolic surfaces with pinched curvature, geometrically finite hyperbolic manifolds).

Since Benoist's 3-manifolds are indecomposable we bypass the definition. In low dimensions, there are few decomposable and divisible properly convex Ω. In dimension three, the 4-simplex is the only example, hence our nonsimplicial assumption for Ω in Theorem 1.3.4 is equivalent to the assumption that Ω is indecomposable. Thus Theorem 1.4.3 implies Γ is Zariski dense for the Benoist 3-manifolds.

An intermediate proposition to Theorem 1.4.3 goes as follows:

Proposition 1.4.4 ([7, Proposition 3.2]). *Suppose G is a Zariski-connected subgroup of $GL(n+1,\mathbb{R})$ which divides a properly convex cone C in $\mathbb{R}^{n+1}$. If G is irreducible and C is nonhomogeneous, then G is Zariski dense in $GL(n+1,\mathbb{R})$.*

We would like a more general proposition to conclude Zariski density of the Baker–Cooper subgroup Σ, which does not divide Ω a Benoist 3-manifold. Crampon and Marquis addressed a similar issue when studying the geodesic flow of a geometrically finite strictly convex Hilbert geometry. They prove the geodesic flow is topologically mixing on the nonwandering set [20, Proposition 6.1] when Γ is nonelementary via Zariski density of Γ, which is no longer acting cocompactly, in the strictly convex case. The strategy follows a remark by Benoist [6, Remark following Corollary 3.2], which the authors prove in the special case where Ω is strictly convex while noting that strict convexity is not a necessary hypothesis. A slight restatement replacing irreducible with strongly irreducible and removing strict convexity produces the appropriate result.

Proposition 1.4.5 (restatement of [20, Proposition 6.5]). *Suppose Γ is a strongly irreducible subgroup of $\mathrm{SL}(n+1,\mathbb{R})$ which preserves a properly convex $\Omega \subset \mathbb{RP}^n$. Let G be the Zariski closure of Γ. Then G is a Zariski-connected real semi-simple Lie group.*

A case-by-case elimination argument then yields that $G = \mathrm{SL}(n+1,\mathbb{R})$ or Σ preserves an ellipse, a contradiction. Thus, it suffices to prove that Σ is strongly irreducible to conclude that Σ is Zariski dense in $\mathrm{SL}(4,\mathbb{R})$.

A subgroup $H < \mathrm{PSL}(4,\mathbb{R})$ is *irreducible* if it does not stabilize a projective point, line, or plane in $\mathbb{RP}^3$, and H is *strongly irreducible* if every finite-index subgroup is irreducible.

Lemma 1.4.6. *The Baker–Cooper subgroup is either strongly irreducible or has dense length spectrum.*

Some notions of duality come up in the proof. See Appendix B.1 for some background definitions and arguments.

Proof. First, since Σ is a surface group, every finite-index subgroup is also a surface subgroup. It suffices to show any surface group in $\mathrm{PSL}(4,\mathbb{R})$ preserving a domain $\Omega \subset \mathbb{RP}^3$ is irreducible. By contradiction, suppose Σ fixes a point $p \in \mathbb{RP}^3$. Clearly $p \notin \Omega$ because Γ actingly discretely without torsion cannot have elliptic elements. Also, $p \notin \partial\Omega$ because all elements of Σ are hyperbolic so they do not stabilize any triangles (Remark 1.3.13) and Γ, hence Σ, are acting properly discontinuously on Ω. If $p \notin \overline{\Omega}$, then we consider the dual case: then Σ^t preserves a projective plane Π which intersects Ω^*. Then Σ^t is acting cocompactly on a totally geodesic hypersurface $\Pi \cap \Omega^*$. By Theorem 1.4.3, Σ^t is either Zariski dense and hence has dense length spectrum by Corollary 1.4.2 or $\Pi \cap \Omega^*$ is homogeneous and Σ^t has dense length spectrum anyways. Then so does Σ since dual groups preserving dual properly convex sets are isospectral. Thus, is Σ preserves Ω and fixes a point, then Σ has dense length spectrum in $\mathbb{R}^+$.

Now suppose Σ preserves a line l. The case where $l \subset \Omega$ or l is disjoint from $\overline{\Omega}$ by duality is impossible because $\mathrm{Aut}(l) = \mathbb{R}$. If l intersects $\partial\Omega$ then either $\Sigma \not< \mathrm{Aut}(\Omega)$ or $\Sigma \subset \mathrm{Stab}(\triangle)$, both a contradiction (Remark 1.3.13).

If Σ stabilizes a plane, then we revisit the dual cases where Σ^t stabilizes a point, unless the plane interests Ω. In this case, we have already seen that Σ has dense length spectrum. □

Proposition 1.4.7. *Let Ω be a properly convex, indecomposable domain in $\mathbb{RP}^3$, and Γ a discrete group of projective transformations preserving Ω with compact quotient. The subset Γ_{hyp} of hyperbolic group elements in Γ has dense length spectrum in $\mathbb{R}^+$.*

Proof. By Lemma 1.4.6, Proposition 1.4.5, and Proposition 1.4.4, the Baker–Cooper subgroup $\Sigma < \Gamma$ is either Zariski dense or has dense length spectrum. By Corollary 1.4.2, density of the length spectrum of Σ holds in both cases. Then by Remark 1.3.13 the length spectrum of $\Gamma_{\text{hyp}} \supset \Sigma$ is dense in $\mathbb{R}$. □

1.5 The Hilbert geodesic flow

1.5.1 Basic definitions

Suppose $\Gamma < \mathrm{PSL}(n+1,\mathbb{R})$ acts properly discontinuously without torsion on $\Omega \subset \mathbb{RP}^n$, so that the quotient $M = \Omega/\Gamma$ is a manifold. The *geodesic flow* of M is defined on SM, the Finsler unit tangent bundle to M, by flowing unit tangent vectors along projective lines at unit Hilbert speed:

$$\begin{aligned} \varphi^t \colon SM &\longrightarrow SM \\ (x,v) &\longmapsto (x+tv, v). \end{aligned}$$

In other words, $(x,v) \in SM$ determines a unique oriented projective line $\ell_v(t)$ parameterized at unit Hilbert speed, with $\ell_v(0) = x$ and $\varphi^t(v)$ the Finsler unit tangent vector to ℓ_v based at $\ell_v(t)$ (see Figure 1.5.1). Geodesics are not always unique for the Hilbert metric, so the choice to flow along projective lines is an important distinction from the classical definition of the geodesic flow. Note also that Γ takes projective lines to projective lines, so we can lift $\varphi^t \colon SM \to SM$ to $\tilde{\varphi}^t \colon S\Omega \to S\Omega$ equivariantly.

In general, ℓ_v will always denote the oriented projective line determined by $(x,v) \in SM$, and we will often abuse notation by supressing the x and referring to $v \in SM$. The intersection of ℓ_v with Ω in positive time is denoted v^+, in negative time denoted v^-. See again Figure 1.5.1.

For strictly convex divisible Hilbert geometries, Benoist showed that the geodesic flow exhibits uniform hyperbolicity [8]. The Benoist 3-manifolds are not strictly convex and it is easy to verify or cite Benoist's dichtomy (Theorem 1.3.2) to conclude that uniform hyperbolicity fails because of the embedded flats.

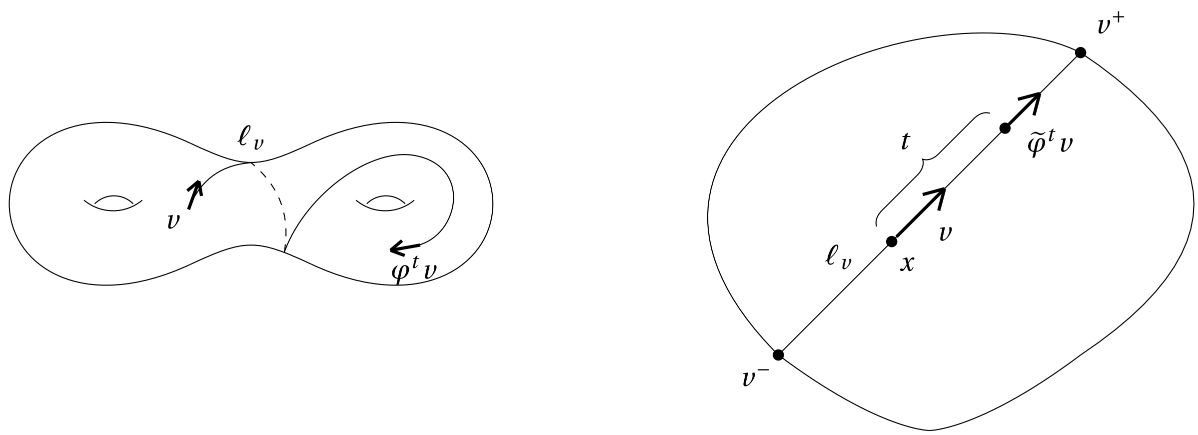


Figure 1.5.1: The geodesic flow moves tangent vectors along projective lines at unit speed. On the left panel we have pictured the geodesic flow of a genus two surface. On the right is the geodesic flow of the universal cover of a Hilbert geometry.

1.5.2 Metrizing SM

In his thesis, Crampon applies Foulon's dynamical formalism to construct an Finslerian analogue for the Riemannian Levi-Civita connection, which is associated to a natural $d\varphi^t$-invariant decomposition of TSM [18]. One then induces a metric on SM by taking the Euclidean norm on TSM over natural norms on the products of the decomposition. More explicitly, if $X = \frac{d}{dt}\Big|_{t=0} \varphi^t$ is the vector field that generates the geodesic flow, then there is a horizontal component denoted $h^X SM$ to TSM and a linear operator J^X such that if VSM is the vertical component,

$$TSM = \mathbb{R}.X \oplus h^X SM \oplus VSM,$$

$$J^X(h^X SM) = VSM,\ J^X(VSM) = h^X SM,$$

$$J^X \circ J^X = -Id|_{h^X SM \oplus VSM}.$$

Then for any $Z = aX + Z_h + Z_v \in \mathbb{R}.X \oplus h^X SM \oplus VSM$, let

$$\bar{F}(z) = \sqrt{a^2 + \frac{1}{2}F(d\pi Z_v)^2 + \frac{1}{2}F(d\pi J^X(Z_v))^2)},$$

where F is the Finsler norm on Ω associated to d_Ω.

It is worth noting that regularity of X is given by regularity of $\partial\Omega$, thus for the Benoist examples X

is only a continuous vector field [8, Section 3.2.2]. Then we can only expect that the decomposition and hence $\bar{F}$ is only continuous. Then we have an induced continuous metric on SM by taking an infimum over lengths of curves:

$$d(v,w) = \inf\left\{\int_0^1 \bar{F}(\dot{c}(t))\; dt \mid c \text{ is a } C^1 \text{ curve }, c(0) = v, c(1) = w\right\}.$$

Lemma 1.5.1 ([19, Lemma 8.3])**.** *For any pair of projective line segments $\sigma, \tau : [0,r] \to \Omega$ and any parameterization, we have the following inequality:*

$$d_\Omega(\sigma(t),\tau(t)) \le d_\Omega(\sigma(0),\tau(0)) + d_\Omega(\sigma(r),\tau(r)) \qquad \textit{for all } t \in [0,r].$$

Corollary 1.5.2. *The Γ-equivariant metric $d\colon\ S\Omega\times S\Omega \to \mathbb{R}^+$ is convex, meaning $d_\Omega(p,q) \le d(v,w)$ for any $(p,v),(q,w) \in S\Omega$.*

1.5.3 Stable and unstable foliations

Define the *weak stable and weak unstable* foliation by

$$W^{os}(x) = \{y \in SM \mid d(\varphi^t x, \varphi^t y) < c \text{ for all } t \ge 0 \text{ and some } c(y) > 0\},$$
$$W^{ou}(x) = \{y \in SM \mid d(\varphi^{-t} x, \varphi^{-t} y) < c \text{ for all } t \ge 0 \text{ and some } c(y) > 0\}.$$

Define the *strong stable and strong unstable sets* to be

$$W^{ss}(x) = \{y \in SM \mid d(\varphi^t x, \varphi^t y) \to 0 \text{ as } t \to +\infty\},$$
$$W^{su}(x) = \{y \in SM \mid d(\varphi^{-t} x, \varphi^{-t} y) \to 0 \text{ as } t \to +\infty\}.$$

In the hyperbolic setting, the weak stable and unstable sets are foliated by strong stable and unstable leaves in the flow direction, and the foliation is flow invariant:

$$W^{os}(x) = \bigcup_{t\in\mathbb{R}} W^{ss}(\varphi^t x), \qquad \varphi^t(W^{ss}(x)) = W^{ss}(\varphi^t x) \text{ for all } t \in \mathbb{R},$$

$$W^{ou}(x) = \bigcup_{t\in\mathbb{R}} W^{su}(\varphi^t x), \qquad \varphi^t(W^{su}(x)) = W^{su}(\varphi^t x) \text{ for all } t \in \mathbb{R}.$$

Moreover, in the uniformly hyperbolic setting the leaves are smooth submanifolds. For a strictly convex Hilbert geometry with C^1-boundary, the strong stable and unstable leaves are global for every point in $S\Omega$ and admit a concrete geometric description [8]. They foliate the weak stable and unstable sets. An explicit proof that stable and unstable sets admit a geometric characterization is available in [19, Section 2.4.1], where Crampon introduces a temporary Finsler metric on SM to prove this geometric characterization of $W^{ss}(v), W^{su}(v)$ for a strictly convex Ω with C^1-boundary. The result is only contingent on whether the geometric characterization is definable.

For now, define the *horosphere* at $\xi \in \partial\Omega$ through $x \in \Omega$ by $\mathscr{H}_\xi(x) = \{y \in \Omega \mid d_\Omega((y\xi)_t, (x\xi)_t) \to 0 \text{ as } t \to +\infty\}$. A horosphere $\mathscr{H}_\xi(x)$ is *global* if and only if for all $\eta \in \partial\Omega$, there exists a point $y \in (\eta\xi) \cap \mathscr{H}_\xi(x)$.

Proposition 1.5.3. *Suppose $v \in SM$ is such that for any lift $\tilde{v} \in \Omega$, the horospheres at v^- and v^+ through $\tilde{x}$, the footpoint of $\tilde{v}$, are global. Then*

$$\widetilde{W}^+(\tilde{v}) := \{w \in S\Omega \mid \pi w \in \mathscr{H}_{v^+}(\pi v),\ w^+ = v^+\}$$

projects to the strong stable set for v: $W^+(v) = W^{ss}(v)$.

Proof. We can analogously define

$$\widetilde{W}^-(\tilde{v}) := \{w \in SM \mid \pi w \in \mathscr{H}_{v^-}(\pi v),\ w^- = v^-\}.$$

If $\mathscr{H}_{v^+}(\tilde{x})$ and $\mathscr{H}_{v^-}(\tilde{x})$ are global, then there is a Γ-equivariant decomposition for $T_v S\Omega = \mathbb{R}.X \oplus T_v\widetilde{W}^+(v) \oplus T_v\widetilde{W}^-(v)$ which thus projects to a decomposition of $T_v SM$ [8]. Define a temporary Finsler metric $\|\cdot\|^W$ on $T_v SM$ by

$$\|Z\|^W = \sqrt{a^2 + F(d\pi Z^+)^2 + F(d\pi Z^-)^2}.$$

Then $\|d\varphi^t|_{T_vSM}Z^-\|^W, \|d\varphi^t_{T_vSM}Z^+\|^W$ are strictly increasing and strictly decreasing, respectively, as t increases [19, Corollary 2.6]. The result follows by the choice of the norm and the definition of horospheres, and equivalence of norms restricted to the compact T_vSM. □

1.5.4 Volume

Remark 1.5.4. The natural volume form on SM is never invariant for the geodesic flow except for the Riemannian case, where Ω is the ellipsoid. For more on the proof in the strictly convex setting, see Appendix A.3.

Chapter 2

Topological Dynamics

From this point onward, unless otherwise noted, we assume Ω is a Benoist 3-manifold with compact quotient M and discrete, torsion-free dividing group Γ.

In this chapter we prove topological results for the geodesic flow of the Benoist 3-manifolds which set the stage for the ergodic theory.

2.1 Regular points

We introduce terminology inspired by the notions of Lyapunov regularity in nonuniformly hyperbolic dynamics. First, recall Definition 1.3.8 of proper and extremal boundary points. For the Benoist 3-manifolds, vertices of properly embedded triangles are the only nonproper points, and all nonextremal points are contained in the side of some properly embedded triangle. Thus, the proper extremal points are the complement of the boundaries of properly embedded triangles.

We will say $v \in S\Omega$ is *forward regular* if v^+ is a proper extremal point, and similarly for *backward regular*. If v^+ is a nonproper or nonextremal point, then v is *forward singular*. If v is both forward and backward regular, then we will say v is *regular* (and similarly for singular vectors). The partition of $\partial\Omega$ by proper extremal, proper nonextremal, and nonproper extremal points is Γ-invariant, so the associated partition of $S\Omega$ by regular, singular points is well-defined modulo Γ and projects naturally to SM.

Lastly, a closed orbit $\varphi \cdot v$ is *hyperbolic* or regular if when lifted to the universal cover, $\ell_{\tilde{v}}$ is preserved by a hyperbolic group element. Note that hyperbolic closed orbits must be regular (Proposition 1.3.10).

Let $S\Omega_{\text{reg}}$ be the collection of regular vectors and the complement, $S\Omega_{\text{sing}}$, the set of $v \in \Omega$ such that v^+ or v^- is in the boundary of some properly embedded triangle. The collection of vectors tangent to projective lines contained in properly embedded triangles is denoted $S\Omega_{\text{flat}}$. In the appendix we prove that $S\Omega_{\text{sing}}$ is asymptotic to $S\Omega_{\text{flat}}$ under the geodesic flow, hence the same is true on the quotient SM

(Lemma A.4.1).

2.2 Recurrent orbits

2.2.1 Density of hyperbolic periodic orbits

Lemma 2.2.1. *Hyperbolic periodic orbits are dense for the geodesic flow of a Benoist 3-manifold.*

Proof. We want to show any $(\xi,\eta) \in \partial\Omega \times \partial\Omega \smallsetminus \mathrm{diag}$ can be approximated by (g^-, g^+) such that $g \in \Gamma_{\mathrm{hyp}}$. Take two noncommuting hyperbolic elements $g, h \in \Gamma$ (Corollary 1.3.14). Construct the sequence $k_n = g^n h^n$. Then there are fixed points k_n^+ and k_n^- in $\partial\Omega$ of k_n such that $k_n^+ \to g^+$ and $k_{-n}^- \to h^-$ as $n \to \infty$. Using the sequence k_n and minimality of the action of Γ on $\partial\Omega$ [9, Proposition 3.10], we conclude that any $(\xi,\eta) \in \partial\Omega \times \partial\Omega \setminus \mathrm{diag}$ is approximable by such k_n. If any k_n admits a projective line as an axis, then this projective line axis corresponds to a periodic orbit for the flow and we conclude that any vector tangent to the projective line $(\xi\eta)$ is approximable by periodic orbits. To prove the lemma, we just need to show there necessarily exists a subsequence k_{n_i} of only hyperbolic elements.

By contradiction, suppose there is no such subsequence. There exists an N such that for all $n \geq N$, each k_n preserves a triangle $\triangle_n$. If we assume k_n's geodesic axis of translation, which is not necessarily a projective line, is also on the triangle, we consider the accumulation of the boundary of triangles in $\partial\Omega$, which will contain h^- and g^+. This set will be the boundary of a triangle, a line, or a point. None of the above are possible since h, g are hyperbolic and do not commute, and Γ acts discretely so $h^- \neq g^+$ are proper extremal points and $(h^- g^+) \not\subset \partial\Omega$. □

2.2.2 Topological transitivity

In this section we prove existence of a dense orbit, which is equivalent to topological transitivity when the phase space is compact, as in the case of the Benoist 3-manifolds.

Definition 2.2.2 (Topological transitivity)**.** A continuous dynamical system $f^t\colon X \to X$ is *topologically transitive* if for every pair of open sets $U, V \subset X$, there exists a time $T \in \mathbb{R}^+$ such that $f^T(U) \cap V \neq \varnothing$. If X

is a metric space then the system is *uniformly transitive* if for all $\delta > 0$, there exists a $T > 0$ such that for all $x, y \in X$, there is some $t \le T$ such that $f^t\big(B(x,\delta)\big) \cap B(y,\delta) \ne \emptyset$.

Fact 2.2.3. *Transitivity implies uniform transitivity when X is a compact metric space.*

Proof. Take a finite subcover $\{B(x_i,\delta/2) \times B(y_i,\delta/2)\}_{i=1}^k$ of $X \times X$. By transitivity, for each i there is a T_i such that $f^{T_i}\big(B(x_i,\delta/2)\big) \cap B(y_i,\delta/2) \ne \emptyset$. Take $T = \max_i T_i$. Then for all x, y, there exists an i such that $x \in B(x_i,\delta/2) \subset B(x,\delta)$, $y \in B(y_i,\delta/2) \subset B(y,\delta)$ and we have the $T_i \le T$ as needed for the intersection. □

We refer the reader to [1] for more on topological transitivity and its characterizations.

To prove transitivity, we study asymptotic behavior of orbits in the universal cover. Two points $v, w \in S\Omega$ are said to be *positively asymptotic* if there exists a C such that $d(\varphi^t v, \varphi^t w) < C$ for all $t \ge 0$. We can define *negatively asymptotic* similarly for $t \le 0$. It is not hard to check that for $v \in S\Omega$ such that v^+ is a proper extremal point, w is positively asymptotic to v if and only if $w^+ = v^+$. This is a special case of Lemma 4.1.1. Moreover, the Hilbert distance between the line segments $\ell_v[0,t]$ and $\ell_w[0,t]$ goes to 0 as $t \to +\infty$.

Proposition 2.2.4. *The geodesic flow of a Benoist 3-manifold is topologically transitive.*

Proof. Take two open sets U and V in SM. By Lemma 2.2.1, there are hyperbolic periodic orbits $u \in U$ and $v \in V$. We now construct a heteroclinic orbit. Lifting to the universal cover, we have $\tilde u \in \tilde U$, $\tilde v \in \tilde V \subset S\Omega$ such that u^- and v^+ are proper extremal points of $\partial\Omega$. See Figure 2.2.1 for the schematic of the argument to follow. Then by Benoist's geometric characterization [9], the open projective line segment $(u^- v^+)$ is contained in Ω and is the footpoint of an orbit of the geodesic flow. Choose some $\tilde z \in S\Omega$ tangent to $(u^- v^+)$ in the direction v^+, so $\tilde z$ is negatively asymptotic to $\tilde u$, positively asymptotic to $\tilde v$. Since u, v are periodic, there are hyperbolic group elements γ_u, γ_v preserving ℓ_u, ℓ_v so $d\gamma_u^n(\tilde U) \cap \varphi \cdot u$ and $d\gamma_v^n(\tilde V) \cap \varphi \cdot v$ each contain lifts of u and v respectively for all $n \in \mathbb{Z}$. Since γ_u, γ_v are isometries and $u^\pm, v^\pm$ are proper extremal points, there is an N such that for all $n \ge N$, $d\gamma_u^{-n}(\tilde U) \cap \varphi \cdot \tilde z \ne \emptyset$ and $d\gamma_v^n(\tilde V) \cap \varphi \cdot \tilde z \ne \emptyset$. Then choosing times t_1, t_2 so that $\varphi^{t_1} \tilde z \in d\gamma_u^{-n}(\tilde U) \cap \varphi \cdot \tilde z$ and $\varphi^{t_2} \tilde z \in d\gamma_v^n(\tilde V) \cap \varphi \cdot \tilde z$, we can project $\varphi^{t_1} \tilde z$ to SM and obtain $T = -t_1 + t_2$ such that $z' := \pi_\Gamma \varphi^{t_1} \tilde z \in U$ and $\varphi^T z' \in v$ as desired. □

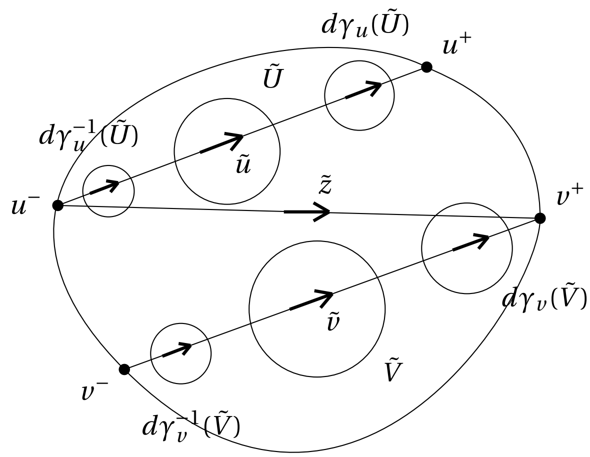


Figure 2.2.1: An illustration to supplement the proof of transitivity in Lemma 2.2.4.

2.3 The Anosov Closing Lemma

In this section, we discuss Anosov closing of recurrent orbits. Should $v \in SM$ return sufficiently close to itself under the flow after time t, Anosov closing says that there is a periodic orbit near v which has orbit length approximately t and shadows the trajectory of v for approximately the length of that orbit. In Appendix C.2, we explore applications of the Anosov Closing Lemma to prove that ergodicity is a generic property for invariant measures of the dynamics.

Define a filtration of $SM \setminus \bigcup S\mathbb{T}$ by compact sets bounded away from flats:

$$\Lambda_\eta := \left\{ v \in SM \mid d(v,w) \geq \eta \text{ for all } w \in \bigcup S\mathbb{T} \right\}.$$

Theorem 2.3.1. *Let Ω be an irreducible, nonstrictly convex domain in $\mathbb{RP}^3$. Suppose $\Gamma < \mathrm{PSL}(4,\mathbb{R})$ is a discrete, torsion free group dividing Ω, with compact quotient $M = \Omega/\Gamma$. Then for all $\eta > 0, \epsilon > 0$, there exists a $\delta > 0$ and $T > 0$ such that:*

For any $t \geq T$, $v \in \Lambda_\eta$ with $d(\varphi^t v, v) < \delta$, there exists a hyperbolic periodic orbit v' of period $t' \in]t - \epsilon, t+\epsilon[$ which ϵ-shadows v for time $\min\{t, t'\}$.

Proof. As in [16] we adapt proof by contradiction following Eberlein. So we have particular $\eta, \epsilon > 0$ and a sequence of $v_n \in \Lambda_\eta$ paired with a sequence $t_n \to \infty$ such that $d(v_n, \varphi^{t_n} v_n) \to 0$, yet any w_n which ϵ-shadows v_n for time t_n is not periodic of any period $t_n' \in]t_n - \epsilon, t_n + \epsilon[$.

By compactness of Λ_η, we can assume up to extraction of subsequences that the v_n converge to some $v \in \Lambda_\eta$. Lifting SM to a compact fundamental domain $SD \ni \tilde{v}$ in $S\Omega$, we have some $\tilde{v} \in S\Omega$ with points v^+, v^- in $\partial\Omega$, and lifts $\tilde{v_n}$ of the v_n which converge to $\tilde{v}$ in SD. Also, since $\varphi \cdot v_n$ almost closes up after time t_n, there are group elements γ_n which take $\tilde{v_n}$ close to $\varphi^{t_n} \tilde{v_n}$.

Again, the contradiction hypothesis is that if w_n ϵ-shadows v_n for time t_n, then w_n cannot close up after time $t_n' \in]t_n - \epsilon, t_n + \epsilon[$. Eberlein's geometric observation is that in the universal cover, if w_n ϵ-shadows v_n for time t_n, then the same γ_n which moves $\tilde{v_n}$ close to $\varphi^{t_n} \tilde{v_n}$ must also be responsible for moving $\tilde{w_n}$ close to $\varphi^{t_n} \tilde{w_n}$, because Γ is acting on Ω properly discontinuously and cocompactly by isometries, the assumption that w_n is not periodic of period approximately t_n is realized in the universal cover as follows: if $d(\tilde{w_n}, \tilde{v_n}) < \epsilon$, then $\gamma_n . \tilde{w_n} \neq \varphi^{t_n'} \tilde{w_n}$ for any $t_n' \in]t_n - \epsilon, t_n + \epsilon[$.

The goal of the following lemmas will be to show that nonexistence of an axis of γ_n ϵ-close to $\ell_{\tilde{v_n}}[0, t_n]$ for infinitely many n is mutually exclusive with the assumption that the v_n and v are in Λ_η, producing the desired contradiction.

Lemma 2.3.2. *Let $p \in \Omega$ be the footpoint of $\tilde{v}$. Then $\gamma_n . p \to v^+$ and $\gamma_n^{-1} . p \to v^-$.*

Proof. Take any convex open neighborhood $\mathcal{N}(v^+)$ in $\overline{\Omega}$. Since $\tilde{v_n} \to \tilde{v}$, we have $v_n^+ \in \mathcal{N}(v^+)$ for all sufficiently large n. Then as $t_n \to +\infty$, $\ell_{\tilde{v_n}}(t_n) \in \mathcal{N}(v^+)$ by convexity of $\mathcal{N}(v^+)$. Since γ_n is chosen so that $d(\gamma_n . \tilde{v_n}, \varphi^{t_n} \tilde{v_n}) \to 0$ as $n \to \infty$, then $d_\Omega(\gamma_n . p_n, \ell_{\tilde{v_n}}(t_n)) \to 0$ with n. Once $\ell_{\tilde{v_n}} \in \mathcal{N}(v^+)$ for all large enough n and $\gamma_n . p_n$ is sufficiently close to $\ell_{\tilde{v_n}}$, we will have $\gamma_n . p_n \in \mathcal{N}(v^+)$.

Finally, as $\tilde{v_n} \to \tilde{v}$ implies $p_n \to p$ and γ_n is an isometry, we can conclude for large n that $\gamma_n . p \in \mathcal{N}(v^+)$.

Now consider a convex open neighborhood of v^-, $\mathcal{N}(v^-) \subset \overline{\Omega}$. As $\gamma_n . \tilde{v_n}$ gets closer to $\varphi^{t_n} \tilde{v_n}$, γ_n^{-1} brings the line segment $\ell_{\tilde{v_n}}[-s_n + t_n, s_n + t_n]$ back very close to the line segment $\ell_{\tilde{v_n}}[-s_n, s_n]$ for some $s_n \to \infty$ with n. Then as s_n is very large, $\ell_{\tilde{v_n}}(-s_n)$ is closer to v_n^-, as will be $\gamma_n . \ell_{\tilde{v_n}}(-s_n + t_n)$ which is converging to $\gamma_n.v_n^-$ with large s_n. Hence, $\gamma_n^{-1}.v_n^-$ is closer to v_n^- in the boundary. Then as $\tilde{v_n} \to \tilde{v}$, for sufficiently large n, $\gamma_n^{-1}.v^- \in \mathcal{N}(v^-)$. Since $\gamma_n^{-1}.p_n$ is a point on the line $\gamma^{-1}.\ell_{\tilde{v_n}}$, it suffices to observe that $d_\Omega(\gamma_n^{-1}.p_n, p_n) = d_\Omega(p_n, \gamma_n.p_n) \sim t_n \to \infty$ as $n \to \infty$ to conclude $\gamma_n^{-1}.p \in \mathcal{N}(v^-)$ for all sufficiently large n. □

We next define open neighborhoods in $\partial\Omega$, $V_k(v^+)$, $V_k(v^-)$ such that for any $\xi \in V_k(v^+)$, $\zeta \in V_k(v^-)$, the projective line $(\zeta\xi)$ is distance less than $\frac{1}{k}$ from $\ell_v(0)$ in the Hilbert metric. The existence of such V_k is immediate in a Hilbert geometry by the built-in boundary at infinity and its relationship to d_Ω. The V_k are also homeomorphic to open balls in $\mathbb{R}^2$. Choose k large enough that $\frac{1}{k} < \frac{\epsilon}{2}$.

Lemma 2.3.3. *For all sufficiently large n, $\gamma_n(\overline{V_k(v^+)}) \subset V_k(v^+)$ and $\gamma_n^{-1}(\overline{V_k(v^-)}) \subset V_k(v^-)$.*

Proof. Note that as $\gamma_n.v_n^+$ is closer to v_n^+ and $\tilde{v_n} \to \tilde{v}$, so will $\gamma_n.v^+ \to v^+$ (and similarly, $\gamma_n^{-1}.v^- \to v^-$). If $\gamma_n.v^+$ is very close to v^+, then γ_n either fixes v^+, is contracting near v^+, or both. The only way that $\gamma_n(\overline{V_k(v^+)}) \not\subset V_k(v^+)$ would be if γ_n stabilized a properly embedded triangle $\triangle_n$ such that $\partial\triangle_n \cap \partial V_k(v^+) \neq \varnothing$. If this happened infinitely often, then v^+ would necessarily be the limit of vertices of $\triangle_n$ which are attracting eigenvectors for the $\gamma_n \in \mathrm{Stab}(\triangle_n)$. Since $\gamma_n^{-1}.v^- \to v^-$ and $\gamma_n^{-1} \in \mathrm{Stab}(\triangle_n)$, we can also conclude that vertices of $\triangle_n$ which are repelling eigenvectors for γ_n must accumulate on v^-. Then in the quotient SM, for large enough n, v must be distance less than η from a quotient torus of one of the $\triangle_n$, contradicting the assumption that $v \in \Lambda_\eta$.

An analogous argument applies to show, up to extraction of subsequences, for all sufficiently large n, $\gamma_n^{-1}(\overline{V_k(v^-)}) \subset V_k(v^-)$. □

So we now have that for large n, $\gamma_n(\overline{V_k(v^+)}) \subset V_k(v^+)$ and similarly $\gamma_n^{-1}(\overline{V_k(v^-)}) \subset V_k(v^-)$, both of which are homeomorphic to open balls in $\mathbb{R}^2$.

Applying Brouwer's fixed point theorem, it follows that γ_n fixes points in $V_k(v^-)$ and $V_k(v^+)$. Then γ_n has an axis distance less than $\frac{1}{k} < \frac{\epsilon}{2}$ from $\ell_{\tilde{v}}(0)$, hence ϵ-close to $\ell_{\tilde{v_n}}(0)$ for all sufficiently large n. We also assume that $\gamma_n.\tilde{v_n}$ is arbitrarily close to $\varphi^{t_n}\tilde{v_n}$, so the axis of γ_n will eventually and thereafter be ϵ-close to $\ell_{\tilde{v_n}}[0, t_n]$ and the translation length of γ_n must be ϵ-close to t_n. And so we have a periodic orbit of period $t_n' \in]t_n - \epsilon, t_n + \epsilon[$ which ϵ-shadows v_n for time $\max\{t_n, t_n'\}$. If we obey our hypothesis that such a periodic orbit is impossible, then we would necessarily have $v \not\in \Lambda_\eta$ as proven in Lemma 2.3.3 – a contradiction.

Note also that for small ϵ, hyperbolicity of the periodic orbit is implicit, since a periodic orbit tangent to a torus is bounded away from v_n by η and thus could not ϵ-shadow v_n for small ϵ. □

2.4 Beyond local product structure

For any topological dynamical system, we can define stable and unstable sets. In particular, we define the *weak stable* and *weak unstable* foliation for a point x to be the set of all points positively or negatively asymptotic to x. Of course, the strong (un)stable set of x always contains at least x, and the weak (un)stable set of x contains at least the orbit of x. A feature of hyperbolicity is the existence of large stable and unstable sets (see Section 1.5.3) and that these sets are manifolds.

Often in the nonuniformly hyperbolic theory, (un)stable sets exist in only a small neighborhood of each point. Pesin theory is one way to make the most of local stable and unstable leaves with exponential expansion and contraction rates. Coudene shows that the presence of large stable and unstable leaves with local product structure yields significant topological dynamical results, such as topological mixing, with only some hyperbolicity [15]. For topological results, uniform exponential expansion and contraction rates are less important than global leaves of the unstable foliation.

We do not have true local product structure for the Benoist 3-manifolds, but we have global stable and unstable leaves for most points. We will see in the next section that with existing hyperbolicity results, this property is sufficient for topological mixing by proving essential applications: the Orbit Gluing Lemma 2.4.11 and density of the unstable leaves (Proposition 2.5.3).

Definition 2.4.1 (Local product structure, Bowen bracket)**.** Let X be any metric space with flow $f^t\colon X \to X$. The flow has a *local product structure* if for all $z \in X$, there exists a neighborhood V of z such that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in V$, if $d(x, y) < \delta$ then there exists a point $\langle x, y\rangle \in X$ and some t with $|t| \le \epsilon$ such that

$$\langle x, y\rangle \in W^{su}_\epsilon(f^t x) \cap W^{ss}_\epsilon(y)$$

where $W^{su}_\epsilon(x) = W^{su}(x) \cap B(x,\epsilon)$, and similarly for the strong stable leaf. The point $\langle x, y\rangle$ is commonly

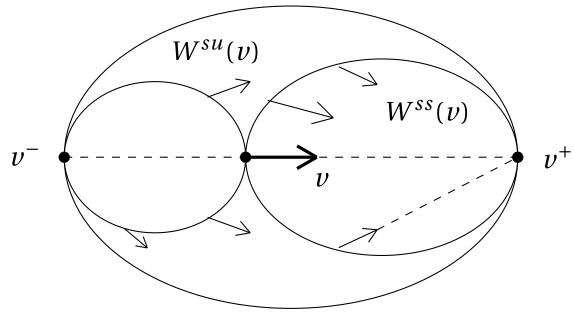


Figure 2.4.1: The stable manifold $W^{ss}(v)$ is contracted under the Hilbert geodesic flow on Ω with C^1-boundary. Note that $W^{ss}(v)$ projects to the horosphere at v^+ passing through πv. Stable manifolds can be defined for v if Ω admits a unique supporting hyperplane at v^+.

known as the *Bowen bracket* of x and y.

Coudene first shows [15, Theorem 4] that if X is a locally complete metric space and f^t is transitive and satisfies the Anosov Closing Lemma, then having noncommensurate period lengths for orbits of f^t implies that f^t is weakly topologically mixing, i.e., $f^t \times f^t$ is transitive on $X \times X$. Combining this result with [15, Theorem 6] gives

Theorem 2.4.2 ([15])**.** *Suppose X is a locally complete metric space such that the flow f^t is transitive, satisfies the Anosov Closing Lemma, and admits a local product structure. If lengths of periodic orbits of f^t generate a dense subgroup of $\mathbb{R}$, then f^t is topologically mixing.*

2.4.1 Global stable and unstable leaves

For M a compact Benoist 3-manifold, (SM, φ^t) admits global stable and unstable leaves on a large set, but with a countable dense set of exceptions. As such, we will not have local product structure for this system. However, we can push Coudene's techniques to our example, in some cases due to the ambient geometry rather than dynamical phenomena.

The strong stable and unstable sets at x are *global* if for all $\xi \neq x^+$ in $\partial\Omega$, there exists a $y \in W^{ss}(x)$ such that $\xi = y^-$.

Proposition 2.4.3. *Suppose $x \in S\Omega$ is forward nonsingular. Then $W^{ss}(x)$ is defined globally. If x is forward singular then $W^{ss}(x)$ is not defined globally.*

Moreover, x is forward regular if and only if W^{os} admits a flow invariant foliation by strong stable leaves.

Proof. The forward direction follows Section 1.5.3 and the work of Benoist [8]. If $x \in S\Omega_{\text{reg}}$ then

$$W^{ss}(x) = \{y \in S\Omega \mid \pi y \in \mathcal{H}_{x^+}(\pi x),\ y^+ = x^+\},$$
$$W^{su}(x) = \{y \in S\Omega \mid \pi y \in \mathcal{H}_{x^-}(\pi x),\ y^- = x^-\},$$

and the strong stable and unstable sets foliate the weak stable and unstable sets

$$\begin{aligned} W^{ou}(x) &= \bigcup_{t\in\mathbb{R}} W^{su}(\varphi^t x) \\ &= \{w \in S\Omega \mid w^- = v^-\} \end{aligned}$$

such that the foliation is both Γ-invariant and flow-invariant.

The converse is clear by studying a non-C^1 transversal to a nonproper point: asymptotically, two orbits $\varphi^t x, \varphi^t y$ meeting at a nonproper $\xi \in \partial\Omega$ will remain bounded distance apart, and thus all such y^- do not correspond to a $y \in W^{ss}(x)$. More explicitly, take a 2-dimensional slice Ω' of Ω such that x^+ is nonproper in Ω'. Then there exists a triangle $\triangle \supset \Omega'$ such that x^+ is a vertex of $\triangle$. Then for all $t \geq 0$ and $y \in S\Omega'$ such that $y^+ = x^+$,

$$d(\pi\varphi^t x, \pi\varphi^t y) = d_\Omega(\ell_x(t), \ell_y(t)) \geq d_\triangle(\ell_x(t), \ell_y(t)) = c.$$

Again, if x^+ is a proper extremal point then it is clear from Benoist's construction that the weak stable manifold is foliated by strong stable leaves, and the foliation is flow invariant. If x^+ is proper but not extremal, then $x^+ \in \sigma$ for some line segment $\sigma \subset \partial\Omega$. Then $\cup\{y \in S\Omega \mid y^+ \in \sigma\} \subset W^{os}(x)$, implying $W^{os}(x)$ cannot be foliated by the flow-equivariant leaves $W^{ss}(\varphi^t x)$. □

By Theorem 1.3.4(g), the vertices of properly embedded triangles in Ω are dense in $\partial\Omega$, and as such the singular points $x \in S\Omega$ in $\partial\Omega$ are dense in $S\Omega$. Since these points do not admit stable and unstable sets, the geodesic flow cannot have local product structure.

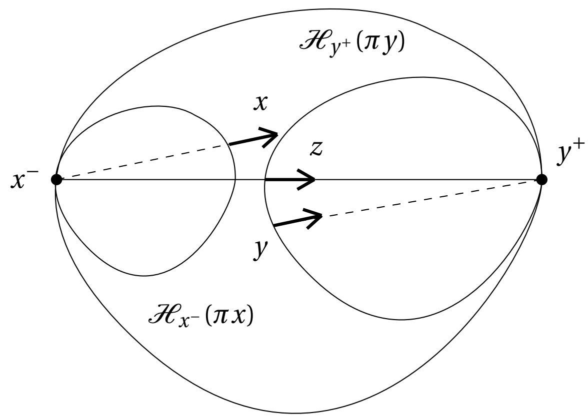


Figure 2.4.2: If x is backward regular and y is forward regular, then the Bowen bracket $z := \langle x, y\rangle \in W^{ou}(x) \cap W^{ss}(y)$ is uniquely defined by $z^- = x^-, z^+ = y^+$ and $\pi z \in \mathscr{H}_{y^+}(\pi y)$.

Fact 2.4.4. *The geodesic flow of the Benoist 3-manifolds does not have local product structure.*

However, the Bowen bracket for regular vectors is well-defined following the characterization in Proposition 2.4.3. If $x, y \in SM_{\text{reg}}$ then $z \in W^{ou}(x) \cap W^{ss}(y)$ is uniquely determined by $z^- = x^-$, $z^+ = y^+$, and $\pi z \in \mathscr{H}_{y^+}(\pi y)$, and all of the above conditions can be satisfied because the proper extremal properties of x^-, y^+ imply $(x^- y^+) \subset \Omega$ and that horospheres at x^-, y^+ are global. See Figure 2.4.2. We will also have a weak local product structure for the geodesic flow on $S\Omega$.

Proposition 2.4.5. *For all $\epsilon > 0$ and $y \in SM_{reg}$, there is a $\delta > 0$ such that for all $x \in B(y,\delta) \cap SM_{reg}$, there exists a $|t| < \epsilon$ such that*

$$\langle x, y\rangle \in W^{su}_\epsilon(\varphi^t x) \cap W^{ss}_\epsilon(y).$$

Proof of Proposition 2.4.5. Lift y to $\tilde{y} \in SD$, with D a fundamental domain for the Γ-action on Ω. For all $\epsilon > 0$, there are neighborhoods $U^+ \ni y^+, U^- \ni y^-$ in $\partial\Omega$ such that $x \in B(y,\epsilon) \implies x^+ \in U^+, x^- \in U^-$. For any such neighborhoods, if x is such that $x^+ \in U^+, x^- \in U^-$ and πx is sufficiently close to y, then $x \in B(y,\epsilon)$. Making U^- small enough that $U^- \subset \{z^- \in \partial\Omega \mid z \in W^{ss}_\epsilon(y)\}$ guarantees that any x with $x^- \in U^-$ satisfies $\langle x, y\rangle \in W^{ss}_\epsilon(y)$. Taking U^+ be as small as needed, we can make these x with $x^- \in U^-$ uniformly arbitrarily close to $\varphi^{t_x}\langle x, y\rangle$ in this local neighborhood of y. It suffices to choose $\delta > 0$ sufficiently small as to ensure

$|t_x| < \epsilon$. □

Remark 2.4.6. An upgrade to uniformity of δ over some filtration by compact sets in Proposition 2.4.5 would be very helpful for future endeavors around uniqueness of equilibrium states.

2.4.2 Orbit gluing in Hilbert geometries

Coudene's first application of local product structure is to show that pieces of orbits which are successively close can be "glued together" by a true orbit [15]. We will make the notion of orbit gluing rigorous and prove the analogous lemma in our context for orbit pieces which are sufficiently regular.

Uniform orbit gluing is better known as shadowing of pseudo-orbits in the literature. Let us introduce that notion here.

Definition 2.4.7 (Pseudo-orbits and the Bowen metric)**.** We can associated to any forward orbit segment $\varphi^{[0,t]}x$ the pair $(x,t) \in SM \times \mathbb{R}_0^+$. An n-length δ-*pseudo-orbit* is a finite collection of finite length orbit segments $\{(x_i,t_i)\}_{i=1}^n \subset SM \times \mathbb{R}_0^+$ such that $d(\varphi^{t_i}x_i, x_{i+1}) < \delta$ for $i = 1,\dots,n-1$. Given any metric space admitting a flow, one can define the *Bowen distance* by

$$d_t(x,y) := \max_{s \le t} d(\varphi^s x, \varphi^s y).$$

Then d_t is a metric on SM, nondecreasing in t. Metric d_t balls are called *Bowen balls*, denoted $B_t(x,\delta)$.

For uniform shadowing of pseudo-orbits, hyperbolicity is required to conclude that the tolerance needed for shadowing does not depend on the number of orbits. Ideally, we would like to glue orbit segments at least uniformly over δ if we cannot shadow uniformly over n:

Definition 2.4.8. A continuous time dynamical system (X,φ) satisfies δ-uniform orbit gluing if for all $\epsilon > 0$ and $n \in \mathbb{N}$, there exists a $\delta > 0$ such that for any δ-pseudo-orbit $\{(x_i,t_i)\}_{i=1}^n \subset X \times \mathbb{R}_0^+$, there exists a $z \in W_\epsilon^{su}(\varphi^t x_1)$ for some $|t| < \epsilon$ such that for all $k = 1,\dots,n-1$,

$$\varphi^{\sum_{i=1}^{k-1} t_i} z \in B_{t_k}(\varphi^{t'_k} x_k, \epsilon) \text{ for some } |t'_k| < \epsilon,$$
$$\varphi^{\sum_{i=1}^{n-1} t_i} z \in W_\epsilon^{ss}(x_n).$$

Conjecture 2.4.9 (δ-Uniform Orbit Gluing Lemma)**.** *The geodesic flow of the Benoist 3-manifolds satisfies δ-uniform orbit gluing over any δ-pseudo orbits with regular points,* $\{(x_i, t_i)\}_{i=1}^n \subset SM_{reg} \times \mathbb{R}_0^+$.

For now, we can shadow pseudo-orbits nonuniformly over position and number of starting points.

Definition 2.4.10. The dynamics satisfies *weak orbit gluing* if for all $\epsilon > 0$ and $\{x_i\}_{i=1}^n$ there exists $\delta > 0$ such that for all times t_i, $i = 1, \ldots, n-1$ with $d(\varphi^{t_i} x_i, x_{i+1}) < \delta$, there is an orbit $z \in W^{su}(x)$ which ϵ-shadows the orbit segments $[x_1, t_1], \ldots, [x_{n-1}, t_{n-1}], [x_n, +\infty]$. More explicitly:

$$|t| < \epsilon \quad z \in W_\epsilon^{su}(\varphi^t x_1) \quad \varphi^{\sum t_i} \in W_\epsilon^{ss}(x_n),$$

and there are numbers $|t_i'| < \epsilon$ for $i = 1, \ldots, n-2$ such that for all $k = 2, \ldots, n-1$,

$$d(\varphi^{t_1+\cdots+t_{k-1}+t}(z), \varphi^{t_k'+t}(x_k)) < \epsilon, \qquad \forall 0 < t < t_k,$$
$$d(\varphi^t(z), \varphi^{t_1'+t}(x_1)) < \epsilon, \qquad \forall 0 < t < t_1.$$

Lemma 2.4.11 (weak orbit gluing)**.** *The geodesic flow of the Benoist 3-manifolds satisfies weak orbit gluing for n pseudo-orbits such that $x_1, \ldots, x_{n-1}$ are backward regular and x_n is forward regular.*

Proof. At x_1, there is some $z_1 \in W^{ou}(\varphi^{t_1} x_1) \cap W^{ss} x_2$. Assuming $\varphi^{t_1} x_1$ is close to x_2, then $\varphi^s z_1$ will be close to $\varphi^s x_2$ for $s \geq 0$ and $\varphi^{t_2} z_1$ is close to $\varphi^{t_2} x_2$ which is close to x_3. Then choose $z_2 \in W^{os}(\varphi^{t_2} z_1) \cap W^{ss}(x_3)$. Continue this construction to find $z_3, \ldots, z_{n-1}$ with each $z_i \in W^{os}(\varphi^{t_i} z_{i-1}) \cap W^{ss}(x_{i+1})$. We let $z = \varphi^{-t_1-\cdots-t_{n-1}} z_{n-1}$. Then $z \in \cap_{i=1}^{n-1} W^{os}(z_i) = W^{os}(x_1)$.

It remains to specify what we require for closeness. Let $\epsilon_n = \frac{\epsilon}{2}$. Then there exists a δ_n such that if $d(\varphi^{t_{n-1}}(z_{n-2}), x_n) < \delta_n$, there is a $|t_{n-1}'| < \frac{\epsilon}{2}$ such that

$$W_{\epsilon_n}^{su}(\varphi^{t_{n-1}+t_{n-1}'} z_{n-2}) \cap W_{\epsilon_n}^{ss}(x_n) \ni z_{n-1}$$

and z_{n-1} is necessarily a regular vector. We then choose recursively $\epsilon_{k-1} \leq \min\{\frac{\epsilon_k}{2}, \frac{\delta_k}{2}\}$, and each step we have a corresponding new δ_{k-1} such that for any t_{k-2} with $d(\varphi^{t_{k-2}} z_{k-3}, x_{k-1}) < \delta_{k-1}$, there is a $|t_{k-2}'| <$

ϵ_{k-1} such that

$$W^{su}_{\epsilon_{k-1}}(\varphi^{t_{k-2}+t'_{k-2}}z_{k-3}) \cap W^{ss}_{\epsilon_{k-1}}(x_{k-1}) \ni z_{k-2}.$$

Note that $d(z_{k-2}, x_{k-1}) \le \epsilon_{k-1} \le \frac{\delta_k}{2}$ and $z_{k-2} \in W^{ss}(x_{k-1})$ so if $\varphi^{t_{k-1}}x_{k-1}$ is $\frac{\delta_k}{2}$-close to x_k the next set of orbits $(-\infty, \varphi^{t_{k-1}}z_{k-2}], [x_k, +\infty)$ can be glued with accuracy ϵ_k.

Repeat until we verify $z_1 \in W^{su}_{\epsilon_1}(\varphi^{t_1+t'_1}x_1) \cap W^{ss}_{\epsilon_1}(x_2)$.

Choose $\delta \le \delta_1$, smaller than all the δ's, and choose the t_i above for this δ when finding the t_i with respect to ϵ_{i+1}. Let $z = \varphi^{-t_1}z_2$.

Let $t''_k = \sum_{i=1}^{k} t'_i$. For $k = 2, \ldots, n$,

$$|t''_k| \le \sum_{i=1}^{n-1} |t'_i| \le \sum_{i=1}^{k} \frac{\delta_i}{2} < \frac{\delta_k}{2}$$

as needed for the recursive orbit gluing. Also,

$$|t''_k| \le |t''_n| \le \sum_{i=1}^{n-1} |t'_i| \le \sum_{i=1}^{n-1} \epsilon_{i+1} < \epsilon$$

as needed for the statement of the Theorem. And lastly,

$$z = \varphi^{-t_1-\cdots-t_{n-1}}z_{n-1} \in W^{su}_{\epsilon_1+\cdots+\epsilon_n}(\varphi^{t'_1+\cdots+t'_n}(x_1)),$$
$$\varphi^{t_1+\cdots+t_k}z \in W^{su}_{\epsilon_k+\cdots+\epsilon_n}(\varphi^{t_k+t'_k+\cdots+t'_n}z_k).$$

□

2.5 Topological mixing

Using Proposition 1.4.7, that the hyperbolic length spectrum of Γ is dense in $\mathbb{R}^+$, we now show that unstable sets for periodic orbits are dense and the geodesic flow is topologically mixing.

2.5.1 Density of unstable sets

When orbit gluing is combined with some hyperbolicity, we can prove that global properties of periodic orbits hold locally. We then apply these results to prove density of unstable sets.

Let $\mathscr{P}$ be the set of periodic orbits of φ modulo orbit equivalence, and let $\mathscr{P}_{\text{hyp}}$ be all the hyperbolic periodic orbits in $\mathscr{P}$. Let T_p denote the length of $p \in \mathscr{P}$. For $S \subset \mathbb{R}$ we denote the additive subgroup generated by S to be $\langle S \rangle$, and $\overline{\langle S \rangle}$ is the topological closure in $\mathbb{R}$.

Lemma 2.5.1. *Given that $\varphi^t : SM \to SM$ is a positively transitive flow satisfying Anosov Closing and weak orbit gluing, if $\overline{\langle T_p \rangle}_{p \in \mathscr{P}_{hyp}} = \mathbb{R}$ then for all open $U \subset SM$, the lengths of periodic orbits passing through U generate a dense subgroup of $\mathbb{R}$.*

Proof. Let $p \in \mathscr{P}_{\text{hyp}}$. Choose $\eta > 0$ such that $U \subset \Lambda_\eta$. Let $\epsilon > 0$. Choose $0 < \delta(\epsilon, \eta) < \epsilon$ small enough to satisfy Anosov Closing on Λ_η. Consider a point $x_0 \in B(x_0, \epsilon) \subset U$ with a dense forward orbit (we can assume ϵ is small enough to guarantee such an x_0). Choose $\delta'(\eta, \frac{\delta}{6}, 3) < \frac{\delta}{3}$ as for $\frac{\delta}{6}$-fine orbit gluing for 3 orbit segments with starting points x_0, p, x_0 (Lemma 2.4.11). Then there exist $s_0, s_1 > 0$ such that $d(\varphi^{s_0} x_0, p) < \delta'$ and $d(\varphi^{s_0+s_1} x_0, x_0) < \delta'$. Thus, the orbit segments $\{(x_0, s_0), (p, nT_p), (\varphi^{s_0} x_0, s_1)\}$ can by glued by some x_n with fineness $\frac{\delta}{6}$:

$$x_n \in B_{s_0}\left(\varphi^{t_1'} x_0, \frac{\delta}{6}\right) \text{ for some } |t_1'| < \frac{\delta}{6},$$
$$\varphi^{s_0} x_n \in B_{nT_p}\left(\varphi^{t_2'} p, \frac{\delta}{6}\right) \text{ for some } |t_2'| < \frac{\delta}{6},$$
$$\varphi^{s_0+T_p} x_n \in B_{s_1}\left(\varphi^{t_3'} \varphi^{s_0} x_0, \frac{\delta}{6}\right) \text{ for some } |t_3'| < \frac{\delta}{6}.$$

Then

$$\begin{aligned} d(x_n, \varphi^{s_0+nT_p+s_1}(x_n)) &\le d(x_n, x_0) + d(x_0, \varphi^{s_0+s_1} x_0) + d(\varphi^{s_0+s_1} x_0, \varphi^{s_0+nT_p+s_1} x_n), \\ &< 2\left(\frac{\delta}{6}\right) + \left(\frac{\delta}{3}\right) + 2\left(\frac{\delta}{6}\right) = \delta. \end{aligned}$$

Note that $x_n \in B(x_0, \delta/3) \subset B(x_0, \epsilon) \subset U \subset \Lambda_\eta$. By Anosov closing on Λ_η, there exists a z_n which has period length $s_0 + nT_p + s_1 + t_n'$ for $|t_n'| < \epsilon$. Since z_n also ϵ-shadows x_n, for small ϵ we have $z_n \in U$. We can repeat the above argument for all n with the same η, ϵ, p and x_0, hence the same δ, δ' and the same s_0, s_1. Then we have $z_n, z_{n+1} \in U$ and thus

$$\langle T_q \rangle_{q \in U \cap \mathscr{P}_{\text{hyp}}} \ni s_0 + (n+1)T_p + s_1 + t'_{n+1} - (s_0 + nT_p + s_1 + t'_n) = T_p + t'_{n+1} - t'_n.$$

Since $|t'_{n+1} - t'_n| \le |t'_{n+1}| + |t'_n| < 2\epsilon$, letting ϵ go to zero we conclude $T_p \in \overline{\langle T_q \rangle}_{q \in U \cap \mathscr{P}_{\text{hyp}}}$ for all $p \in \mathscr{P}_{\text{hyp}}$, which proves the lemma because $\overline{\langle T_p \rangle}_{p \in \mathscr{P}_{\text{hyp}}} = \mathbb{R}$. □

We now prove an essential proposition for topological mixing: the density of stable and unstable sets in SM. For the proof of mixing we will use the following fact, a consequence of the density of unstable sets for periodic orbits:

Fact 2.5.2. *For p periodic, density of $W^{su}(p)$ implies that for all $\delta > 0$ and for all $x \in SM$, there is a $T(p, \delta, x) > 0$ such that for all $t \ge T$, $\varphi^t W^{su}_\delta(p) \cap B(x, \delta) \neq \varnothing$.*

Proof. By assumption, there exists some $z \in W^{su}(p) \cap B(x, \delta/2)$. Then $d(\varphi^{-t} z, \varphi^{-t} p) \to 0$ as $t \to +\infty$ so there exists an $S > 0$ such that $s \ge S \implies d(\varphi^{-s} p, \varphi^{-s} z) < \delta$. For all $n \in \mathbb{N}$ such that $nT_p \ge S$, then $d(p, \varphi^{-nT_p} z) = d(\varphi^{-nT_p} p, \varphi^{-nT_p} z) < \delta$, and $\varphi^{-nT_p} z \in \varphi^{-nT_p} W^{su}(p) = W^{su}(p)$. Hence $\varphi^{-nT_p}(z) \in W^{su}_\delta(p)$ and $z \in \varphi^{nT_p}(W^{su}_\delta(p)) \cap B(x, \delta/2) \neq \varnothing$.

Take a finite $\delta/2$-cover $\{t_1, \dots, t_k\}$ of $[0, T_p]$. Repeat for each periodic point $\varphi^{t_i} p$ of period T_p the above argument to produce a $z_i \in W^{su}(\varphi^{t_i} p) \cap B(x, \delta/2)$ and minimum $n_i \in \mathbb{N}$ with $n \ge n_i \implies z_i \in \varphi^{nT_p}(W^{su}_\delta(\varphi^{t_i} p)) \cap B(x, \delta/2) \subset \varphi^{nT_p + t_i}(W^{su}_\delta(p)) \cap B(x, \delta/2) \neq \varnothing$. Let $N = \max_{1 \le i \le k} n_i$ and $T = (N+1)T_p$. Then for all $t \ge T$, there is some $M_t \ge N+1$, $i \in \{1, \dots, k\}$, and $0 \le \epsilon \le \delta/2$ such that $t = M_t T_p + t_i + \epsilon$ and thus

$$z_i \in \varphi^{M_t T_p + t_i}(W^{su}_\delta(p)) \cap B(x, \delta/2) \implies \varphi^\epsilon z_i \in \varphi^t(W^{su}_\delta(p)) \cap B(x, \delta) \neq \varnothing$$

as desired. □

Proposition 2.5.3. *If $x \in SM$ is a hyperbolic periodic orbit, $W^{su}(x)$ is dense in SM.*

Proof. Let $U \subset SM$ be open, $x \in \mathscr{P}_{\text{hyp}}$. By Proposition 1.4.7 and Lemma 2.5.1 there exists a $y \in \mathscr{P}_{\text{hyp}}$ such that $\overline{\langle T_x, T_y \rangle} = \mathbb{R}$. Let $\epsilon > 0$ such that $B(y, \epsilon) \subset U$. Since $x, y \in SM_{\text{reg}}$ there exists a $T \in \mathbb{R}$ such that $z \in W^{su}(x) \cap W^{ss}(\varphi^T(y))$. Then $\varphi^{-T} z \in W^{ss}(y)$ so choose $M \in \mathbb{N}$ large enough that for any $m \ge M$,

$d(\varphi^{mT_y - T} z, y) < \epsilon/2$. Because $\overline{\langle T_x, T_y \rangle} = \mathbb{R}$, there are large enough $m, n \in \mathbb{N}$ with $m \geq M$ such that

$$-\epsilon/2 < |-nT_x + mT_y - T| < \epsilon/2$$

implying that $d(\varphi^{-nT_x} z, \varphi^{mT_y - T} z) < \epsilon/2$. Note that $\varphi^{-nT_x} z \in \varphi^{-nT_x} W^{su}(x) = W^{su}(x)$ and $d(\varphi^{-nT_x} z, y) < \epsilon$ by the triangle inequality implies $\varphi^{-nT_x} z \in B(y, \epsilon) \subset U$. □

Conjecture 2.5.4. *If $W^{su}(p)$ is dense for all hyperbolic periodic orbits p, then $W^{su}(z)$ is dense for all regular $z \in SM$.*

We now prove topological mixing for the geodesic flow of the Benoist 3-manifolds.

Definition 2.5.5 (Topological mixing)**.** The geodesic flow is *topologically mixing* if for all U, V open, there exists a $T \in \mathbb{R}^+$ such that $t \geq T \Longrightarrow \varphi^t U \cap V \neq \varnothing$.

Theorem 2.5.6. *The geodesic flow on SM is topologically mixing.*

Proof. Since U is open there exists a hyperbolic periodic point $p \in U$ (Lemma 2.2.1). Let $\delta > 0$ be small enough that $W^{su}_{\delta}(p) \subset B(p, \delta) \subset U$ and $B(x, \delta) \subset V$ for some $x \in V$. By Fact 2.5.2, a consequence of Proposition 2.5.3, there is a $T(U, V) > 0$ such that for all $t \geq T$,

$$\varnothing \neq \varphi^t W^{su}_{\delta}(p) \cap B(x, \delta) \subset \varphi^t U \cap V.$$

□

Chapter 3

The Bowen program in a nonuniform setting

R. Bowen famously introduced and proved that specification, a property of uniformly hyperbolic systems, is sufficient for uniqueness of equilibrium states. He also introduced entropy-expansivity, more general than uniform expansivity, to conclude existence of equilibrium states.

In this chapter, we explore a variety of presentations of weaker specification properties in search of one which will help us characterize the topological entropy for the geodesic flow of the Benoist 3-manifolds by exponential orbit growth rates. Ideally, we would like to find a specification property which applies in large generality, so we forgo any hyperbolicity assumptions. We also prove entropy-expansiveness for any compact Hilbert geometry. As a consequence, a measure of maximal entropy exists for the geodesic flow of any Hilbert geometry. A consequence for the Benoist 3-manifolds is that we can prove an entropy inequality which suffices for positive topological entropy.

3.1 Specification without hyperbolicity control

First, as a motivation, we state the original specification property due to Bowen on his hunt for systems with unique equilbrium states. Recall that we characterize orbit segments by pairs of points in the phase space and times larger than 0.

Definition 3.1.1 (Uniform specification)**.** For a flow $\varphi^t\colon X \to X$ of a metric space (X, d), we say $x \in X$ ϵ-shadows the orbit segments $\{(x_i, t_i)\}_{i=1}^n \subset X \times \mathbb{R}_0^+$ with specification spacing $s_i > 0$, $i = 1, \dots, n-1$ if there are $|t_i'| < \epsilon$ such that

$$x \in W_\epsilon^{su}(\varphi^{t_1'} x_1),$$
$$\varphi^{\sum_{i=1}^{k-1} t_i + s_i} x \in B_{t_k}(\varphi^{t_k'} x_k, \epsilon) \text{ for } k = 1, \dots, n-1,$$
$$\varphi^{\sum_{i=1}^{n-1} t_i + s_i} x \in W_\epsilon^{ss}(x_n).$$

The dynamical system (X,φ) satisfies *uniform specification* if for all $\epsilon > 0$ there is an $S > 0$ such that for any finite set of orbit segments $\mathscr{X} \subset X \times \mathbb{R}_0^+$ and specification spacing $s_i > S$, $i = 1,\dots,|\mathscr{X}|-1$, there is an $x \in X$ which ϵ-shadows $\mathscr{X}$ with spacing s_i.

Uniform specification follows from uniform shadowing of pseudo-orbits, uniform transitivity, and uniform control over exponential contraction rates of stable and unstable sets.

Definition 3.1.2 (Weak mixed specification)**.** The dynamics (X,φ) satisfies *weak mixed specification* over $\mathscr{G} \subset X \times \mathbb{R}^+$ if for all $\epsilon > 0$ and $n \in \mathbb{N}$, there exists a $S > 0$ such that for all $\mathscr{X} := \{(x_i, t_i)\}_{i=1}^n \subset \mathscr{G}$ and all $s_i > S$, $i = 1,\dots,n-1$, there exists an $x \in X$ which ϵ-shadows $\mathscr{X}$ with specification spacing s_i.

This is transverse to the Climenhaga–Thompson nonuniform specification property [13]. The weak mixed specification property is stronger than mixing, but the nonuniformity over number of orbit segments comes from an absence of hyperbolicity control over stable and unstable sets. The Climenhaga–Thompson nonuniform specification property is uniform over n but does not imply topological mixing.

Theorem 3.1.3. *Let (X,d) be a compact metric space with a continuous time dynamics given by φ. If (X,φ) satisfies δ-uniform orbit gluing over $\mathscr{G} \subset X \times \mathbb{R}^+$ and topological mixing, then the system has the weak mixed specification property over $\mathscr{G}$.*

Proof. For $\epsilon > 0$, choose $\delta(\epsilon, 2n-1) > 0$ as for ϵ-fine orbit gluing of $2n-1$ orbit segments in $\mathscr{G}$ (Definition 2.4.8 over $\mathscr{G}$). Take $\mathscr{U} := \{B(y_i, \delta/2)\}_{i=1}^k$ a finite cover of X. Then by topological mixing there is an $M > 0$ such that for all $m \geq M$ and all $U, V \in \mathscr{U}$, $\varphi^m(U) \cap V \neq \varnothing$. Now consider $\mathscr{X} = \{(x_i, t_i)\}_{i=1}^n \subset \mathscr{G}$ with specification spacings $s_1,\dots,s_{n-1} \geq M$. For each $i = 1,\dots,n-1$, there are y_1, y_2 such that $\varphi^{t_i} x_i \in B(y_1, \delta/2)$ and $x_{i+1} \in B(y_2, \delta/2)$. Then

$$\varnothing \neq \varphi^{s_i} B(y_1, \delta/2) \cap B(y_2, \delta/2) \subset \varphi^{s_i} B(\varphi^{t_i} x_i, \delta) \cap B(x_{i+1}, \delta).$$

Let $z_i \in B(\varphi^{t_i} x_i, \delta)$ such that $\varphi^{s_i} z_i \in B(x_{i+1}, \delta)$. Then the $2n-1$-length δ-pseudo orbit

$$\{(x_i, t_i), (z_i, s_i), (x_n, t_n)\}_{i=1}^{n-1} \subset \mathscr{G}$$

can be ϵ-shadowed by a true orbit x with $|t_i'|,|s_i'| < \epsilon$ for $i = 1,\ldots,n-1$, hence

$$x \in W^{su}_\epsilon(\varphi^{t_i'} x_1), \tag{3.1.1}$$

$$\varphi^{\sum_{i=1}^{k-1} t_i + s_i} x \in B_{t_k}(\varphi^{t_k'} x, \epsilon) \text{ for } k = 1,\ldots,n-1, \tag{3.1.2}$$

$$\varphi^{t_k + \sum_{i=1}^{k-1} t_i + s_i} x \in B_{s_k}(\varphi^{s_k'} z_k, \epsilon) \text{ for } k = 1,\ldots,n-2, \tag{3.1.3}$$

$$\varphi^{\sum_{i=1}^{n-1} t_i + s_i} x \in W^{ss}_\epsilon(x_n). \tag{3.1.4}$$

Note that (3.1.1, 3.1.3, 3.1.4) are as needed for weak mixed specification. □

A specification property which is stronger than topological mixing is likely not needed for the goals of this chapter. Thus we introduce another specification property. This property is strictly weaker than the Climenhaga–Thompson nonuniform specification property.

Definition 3.1.4 (Weak specification)**.** The dynamics (X,φ) satisfies *weak specification* if for all $\epsilon > 0$, $n \in \mathbb{N}$ there exists an $S > 0$ such that for all $\mathscr{X} := \{(x_i, t_i)\}_{i=1}^n \subset X \times \mathbb{R}_0^+$ there exist $s_i \le S$, $i = 1,\ldots,n-1$, and an $x \in X$ which ϵ-shadows $\mathscr{X}$ with specification spacing s_i.

Theorem 3.1.5. *Let (X,d) be a compact metric space with a continuous time dynamics given by φ. If (X,φ) satisfies δ-uniform orbit gluing over $\mathscr{G} \subset X \times \mathbb{R}^+$ and uniform transitivity then the system satisfies the weak specification property over $\mathscr{G}$.*

Proof. For $\epsilon > 0$, choose $\delta(\epsilon, 2n-1) > 0$ as for ϵ-fine orbit gluing of $2n-1$ orbit segments in $\mathscr{G}$ (Definition 2.4.8 over $\mathscr{G}$). Take $\mathscr{U} := \{B(y_i, \delta/2)\}_{i=1}^k$ a finite cover of X. Then by uniform transitivity there is an $M(\epsilon, n)$ and times $s_{ij} < M$ such that $\varphi^{s_{ij}}(B(y_i,\delta/2)) \cap B(y_j,\delta/2) \neq \varnothing$. Now consider $\mathscr{X} = \{(x_i, t_i)\}_{i=1}^n \subset \mathscr{G}$. For each $i = 1,\ldots,n-1$, there are y_{a_i}, y_{b_i} with $a_i, b_i \in \{1,\ldots,k\}$ such that $\varphi^{t_i} x_i \in B(y_{a_i}, \delta/2)$ and $x_{i+1} \in B(y_{b_i}, \delta/2)$. Then

$$\varnothing \neq \varphi^{s_{a_i b_i}} B(y_{a_i}, \delta/2) \cap B(y_{b_i}, \delta/2) \subset \varphi^{s_i} B(\varphi^{t_i} x_i, \delta) \cap B(x_{i+1}, \delta).$$

Denote $s_i := s_{a_i b_i}$. Let $z_i \in B(\varphi^{t_i} x_i, \delta)$ such that $\varphi^{s_i} z_i \in B(x_{i+1}, \delta)$. Then there exist $s_i \le M$ such that the $2n-1$-length δ-pseudo orbit

$$\{(x_i, t_i), (z_i, s_i), (x_n, t_n)\}_{i=1}^{n-1} \subset \mathscr{G}$$

can be ϵ-shadowed by a true orbit x with $|t_i'|,|s_i'| < \epsilon$ for $i = 1,\ldots,n-1$, hence

$$x \in W_\epsilon^{su}(\varphi^{t_i'} x_1), \tag{3.1.5}$$

$$\varphi^{\sum_{i=1}^{k-1} t_i + s_i} x \in B_{t_k}(\varphi^{t_k'} x, \epsilon) \text{ for } k = 1,\ldots,n-1, \tag{3.1.6}$$

$$\varphi^{t_k + \sum_{i=1}^{k-1} t_i + s_i} x \in B_{s_k}(\varphi^{s_k'} z_k, \epsilon) \text{ for } k = 1,\ldots,n-2, \tag{3.1.7}$$

$$\varphi^{\sum_{i=1}^{n-1} t_i + s_i} x \in W_\epsilon^{ss}(x_n). \tag{3.1.8}$$

Note that (3.1.5, 3.1.7, 3.1.8) are as needed for weak specification. □

Given the lack of uniformity in Lemma 2.4.11, we cannot yet prove weak specification. Instead, we have a weaker version.

Definition 3.1.6 (Weaker specification)**.** The dynamics (X,φ) satisfies *weaker specification* if for all $\epsilon > 0$, $n \in \mathbb{N}$, and $\{x_i\}_{i=1}^n \subset X$, there exists an $S > 0$ such that for all $\mathscr{X} := \{(x_i, t_i)\}_{i=1}^n \subset X \times \mathbb{R}_0^+$, there are $s_i \le S$, $i = 1,\ldots,n-1$, and an $x \in X$ which ϵ-shadows $\mathscr{X}$ with specification spacings s_i.

Proposition 3.1.7. *The geodesic flow $\varphi^t : SM \to SM$ satisfies the weaker specification property.*

Proof. The proof is virtually the same as earlier proofs. The only assumptions we need are the weak orbit gluing property and uniform transitivity. □

3.2 Entropy-expansiveness of Hilbert geometries

In this section we prove entropy-expansiveness for the geodesic flow of any divisible Hilbert geometry. First we review some preliminary notions from entropy theory following the work of Rufus Bowen [11].

We let d_t again refer to the Bowen metric (Definition 2.4.7). A (t,δ)*-spanning set* for $K \subset SM$ is one which is δ-dense in K for the d_t metric. For any compact $K \subseteq SM$, we can choose a minimal finite (t,δ)-spanning set and denote the cardinality by $S(t,\delta,K)$. Then we define the *topological entropy* of φ^t on K by

$$h_{top}(\varphi, K) := \lim_{\delta\to 0} \lim_{t\to\infty} \frac{1}{t} \log S(t,\delta,K).$$

There are many equivalent definitions of h_{top} [26], and we include one other here. For $K \subseteq SM$ compact, we define a (t,δ)-separated set $R \subset K$ such that for all $u, v \in R$, $d_t(v,u) \geq \delta$. Let $R(t,\delta,K)$ denote the maximal cardinality for (t,δ)-separated sets, which is again finite by compactness of K. Then

$$h_{top}(\varphi, K) = \lim_{\delta\to 0}\lim_{t\to\infty}\frac{1}{t}\log R(t,\delta,K).$$

When $K = SM$, we abbreviate $S(t,\epsilon) := S(t,\epsilon,SM)$, $R(t,\epsilon) := R(t,\epsilon,SM)$, and $h_{top}(\varphi) := h_{top}(\varphi, SM)$.

For the purposes of applying Bowen's work, we take

$$h_{top}(\varphi, K, \delta) := \lim_{t\to\infty}\frac{1}{t}\log S(t,\delta,K).$$

so that $h_{top}(\varphi,K) = \lim_{\delta\to 0} h_{top}(\varphi,K,\delta)$, and for $K = SM$ we have $h_{top}(\varphi) := \lim_{\delta\to 0} h_{top}(\varphi,\delta)$.

For any point x in SM, we define the *infinite Bowen balls* about x in positive or negative time:

$$\Phi_\epsilon(x) := \bigcap_{t\in\mathbb{R}^+}\varphi^{-t}\overline{B(\varphi^t x,\epsilon)} = \{y \in M \mid d(\varphi^t y, \varphi^t x) \leq \epsilon \text{ for all } t \in \mathbb{R}^+\}.$$

Intuitively, we should think of the $\Phi_\epsilon(x)$ as the exceptions to expansivity. An expansive map (not flow) is defined by the existence of an $\epsilon > 0$ such that $\Phi_\epsilon(x) = \{x\}$ for all x. An expansive flow would satisfy that $\Phi_\epsilon(x) = W^{os}_\epsilon(x)$ for all x.

Expansive maps and flows are special cases of entropy expansive systems. Define

$$h^*(\epsilon) := \sup_{x\in SM} h_{top}(\varphi, \Phi_\epsilon(x)).$$

Then φ is *h-expansive* with expansivity constant $\epsilon > 0$ if $h^*(\epsilon) = 0$. In other words, there is an $\epsilon > 0$ such that the ϵ-exceptions to expansivity have no influence on the entropy of the system.

For an h-expansive system, Bowen proves that we can bypass the cumbersome limit over $\delta \to 0$ of $h_{top}(\varphi,\delta)$ to compute $h_{top}(\varphi)$.

Theorem 3.2.1 ([11, Theorem 2.4])**.** *If ϵ is an h-expansive constant for φ, then*

$$h_{top}(\varphi) = h_{top}(\varphi,\epsilon).$$

Moreover, to compute the measure-theoretic entropy of a system, one can simply take a sufficiently fine measurable partition rather than an infimum over all possible partitions. We will save the definitions of measure-theoretic entropy for Chapter 6 where they are relevant, but include Bowen's theorem here for the familiar reader.

Theorem 3.2.2 ([11, Theorem 3.5])**.** *For a finite-dimensional phase space: If ϵ is an h-expansive constant for φ, and $\mathscr{A}$ a finite measurable partition for a measure μ with* $\mathrm{diam}(\mathscr{A}) \leq \epsilon$*, then $h_\mu(\varphi) = h_\mu(\varphi, \mathscr{A})$.*

One immediate consequence of Bowen's theorems is existence of a measure of maximal entropy.

Fact 3.2.3. *If φ is h-expansive, then there exists a measure of maximal entropy.*

Proof. This result is due to a standard construction in [41, Theorem 8.6 (2)] and Bowen's theorems. For all $\epsilon > 0$, there is a measure μ such that $h_\mu(\varphi) = h_{top}(\varphi, \epsilon)$. The construction of μ is as follows: take for each $t \in \mathbb{R}^+$ a finite set E_t which (t, ϵ)-spans M with minimal cardinality. Then define probability measures

$$\nu_t = \frac{1}{|E_t|} \sum_{x \in E_t} \delta_x$$

and take their averages along orbits

$$\mu_t = \frac{1}{t} \int_0^t \varphi^s * \nu_t \, ds.$$

The weak* accumulation points of μ_t are φ-invariant. Then any such accumulation point μ is a measure of maximal entropy by computing the entropy as a supremum over measurable partitions of cardinality $|E_n|$ for each n such that every point in E_n is associated to a unique element of the partition.

□

Proof of entropy expansiveness

For any manifold, the *injectivity radius* of $x \in M$ is defined to be

$$\mathrm{inj}(x) := \frac{1}{2} \inf_{\ell} \{\mathrm{length}(\ell)\},$$

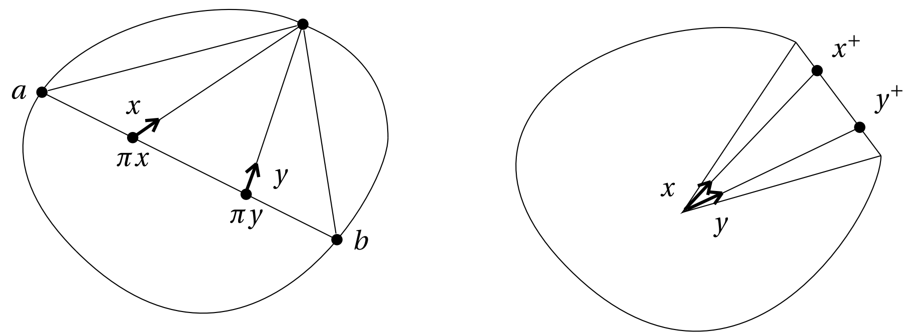


Figure 3.2.1: For Lemma 3.2.4. On the left, $d_\Omega(\ell_x(t), \ell_y(t)) \leq d_\triangle(\ell_x(t), \ell_y(t)) = d_\Omega(\pi x, \pi y)$ for all $t \geq 0$, thus $d(\varphi^t x, \varphi^t y) \leq d(x, y)$ for $t \geq 0$. On the right, the set of all x such that $x^+ \in C$ is bounded, and such points only separate to distance $d_C(x^+, y^+)$ under the geodesic flow.

where ℓ varies over all homotopically nontrivial loops through x. Then define the *injectivity radius of* M to be

$$\mathrm{inj}(M) := \inf_{x \in M} \mathrm{inj}(x).$$

Lemma 3.2.4. *Let* (Ω, d_Ω) *be any Hilbert geometry and* $x, y \in S\Omega$*.*

(a) If $x^+ = y^+$ *then* $d(\varphi^t x, \varphi^t y) \leq d(x, y)$ *for all* $t \geq 0$*.*

(b) If $x^+, y^+ \in C$ *an open convex set in* $\partial\Omega$ *and* $\pi x = \pi y$*, then* $d(\varphi^t x, \varphi^t y) \leq d_C(x^+, y^+)$ *for all* $t \geq 0$*.*

Proof. Since C is a properly convex domain, the Hilbert distance d_C on C is a valid metric. The rest of the argument is illustrated in Figure 3.2.1. □

Let $\mathrm{diam}(D)$ denote the metric diameter of a compact set in a metric space. In other words, $\mathrm{diam}(D)$ is the largest distance between any two points in D.

Theorem 3.2.5. *The geodesic flow* φ^t *on any compact Hilbert geometry is* h*-expansive.*

Proof. Choose a fundamental domain $D \subset \Omega$ for the Γ-action with compact quotient M. Note that $\mathrm{inj}(M) = \mathrm{diam}(D)$. We can represent $x \in SM$ as a unique lift $x \in SD$ and we can choose our tiling so that $d(\pi x, \partial D) > \operatorname{diam} D/2$. For $x \in SD$ define

$$\Phi^+(x,\epsilon) := \{y \in \overline{B(x,\epsilon)} \mid y^+ = x^+\}.$$

If $x \in S\Omega$ such that x^+ is an extremal point in $\partial\Omega$, then $\Phi_\epsilon(x) = \Phi^+(x,\epsilon)$. Then by Lemma 3.2.4(a) if E is any minimal $(0,\delta)$-spanning set for $\Phi_\epsilon(x)$ then E is also a (t,δ)-spanning set for $\Phi_\epsilon(x)$ for all $t \geq 0$. Thus

$$h_{top}(\Phi_\epsilon(x),\delta) \leq \lim_{t\to\infty} \frac{1}{t} \log|E| = 0$$

and $h_{top}(\Phi_\epsilon(x)) = 0$.

Suppose now that x^+ is not extremal. Let $C \subset \partial\Omega$ be the maximal convex domain containing x^+. Now define

$$\Phi_C^+(x,\epsilon) := \{y \in \overline{B(x,\epsilon)} \mid y^+ \in \overline{B_C(x^+,\epsilon)}\}.$$

Then $\Phi_\epsilon(x) \subset \Phi_C^+(x,\epsilon)$. For all $\eta \in B_C(x^+,\epsilon)$ let v_η be such that $\pi v_\eta = \pi x$ and $v_\eta^+ = \eta$. By Lemma 3.2.4(b), $d(v_\eta, x) \leq d_C(\eta, x^+) \leq \epsilon$. Then for all $w \in \Phi_C^+(x,\epsilon)$, there is an $\eta = w^+$ implying $d(w, v_\eta) \leq d(w,x) + d(x, v_\eta) = \epsilon + \epsilon = 2\epsilon$, hence

$$\Phi_C^+(x,\epsilon) \subset \bigcup_{\eta \in \overline{B_C(x^+,\epsilon)}} \Phi^+(v_\eta, 2\epsilon).$$

Choose a finite $\delta/2$-cover of $B_C(x^+,\epsilon/2)$ by $\{\eta_i\}_{i=1}^k$ and $v_i := v_{\eta_i}$. Then for all $y \in \Phi^+(v_\eta, 2\epsilon)$, there is an η_i such that $d_C(\eta,\eta_i) < \delta/2$ and $d(y, v_i) \leq d(y, v_\eta) + d(v_\eta, v_i) < 2\epsilon + d_C(\eta,\eta_i) < \frac{5\epsilon}{2}$ for δ small. Describe all such y by

$$\Phi_C^+(v_i, 5\epsilon/2, \delta/2) := \{y \in \overline{B(x, 5\epsilon/2)} \mid y^+ \in \overline{B_C(x^+,\delta/2)}\}.$$

Thus, by choice of $D \ni \pi x$ so that $d(\pi x, \partial D) > \operatorname{diam} D/2$ we can make ϵ small enough that

$$\Phi_\epsilon(x) \subset \bigcup_{i=1}^k \Phi_C^+(v_i, 5\epsilon/2, \delta/2) \subset SD.$$

Note also for each compact $\Phi_C^+(v_i, 5\epsilon/2, \delta/2)$, a minimal $(0,\delta)$-spanning set E_i will be a (t,δ)-spanning set for all $t \geq 0$. Thus,

$$h_{top}(\Phi_\epsilon(x),\delta) \leq \lim_{t\to\infty} \frac{1}{t} \log\left(\sum_{i=1}^k |E_i|\right) = 0.$$

□

3.3 Equivalent characterizations of topological entropy

In this section we combine results from earlier sections in the chapter to characterize topological entropy of the Benoist 3-manifolds by the exponential growth rate of periodic orbits. A consequence is positive topological entropy.

Definition 3.3.1. Let $P_t(\varphi)$ denote the collection of *isolated* φ-periodic orbits of period at most t, modulo orbit equivalence, and

$$\rho(\varphi) := \lim_{t\to\infty} \frac{1}{t} \log \# P_t(\varphi).$$

The effect of defining P_t by isolated orbits is that we neglect the flat orbits on $S\mathbb{T}$. Any periodic orbit on $S\mathbb{T}$ corresponds to a continuous 1-parameter family of periodic orbits of the same homotopy type, so these are not isolated and not counted. By contrast, the hyperbolic periodic orbits are isolated and countable.

Proposition 3.3.2. *Since the geodesic flow of a Benoist 3-manifold is h-expansive,*

$$\rho(\varphi) \le h_{top}(\varphi).$$

Proof. Choose $\epsilon \le \operatorname{inj} M/3$ the h-expansivity constant for the geodesic flow on SM. We show that P_t is a (t,ϵ)-separated set. If $x, y \in P_T$ such that $d_T(x, y) < \epsilon$, then $d_t(x, y) < \epsilon$ for all $t \in \mathbb{R}$. Since Γ acts discretely and $\epsilon < \operatorname{inj}(M)/3$, this is only possible if $x = y$ or if x and y lift to tangent vectors to a properly embedded triangle $\triangle$ such that $\ell_{\tilde{x}}, \ell_{\tilde{y}} \subset \triangle$. Then $x, y \in S\mathbb{T}$ so they are not counted in P_T.

Thus, P_t is (t,ϵ)-separated and has cardinality less than or equal to $R(t,\epsilon)$, the cardinality of a maximal (t,ϵ)-separated set. We conclude by h-expansivity (Theorem 3.2.5) and its consequence Theorem 3.2.1 that

$$\rho(\varphi) = \lim_{t\to\infty} \frac{1}{t} \log \# P_t(\varphi) \le \lim_{t\to\infty} \frac{1}{t} \log R(t,\epsilon) = h_{top}(\varphi).$$

□

Proposition 3.3.3. *The geodesic flow of a compact Benoist 3-manifold has positive topological entropy.*

Proof. Invariant lines for elements of the Baker–Cooper subgroup Σ correspond to hyperbolic orbits in SM. Then by Proposition 3.3.2,

$$h_{top}(\varphi) \geq \rho(\varphi) \geq \beta(\Sigma) > 0,$$

where β is the growth function of Σ. □

We seek equality for Proposition 3.3.2, but such a result hinges on a uniformity which we do not currently have.

Proposition 3.3.4. *If $\varphi^t : SM \to SM$ satisfies the Anosov Closing Lemma and weak specification then*

$$\rho(\varphi) = h_{top}(\varphi).$$

Proof. For the following proof, let $P_t(K)$ count the number of isolated periodic orbits passing through $K \subset SM$, and $\rho(K)$ the exponential growth of orbits through K over length. Let $\epsilon > 0$ be the h-expansivity constant for (SM, φ) and η small. Let $E(t,\epsilon)$ be a maximal (t,ϵ)-separated set for $\Lambda_{2\eta}$ with cardinality $R(t, \Lambda_{2\eta}, \epsilon)$ and $t > 0$. Choose $0 < \delta < \min\{\epsilon/2, \eta\}$ as for the Anosov Closing Lemma for closing orbits in Λ_η with fineness $\epsilon/4$. For the finitely many $x \in E$, construct a specification $\{(x,t),(x,0)\}$ with spacings $s_x \leq M(\delta/2, 2)$ Then there is a $z \in B_t(x, \delta/2)$ with $d(\varphi^{t+s_x} z, x) < \delta/2$, implying

$$d(z, \varphi^{t+s_x} z) \leq d(z,x) + d(\varphi^{t+s_x} z, x) < \delta$$

and $d(z, S\mathbb{T}) \geq d(x, S\mathbb{T}) - d(x,z) \geq 2\eta - \delta/2 > 2\eta - \eta/2 > \eta$ for all immersed tori $\mathbb{T} \subset M$. Thus we have for $x \in E$ a periodic orbit $p(x)$ such that

$$\varphi^{t'}(p(x)) = p(x) \text{ for some } t' \in]t + s_x - \epsilon/4, t + s_x + \epsilon/4[,$$

$$p(x) \in B_{\min\{t', t+s_x\}}(y, \epsilon/4),$$

and $p \in P_{t+M+\epsilon/4}$ since $s_x \leq M$. Note that

$$d_t(p,x) \leq d_t(p,z) + d_t(z,x) < \epsilon/4 + \delta/2 < \epsilon/4 + \epsilon/4 = \epsilon/2.$$

Then the mapping $p\colon E \to P_{t+M+\epsilon/4}$ is one-to-one: for $x \neq y \in E$,

$$d_t(p(x), p(y)) \geq d_t(x, y) - d_t(y, p(y)) - d_t(x, p(x)) > \epsilon - \epsilon/2 - \epsilon/2 = 0.$$

The number of periodic orbits constructed did not depend on M or ϵ, only E which depends only on t for fixed ϵ. Thus for each t and any $\eta > 0$,

$$|P_{t+M+\epsilon/4}(\varphi, \Lambda_\eta)| \geq R(t, \Lambda_{2\eta}, \epsilon)$$

and since M only depends on ϵ for fixed η we conclude

$$\rho(\varphi, \Lambda_\eta) = \lim_{t\to\infty} \frac{1}{t} \log|P_{t+M+\epsilon/4}(\varphi, \Lambda_\eta)| \geq \lim_{t\to\infty} \frac{1}{t} \log R(t, \Lambda_{2\eta}, \epsilon) = h_{top}(\varphi, \Lambda_{2\eta}).$$

The leftmost inequality is verifiable using that $\lim_{t\to\infty} \frac{1}{t} \log|P_{t+s}| = \lim_{t\to\infty} \frac{1}{t+s} \log|P_{t+s}| + \frac{s}{t(t+s)} \log|P_{t+s}|$.

Since $\cup_{\eta>0}\Lambda_\eta$ is an open dense set in SM, we conclude the proposition. □

3.4 Volume entropy

We remark in this section that A. Manning's proof that volume entropy and topological entropy agree for compact nonpositively curved Riemannian manifolds generalizes to this context immediately [33]. Taking vol to be the Busemann-Hausdorff volume on Ω, we have

Proposition 3.4.1. *Let* $h_{vol} = \lim_{r\to\infty} \frac{1}{r} \log \mathrm{vol}(B_\Omega(x, r))$, *where* Ω *is a divisible, indecomposable, properly convex domain in* $\mathbb{RP}^3$. *Then*

$$h_{vol} = h_{top}(\varphi).$$

Proof. First, the statement in [33, Theorem 1] that $h_{vol} \leq h_{top}(\varphi)$ holds as long as M is compact and (Ω, d_Ω) is complete. The proof of the opposite inequality in Theorem 2 uses sectional nonpositive curvature to prove a technical lemma. We can bypass curvature and prove the lemma immediately in Hilbert geometries. The rest of the proof follows in the same way.

This lemma has already been proven by Crampon in the strictly convex case, but the proof does not depend on strict convexity.

Lemma 3.4.2 ([17], Lemma 8.3)**.** *For any two geodesics* $\sigma, \tau : [0, r] \to M$*,* $r > 0$*,*

$$d_\Omega(\sigma(t), \tau(t)) \leq d_\Omega(\sigma(0), \tau(0)) + d_\Omega(\sigma(r), \tau(r)).$$

□

Chapter 4

Asymptotic geometry and applications

The goal for this chapter is to study the asymptotic geometry of the Benoist 3-manifolds. The end goal is to construct a measure of maximal entropy as a product measure over measures at infinity.

In Section 4.1, we introduce the visual boundary constructed by geodesic rays, which is topologically distinct from the Hilbert boundary $\partial\Omega$ (Proposition 4.1.3) but is well-defined up to a minor restriction (Theorem 4.1.5). On this boundary, the Busemann function is well-defined and continuous for most points and Γ acts minimally by homeomorphisms (Lemmas 4.2.2, 4.4.1, 4.4.2).

4.1 The visual boundary

In an affine chart, Ω is naturally endowed with a Hilbert boundary, which we have denoted $\partial\Omega$. To study the ergodic geometry of a nonstrictly convex Hilbert geometry, we will need the visual boundary, a classical compactification for rank one manifolds (cf. [5, Chapter 2]). Many existing tools are compatible with the visual boundary when they are not necessarily well-defined or continuous on the Hilbert boundary. The visual boundary and the Hilbert boundary are homeomorphic when Ω is a strictly convex domain with C^1-boundary. There is a proper distinction between these boundaries when Ω is not strictly convex with C^1-boundary, which we explore in greater detail in this section.

In this section, we introduce and define a natural topology on the visual boundary at $x \in \Omega$, denoted $\partial_V^x\Omega$, for any Hilbert geometry. Of significance in this setting is whether the Hilbert geometry of interest is CAT(0). In general, when a metric space (X, d_X) is CAT(0), the topology does not depend on the basepoint: $\partial_V^x X \cong \partial_V^y X$ for all $x, y \in X$. When X is not CAT(0) this property no longer holds. The moral of this section is that, nonetheless, the visual boundary for the Benoist 3-manifolds is almost unique and "sufficiently nice" almost everywhere. The precise statement is found in Theorem 4.1.5 and follows from a series of technical lemmas (Lemma 4.1.1 and 4.1.2).

4.1.1 Basic set-up

The *visual boundary* of a geodesic metric space X is the collection of geodesic rays based at a point x modulo bounded equivalence. More explicitly,

$$\partial_V^x X := \{\xi : [0,\infty) \to X \mid \xi(0) = x,\ \xi \text{ geodesic}\}/\sim,$$

where $\xi \sim \eta$ if and only if there exists a $C > 0$ such that $d_X(\xi(t), \eta(t)) \leq C$ for all $t \geq 0$. We will also say that two geodesics $\xi, \eta : (-\infty, \infty) \to X$ are *positively (negatively) asymptotic* if there exists a constant $c > 0$ such that $d_\Omega(\xi(t), \eta(t)) < c$ for all $t \geq 0$ ($t \leq 0$), and denote this by $\xi \sim_+ \eta$ ($\xi \sim_- \eta$) when ξ, η are biinfinite. Note that asymptoticness and bounded equivalence are equivalence relations. Reflexivity and symmetry are immediate, and transitivity follows from the triangle inequality. We denote the compactification by $\overline{X} = X \cup \partial_V^x X$. We will also relate the collection of pairs of points in the visual boundary by

$$\partial_{V,x}^2 \Omega := \{(\xi, \eta) \in \partial_V^x \Omega \times \partial_V^x \Omega \mid \exists \text{ projective line } c \subset \Omega,\ c \sim_+ \xi \text{ and } c \sim_- \eta\}.$$

Note that $\partial_{V,x}^2 \Omega$ is well-defined for representations of ξ, η by transitivity of $\sim$.

Let $B_\Omega(x, r)$ be the open metric d_Ω-ball about x of radius r. For Ω a Hilbert geometry we define the Ω-*shadow* of radius r from x to y in Ω as

$$\mathscr{F}_r(x, y) := \{z \in \Omega \setminus B_\Omega(x, r) \mid \text{ there exists a geodesic } x \text{ to } z \text{ such that } \xi \cap B_\Omega(y, r) \neq \varnothing.$$

The *shadow* of radius r from x to y at infinity is defined to be

$$\widetilde{\mathscr{O}}_r(x, y) := \{\xi \in \partial_V^x \Omega \mid \xi \cap B_\Omega(y, r) \neq \varnothing\}.$$

Note that $\mathscr{F}_r(x, y) \subset \Omega$ and $\widetilde{\mathscr{O}}_r(x, y) \subset \partial_V^x \Omega$. Shadows in Ω and at infinity are pictured in Figure 4.1.1.

We define a topology on the space of geodesic rays based at x generated by $\widetilde{\mathscr{O}}_r(x, y)$ over $r > 0, y \in \Omega$. Note that fixing r, for all $y \in \Omega$ we can find a $y_r \in \Omega$ such that $\widetilde{\mathscr{O}}_r(x, y_r) \subset \widetilde{\mathscr{O}}_r(x, y)$, so we can generate this topology with $r > 0$ fixed. A priori, $\partial_V^x \Omega$ could be a proper quotient of the space of geodesic rays at x,

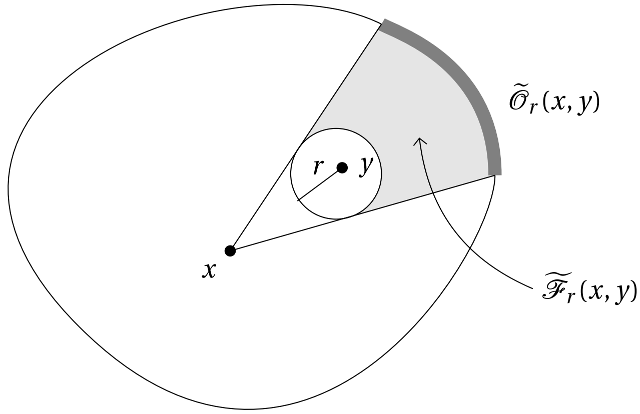


Figure 4.1.1: Images of shadows in Ω, $\widetilde{\mathscr{F}}_r(x,y)$, and shadows at infinity, $\widetilde{\mathscr{O}}_r(x,y)$, for a strictly convex Ω in which geodesics are uniquely projective lines.

hence it is natural to generate the topology on $\partial_V^x\Omega$ by quotient shadows:

$$\mathscr{O}_r(x,y) = \widetilde{\mathscr{O}}_r(x,y)/\text{bounded equivalence} = \{\xi \in \partial_V^x\Omega \mid \xi \cap B_\Omega(y,r) \neq \varnothing\}/\xi \sim \eta \text{ if } d_\Omega(\xi(t),\eta(t)) \leq c.$$

for some $c > 0$ and all $t \geq 0$.

There is an interesting dichotomy for the visual boundary of Hilbert geometries which parallels that of Benoist's. Since any strictly convex, divisible Hilbert geometry is δ-hyperbolic [8], $\partial_V^x\Omega$ is independent of x. This fact can be made even more explicit for Hilbert geometries since the geometry is very concrete, and such a characterization is useful outside the strictly convex context.

For a strictly convex Hilbert geometry, projective lines are the unique geodesics for (Ω, d_Ω) (Proposition 1.1.6). Then for any point x there is a one-to-one correspondence between projective rays at x and their intersections with the Hilbert boundary $\partial\Omega$. Thus, when Ω is strictly convex, $\partial_V^x\Omega = \partial\Omega$ for all x, and one can check that the topology on the visual boundary agrees with the Euclidean topology on $\partial\Omega$ as a subspace of an affine space. Evidently, this result holds regardless of choice of x for the strictly convex case. There is no such natural correspondence for Ω a nonstrictly convex Hilbert geometry.

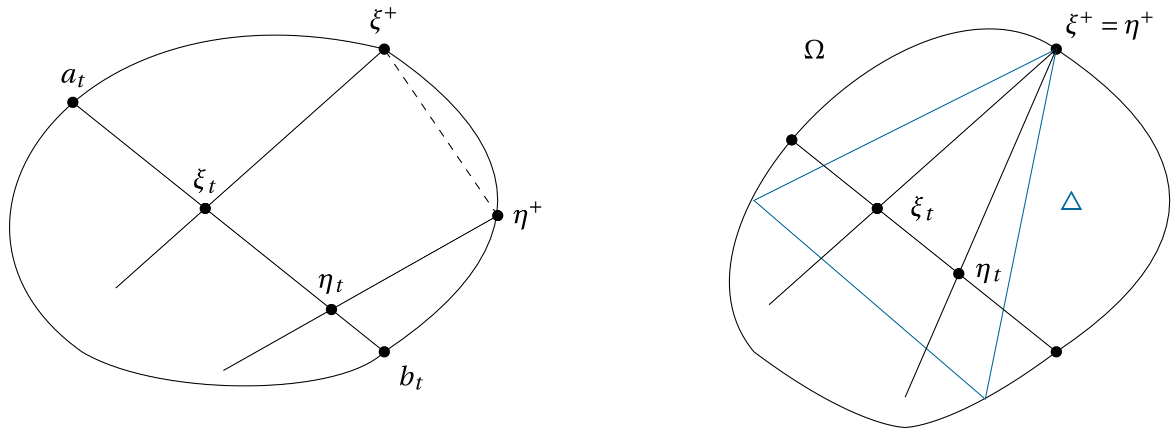


Figure 4.1.2: Images illustrating the construction of relative triangles from the proof of Lemma 4.1.1.

4.1.2 Technical lemmas

To prove the main theorem of this subsection we will need some technical lemmas for Hilbert geodesics of a Benoist 3-manifold. Define H to be a *supporting hyperplane* to a properly convex $\Omega \subset \mathbb{RP}^n$ if H is a codimension 1 projective subspace of $\mathbb{RP}^n$ which intersects $\partial\Omega$ but not Ω.

Lemma 4.1.1. *If ξ, η are projective lines, hence geodesics of (Ω, d_Ω), then ξ and η are positively asymptotic if and only if either $\xi^+ = \eta^+$ or there exists a supporting hyperplane H to Ω such that ξ^+, η^+ are in the interior of $H \cap \partial\Omega$.*

Proof. First, if ξ^+, η^+ are not contained in the same supporting hyperplane then $\xi^+ \neq \eta^+$ and there is no open line segment in $\partial\Omega$ containing both ξ^+ and η^+. Then there are two subcases: first, suppose the open projective line segment $(\xi^+\eta^+)$ is contained completely in Ω. Letting ξ_t, η_t parameterize ξ and η for $t \to +\infty$ and a_t, b_t parameterize the intersection of $\overline{\xi_t\eta_t}$ with $\partial\Omega$ as in Figure 4.1.2, we have

$$|\xi_t b_t|, |\eta_t a_t| \xrightarrow{t\to+\infty} |\xi^+\eta^+| < \infty,$$
$$|\eta_t b_t|, |\xi_t a_t| \xrightarrow{t\to+\infty} 0.$$

Then ξ and η cannot remain a bounded d_Ω-distance apart for all $t \geq 0$:

$$d_\Omega(\xi_t, \eta_t) = \frac{1}{2}\log\frac{|\xi_t b_t|}{|\eta_t b_t|}\frac{|\eta_t a_t|}{|\xi_t a_t|} \xrightarrow{t\to+\infty} \infty.$$

Suppose ξ^+ is an endpoint of an edge $\sigma \subset \partial\Omega$ such that $\eta^+ \neq \xi^+$, and $\eta^+ \in \overline{\sigma}$. Refer to Figure 4.1.2 for the construction of unit-speed parameterizations for ξ, η. Then $d_\Omega(\xi_t, \eta_t) = \dfrac{1}{2}\log\dfrac{|a_t\eta_t||b_t\xi_t|}{|a_t\xi_t||b_t\eta_t|} \sim \dfrac{1}{2}\log\dfrac{|a\eta^+||b\xi^+|}{|a_t\xi_t||by|} \to \infty$ since $|a_t\xi_t| \to 0$ as $t \to \infty$.

Conversely, consider $\xi^+, \eta^+ \in (H \cap \partial\Omega)^o$ for some supporting hyperplane H. If $\xi^+ = \eta^+$, then we can construct a triangle $\triangle \subset \Omega$ such that $\xi[0,\infty), \eta[0,\infty) \subset \triangle$ and $\xi^+ = \eta^+$ is a vertex of $\triangle$ (see Figure 4.1.2). Since the cross ratio of four lines is well-defined, $d_\triangle(\xi_t, \eta_t) = c$ some constant for all $t \geq 0$. By a property of the cross ratio, we then have

$$d_\Omega(\xi_t, \eta_t) \leq d_\triangle(\xi_t, \eta_t) = c$$

for all $t \geq 0$: notably, by the cross-ratio of four lines, this will not depend on parameterization of t. Thus, ξ and η are positively asymptotic.

If $\xi^+ \neq \eta^+$ then by convexity of Ω, $C = H \cap \partial\Omega$ is convex. Take a cross-section of Ω with a 2-dimensional projective plane P such that $\overline{\xi^+\eta^+} \subset P$ and $\xi \subset P$. Note that η may not be contained in P, but η is asymptotic to $\eta' \subset P$ where $(\eta')^+ = \eta^+$ and $\eta^- = \xi^-$. Let a, b be the intersection points of $\overline{\xi^+\eta^+}$ with ∂C. We can again construct a triangle $\triangle \subset P \cap \Omega$ such that $\xi_t, \eta'_t \subset \triangle$ for $t \geq 0$ and a, b, ξ^- are the vertices of $\triangle$ (see Figure 4.1.3). Then

$$d_\Omega(\xi_t, \eta'_t) \leq d_\triangle(\xi_t, \eta'_t) = c = \frac{1}{2}\log[a;\xi^+;\eta^+;b]$$

for all $t \geq 0$ by the cross-ratio of four lines and properties of the cross-ratio. Hence ξ and η' are positively asymptotic, and by transitivity of the asymptotic property ξ is positively asymptotic to η. □

Lemma 4.1.2. *Let Ω be a Benoist 3-manifold. Then every point in $\partial_V^x\Omega$ can be represented as a projective ray at x if and only if $x \in \Omega_{hyp}$. Moreover, if $\xi^+ \in \partial\Omega$ is proper and extremal then the projective ray representation is unique.*

Proof. There are two main arguments in the proof.

Part 1. First, if $x \in \triangle$ for some properly embedded $\triangle$ then there are infinitely many geodesic conics in $\triangle$ which are not asymptotic to any projective ray. We like show these are the only exceptions. Assume that $\eta\colon [0,\infty) \to \Omega$ is a geodesic ray but not a projective ray based at x, and that there exists an increasing sequence $t_n \to \infty$ as $n \to \infty$ such that, letting $\eta_n = \eta(t_n)$, each triple $\eta_{n-1}, \eta_n, \eta_{n+1}$ is noncollinear. Then these triples determine a projective hyperplane denoted H_n for all n.

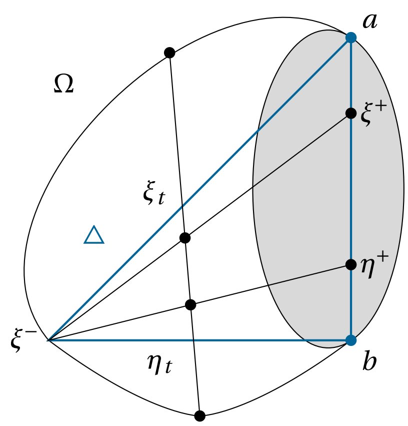


Figure 4.1.3: An image illustrating the construction of relative triangles from the proof of Lemma 4.1.1.

Let σ_n^-, σ_n^+ be the intersection points of $\overline{\eta_n\eta_{n+1}}$ with $\partial\Omega$. Since η is geodesic, the path $[\eta_{n-1}\eta_n] \cup [\eta_n, \eta_{n+1}] \neq [\eta_{n-1}\eta_{n+1}]$ is also a geodesic path, and by Proposition 1.1.6, the line segments $[\sigma_n^-\sigma_{n-1}^-]$ and $[\sigma_{n-1}^+\sigma_n^+]$ are contained in $H_n \cap \partial\Omega$. Then by Benoist's Rigidity Theorem 1.3.4(f) there are properly embedded triangles $\triangle_n^-, \triangle_n^+$ such that $[\sigma_{n-1}^-\sigma_n^-] \subset \partial\triangle_n^-$ and $[\sigma_{n-1}^+\sigma_n^+] \subset \triangle_n^+$. Then $\sigma_n^- \in \overline{\triangle_n^-} \cap \overline{\triangle_{n+1}^-}$ and $\sigma_n^+ \in \overline{\triangle_n^+} \cap \overline{\triangle_{n+1}^+}$ implies $\triangle_{n+1}^\alpha = \triangle_n^\alpha = \triangle^\alpha$ for all n and each $\alpha \in \{-,+\}$ (see Figure 4.1.4).

Let s^α be the side of $\triangle^\alpha$ such that each $[\sigma_n^\alpha\sigma_{n+1}\alpha] \subset s^\alpha$, and let $H = H_n$ be the hyperplane such that $s^-, s^+ \subset H \cap \partial\Omega$. We can argue that $\eta \subset H \cap \Omega$ by considering any infinitesimal noncollinear triples of points $\eta_{t+\epsilon}, \eta_t, \eta_{t-\epsilon}$ for $\epsilon > 0$, and proving these triples must sit in the same hyperplane as all the points $\eta_n \in H \cap \Omega$ by an identical argument. Note that at this point, we have proven that if ξ is a projective ray and ξ^+ is proper extremal, then ξ is the unique geodesic ray in its class.

Part 2. By Part 1. we can assume that if η is geodesic and not asymptotic to any projective line, then η is completely contained in a hyperplane slice Ω' of Ω which has two open line segments in the boundary. The goal is to show that $\eta \subset \triangle$ for some properly embedded $\triangle \subset \Omega$. Denote the embedded open line segments $\sigma, \sigma' \subset \partial\Omega'$ which we can assume to be maximal with endpoints a, b and a', b' respectively. If either $a = a'$ or $b = b'$ then σ, σ' must be contained in the same properly embedded triangle and we are done, so we assume they are distinct.

The rest of the proof breaks into two steps. In the first, we show without loss of generality that if η is not asymptotic to any projective line, then $\eta_t \to a \in \partial\Omega'$. By Part 1. and that η is nonbacktracking and

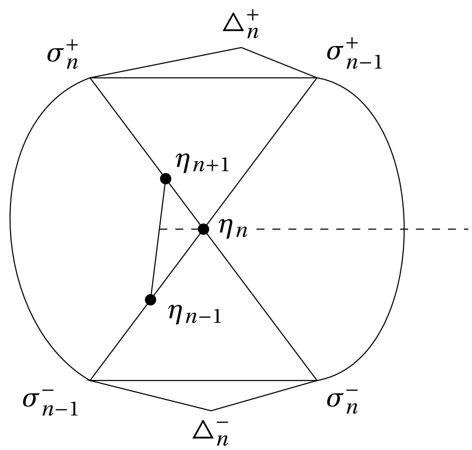


Figure 4.1.4: For the proof of Lemma 4.1.2

$\eta_t \to \partial\Omega$ as a geodesic ray, we conclude without loss of generality that η_t accumulates on $[ab] \subset \partial\Omega'$. Note that η_t cannot accumulate on both a and b as a geodesic. Suppose there is a point $c \in (ab)$ which is a limit point for η_{t_n}, where $t_n \to \infty$ as $n \to \infty$. Consider an open line segment $(c_1 c_2) \subset (ab)$ containing c. Let U denote all points in Ω' between the projective lines $(a'c_1)$ and $(b'c_2)$ (see Figure 4.1.5). For all large n, $\eta_{t_n} \in U$. But then η_{t_n} remains bounded distance from a projective ray to c. So there must exists a $t \neq t_n$ such that $\eta_t \not\in U$: then repeat the Part 1. argument to conclude σ' is not maximal, a contradiction. Therefore, η_t cannot accumulate on any point other than the extreme points a and b of σ.

In the next step, we show if $\eta_t \to a$ is geodesic but asymptotic to no projective line and $b \neq b'$, then σ' cannot be maximal. Then we must have $b = b'$ and $\Omega' = \triangle$. Let $H_\sigma, H_{\sigma'}$ denote the supporting hyperplanes to Ω' at σ, σ'. Let $\{q\} = H_\sigma \cap H_{\sigma'}$ and note that $q \neq b'$. By a cross ratio of four lines argument, there must exist a point $p \in (b'q)$ such that $(ap) \cap \eta \neq \varnothing$ (see Figure 4.1.6). Let t_0 be such that $\eta_{t_0} \in (ap) \cap \eta$. Then for all $p' \in (b'p)$, there exists a $t_{p'}$ such that $\eta_{t_{p'}} \in (p'\eta_{t_0})$, implying by Part 1. that either $p' \in \sigma'$ (contradiction) or $(\eta_{t_0}\eta_{t_{p'}}) = \eta_{(t_0 t_{p'})}$ (see Figure 4.1.7). Repeating for all $p' \to p$, η must be a piecewise projective ray, but as soon as η is a nontrivial piecewise projective ray we have $p' \in \sigma'$, the last contradiction. □

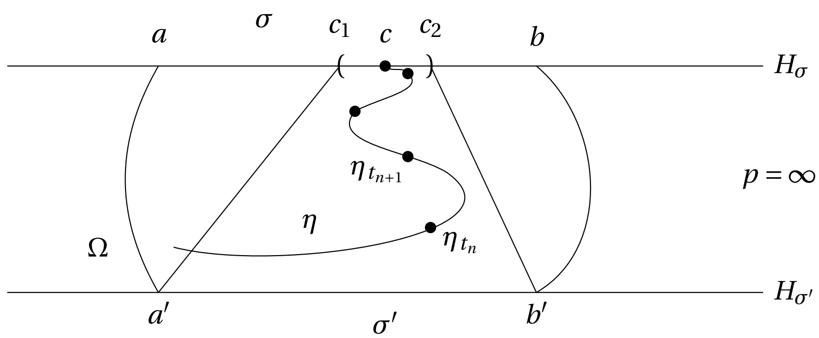


Figure 4.1.5: For the proof of Lemma 4.1.2

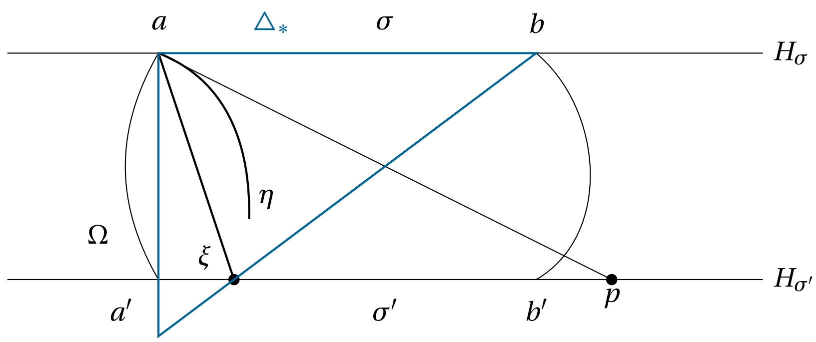


Figure 4.1.6: For the proof of Lemma 4.1.2: note that $d_\Omega(\xi_t, \eta_t) \le d_{\triangle_*}(\xi_t, \eta_t) = c$ for all t.

4.1.3 A dichotomy for the visual boundary of a Hilbert geometry

For any Hilbert geometry, the properties of the visual boundary are related to the structure of the Hilbert boundary of Ω.

Proposition 4.1.3. *The following are equivalent for any Hilbert geometry* Ω*:*

1. *The map from* $\partial_V^x\Omega$ *to* $\partial\Omega$ *which sends geodesic rays* ξ *to their intersection with the Hilbert boundary* ξ^+ *is a homeomorphism.*

2. $\partial_V^x\Omega$ *is Hausdorff for all* $x \in \Omega$*.*

3. Ω *is strictly convex.*

Proof. First, 3. $\Longrightarrow$ 1. $\Longrightarrow$ 2. is clear. If Ω is strictly convex then geodesics are unique by Proposition 1.1.6, and therefore the only geodesics are projective lines. Applying Lemmas 4.1.1 and 4.1.2, every $[\xi] \in \partial_V^x\Omega$ must be represented uniquely by a projective ray based at x. So there is a one-to-one correspondence

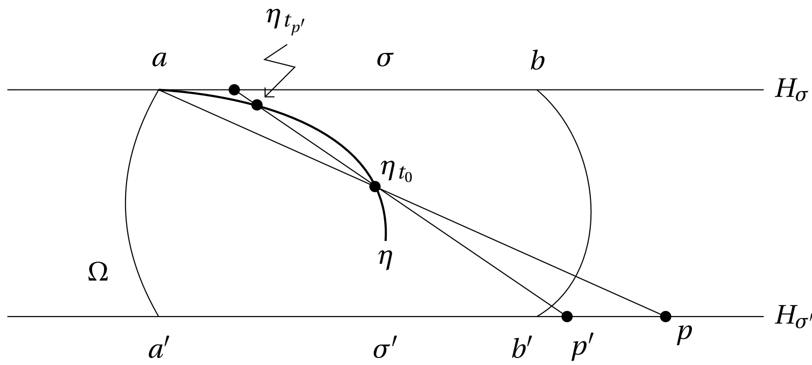


Figure 4.1.7: For the proof of Lemma 4.1.2

between $\partial_V^x\Omega$ and $\partial\Omega$ given by the indicated map, which is clearly a homeomorphism. Moreover, $\partial\Omega$ with the Euclidean topology is Hausdorff as a topological sphere.

It suffices to show that 2. $\Longrightarrow$ 3., which we will do by contraposition. If Ω is not strictly convex, then there is a maximal open line segment $\sigma \subset \partial\Omega$. Let a, b be the endpoints of σ: $\{a, b\} = \overline{\sigma} \setminus \sigma$. Then (xa) is not asymptotic to any geodesic from x to σ. But any open collection of geodesic rays at x which contains (xa) will necessarily contain a representative ξ of the class of rays from x to σ. Thus, the equivalence classes for (xa) and ξ cannot be separated by disjoint open sets in $\partial_V^x\Omega$. □

4.1.4 The visual boundary of Benoist's 3-manifolds

The visual boundary of a Benoist 3-manifold is not Hausdorff by Proposition 4.1.3, but we will see in the following that the visual boundary is topologically "almost regular" and "almost unique".

Proposition 4.1.4. *Let Ω be a Benoist 3-manifold. Then $\partial_V^x\Omega$ is T_1 if and only if $x \in \Omega_{hyp}$.*

Proof. A topology is T_1 if $x \neq y$ can be partially separated by one open set containing x but not y. It is an exercise to verify that a quotient topology is T_1 if and only if every equivalence class is closed in the topology (cf. Folland exercise 28(c)). By Lemma 4.1.2, we only need to check that $\{\xi\}$ such that ξ^+ is in the side of a properly embedded triangle is closed if and only if $x \in \Omega_{\mathrm{hyp}}$.

First assume $x \in \Omega_{\mathrm{hyp}}$. If ξ is extremal then by Lemma 4.1.2, ξ is uniquely represented by $(x\xi^+)$ in $\partial_V^x\Omega$.

Take any η in the complement of $\{\xi\}$. Then η^+ cannot be contained in the same side of a triangle as

ξ^+ for any representations of η, ξ, and hence there is no line segment embedded in $\partial\Omega$ connecting η^+ to ξ^+.

Now, parameterize η at unit speed and pick your favorite time t. Let r be the finite minimum Ω-distance between $\eta(t)$ and all possible $\xi(t)$ for all representations of $\xi \in \partial_V \Omega$. Then $\mathcal{O}_{\frac{r}{2}}(x, \eta(t))$ contains η but not ξ, so $\{\xi\}^c$ is open in $\partial_V^x \Omega$.

Suppose now that $x \in \triangle$ for some properly embedded triangle $\triangle$. Let v_i be the vertices of $\triangle$ for $i = 1, 2, 3$. Then $\{v_1\}$ is not closed, because for any open ball containing (xv_2), there is a conic based at x which passes arbitrarily close to any point on (xv_2) but meets v_1 at infinity. Thus there is no open set containing (xv_2) which is contained in $\{v_1\}^c$, and $\{v_1\}$ is not closed. □

We now prove the main theorem of the section: although the visual boundary does depend on the basepoint, it is unique up to homeomorphism for $x \in \Omega_{\text{hyp}}$, and by Proposition 4.1.4 is it also quasi-nice. We also have a straightforward characterization for $x \in \partial_V^x \Omega$ which will make the visual boundary feel more concrete.

Define an equivalence class on $\partial\Omega$ by $\xi^+ \sim_\sigma \eta^+$ if and only if $\{\xi^+, \eta^+\} \subset \sigma$ a side of a properly embedded triangle or $\xi^+ = \eta^+$.

Theorem 4.1.5. *Let x be the basepoint for the visual boundary of a Benoist 3-manifold. Then $\partial_V^x \Omega \cong \partial\Omega / \sim_\sigma$ if and only if $x \in \Omega_{hyp}$.*

In particular, $\partial_V^x \Omega \cong \partial_V^y \Omega$ for all $x, y \in \Omega_{hyp}$.

Proof. First we define a mapping

$$\psi \colon \partial\Omega / \sim_\sigma \to \partial_V^x \Omega,$$

$$[p] \mapsto [(xp)].$$

Note that by Lemma 4.1.1, $p \sim_\sigma q \iff p = q$ or $\{p, q\} \subset \sigma \subset \partial\triangle \iff (xp) \sim (xq)$. Thus ψ is both well-defined and one-to-one. Moreover, ψ is onto if and only if x is not in any properly embedded triangle if and only if all geodesic rays based at x are asymptotic to projective rays by Lemma 4.1.2.

It is evident that once ψ is a bijection, then ψ is also a homeomorphism. Moreover, if $x \in \Omega_{\text{hyp}}$ and

$y \in \triangle$ then by Proposition 4.1.4, $\partial_V^x\Omega$ is T_1 and $\partial_V^y\Omega$ is not, so $\partial_V^y\Omega \not\cong \partial_V^x\Omega \cong \partial\Omega/\sim_\sigma$. □

Given Theorem 4.1.5, we define $\partial_V\Omega = \partial_V^x\Omega$ for some point x which is not in any properly embedded triangle. We also drop the brackets in the notation for $[\xi] \in \partial_V\Omega$ and will be explicit when careful consideration of equivalence classes is needed.

4.2 The Busemann function

The first essential tool we discuss which is compatible with the visual boundary by design is the Busemann function. For any three points $x, y, z \in \Omega$, we define the *Busemann cocycle* to be

$$\beta_z(x, y) = d_\Omega(x, z) - d_\Omega(y, z).$$

Evidently, β is antisymmetric and satisfies the property of a cocycle, that is

$$\beta_z(x, y) + \beta_z(y, w) = \beta_z(x, w)$$

for all $x, y, z, w \in \Omega$. Also, since Γ is acting on Ω by isometries, β is quasi-Γ-invariant, meaning $\beta_{\gamma z}(\gamma x, \gamma y) = \beta_z(x, y)$ for all $\gamma \in \Gamma$. When thinking about the Busemann cocycle, note that for fixed $x, z \in \Omega$, the set of all y such that $\beta_z(x, y) = 0$ is the Hilbert unit sphere centered at z passing through x. Geometrically, $\beta_z(x, y)$ describes the distance between the Hilbert spheres centered at z passing through x and y. The level sets $\beta_z(x, \cdot) = c$ are equivalence classes for each c.

We would like to extend the Busemann cocycle to $\partial_V\Omega$. By the triangle inequality one can check that $-d_\Omega(x, y) \le \beta_z(x, y) \le d_\Omega(x, y)$ for all $z \in \Omega$. Choosing a projective line $\xi_t \to \xi$, where ξ is the projective ray at x representing $[\xi] \in \partial_V\Omega$, is a natural path to take to infinity. Following this path, by the triangle inequality we have for $s > 0$

$$\begin{aligned}
\beta_{t+s}(x, y) &= d_\Omega(x, \xi_{t+s}) - d_\Omega(y, \xi_{t+s}) \\
&\ge d_\Omega(x, \xi_{t+s}) - (d_\Omega(y, \xi_t) + s) \\
&= d_\Omega(x, \xi_t) + s - d_\Omega(y, \xi_t) - s = \beta_{\xi_t}(x, y).
\end{aligned}$$

Thus, $\beta_{\xi_t}(x,y)$ is nondecreasing as t increases and is bounded above by $d_\Omega(x,y)$, and so acheives a limit as $t \to +\infty$.

Our choice of path was particular, and even depended on our choice of boundary. If $[\xi] \in \partial_V \Omega$ is represented by ξ such that $\xi^+ \in \triangle$, then the limit does in fact depend on choice of path to $[\xi]$. Since there are countably many triangles of codimension 1 properly embedded in Ω, most points in $\partial\Omega$ are not contained in the boundary of a properly embedded triangle and hence by Benoist [9] are extremal points which admit a unique supporting hyperplane, which we call *proper*. The complement of boundaries of properly embedded triangles in $\partial\Omega$ is the set of *proper extremal* points.

If ξ^+ is a proper extremal point, then much like in the hyperbolic case, $\beta_\xi(x,y)$ is well-defined for any $x,y \in \Omega$ and any path to ξ. This issue motivates defining all $\xi \in \partial_V \Omega$ which are represented by projective rays meeting $\partial\Omega$ at a proper extremal point as proper extremal points of $\partial_V \Omega$ in their own right.

Lemma 4.2.1. *The Busemann function is well-defined on proper extremal points of $\partial_V \Omega$, meaning $\lim_{z\to\xi} \beta_z(x,y)$ exists and is unique over all continuous paths $z \to \xi \in \partial_V \Omega = \partial_V^o \Omega$ where $o \in \Omega_{hyp}$ and ξ is a proper extremal point.*

Proof. We will show that any choice of path $z_n \to \xi$ will give the same limit as choosing z_n on the canonical projective ray representation for ξ (Lemma 4.1.2). By the cocycle property, for any $x,y,z \in \Omega$

$$\beta_z(x,y) = \beta_z(x,o) + \beta_z(o,y) = -\beta_z(o,x) + \beta_z(o,y),$$

so it suffices to prove the lemma for $\beta_\xi(o,y)$. First, we show that if y is on ξ, then $|\beta_\xi(o,y)| = d_\Omega(o,y)$ for all paths to ξ. Let ξ_n be the closest point projection of z_n to ξ. Then for n large, since $z_n \to \xi$, $d_\Omega(o,z_n)$ is ϵ-close to $d_\Omega(o,\xi_n)$ because ξ is based at o, and similarly $d_\Omega(y,z_n)$ is ϵ-close to $d_\Omega(y,\xi_n)$ because y is a point chosen on ξ. Then

$$d_\Omega(o,y) - 2\epsilon = d_\Omega(o,\xi_n) - d_\Omega(y,\xi_n) - 2\epsilon \le d_\Omega(o,z_n) - d_\Omega(y,z_n) \le d_\Omega(o,\xi_n) - d_\Omega(y,\xi_n) + 2\epsilon = d_\Omega(o,y) + 2\epsilon$$

since o, y, ξ_n are all points on ξ, and for large n, y is between o and ξ_n on ξ since $z_n \to \xi \implies d_\Omega(o,\xi_n) \to \infty$. (Note that we could extend the projective ray ξ to a projective line through o and repeat this argument for all points y on the line: then the limit is $\pm d_\Omega(o,y)$).

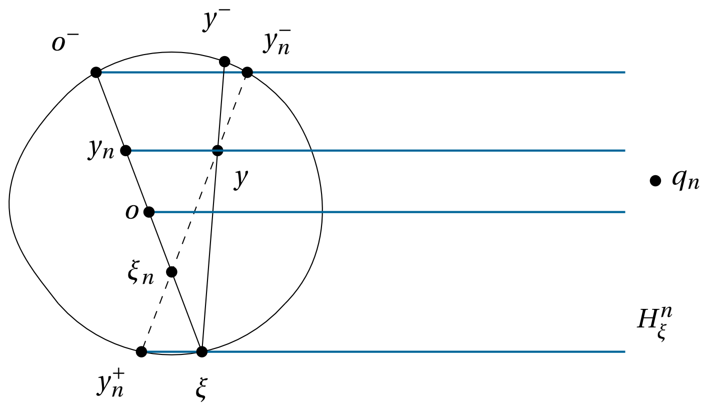


Figure 4.2.1: Proof of the last part of Lemma 4.2.1. By the cross-ratio of four lines, $d_\Omega(y,\xi_n) = d_\Omega(y_n,\xi_n)$ implies $\beta_{\xi_n}(y,o) = d_\Omega(y_n,o)$.

Now suppose $y \not\in \xi$. Since ξ is a proper extremal point, the projective line η through y asymptotic to ξ is the unique geodesic at y asymptotic to ξ, and $z_n \to \xi \implies z_n \to \eta$ because η, ξ are proper extremal points. By a cross ratio of four lines argument, we show that $\beta_\xi(o,y)$ is unique. See Figure 4.2.1. Since η,ξ are proper and extremal, $|\beta_\xi(\bar{o},y)| = |\beta_\eta(\bar{o},y)| = d_\Omega(\bar{o},y) = d_\Omega(o,\bar{y}) = |\beta_\xi(o,\bar{y})|$. To conclude uniqueness, it suffices to show $\beta_\xi(o,y) = \beta_\xi(o,\bar{y})$. This follows from the cross ratio argument and uniqueness of the supporting hyperplane at ξ (see Figure 4.2.1). □

Given the uniqueness of the limit for $\xi \in \partial_V\Omega$, one easily checks that

$$\beta_{\gamma\xi}(\gamma x, \gamma y) = \beta_\xi(x,y).$$

So β is quasi-Γ-invariant on $\partial\Omega \cup \partial_V\Omega$. Also, the cocycle property clearly extends to $\xi \in \partial_V\Omega$:

$$\beta_\xi(x,w) = \beta_\xi(x,y) + \beta_\xi(y,w).$$

Lemma 4.2.2. *The Busemann function restricted to proper extremal points in $\partial_V\Omega$ is continuous.*

Proof. We can make all the arguments in $\mathbb{RP}^2$. Suppose $\xi \in \partial_V\Omega$ is a proper extremal point. Choose x, y in Ω such that $\beta_\xi(x,y) = 0$. Then the setup is as in Figure 4.2.2: the unique hyperplane H_ξ supporting Ω at ξ intersects $\overline{x^- y^-}$ at some point p, and x, y, p must be collinear.

Consider the following for some sequence $\xi_n \to \xi$ with ξ_n proper extremal points in $\partial_V\Omega$: let H_{ξ_n} be

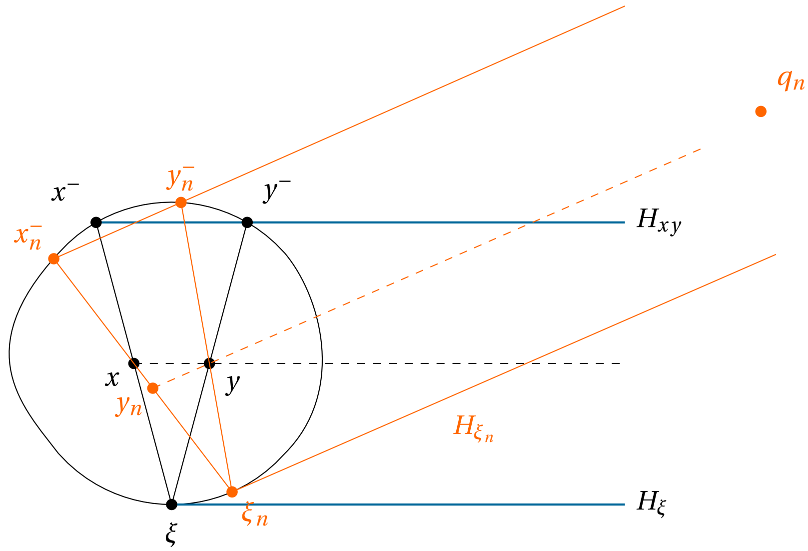


Figure 4.2.2: Choose x, y so that $\beta_\xi(x, y) = 0$. If there exists a sequence ξ_n such that $y_n \to \bar{x} \neq x$ on $(x\xi)$, then ξ admits distinct supporting hyperplanes, a contradiction.

the unique supporting hyperplanes at ξ_n, x_n^- the intersection of $\overline{x\xi_n}$ with $\partial\Omega$ and y_n^- the intersection of $\overline{y\xi_n}$ with $\partial\Omega$. Let $H_{xy}^n = \overline{x_n^- y_n^-}$ and $\{q_n\} = H_{xy}^n \cap H_{\xi_n}$.

Then we construct a sequence y_n by the intersection of $\overline{yq_n}$ with $\overline{x\xi_n}$. Note that this sequence y_n has been particularly constructed so that for each n, $|\beta_{\xi_n}(x, y)| = d_\Omega(x, y_n)$.

We first claim if $\beta_\xi(x, y)$ is not continuous over proper extremal points near ξ, then there exists a sequence of ξ_n such that $y_n \to \bar{x} \neq x$ for some $\bar{x} \in (x\xi)$. To make the argument, it suffices to justify that for any sequence of y_n chosen this way, the y_n must accumulate on a point in $(x\xi)$. Since $x_n^- \to x^-$ and $\xi_n \to \xi$, its clear that $y_n \in (x_n^- \xi_n)$ must accumulate on $[x^- \xi]$. This is impossible because by construction, $\overline{y_n y} \cap H_{xy}^n = \{q_n\}$ so $q_n \to x^-$ as well implying $H_{\xi_n} \supset [\xi_n q_n] \to [\xi x^-]$ which implies $x \in [\xi x^-] \subset \partial\Omega$, a contradiction. Similarly, if $y_n \to \xi$ then $[y^- \xi] \subset \partial\Omega$, another contradiction. Thus, up to extraction of subsequences, $y_n \to \bar{x} \in (x\xi)$, and if $\beta_\xi(x, y)$ is not continuous then the sequence can be constructed so that $0 < \epsilon \leq |\beta_{\xi_n}(x, y)| = d_\Omega(x, y_n)$. Hence up to extraction, $\bar{x} \neq x$.

Lastly, since $\mathbb{RP}^2 \setminus \Omega$ is compact, up to extraction we can assume $q_n \to q \notin \Omega$. Then since $H_{\xi_n} \supset [\xi_n q_n]$ is a supporting hyperplane to Ω for all n and $\xi_n \to \xi$, $[\xi q]$ is contained in a supporting hyperplane $H_{\xi q}$ at ξ. Moreover, $\bar{y}$, y, and q must all be collinear, since y_n, y and q_n were collinear. Assuming $\bar{y} \neq x$, then $q \neq p$ and hence $H_\xi, H_{\xi q}$ are distinct supporting hyperplanes at ξ. $\square$

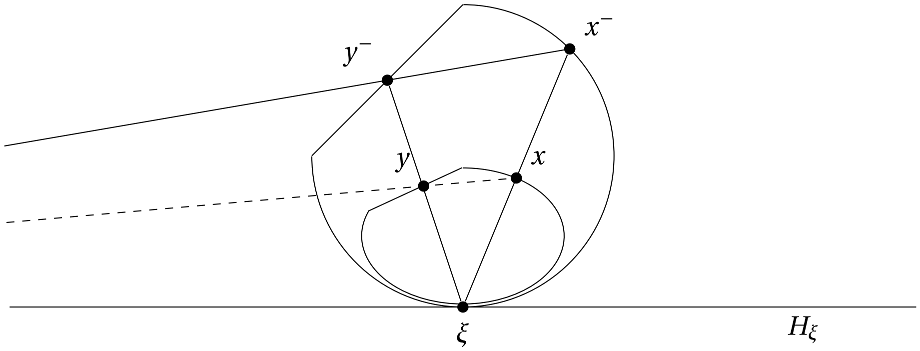


Figure 4.2.3: If ξ is proper and extremal, then $\beta_\xi(x, y)$, hence $\mathscr{H}_\xi(x)$, is defined and continuous near ξ for all points $x, y \in \Omega$.

Question 4.2.3. Can $\beta_\xi(x, y)$ be extended to a continuous function on $\partial_V\Omega$? In Chapter 5 we will need the Busemann function to at least be defined over all of $\partial_V\Omega$, but we can make appropriate choices when needed.

4.2.1 Horospheres

A *horosphere* through $x \in \Omega$ based at $\xi \in \partial_V\Omega$ is the zero set of $\beta_\xi(x, \cdot)$, denoted by $\mathscr{H}_\xi(x)$. We can understand horospheres as spheres centered at infinity. Geometrically, $\beta_\xi(x, y)$ is the distance between $\mathscr{H}_\xi(x)$ and $\mathscr{H}_\xi(y)$ for a proper extremal point ξ.

Lemma 4.2.4. *Horospheres are globally defined and continuous for proper extremal points.*

Proof. Lemmas 4.2.1 and 4.2.2. See Figure 4.2.3. □

4.3 The Gromov product

The *Gromov product* at $x \in \Omega$ between $a, b \in \Omega$ is

$$\langle a, b\rangle_x := \frac{1}{2}[d_\Omega(a, x) + d_\Omega(x, b) - d_\Omega(a, b)].$$

To extend the Gromov product to $a, b \in \partial_V\Omega$, one takes the limit of $\langle a_t, b_t\rangle_x$ as a_t, b_t trace out geodesic rays converging to a, b with $t \to \infty$.

Note that for every $p \in [a, b]$ for $a, b \in \Omega$,

$$\begin{aligned}\beta_a(x,p) + \beta_b(x,p) &= d_\Omega(x,a) - d_\Omega(a,p) + d_\Omega(x,b) - d_\Omega(b,p)\\ &= d_\Omega(x,a) + d_\Omega(x,b) - d_\Omega(a,b)\\ &= 2\langle a,b\rangle_x.\end{aligned}$$

Taking limits $a \to \xi, b \to \eta$ we still have

$$\frac{1}{2}(\beta_\xi(x,p) + \beta_\eta(x,p)) = \langle \xi,\eta\rangle_x$$

for any $p \in (\xi\eta)$, as long as there exists a geodesic from ξ to η, ie; if and only if ξ,η are not in the same flat. Modulo this issue, the Gromov product is well defined at infinity as long as the Busemann function is well-defined for ξ and η. Thus the Gromov product is well defined if ξ, η are proper extremal points in $\partial_V\Omega$. Such ξ, η have well defined horospheres, so we also have geometric intuition for the Gromov product at infinity: $|\langle\xi,\eta\rangle_x|$ is half the length of the portion of $(\xi\eta)$ between the horospheres $\mathscr{H}_\xi(x)$ and $\mathscr{H}_\eta(x)$ (see Figure 4.3.1). So we can also observe that $\langle\xi,\eta\rangle_x = 0$ if and only if $x \in (\xi\eta)$.

Lemma 4.3.1. *The Gromov product is well-defined on proper extremal pairs in* $\partial^2_{V,x}\Omega$.

Using the Gromov product, we can define a family of metrics on the visual boundary, quasi-invariant under isometry: for each $x \in \Omega$,

$$d_x(\xi,\eta) := e^{-\langle\xi,\eta\rangle_x}$$

then $d_x(\xi,\xi) = 0$ and the maximum d_x distance from ξ is 1, attained for the antipodal point of ξ in $\partial^x_V\Omega$. The Gromov product satisfies Γ-quasi-invariance as well as the Busemann functions, and hence so will

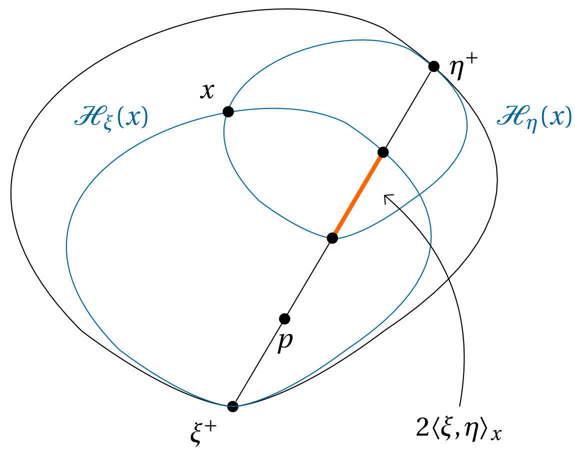


Figure 4.3.1: The Gromov product $\langle \xi, \eta \rangle_x = \frac{1}{2}\left(\beta_\xi(x,p) - \beta_\eta(x,p)\right)$ is half the length of $(\xi^+\eta^+)$ between $\mathcal{H}_{\xi^+}(x)$ and $\mathcal{H}_{\eta^+}(x)$.

the visual metric:

$$\begin{aligned} d_x(\xi,\eta) &= e^{-\langle \xi,\eta \rangle_x} \\ &= e^{-\frac{1}{2}(\beta_\xi(x,p)+\beta_\eta(x,p))} \\ &= e^{-\frac{1}{2}(\beta_{\gamma\xi}(\gamma x,\gamma p)+\beta_{\gamma\eta}(\gamma x,\gamma p))} \\ &= d_{\gamma x}(\gamma\xi,\gamma\eta), \end{aligned}$$

since $\gamma p \in (\gamma\xi\ \gamma\eta) = \gamma(\xi\eta)$.

Lemma 4.3.2. *The visual metric $d_x \colon \partial^2_{V,x}\Omega \to \mathbb{R}^+$ is continuous over proper extremal pairs.*

Proof. This result immediately follows from Lemmas 4.2.2 and 4.3.1. □

4.4 The induced Γ-action on $\partial_V\Omega$

We need to define the Γ-action on $\partial_V\Omega = \partial_V^x\Omega$. Since Γ preserves Ω, acts by projective transformations, and preserves projective lines and incidences of projective lines, Γ takes properly embedded triangles to properly embedded triangles. Thus, we define $\gamma.\xi$ to be the ray $[x\ \gamma\xi^+)$, and this action is well-defined for representations of $\xi \in \partial_V\Omega$.

Lemma 4.4.1. Γ *acts on* $\partial_V \Omega$ *by homeomorphisms.*

Proof. Note that

$$\begin{aligned}\gamma.\mathscr{O}_r(x,y) &= \{\gamma\xi : [0,\infty) \to \Omega \mid \xi(0) = x,\ d_\Omega(\xi,y) < r\}/\sim \\ &= \{\gamma\xi : [0,\infty) \to \Omega \mid \gamma\xi(0) = \gamma x,\ d_\Omega(\gamma\xi,\gamma y) < r\}/\sim \\ &= \{\eta : [0,\infty) \to \Omega \mid \eta(0) = \gamma x,\ d_\Omega(\eta,\gamma y) < r\}/\sim \\ &= \mathscr{O}_r(\gamma x,\gamma y).\end{aligned}$$

Because γ preserves the collection of properly embedded triangles, $x \in \Omega_{\text{hyp}} \implies \gamma x \in \Omega_{\text{hyp}}$. Then since $\mathscr{O}_r(\gamma x,\gamma y)$ is open in $\partial_V^{\gamma x}\Omega$, it is also open in $\partial_V^{x}\Omega$ by Theorem 4.1.5. Then γ^{-1} and hence all of Γ acts by homeomorphisms on $\partial_V\Omega = \partial_V^{x}\Omega$. □

By [9, Proposition 3.10], Γ acts on the Hilbert boundary $\partial\Omega$ minimally. We have already seen that $\partial\Omega$ does not agree with $\partial_V\Omega$ for the nonstrictly convex Benoist 3-manifolds, so the lemma still needs proof.

Lemma 4.4.2. *The* Γ*-action on* $\partial_V\Omega$ *is minimal.*

Proof. First, by Theorem 4.1.5, $\partial_V\Omega \cong \partial\Omega/\sim_\sigma$. It suffices to verify that Γ acts equivariantly with respect to this projection, and that minimality descends to factors. Clearly Γ is equivariant, since each γ preserves equivalence classes of $\sim_\sigma$ as a projective transformation. Suppose there exists a closed nontrivial Γ-invariant $A \subset \partial\Omega/\sim_\sigma$. Then by definition of the projection map and equivariance, the lift $\tilde{A}$ in $\partial\Omega$ is also closed, Γ-invariant, and nontrivial (neither $\tilde{A}$ nor $\tilde{A}^c$ can be empty when lifting a nontrivial A under a projection). Thus, if the Γ-action on the factor is not minimal, neither is the Γ-action upstairs. □

Chapter 5

Patterson–Sullivan Theory

In this chapter we construct a conformal family of measures on $\Omega \cup \partial_V \Omega$ parameterized by $x \in \Omega_{\mathrm{hyp}}$ which are compatible with the group action Γ. We use a modified divergent Poincaré series to push the mass of these measures to $\partial_V \Omega$. The theory is originally due to the independent work of Patterson and Sullivan for geometrically finite uniformly hyperbolic surfaces [34, 38].

In Section 5.1, we study properties of the Poincaré series and the critical exponent δ_Γ, and prove that $\delta_\Gamma \geq h_{top}(\varphi)$ for geodesic flows of the Benoist 3-manifolds (Theorem 5.1.2). We then construct in Section 5.2 a conformal density of dimension δ_Γ (Proposition 5.2.1) which is unique up to a constant (Theorem 5.4.2) and assigns trivial measure to flats (Proposition 5.3.9). Consequences of uniqueness include sharp asymptotics for sphere growth (Theorem 5.5.1) and divergence of the group Γ (Corollary 5.5.2).

5.1 Poincaré Series and the critical exponent

The *critical exponent* δ_Γ of a group Γ acting geometrically[1] on (Ω, d_Ω) is the critical value of $s \in \mathbb{R}$ for the *Poincaré series*

$$P(x, y, s) = \sum_{\gamma \in \Gamma} e^{-s d_\Omega(x, \gamma . y)}$$

The group Γ is of *divergent type* if $P(x, y, \delta_\Gamma)$ diverges and *convergent type* if $P(x, y, \delta_\Gamma)$ converges.

5.1.1 Basic properties

Lemma 5.1.1. *The following are basic properties of the Poincaré series.*

(a) Convergence of $P(x, y, s)$ *does not depend on* x *or* y.

(b) $\delta_\Gamma = \limsup_{t \to \infty} \frac{1}{t} \log N_\Gamma$ *where* $N_\Gamma := \#\{\gamma \in \Gamma \mid d_\Omega(x, \gamma x) \leq t\}$.

[1] discretely, properly discontinuously, and by isometries

(c) $\delta_\Gamma < \infty$ *when* Γ *is cocompact.*

Proof. (a) Suppose for a favorite point o that $P(o,o,s)$ converges. Then $P(x,o,s)$ converges because $d_\Omega(o,\gamma o) \leq d_\Omega(o,\gamma x) + d_\Omega(\gamma x,\gamma o) = d_\Omega(o,\gamma x) + d_\Omega(x,o)$, so

$$P(o,o,s) \geq \sum_\Gamma e^{-sd_\Omega(o,\gamma x)} e^{-sd_\Omega(x,o)} = e^{-sd_\Omega(x,o)} P(o,x,s)$$

Similarly, $d_\Omega(o,\gamma x) \leq d_\Omega(o,y) + d_\Omega(y,\gamma x)$ gives that $P(y,x,s)$ will converge when $P(o,x,s)$ converges.

(b) See for example Lemma 2.1 of [35]

(c) Any compact topological manifold has the homotopy type of a CW-complex [27]. Thus $\Gamma \cong \pi_1(M)$ must be finitely generated. Moreover, Γ is quasi-isometric to its Cayley graph, which embeds isometrically in Ω since all group elements are hyperbolic or flat and Γ has at most exponential growth. So we can conclude $\delta_\Gamma < \infty$.

□

5.1.2 The critical exponent and entropy

Theorem 5.1.2. *For* Γ *acting discretely and cocompactly on a Benoist 3-manifold,* $\delta_\Gamma \geq \rho(\varphi)$*, the exponential growth rate of periodic orbits.*

Proof. Crucial to the proof is that φ^t was carefully chosen to be the projective line flow, since geodesics are not unique in the nonstrictly convex case. Any periodic orbit for φ^t is only periodic if, projecting to a curve $c(t)$ in M, the curve c lifts to a projective line in Ω which is invariant under some hyperbolic $\gamma \in \Gamma$. So there are at least as many $\gamma \in \Gamma$ with translation length $\leq t$ as there are periodic orbits for the geodesic flow of length $\leq t$ and we conclude the result. □

Corollary 5.1.3. *By Proposition 3.3.3,* $\delta_\Gamma \geq \rho(\varphi) > 0$.

Theorem 5.1.4. *For the Benoist 3-manifolds,*

$$\delta_\Gamma = h_{vol} = h_{top}$$

Proof. It suffices to assume Γ is acting discretely and cocompactly on (Ω, d_Ω) by isometries. Let vol be the Busemann-Hausdorff volume, an isometry invariant volume form on Ω. We can take as a definition for the critical exponent of Γ to be

$$\delta_\Gamma := \lim_{r\to\infty} \frac{1}{r} \log \#\{\gamma \in \Gamma \mid d(o, \gamma o) \le r\}.$$

Let D be a fundamental domain for the Γ-action on Ω such that $o \in D$. Since Γ acts discretely and cocompactly, we can count the finitely many images of D under Γ which remain inside any ball of radius r about o. In particular, we note that

$$\begin{aligned}\#\{\gamma \in \Gamma \mid \gamma D \subset B_\Omega(o, r - \operatorname{diam} D)\} &\le \#\{\gamma \in \Gamma \mid d(o, \gamma o) \le r\} \\ &\le \#\{\gamma \in \Gamma \mid \gamma D \subset B_\Omega(o, r + \operatorname{diam} D)\}.\end{aligned}$$

Then at the limit,

$$\delta_\Gamma = \lim_{r\to\infty} \frac{1}{r} \log \#\{\gamma \in \Gamma \mid \gamma D \subset B_\Omega(o, r)\}.$$

Now we connect this characterization of δ_Γ with h_{vol}. For each $r \gg \operatorname{diam} D$,

$$B_\Omega(o, r - \operatorname{diam} D) \subset \cup\{\gamma D \mid \gamma D \subset B_\Omega(o, r)\} \subset B_\Omega(o, r + \operatorname{diam} D)$$

For all r, $\operatorname{diam} D$ is fixed, so

$$\begin{aligned}\operatorname{vol}(B_\Omega(o, r + \operatorname{diam} D)) &= \operatorname{vol}(B_\Omega(o, \operatorname{diam} D)) + \operatorname{vol}(B_\Omega(o, \operatorname{diam} D + r) \setminus B_\Omega(o, \operatorname{diam} D)) \\ &\sim \operatorname{vol}(B_\Omega(o, \operatorname{diam} D)) + \operatorname{vol}(B_\Omega(o, R))\end{aligned}$$

for a reparameterization $R = \operatorname{diam} D + r$. We can say the same for $\operatorname{vol}(B_\Omega(o, r - \operatorname{diam} D))$ and conclude

$$h_{vol} = \lim_{r\to\infty} \frac{1}{r} \log \operatorname{vol}\left(\{\gamma D \mid \gamma D \subset B_\Omega(o, r)\}\right).$$

Since vol is an isometry invariant and D is a fundamental domain for Γ,

$$\operatorname{vol}\left(\cup\{\gamma D \mid \gamma D \subset B_\Omega(o,r)\}\right) = \operatorname{vol} D \times \#\{\gamma \in \Gamma \mid \gamma D \subset B_\Omega(o,r)\}.$$

Therefore

$$\begin{aligned} h_{vol} &= \lim_{r\to\infty} \frac{1}{r} \log\left(\operatorname{vol} D \times \#\{\gamma \in \Gamma \mid \gamma D \subset B_\Omega(o,r)\}\right) \\ &= \lim_{r\to\infty} \frac{\log \operatorname{vol} D}{r} + \frac{1}{r}\log\left(\#\{\gamma \in \Gamma \mid \gamma D \subset B_\Omega(o,r)\}\right) \\ &= \delta_\Gamma. \end{aligned}$$

□

5.2 Conformal densities

A *density at infinity* μ is a collection of finite measures $\{\mu_x\}_{x\in\Omega}$ on $\partial_V\Omega$, and is called a *conformal density of dimension* α if the Radon–Nikodym derivative is α-Hölder. Of particular interest to our study are *Busemann densities*: these are α-conformal densities which satisfy

- (*quasi-*Γ*-invariance*) For all $\gamma \in \Gamma$, $\gamma_*\mu_x = \mu_{\gamma x}$
- (*transformation rule*) For all $x, y \in \Omega$, the Radon–Nikodym derivative is given by

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-\alpha\beta_\xi(x,y)}.$$

We remark here that the transformation rule implies the Busemann measures are mutually absolutely continuous. The only obstruction to constructing such a family of measures is that the Busemann function is not continuous when ξ is in the boundary of a properly embedded triangle. Thus we will tweak our definition of a conformal density to be parameterized by x outside of properly embedded triangles, and restrict the transformation rule to ξ proper extremal points. For the upcoming section, we let Ω_{hyp} and $\partial_V\Omega_{\text{hyp}}$ be the points of Ω outside of properly embedded triangles and the proper extremal points on $\partial_V\Omega$, respectively. We also let $\overline{\Omega} = \partial\Omega \cup \partial_V\Omega$.

5.2.1 Patterson–Sullivan Measures

The Patterson–Sullivan density was originally due to Patterson for Fuchsian groups acting on $\mathbb{H}^2$, was generalized by Sullivan to geometrically finite actions on CAT(-1) spaces, and has been studied by many thereafter, most notably Knieper in his work for rank one manifolds [34, 38, 28].

Proposition 5.2.1. *There exists a nontrivial δ_Γ-conformal density on $\partial_V\Omega$, called a Patterson–Sullivan density.*

Proof. The construction goes as follows: for $s > \delta_\Gamma$, choose an observation point $o \in \Omega_{\text{hyp}}$ for the measures and for the visual boundary. For each $x \in \Omega$ define a measure on $\overline{\Omega} = \Omega \cup \partial_V^o \Omega$ as follows:

$$\mu_{x,s} = \frac{1}{P(o,o,s)} \sum_{\gamma\in\Gamma} e^{-sd_\Omega(x,\gamma o)} \delta_{\gamma o}$$

where δ_p is the Dirac mass at p. Note that for $s > \delta_\Gamma$, $\mu_{x,s}$ is supported inside Ω. Also, by definition of the critical exponent, if $s > \delta_\Gamma$ then $P(x,y,s)$ is finite for all $x, y \in \Omega$ so $\mu_{x,s}(\overline{\Omega}) = P(x,o,s)/P(o,o,s) < \infty$. By compactness of $\overline{\Omega}$ we may take a weak limit as s decreases to δ_Γ to obtain a finite nontrivial measure,

$$\mu_x = \lim_{s\to\delta_\Gamma^+} \mu_{x,s}.$$

If Γ is of divergent type, then the total mass of μ_x^s is pushed to $\Lambda_\Gamma = \partial_V \Omega$ as $s \to \delta_\Gamma$ and $P(o,o,s) \to \infty$[2]. At the limit, supp $\mu_x \subset \partial_V\Omega$. It suffices to check that the μ_x satisfy the definition of a Busemann density.

If the Poincaré series converges at δ_Γ (Γ is of convergent type), then we follow Patterson's method for Fuchsian groups which generalizes to any manifold group [34]. First, he showed it is possible to constuct an increasing function $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ with subexponential growth: that is, for all $\epsilon < 0$, there exists an $x_0(\epsilon) > 0$ such that for all $x > x_0$, $y > 0$

$$f(x+y) \le f(x) e^{y\cdot\epsilon},$$

and the modified Poincaré series

$$P_f(x,y,s) = \sum_{\gamma\in\Gamma} f(d_\Omega(x,\gamma y)) e^{-sd_\Omega(x,\gamma y)}$$

[2]Exercise: if Γ is of divergent type then any compact $K \subset \Omega$ has trivial measure.

diverges at $s = \delta_\Gamma$ (see Lemma 3.1 of [34]). Then we denote by μ_x^f a weak limit as $s \to \delta_\Gamma^+$ of

$$\mu_{x,s}^f = \frac{1}{P_f(o,o,s)} \sum_{\gamma\in\Gamma} f(d_\Omega(x,\gamma o)) e^{-sd_\Omega(x,\gamma o)} \delta_{\gamma o}$$

and $\{\mu_x^f\}_{x\in\Omega}$ will satisfy the definition of a Busemann density. Taking $f \equiv 1$ recovers μ_x, so we will check that these measures satisfy the definition of a Busemann density for the case that Γ is convergent.

We remark first that $P_f(o,o,s)$ exhibits the same convergence and divergence behavior as $P(o,o,s)$ for $s \neq \delta_\Gamma$ so $\mu_{x,s}^f$ will be a finite nontrivial measure supported on point masses in Ω much like $\mu_{x,s}$. Taking a weak-limit then produces a finite nontrivial measure μ_x^f supported on $\partial_V\Omega$ by the divergence of $P_f(o,o,s)$. Moreover,

1. (quasi-Γ-invariance) For any Borel measurable set $A \subset \overline{\Omega}$,

$$\begin{aligned}\mu_{x,s}^f(\gamma^{-1}A) &= \frac{1}{P_f(o,o,s)} \sum_{g\in\Gamma} f(d_\Omega(x,g.o)) e^{-sd_\Omega(x,g.o)} \delta_{g.o}(\gamma^{-1}A)\\ &= \frac{1}{P_f(o,o,s)} \sum_{\gamma g\in\Gamma} f(d_\Omega(\gamma x,\gamma g o)) e^{-sd_\Omega(\gamma x,\gamma g.o)} \delta_{\gamma g.o}(A) = \mu_{\gamma x,s}^f(A).\end{aligned}$$

   Then the quasi-Γ-invariance property clearly extends to the weak limit μ_x^f.

2. (transformation rule)

   Since $\mu_{x,s}^f$ is supported on countably many point masses in Ω for $s > \delta_\Gamma$, we compute

$$\frac{d\mu_{x,s}^f}{d\mu_{y,s}^f}(\gamma o) = \frac{\mu_{x,s}^f(\gamma o)}{\mu_{y,s}^f(\gamma o)} = \frac{f(d_\Omega(x,\gamma o)) e^{-sd_\Omega(x,\gamma o)}}{f(d_\Omega(y,\gamma o)) e^{-sd_\Omega(y,\gamma o)}} = \frac{f(d_\Omega(x,\gamma o))}{f(d_\Omega(y,\gamma o))} e^{-s\beta_{\gamma o}(x,y)}.$$

   As $s \to \delta_\Gamma$ and $\operatorname{supp}\mu_{x,s}^f, \operatorname{supp}\mu_{y,s}^f$ is pushed to $\partial_V\Omega$, we hope that $\frac{f(d_\Omega(x,\gamma o))}{f(d_\Omega(y,\gamma o))} \to 1$ with $\gamma o \to \partial_V\Omega$. By the increasing and subexponential properties of f, for all $\epsilon > 0$ we have that for all γo such that $d_\Omega(\gamma o, y)$ is sufficiently large,

$$f(d_\Omega(x,\gamma o)) \le f(d_\Omega(y,\gamma o) + d_\Omega(x,y)) \le f(d_\Omega(y,\gamma o))\, e^{d_\Omega(x,y)\cdot\epsilon}.$$

   Since this property holds for all $\epsilon > 0$, if $\gamma_n o \to \xi$ a proper extremal point the the transformation rule will hold for the weak limits μ_x^f, μ_y^f at ξ.

Remark 5.2.2. For nonproper or nonextremal points, we can define the Busemann function for $\xi \in \partial_V \triangle$ using a weak limit of the Radon–Nikodym derivatives which exists since the μ_x are locally uniformly finite:

$$\beta_\xi(x, y) = \frac{1}{\delta_\Gamma} \log\left(\lim_{n\to\infty^*} e^{-\delta_\Gamma \beta_{\gamma_n o}(x,y)} \right)$$

for $\xi \in \partial_V \triangle, \gamma_n o \to \xi$ as $n \to \infty$.

□

5.3 The Shadow Lemma and applications

In this section we prove a classical lemma attributed to Sullivan which yields estimates for the Patterson–Sullivan measure of shadows. We begin with a series of logically independent lemmas which are essential for the proof of the Shadow Lemma. Then we prove a few immediate consequences of the Shadow Lemma: the boundaries of triangles are null sets, the Patterson–Sullivan measures have no atoms, and, using cocompactness of the Γ-action, we prove strong local estimates for the measure of shadows.

5.3.1 Some geometric lemmas

Let $\mathscr{C} A$ be the convex closure of a subset A of projective space. Note that by convexity of any properly convex Ω, if $A \subset \overline{\Omega}$ then $\mathscr{C} A \subset \overline{\Omega}$.

Lemma 5.3.1. *If h is a hyperbolic projective transformation preserving a properly convex domain Ω, then for any open sets $O^+ \subset \partial\Omega$ containing h^+ and $\mathscr{O}^-$ containing h^-, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$h^n(\overline{\Omega} \setminus \mathscr{C} O^-) \subset O^+ \text{ and } h^{-n}(\overline{\Omega} \setminus \mathscr{C} O^+) \subset O^-.$$

Proof. If h is hyperbolic then the only fixed points of h in $\partial\Omega$ are h^+ and h^-: any neutral eigenvectors must be outside of Ω, else the projective triangle determined by the eigenspace generated by three eigenvectors would be a properly embedded triangle in Ω which is preserved by h. Then h will contract the closed topological ball $\overline{\Omega} \setminus \mathscr{C}O^-$ containing h^+ but not any other eigenvectors of h as a projective transformation, and there will be an $N^+ \in \mathbb{N}$ such that for $n \geq N^+$, $h^n(\overline{\Omega} \setminus \mathscr{C}O^-) \subset O^+$, an open neighborhood of h^+. Similarly we can find an N^- for h^- and the other inclusion, and choose $N = \max\{N_1, N_2\}$ to complete the proof of the lemma. □

Lemma 5.3.2. *Suppose h, g are hyperbolic projective transformations preserving Ω such that $g^+ \neq h^+$. Then there exist neighborhoods V_g, V_h of g^+, h^+ such that $\mathscr{C}\overline{V_g} \cap \mathscr{C}\overline{V_h} = \varnothing$ and there is no properly embedded triangle which intersects both $\mathscr{C}\overline{V_g}$ and $\mathscr{C}\overline{V_h}$.*

Proof. First, since g, h are hyperbolic, g^+, h^+ are proper extremal points. By Theorem 4.1.5, there are disjoint open neighborhoods V_g, V_h around g^+, h^+ respectively whose closures are also disjoint. If the lemma was false, by convexity of $\mathscr{C}g^nV_g, \mathscr{C}h^nV_h$, there would exist a sequence of properly embedded triangles $\triangle_n$ such that $\overline{g^nV_g} \cap \partial_V\triangle \neq \varnothing$ and $\overline{h^nV_h} \cap \partial_V\triangle \neq \varnothing$ for all n. Since the collection of properly embedded triangles is closed in Ω [9, Proposition 3.2], the $\triangle_n$ accumulate on some $\triangle$ properly embedded in Ω. Because g, h are hyperbolic, $\cap_{n=1}^{\infty}\overline{g^nV_g} = \{g^+\}$ and $\cap_{n=1}^{\infty}\overline{h^nV_h} = \{h^+\}$. Then $(g^+h^+) \subset \overline{\triangle}$ which contradicts the proper extremal property for fixed points of hyperbolic isometries. □

Proposition 5.3.3. *Let $x \in \Omega_{hyp}$ be the basepoint for $\partial_V\Omega = \partial_V^x\Omega$. For any two noncommuting hyperbolic isometries g, h preserving Ω and O a sufficiently small neighborhood of h^+, there exists an $R \gg 0$ and $M \in \mathbb{N}$ such that for all $r \geq R$ and all $y \in \Omega$, either $h^MO \subset \mathscr{O}_r(y, x)$ or $g^Mh^MO \subset \mathscr{O}_r(y, x)$.*

Proof. Applying Lemma 5.3.2, there are pairwise disjoint neighborhoods $V_h^{\pm}, V_g^{\pm}$ of $h^{\pm}, g^{\pm}$ respectively such that no properly embedded triangle intersects any pair of convex hulls of these neighborhoods in $\overline{\Omega}$. In particular, this means for $V_i, V_j \in \{V_h^{\pm}, V_g^{\pm}\}$ with $V_i \neq V_j$, for any $x \in \mathscr{C}V_i$ and $y \in \mathscr{C}V_j$, the projective line (xy) is contained in Ω and is not contained in any single properly embedded triangle.

By Fact 5.3.1, there exists an N_1 such that $h^{-n}(\overline{\Omega} \setminus \mathscr{C}V_h^+) \subset V_h^-$ for all $n \geq N_1$. Moreover, there exists an N_2 such that $g^{-n}(\overline{\Omega} \setminus \mathscr{C}V_g^+) \subset V_g^-$, implying

$$g^{-n}(\mathscr{C}V_h^+) \subset g^{-n}(\overline{\Omega} \setminus \mathscr{C}V_g^+) \subset V_g^- \subset \overline{\Omega} \setminus \mathscr{C}V_h^+.$$

Then for all $y \in \Omega$ and all $n \geq \max\{N_1, N_2\}$, either $h^{-n}y \in \mathscr{C}V_h^-$ or $h^{-n}g^{-n}y \in \mathscr{C}V_h^-$. Let $M = \max\{N_1, N_2\}$.

Next, we claim that for all $r>0$, for $\gamma \in \{h^{-M}, h^{-M}g^{-M}\}$ and $R = d_\Omega(x,\gamma x)$, if $\gamma y \in \mathscr{C}V_h^-$ then

$$\gamma^{-1}V_h^+ \subset \gamma^{-1}\mathscr{O}_r(\gamma y, x) \subset \mathscr{O}_{r+R}(y,x).$$

which completes the proof of the lemma. Note first that the rightmost inclusion is true for all $\gamma \in \Gamma$: if $p \in B_\Omega(x,r)$, then $d_\Omega(\gamma^{-1}p, x) \le d_\Omega(\gamma^{-1}p, \gamma^{-1}x) + d_\Omega(x, \gamma^{-1}x) \le r + R$. So if a geodesic ray ξ with $\xi(0) = \gamma y$ intersects $B_\Omega(x,r)$, then $\gamma^{-1}\xi$ is a geodesic ray with $\gamma^{-1}\xi(0) = y$ which intersects $B_\Omega(x, r+R)$.

For the leftmost inclusion, we show that $V_h^+ \subset \mathscr{O}_r(\gamma y, x)$ for sufficiently large r. First, for any $\eta \in \overline{V_h^+}$ the projective ray $(\gamma y\ \eta)$ is contained in Ω but not any properly embedded triangle by choice of V_h^-, V_h^+ (Fact 5.3.1). Then take $r \ge \max_{\eta \in \overline{V_h^+}} d_\Omega(x, (\gamma y\ \eta))$ and the leftmost containment is satisfied. □

Lemma 5.3.4. *For all $\xi \in \mathscr{O}_r(x,y)$,*

$$d_\Omega(x,y) - 2r \le \beta_\xi(x,y) \le d_\Omega(x,y).$$

Proof. For the moment, we replace $\xi \in \mathscr{O}_r(x,y)$ with $z \in \mathscr{F}_r(x,y)$. Letting $z \to \xi$ along the projective line from x to ξ, we have the result The right inequality is immediate from the triangle inequality. To see the left inequality, divide the projective line from x to z into two segments by its first intersection with the closed $B_\Omega(y,r)$, which we denote p. Note the existence of such a p is given by the assumption that $z \in \mathscr{F}_r(x,y)$. Then by the triangle inequality, $d_\Omega(x,y) \le d_\Omega(x,p) + r$ and $d_\Omega(y,z) \le d_\Omega(p,z) + r$, so

$$d_\Omega(x,y) - 2r \le d_\Omega(x,p) + d_\Omega(p,z) - d_\Omega(y,z) = d_\Omega(x,z) - d_\Omega(y,z) = \beta_z(x,y).$$

If $\xi \in \partial_V \triangle$ and we define $\beta_\xi(x,y)$ as in Remark 5.2.2, then the proof still works because if $\gamma_n x \to \xi$ and $\xi \in \mathscr{O}_r(x,y)$ then there exists an N such that $n \ge N \implies (x\gamma_n x) \cap B_\Omega(y,r) \ne \varnothing$. □

The Shadow Lemma

Lemma 5.3.5. *Let μ be a nontrivial Busemann density on $\partial_V\Omega$. Then* $\operatorname{supp}\mu_x = \partial_V\Omega$.

Proof. Suppose there exists a $\xi \in O \subset \partial_V \Omega$, with O open, such that $\mu_x(O) = 0$. By quasi-Γ-equivariance of μ_x and absolute continuity of the density, for all $\gamma \in \Gamma$,

$$\mu_x(O) = \mu_{\gamma x}(\gamma O) = 0 \Longrightarrow \mu_x(\gamma O) = 0.$$

Since O is open, $\Gamma.O$ covers $\partial_V \Omega$ because Γ acts on $\partial_V \Omega$ minimally (Lemma 4.4.2). Therefore,

$$\mu_x(\partial_V \Omega) \leq \sum_{\gamma \in \Gamma} \mu_x(\gamma O) = 0$$

which proves the lemma by contrapositive. □

Lemma 5.3.6 (Shadow Lemma)**.** *Let μ be a Busemann density of dimension δ on $\partial_V \Omega$. Then for every $x \in \Omega$ and all suffiently large r, there exists a $C > 0$ such that for all $\gamma \in \Gamma$,*

$$\frac{1}{C} e^{-\delta d_\Omega(x,\gamma x)} \leq \mu_x\big(\mathscr{O}_r(x, \gamma x)\big) \leq C e^{-\delta d_\Omega(x,\gamma x)}.$$

Proof. We recall the elegant proof of Roblin [36].

Since γ is an isometry and the μ_x are quasi-Γ-invariant,

$$\mu_x(\mathscr{O}_r(x, \gamma x)) = \mu_x(\gamma \mathscr{O}_r(\gamma^{-1} x, x)) \tag{5.3.1}$$

$$= \mu_{\gamma^{-1} x}(\mathscr{O}_r(\gamma^{-1} x, x)) \tag{5.3.2}$$

$$= \int_{\mathscr{O}_r(\gamma^{-1} x, x)} e^{-\delta \beta_\xi(\gamma^{-1} x, x)} d\mu_x(\xi) \tag{5.3.3}$$

by the transformation rule.

Applying Lemma 5.3.4,

$$\begin{aligned} \int_{\mathscr{O}_r(\gamma^{-1} x, x)} e^{-\delta d_\Omega(\gamma^{-1} x, x)} d\mu_x(\xi) &\leq \int_{\mathscr{O}_r(\gamma^{-1} x, x)} e^{-\delta \beta_\xi(\gamma^{-1} x, x)} d\mu_x(\xi) \\ &\leq \int_{\mathscr{O}_r(\gamma^{-1} x, x)} e^{-\delta (d_\Omega(\gamma^{-1} x, x) - 2r)} d\mu_x(\xi), \end{aligned}$$

so in particular, for $\|\mu_x\| := \mu_x(\partial_V\Omega) < \infty$,

$$e^{-\delta d_\Omega(\gamma^{-1}x,x)}\mu_x(\mathscr{O}_r(\gamma^{-1}x,x)) \le \mu_x(\mathscr{O}_r(x,\gamma x)) \le e^{-\delta d_\Omega(\gamma^{-1}x,x)}e^{2\delta r}\|\mu_x\|. \tag{5.3.4}$$

Then the rightmost inequality of 5.3.4 gives us the rightmost inequality of the lemma by taking $C \ge e^{2\delta r}\|\mu_x\|$, since γ is an isometry. For the leftmost inequality of 5.3.4, we invoke the geometric Proposition 5.3.3 which will apply only for sufficiently large r. There are open sets $O_1, O_2 \subset \partial_V\Omega$ such that for all $\gamma \in \Gamma$, either $O_1 \subset \mathscr{O}_r(\gamma^{-1}x,x)$ or $O_2 \subset \mathscr{O}_r(\gamma^{-1}x,x)$. By Lemma 5.3.5, let $0 < \frac{1}{C} < \min\{\mu_x(O_i)\}$ and the proof of the Shadow Lemma is complete. □

Corollary 5.3.7 (of Lemma 5.3.6 and Corollary 5.1.3, [36])**.** *For* $\Gamma < \mathrm{PSL}(4,\mathbb{R})$ *acting discretely and cocompactly on a Benoist 3-manifold* Ω*, with critical exponent* δ_Γ*,*

(a) If there exists a Busemann density of dimension δ*, then* $\delta \ge \delta_\Gamma$*.*

(b) For each $x \in \Omega$*, there exists a* C *such that* $N_\Gamma(x,r) \le Ce^{\delta_\Gamma r}$*.*

5.3.2 Flats are null sets for the Patterson–Sullivan measures

When constructing the Bowen–Margulis measure from the Patterson–Sullivan density, it is clear for ergodicity and uniqueness that the φ^t-invariant, nonexpanding flats need to have trivial measure. In this section we prove a crucial lemma for this purpose.

Lemma 5.3.8. *Let* $S_\triangle(x,r)$ *denote the sphere of Hilbert radius* r *about* x *restricted to* $\triangle$*:*

$$S_\triangle(x,r) := \{y \in \triangle : d_\Omega(x,y) = r\}.$$

Pick a tiling of $\triangle$ *by* $\mathrm{Stab}_\Gamma(\triangle)$ *such that* x *is in the interior of a fundamental domain in the tiling. Choose* R *so that the open* $B_\triangle(x,R)$ *covers the compact fundamental domain containing* x*, but does not completely cover any other fundamental domain in the tiling. Let* N_r *denote the minimal number of* $\gamma.B_\triangle(x,R)$ *which cover* $S_\triangle(x,r)$*, where* $\gamma \in \mathrm{Stab}_\Gamma(\triangle)$*. Then* N_r *is linear in* r*.*

Proof. We will use the planar model for $(\triangle, d_\Omega)$ - choose the unit hexagon in $\mathbb{R}^2$ to have positive coordinates at $(1,0), (1,1), (0,1)$. Representing $\triangle$ as the projection of the first octant in $\mathbb{R}^3$, the induced isometric action of $\gamma = \text{diag}(\lambda, \nu, \eta) \in \text{Stab}_\Gamma(\triangle)$ on $(\mathbb{R}^2, \hexagon)$ under de la Harpe's isometry [21] is

$$(p,q) \mapsto (p,q) + \frac{1}{6}(\log\lambda - \log\eta, \log\nu - \log\eta).$$

In particular, such γ are acting on $(\mathbb{R}^2, \hexagon)$ by translation. A fundamental domain for this action is a Euclidean cylinder.

Up to finite-index, the action of $\text{Stab}_\Gamma(\triangle)$ is a $\mathbb{Z}^2$-action, so we choose two translations g, h acting on $(\mathbb{R}^2, \hexagon)$ to represent this action. Now we can compute how many fundamental domains are needed to cover the six sides of the r-hexagon given by coordinates $(r,0), (r,r), (0,r)$. Note that by the nature of the hexagonal geometry, it suffices to verify how many fundamental domains are needed to cover the lines from the origin to each of $(r,0), (r,r), (0,r)$. In particular, the symmetry to each of these three axes allows us to restrict attention to a favorite: let ℓ_r denote the line segment from the origin to $(r,0)$, and let $N_r(\ell_r)$ denote the number of $\gamma \in \langle g, h\rangle$ needed so that the union of the $\gamma.B_\triangle(x,R)$ cover ℓ_r.

For a given $\mathbb{Z}^2$-lattice generated by two translations g, h, we can connect vertices by lines which are invariant for one of the generators. If the lattice begins with a vertex at the origin, to count the number of fundamental domains needed to cover ℓ_r we need only count the number of times ℓ_r crosses a line in the lattice. The g-invariant lines have a constant horizontal translation distance t_g, as do the h-invariant lines (denoted t_h). The horizontal displacements are both determined by the g and h action together, but they are constant between adjacent parallel lines in the lattice. Thus, the number of crossings that ℓ_r has with g-invariant lines is approximately r/t_g, and similarly for h. Then we get a linear approximation for the number of balls of radius R needed to cover ℓ_r:

$$N_r(\ell_r) \approx \frac{r}{t_g} + \frac{r}{t_h}.$$

The argument is identical for the other sides of $S_\triangle(x,r)$, and so we conclude that N_r is linear in r. □

Proposition 5.3.9. *The visual boundary of any properly embedded triangle is a null set for the Patterson–Sullivan measures.*

Proof. Choose a fundamental domain T for the action of $\text{Stab}_\Gamma(\triangle)$ on $\triangle$, and let $x \in \Omega_{\text{hyp}}$, our reference point for the Patterson–Sullivan measure, be in a fundamental domain D for the Γ action on Ω such that

$T \subset D$. By compactness, there is a point $p \in T \subset \triangle$ which minimizes $d_\Omega(x, T)$.

Now choose R similar to Lemma 5.3.8, but large enough so that $B_\Omega(x, R)$ covers D, but $B_\Omega(x, R)$ does not contain any other fundamental domains in the Γ-tiling. Then the Γ-orbit of $B_\Omega(x, R)$ covers Ω, and we can choose a finite number of γ such that $\cup_{i=1}^{N} \gamma_i . B_\Omega(x, R)$ covers $S_\triangle(p, r)$, the compact r-sphere in $\triangle$ about p. If we denote the minimal such number by $N_r(\Gamma, x)$, then $N_r(\Gamma, x)$ is bounded above by the N_r in Lemma 5.3.8 which is counting only $\gamma \in \mathrm{Stab}_\Gamma(\triangle)$. We can thus allow N_r to be the cardinality of our chosen finite covering of $S_\triangle(x, r)$ by $\gamma_i B_\Omega(x, R)$ and we will assume that $\gamma_i \in \mathrm{Stab}_\Gamma(\triangle)$ for $i = 1, \ldots, N_r$.

Next, we show for all large enough r, $\partial_V^x \triangle \subset \bigcup_{i=1}^{N_r} \mathscr{O}_{2R}(x, \gamma_i x)$. Let $r \gg 2R$. Consider any projective ray η based at p such that $\eta^+ \in \partial \triangle$, the Hilbert boundary. There is a natural associated projective ray ξ based at x such that $\xi^+ = \eta^+$, and all elements of $\partial_V^x \triangle$ can be represented by such ξ by Lemma 4.1.2. Then parameterizing ξ, η at unit speed, we have that $d_\Omega(\xi_t, \eta_t) \le d_\Omega(x, p) \le R$ for all $t \ge 0$. Since $p, \eta^+ \in \overline{\triangle}$, $\eta \cap S_\triangle(p, r) \neq \varnothing$ and there exists a γ_i such that $\eta \cap B_\Omega(\gamma_i x, R) \neq \varnothing$. Let $t \ge 0$ be such that $d_\Omega(\eta_t, \gamma_i x) < R$. Then $d_\Omega(\xi_t, \gamma_i x) \le d_\Omega(\xi_t, \eta_t) + d_\Omega(\eta_t, \gamma_i x) \le 2R$, and $\xi \in \mathscr{O}_{2R}(x, \gamma_i x)$.

Lastly, for each i let $q_i \in S_\triangle(p, r) \cap B_\Omega(\gamma_i x, R) \neq \varnothing$. Then by the triangle inequality,

$$r = d_\Omega(p, q_i) \le d_\Omega(p, x) + d_\Omega(x, \gamma_i x) + d_\Omega(\gamma_i x, q_i) \le d_\Omega(x, \gamma_i x) + 2R$$

implying $-d_\Omega(x, \gamma_i x) \le 2R - r$ for all $i = 1, \ldots, N_r$. Then by Lemma 5.3.6,

$$\mu_x(\partial_V^x \triangle) \le \sum_{i=1}^{N_r} \mu_x(\mathscr{O}_{2R}(x, \gamma_i x)) \le \sum_{i=1}^{N_r} C_{2R} \cdot e^{-\delta_\Gamma d_\Omega(x, \gamma_i x)} \le C_{2R} \cdot e^{\delta_\Gamma 2R} \cdot e^{-\delta_\Gamma r} \cdot N_r. \tag{5.3.5}$$

Given that N_r is linear in r by Lemma 5.3.8, that $\delta_\Gamma > 0$ by Corollary 5.1.3, and that Inequality 5.3.5 holds for all $r \to +\infty$, we conclude that $\mu_x(\partial_V^x \triangle) = \mu_x(\partial_V \triangle) = 0$. □

The Patterson–Sullivan measures are nonatomic

Lemma 5.3.10. *The Patterson–Sullivan measures on $\partial_V \Omega$ have no atoms.*

Proof. It suffices to check for proper extremal points $\xi \in \partial_V \Omega$ by Proposition 5.3.9. By Proposition 4.1.2, we represent ξ by a projective ray at x: If $\xi \in \partial_V \Omega = \partial_V^x \Omega$, choose a fundamental domain D for the Γ-action on Ω such that $x \in D$. Then since D is compact, Γ acts by isometries, and $\Gamma.D$ covers Ω, we can cover ξ with $\gamma_n D$, and sort γ_n such that $d_\Omega(x, \gamma_n x)$ is increasing with n. Let $r \geq 2\operatorname{diam} D$. Then $\xi \in \mathcal{O}_r(x, \gamma_n x)$ for all n. Since the Patterson–Sullivan measures satisfy Lemma 5.3.6,

$$\mu_x(\{\xi\}) \leq \mu_x(\mathcal{O}_r(x, \gamma_n.x)) \leq Ce^{-\delta d_\Omega(x,\gamma_n.x)}$$

and $e^{-\delta d_\Omega(x,\gamma_n.x)} \to 0$ as $n \to \infty$ because $\delta > 0$ and $d_\Omega(x, \gamma_n.x) \to \infty$ with n. □

Local estimates and equivalent densities

Following Knieper [28], we prove a lemma on local estimates for the μ_x-measure of shadows using the cocompact action of Γ and Sullivan's Shadow Lemma (Lemma 5.3.6). An important corollary will be that any two δ-conformal Busemann densities are equivalent, which we can use to prove uniqueness. This local estimate lemma is also generally used to prove that a δ-conformal Busemann density behaves like a δ-dimensional Hausdorff measure (cf. [38, 28]).

Lemma 5.3.11 (Local estimates)**.** *If $\{\mu_x\}$ is a δ-conformal density on $\partial_V\Omega$, then for all x and all sufficiently large r there exists a constant $b(r)$ such that for $y \in \Omega$ with $d_\Omega(x, y)$ large,*

$$\frac{1}{b(r)}(e^{-d_\Omega(x,y)})^\delta \leq \mu_x(\mathcal{O}_r(x, y)) \leq b(r)(e^{-d_\Omega(x,y)})^\delta.$$

The precise arrangement of the bounds is to encourage the interpretation of $\mathcal{O}_r(x, y)$ as a ball of radius approximately $e^{-d_\Omega(x,y)}$ centered at the projection of y to $\partial_V\Omega$ from x. In this way, we see μ_x is behaving like a δ-dimensional Hausdorff measure. This interpretation is originally due to Sullivan, who proves that the μ_x are in fact Lipschitz-equivalent to a δ-dimensional Hausdorff measure for geometric actions on hyperbolic spaces [38].

Proof. Note that if $y = \gamma x$, then we apply Lemma 5.3.6 to obtain the result. If not: choose r large enough that, given a Γ-tiling by isometric fundamental domains, if we pick any fundamental domain D then for all $x \in D$, $D \subset B_\Omega(x, \frac{r}{2})$. This is clearly possible because Γ is acting cocompactly on Ω.

Now, if D is chosen so that $y \in D$, then there exists a $\gamma \in \Gamma$ such that $\gamma x \in D \subset B_\Omega(y, \frac{r}{2})$ as well. It is easy to check by the triangle inequality that

$$\mathscr{O}_{\frac{r}{2}}(x,\gamma x) \subset \mathscr{O}_r(x,y) \subset \mathscr{O}_{\frac{3r}{2}}(x,\gamma x)$$

and we will check it here because thesis. First, if $\xi \in \mathscr{O}_{\frac{r}{2}}(x,\gamma x)$ then there exists $p \in (x,\xi) \cap B_\Omega(\gamma x, r/2)$ and $d_\Omega(p,y) \le d_\Omega(p,\gamma x) + d_\Omega(\gamma x, y) \le \frac{r}{2} + \frac{r}{2} = r$. Thus $\mathscr{O}_{\frac{r}{2}}(x,\gamma x) \subset \mathscr{O}_r(x,y)$. Similarly, if $\eta \in \mathscr{O}_r(x,y)$ then there exists $q \in (x\eta) \cap B_\Omega(y,r)$ and $d_\Omega(q,\gamma x) \le d_\Omega(q,y) + d_\Omega(y,\gamma x) \le \frac{3r}{2}$, so we have $\mathscr{O}_r(x,y) \subset \mathscr{O}_{\frac{3r}{2}}(x,\gamma x)$.

Applying Lemma 5.3.6, if r is sufficiently large then there is a uniform constant C such that

$$\frac{1}{C} e^{-\delta d_\Omega(x,\gamma x)} \le \mu_x(\mathscr{O}_{\frac{r}{2}}(x,\gamma x)) \le \mu_x(\mathscr{O}_r(x,y)) \le \mu_x(\mathscr{O}_{\frac{3r}{2}}(x,\gamma x)) \le C e^{-\delta d_\Omega(x,\gamma x)}.$$

Our final observation is that since $\gamma x \in B_\Omega(y, r/2)$,

$$-d_\Omega(x,y) - \frac{r}{2} \le -d_\Omega(x,\gamma x) \le -d_\Omega(x,y) + \frac{r}{2}$$

and we conclude

$$\frac{1}{Ce^{\delta r/2}} e^{-\delta d_\Omega(x,y)} \le \mu_x(\mathscr{O}_r(x,y)) \le Ce^{\delta r/2} e^{-\delta d_\Omega(x,y)}.$$

□

For fixed x and fixed sufficiently large r, the $\mathscr{O}_r(x,y)$ generate the topology of $\partial_V\Omega$ over all $y \in \Omega$. Thus we have an immediate corollary of the local estimates lemma (Lemma 5.3.11):

Corollary 5.3.12. *Any two δ-conformal Busemann densities $\{\mu\}_\Omega, \{\nu\}_\Omega$ are equivalent, meaning $\nu_x \ll \mu_x \ll \nu_x$ for all x.*

Proof. Let $\xi \in \partial_V\Omega$ be a proper extremal point by Lemma 5.3.9, and take a sequence $y_n \to \xi$. Then for all sufficiently large n, $d_\Omega(x,y_n)$ is large enough to apply Lemma 5.3.11 to both μ and ν and conclude:

$$\frac{1}{b_\mu(r)b_\nu(r)} \le \frac{\mu_x(\mathscr{O}_r(x,y_n))}{\nu_x(\mathscr{O}_r(x,y_n))} \le b_\mu(r)b_\nu(r).$$

In particular, $\mathscr{O}_r(x,y_n) \to \{\xi\}$ as $y_n \to \xi$, because ξ is proper and extremal. Since proper extremal points form a set of full measure for any δ-conformal density, we conclude that μ_x, ν_x are equivalent. □

5.4 The Patterson–Sullivan density is unique

We now have that for any two Busemann densities $\{\mu_x\}_\Omega$, $\{\nu_x\}_\Omega$ of the same dimension, the measures μ_x and ν_x are equivalent. Now we can proceed with the standard uniqueness proof.

Recall that a measure μ is ergodic with respect to the action by some group Γ if every measurable, Γ-invariant set A has trivial measure: meaning either $\mu(A) = 0$ or $\mu(A^c) = 0$. Classically, the group Γ is either real or integer-valued, but the notion of ergodicity is not restricted to those systems. We prove here ergodicity of the Patterson–Sullivan measures for Γ acting on $\partial_V\Omega$, which has an immediate application to uniqueness.

Proposition 5.4.1. *If μ is a δ–conformal density on $\partial_V\Omega$, then each μ_x is ergodic for the Γ–action on $\partial_V\Omega$.*

Proof. Let $A \subset \partial_V\Omega$ be a Borel, Γ–invariant set with positive μ_x–measure. Define a new density $\bar{\mu}_x(B) := \mu_x(A \cap B)$. Since A is Γ–invariant and has positive measure, we will see that $\bar{\mu}_x$ is a Busemann density in its own right, also of dimension δ. Then μ_x is equivalent to $\bar{\mu}_x$, and we conclude that $\mu_x(\partial_V\Omega \setminus A) = \bar{\mu}_x(\partial_V\Omega \setminus A) = 0$, proving ergodicity of μ_x for Γ.

Now we verify that $\{\bar{\mu}_x\}$ is a Busemann density. First, $\bar{\mu}_x$ is clearly nontrivial and finite, since $\bar{\mu}_x(\partial_V\Omega) = \mu_x(A) > 0$. Moreover, the two defining properties of a δ-dimensional Busemann density are satsified:

1. (quasi-Γ-equivariance) For any Borel set $B \subset \partial_V\Omega$,

   $$\bar{\mu}_{\gamma x}(B) = \mu_{\gamma x}(A \cap B) = \mu_x(\gamma^{-1}(A \cap B)) = \mu_x(A \cap \gamma^{-1}B) = \bar{\mu}_x(\gamma^{-1}B).$$

   Note that for all $g \in \Gamma$, $g(A \cap B) \supseteq gA \cap gB$ because g is invertible. The reverse containment is obvious, allowing us to conclude by Γ-invariance of A that $\gamma^{-1}(A \cap B) = \gamma^{-1}A \cap \gamma^{-1}B = A \cap \gamma^{-1}B$.

2. (transformation rule) This is immediate by the transformation rule for $\{\mu_x\}_\Omega$:

   $$\begin{aligned}\bar{\mu}_x(B) = \mu_x(A \cap B) &= \int_{A\cap B} e^{-\delta\beta_\xi(x,y)}\, d\mu_y(\xi)\\ &= \int_B e^{-\delta\beta_\xi(x,y)} I_A(\xi)\, d\mu_x(\xi) = \int_B e^{-\delta\beta_\xi(x,y)}\, d\bar{\mu}_y(\xi).\end{aligned}$$

   where $I_A \colon \partial_V\Omega \to \{0,1\}$ is the indicator function on A. It is evident that $d\bar{\mu}_x / d\mu_x(\xi) = I_A(\xi)$.

□

Theorem 5.4.2. *Conformal densities of dimension δ on $\partial_V\Omega$ are unique up to a constant.*

Proof. By Proposition 5.4.1, the Patterson–Sullivan densities of dimension δ_Γ are ergodic, so the result follows from Γ–invariance. It is a fact that μ_x is ergodic if and only if every function which is Γ-invariant μ_x-almost everywhere is constant on a set of full μ_x-measure. Let $\mu_x, \bar{\mu}_x$ be two δ-conformal densities. We will show that $d\mu_x/d\bar{\mu}_x$ is Γ-invariant. Ergodicity of μ_x then implies that the Radon–Nikodym derivative is constant μ_x-almost everywhere. Since $\mu_x, \bar{\mu}_x$ both have support $\partial_V\Omega$ by Lemma 5.3.5, the measures would then have to agree up to a constant.

Now we show Γ-invariance of the Radon–Nikodym derivative for two δ-conformal densities. Let A be a Borel measurable set. Then

$$\begin{aligned}
\int_A \frac{d\mu_x}{d\bar{\mu}_x}(\gamma^{-1}\xi)\,d\bar{\mu}_x(\xi) &= \int_{\gamma A} \frac{d\mu_x}{d\bar{\mu}_x}(\xi)\,d\bar{\mu}_x(\gamma\xi) && \Gamma \text{ acts by homeos on } \partial_V\Omega \\
&= \int_{\gamma A} \frac{d\mu_x}{d\bar{\mu}_x}(\xi)\,d\bar{\mu}_{\gamma^{-1}x}(\xi) && \Gamma\text{-quasi-invariance of } \bar{\mu} \\
&= \int_{\gamma A} e^{-\delta\beta_\xi(\gamma^{-1}x,x)}\,d\mu_x(\xi) && \text{transformation rule of } \bar{\mu} \\
&= \int_{\gamma A} e^{-\delta\beta_{\gamma\xi}(x,\gamma x)}\,d\mu_x(\xi) && \text{uniqueness of } \beta \text{ over limit paths} \\
&= \int_{A} e^{-\delta\beta_\xi(x,\gamma x)}\,d\mu_{\gamma x}(\xi) && \Gamma\text{-quasi-invariance of } \mu \\
&= \mu_x(A) && \text{transformation rule of } \mu.
\end{aligned}$$

By uniqueness of the Radon–Nikodym derivative for L^1 functions up to a null set, we conclude that $d\mu_x/d\bar{\mu}_x(\gamma^{-1}\xi) = d\mu_x/d\bar{\mu}_x(\xi)$ almost everywhere. □

5.5 Volume growth and divergence of Γ

In this section we will see that existence of a δ_Γ-conformal density and the local property of that density yields fine asymptotics for volume growth of spheres in Ω. As a corollary of these aymptotics, we prove that Γ is divergent.

Theorem 5.5.1. *Let Ω be a Benoist 3-manifold with Γ acting cocompactly by projective transformations. Then for all $x \in \Omega$, there exists a constant $a(x) > 1$ such that*

$$\frac{1}{a} \leq \frac{\operatorname{vol} S_\Omega(x,t)}{e^{\delta_\Gamma t}} \leq a.$$

Proof. Let $\delta = \delta_\Gamma$ throughout the proof. Let R be sufficiently large to apply Sullivan's Shadow Lemma (Lemma 5.3.6) and consequently the local estimate in Lemma 5.3.11. Consider $r \geq 6R$. By compactness of $S_\Omega(x,t)$, we can take $\{x_i\}_{i=1}^{N_t}$ to be a maximal r-separating set in $S_\Omega(x,t)$. In particular, for all $i \neq j$, $d_\Omega(x_i,x_j) \geq r$, $S_\Omega(x,t) \subset \bigcup_{i=1}^{N_t} B_\Omega(x_i,r)$, and $B_\Omega(x_i,r/3) \cap B_\Omega(x_j,r/3) = \varnothing$ for $i \neq j$.

By the local estimate, there exists a $b(r)$ such that for all x_i, each of which is distance t from x, we have

$$\frac{1}{b}e^{-\delta t} \leq \mu_x(\mathscr{O}_r(x,x_i)) \leq be^{-\delta t}.$$

We can moreover choose the b so that this estimate applies to all $r \in [2R,6R]$. Then

$$\mu_x(\partial_V\Omega) \leq \mu_x(\bigcup_{i=1}^{N_t}\mathscr{O}_r(x,x_i)) \leq \sum_{i=1}^{N_t}\mu_x(\mathscr{O}_r(x,x_i)) \leq N_t be^{-\delta t},$$
$$\mu_x(\partial_V\Omega) \geq \mu_x(\bigcup_{i=1}^{N_t}\mathscr{O}_{r/3}(x,x_i)) = \sum_{i=1}^{N_t}\mu_x(\mathscr{O}_{r/3}(x,x_i)) \geq \frac{N_t}{b}e^{-\delta t}.$$

So the local estimate for our conformal density yields a fine approximation for the growth of N_t with t:

$$\frac{N_t}{b}e^{-\delta t} \leq \mu_x(\partial_V\Omega) \leq N_t be^{-\delta t},$$
$$\frac{be^{\delta t}}{\mu_x(\partial_V\Omega)} \geq N_t \geq \frac{e^{\delta t}}{b\mu_x(\partial_V\Omega)}.$$

We can therefore find a b' such that

$$\frac{1}{b'}e^{\delta t} \leq N_t \leq b'e^{\delta t}.$$

Now, by compactness of M and Γ-equivariance of vol on Ω, we claim there exists an $\ell(r)$ such that for all $x \in \Omega$,

$$\frac{1}{\ell} \leq \mathrm{vol}(B_\Omega(x,r) \cap S_\Omega(x,t)) \leq \ell.$$

Then we may just as easily arrange for an ℓ' such that

$$\mathrm{vol}(S_\Omega(x,t)) \leq \sum_{i=1}^{N_t} \mathrm{vol}(B_\Omega(x_i,r) \cap S_\Omega(x,t)) \leq N_t \cdot \ell' \leq \ell' b' e^{\delta t}$$

and

$$\mathrm{vol}(S_\Omega(x,t)) \geq \sum_{i=1}^{N_t} \mathrm{vol}(B_\Omega(x_i,r/3) \cap S_\Omega(x,t)) \geq \frac{N_t}{\ell'} \geq \frac{1}{\ell' b'} e^{\delta t},$$

which concludes the proof of the theorem. □

Corollary 5.5.2. Γ *acting cocompactly on a Benoist 3-manifold is of divergent type.*

Proof. Let D be a fundamental domain for a Γ-tiling of Ω. Then for $s > \delta_\Gamma$, $P(x,y,s)$ converges so we can apply Fubini's Theorem to the following integral:

$$\int_D \sum_{\gamma\in\Gamma} e^{-sd_\Omega(x,\gamma y)} d\,\mathrm{vol}(y) = \sum_{\gamma\in\Gamma} \int_{\gamma(D)} e^{-sd_\Omega(x,\gamma y)} d\,\mathrm{vol}(y) = \int_0^\infty e^{-st} \mathrm{vol}(S_\Omega(x,t))\,dt$$

As s goes to δ_Γ^+, the right hand side diverges because by Theorem 5.5.1,

$$\int_0^\infty e^{-\delta_\Gamma t} \mathrm{vol}(S_\Omega(x,t))\,dt \geq \int_0^\infty \frac{1}{a}\,dt$$

which is a divergent integral. We can now conclude that the Poincaré Series diverges for $s = \delta_\Gamma$. □

Chapter 6

The Bowen–Margulis measure

In this chapter we introduce the Γ-invariant Bowen–Margulis measure on $S\Omega$, denoted $\widetilde{\mu}_{BM}$. We first induce a measure $\bar{\mu}_x$ on $\partial^2_{V,x}\Omega = \{(\xi,\eta) \in \partial_V\Omega \mid \exists \sigma \subset \Omega$ a projective ray $, \sigma \sim_+ \xi, \sigma \sim_- \eta\}$ using the Patterson–Sullivan measures. By construction and Proposition 5.3.9, $\bar{\mu}_x$ will assign full measure to proper extremal pairs in $\partial^2_{V,x}\Omega$, hence $\widetilde{\mu}_{BM}$ will assign full measure to the regular set, $S\Omega_{\text{reg}} = \{v \in S\Omega \mid v^+, v^-$ proper extremal$\}$. We will denote the singular set by $S\Omega_{\text{sing}} = S\Omega \smallsetminus S\Omega_{\text{reg}}$, and the flat vectors are $S\Omega_{\text{flat}} = \{v \in S\Omega \mid \ell_v \subset \triangle$ for some properly embedded $\triangle \subset \Omega\}$.

On $S\Omega$, $\widetilde{\mu}_{BM}$ is a Γ-invariant, $\widetilde{\varphi}^t$-invariant Radon measure, which projects to a finite, φ^t-invariant measure μ_{BM} on the compact SM.

The chapter culminates in Sections 6.2 and 6.3, where we show that μ_{BM} is an ergodic and mixing measure of maximal entropy for the geodesic flow of a Benoist 3-manifold (Lemma 6.1.1, Theorem 6.2.4, Theorem 6.3.2).

6.1 Definition and properties

Define a measure on $\partial^2_{V,x}\Omega$ by

$$d\bar{\mu}_x(\xi,\eta) = e^{2\delta\langle\xi,\eta\rangle_x} d\mu_x(\xi) d\mu_x(\eta).$$

where, for ξ,η proper extremal, we can reinterpret the Gromov product as

$$\langle\xi,\eta\rangle_x = \beta_\xi(x,p) + \beta_\eta(x,p) \tag{6.1.1}$$

for any $p \in (\xi\eta)$ (see Subsection 4.3). For the purposes of defining the measure, we choose p to be the closest point projection of x to the projective line $(\xi\eta)$, which exists for pairs $(\xi,\eta) \in \partial^2_{V,x}\Omega$ and define the Gromov product for nonproper or nonextremal points by (6.1.1) and Remark 5.2.2.

Then $\bar{\mu}_x$ is Γ-invariant on $\partial^2_{V,x}\Omega$. Let p be as above and $\delta = \delta_\Gamma$.

$$\begin{aligned}
&\gamma_* d\bar{\mu}_x(\xi,\eta) \\
&= e^{2\delta\langle\gamma^{-1}\xi,\gamma^{-1}\eta\rangle_x}\, d\mu_x(\gamma^{-1}\xi)d\mu_x(\gamma^{-1}\eta) && \text{quasi-}\Gamma\text{-invariance} \\
&= e^{2\delta\langle\xi,\eta\rangle_{\gamma x}}\, d\mu_{\gamma x}(\xi)d\mu_{\gamma x}(\eta) && \text{property of Gromov product} \\
&= e^{\delta\left(\beta_\xi(\gamma x,p)+\beta_\eta(\gamma x,p)\right)}\, d\mu_{\gamma x}(\xi)d\mu_{\gamma x}(v^+) \\
&= e^{\delta\left(\beta_\xi(\gamma x,p)+\beta_\eta(\gamma x,p)-\beta_\xi(\gamma x,x)-\beta_\eta(\gamma x,x)\right)}\, d\mu_x(\xi)d\mu_x(\eta) && \text{transformation rule} \\
&= e^{\delta\left(\beta_\xi(x,p))+\beta_\eta(x,p)\right)}\, d\mu_x(\xi)d\mu_x(\eta) && \text{cocyle property of } \beta \\
&= e^{2\delta\langle\xi,\eta\rangle_x}\, d\mu_x(\xi)d\mu_x(\eta) = d\bar{\mu}_x(\xi,\eta).
\end{aligned}$$

We can thus project $\bar{\mu}_x$ to the quotient $\partial^2_{V,x}\Omega/\Gamma$. It remains to show that there is a canonical measure on SM: $\bar{\mu}_x$ induces a flow invariant measure on $S\Omega$, and hence SM by Γ-invariance,

$$\widetilde{\mu}^x_{BM}(A) = \int_A \text{length}_\Omega(\ell_v \cap \pi A)d\bar{\mu}_x(v^-,v^+) = \int_A \text{length}_\Omega(\ell_v \cap \pi A)\, e^{2\delta_\Gamma\langle v^-,v^+\rangle_x} d\mu_x(v^-)d\mu_x(v^+) \quad (6.1.2)$$

where $\pi\colon S\Omega \to \Omega$ is the footpoint projection and length_Ω is the Hilbert length of the geodesic segment $(\xi\eta)\cap\pi A$. On SM the measure is finite, and we may normalize it so $\mu^x_{BM}(SM) = 1$.

A posteriori, the μ^x_{BM}, μ^y_{BM} will be equal up to a constant because they are ergodic and equivalent, the latter of which we verify below. Thus, for the ergodicity proof, we will let $\mu_{BM} := \mu^x_{BM}$ for some $x \in \Omega_{\text{hyp}}$.

Lemma 6.1.1. *The flat vectors $v \in SM_{flat}$ form a null set for μ^x_{BM}.*

Proof. This is an immediate corollary of Proposition 5.3.9 and the definition of μ^x_{BM}. The set A of all $v \in S\Omega$ with $v^- \in \partial\triangle^-$, $v^+ \in \partial\triangle^+$ projects to a set contained in the product set $A^- \times A^+ \subset \partial\triangle^- \times \partial\triangle^+ \setminus \text{diag}$, where $A^\pm = \{v^\pm \mid v \in A\}$.

Since $\partial\triangle^- \times \partial\triangle^+$ is a null set for $\bar{\mu}_x \ll \mu_x \otimes \mu_x$, A has μ^x_{BM}-measure 0 and by finiteness of μ^x_{BM}, so does the union of all such A over the countably many disjoint pairs of properly embedded triangles. □

Lemma 6.1.2. *μ^x_{BM} and μ^y_{BM} are equivalent.*

Proof. For $\mu_x \otimes \mu_x$-almost every $(v^-, v^+) \in \partial^2_{V,x}\Omega$, we can write

$$\begin{aligned} 2\langle v^-, v^+\rangle_x &= \beta_{v^-}(x,p) + \beta_{v^+}(x,p) \\ &= \beta_{v^-}(y,p) - \beta_{v^-}(y,x) + \beta_{v^+}(y,p) - \beta_{v^+}(y,x) \\ &= -\beta_{v^-}(y,x) - \beta_{v^+}(y,x) + \langle v^-, v^+\rangle_y. \end{aligned}$$

So by uniqueness of the Patterson–Sullivan density up to a constant $C(x,y)$,

$$\begin{aligned} d\bar{\mu}_x(v^-, v^+) &= e^{2\delta\langle v^-,v^+\rangle_x} d\mu_x(v^-) d\mu_x(v^+) \\ &= C^2 e^{-\delta(\beta_{v^-}(y,x)+\beta_{v^+}(y,x))} d\bar{\mu}_y. \end{aligned}$$

Since $\beta_{v^-}(y,x) + \beta_{v^+}(y,x)$ is bounded below by $-2d_\Omega(x,y)$, we conclude that $d\bar{\mu}_x \ll d\bar{\mu}_y$. The same arguments give equivalence, which clearly descends to the associated Bowen–Margulis measures. □

6.2 Ergodicity

In this section, we apply a variant of the classical Hopf argument. Originally, Hopf applied smooth stable and unstable foliations to prove ergodicity of the geodesic flow of constant curvature hyperbolic surfaces [24]. This proof has been extended well beyond geodesic flows, but there are many classes of manifolds for which ergodicity of the geodesic flow can be proven by a geometric generalization of the original proof: compact manifolds of pinched negative curvature [37] was an early extension, then later compact rank one manifolds [29, Theorem 4.3], and most recently nonelementary rank one manifolds [30, Section 7].

An important tool for the geometric Hopf argument is the existence of global stable and unstable sets for μ_{BM}-almost every point in SM. This follows from the contruction of horospheres for proper extremal points in $\partial\Omega$ (see Figure 4.2.3 and Lemma 4.2.4). Recall that

$$W^{ss}(v) = \{w \in SM \mid d(\varphi^t v, \varphi^t w) \to 0 \text{ as } t \to +\infty\},$$

$$W^{su}(v) = \{w \in SM \mid d(\varphi^{-t} v, \varphi^{-t} w) \to 0 \text{ as } t \to +\infty\}.$$

The weak stable manifold $W^{os}(v)$ is the union of $W^{ss}(\varphi^t v)$ over $t \in \mathbb{R}$, and similarly for the weak unstable manifold $W^{ou}(v)$.

Lemma 6.2.1. *Stable and unstable sets for regular vectors are global: $W^{ss}(v) \cap W^{ou}(w) \neq \emptyset$ for all forward regular v and backward regular w with $v \neq -w$.*

Proof. If v is forward regular then $W^{ss}(v)$ is defined by a geometric characterization on the universal cover:

$$\widetilde{W}^{ss}(v) = \{w \in S\Omega \mid v^+ = w^+,\ \pi w \in \mathscr{H}_{v^+}(\pi v)\},$$

$$\widetilde{W}^{os}(v) = \{w \in S\Omega \mid v^+ = w^+\}.$$

Since Γ, hence $d\Gamma$, acts by isometries on $S\Omega$, $\widetilde{W}^{ss}(\tilde{v})$ projects to $W^{ss}(v)$ in SM for any lift $\tilde{v}$ of v.

Choose lifts $\tilde{v}, \tilde{w}$ in $S\Omega$ with endpoints v^+, w^- proper extremal points. Then there exists a $\tilde{u} \in \widetilde{W}^{ss}(\tilde{v})$ such that $u^- = w^-$ as long as $w^- \neq v^+$, which is guaranteed by the assumption that $v \neq -w$. Since $u^- = w^-$, $\tilde{u} \in \widetilde{W}^{ou}(\tilde{w})$. Project $\tilde{u}$ to SM and we have the desired $u \in W^{ss}(v) \cap W^{ou}(w)$. □

The Hopf argument

For the original proof, Hopf applied Fubini's theorem using a smooth change of coordinates over the uniformly transverse, smooth stable and unstable foliations (cf. [26, Theorem 5.4.16]). In the general theory, the stable and unstable foliations may be smooth but the smooth volume might not immediately carry the desired product structure. Thus, the proof requires absolute continuity properties of the foliations.

In our case, the Bowen–Margulis measure admits a product structure by construction. As for rank one manifolds, we capitalize on this feature to apply the Hopf argument in a situation where the stable and unstable foliations are not smooth at all.

Let $f\colon SM \to \mathbb{R}$ be integrable. Then the forward Birkhoff averages of f for φ are

$$f^+(v) = \lim_{t\to+\infty} \frac{1}{t} \int_0^t f \circ \varphi^s(v)\, ds. \tag{6.2.1}$$

By the Birkhoff ergodic theorem, f^+ exists for μ_{BM}-almost every $v \in SM$ (cf. [26, Theorem 4.1.2]).

Lemma 6.2.2. *Forward Birkhoff averages of continuous functions are constant on stable sets for forward regular vectors.*

Proof. Suppose f is continuous, hence uniformly continuous on the compact SM and thus L^1-integrable. If v is forward regular, then for all $w \in W^{ss}(v)$, $\lim_{t\to+\infty} |f(\varphi^t(v) - f(\varphi^t w)| = 0$, so $f^+(v) = f^+(w)$:

$$\left|\lim_{t\to+\infty} \frac{1}{t}\int_0^t f\circ\varphi^s(v)\, ds - \lim_{t\to+\infty} \frac{1}{t}\int_0^t f\circ\varphi^s(w)\, ds\right| \le \lim_{t\to+\infty} \frac{1}{t}\int_0^t |f(\varphi^s v) - f(\varphi^s w)|\, ds = 0. \qquad (6.2.2)$$

□

As in (6.2.1), let f^- be the Birkhoff averages for φ^{-1}, which we call the backward Birkhoff averages for f. The argument in Lemma 6.2.2 shows that f^- is constant on unstable sets for f continuous. By the Birkhoff ergodic theorem, since φ is invertible, $f^+ = f^-$ μ_{BM}-almost everywhere (cf. [26, Proposition 4.1.3]).

The next piece of the Hopf argument is to note that it suffices to check that f^+ is constant μ_{BM} almost everywhere for continuous f.

Lemma 6.2.3. *If f^+ is constant μ_{BM}-almost everywhere for all continuous f, then every φ^t-invariant L^1-integrable function is constant μ_{BM}-almost everywhere.*

Proof. Let $L^1(\mu_{BM})$ be the L^1 μ_{BM}-integrable functions on SM and $L^1(\mu_{BM},\varphi)$ the φ^t-invariant ones. By the Birkhoff Ergodic Theorem, Birkhoff averaging is a projection from $L^1(\mu_{BM})$ to $L^1(\mu_{BM},\varphi)$. Suppose f^+ is constant μ_{BM}-almost everywhere for all continuous f. Since continuous functions are dense in L^1, any $g \in L^1(\mu_{BM},\varphi)$ can be approximated by continuous f_n. Let A_n be the sets of full measure such that f_n^+ is constant on A_n. By continuity of the projection, $f_n^+ \to g^+ = g$ implies g must be constant on $\bigcap_{n=1}^\infty A_n$, also a set of full measure. □

Now we can prove the main theorem of the section. This proof, like many that have come before it, is an adaptation of Hopf's original proof for uniformly hyperbolic dynamical systems with C^1-foliations by stable and unstable sets. In lieu of smoothness, we immediately have two necessary properties which we will call absolute continuity. These properties are consequences of the original definition of absolute continuity of a foliation.

Recall the definition of $\widetilde{\mu}_{BM}$ (6.1.2):

$$\widetilde{\mu}_{BM}(A) = \int_A \text{length}_\Omega(\ell_v \cap \pi A)\ e^{2\delta_\Gamma \langle v^-, v^+\rangle_x} d\mu_x(v^-) d\mu_x(v^+)$$

Recall also that $2\langle v^-, v^+\rangle_x = \beta_{v^-}(x,p) + \beta_{v^+}(x,p)$. Then by construction, there is a natural induced measure on weak stable and unstable sets given by the decomposition of the products:

$$\widetilde{\mu}_{BM}(A) = \int_{w\in A} \int_{\{v\in A \mid v^+ = w^+\}} \text{length}_\Omega(\ell_v \cap \pi A) e^{\delta_\Gamma \beta_{v^-}(x,p)} d\mu_x(v^-) e^{\delta_\Gamma \beta_{w^+}(x,q)} d\mu_x(w^+)$$

Then if $\widetilde{\mu}_{BM}(A) = 0$, either $\text{vol}_\Omega(\pi A) = 0$ or $\int_{\{v\in A \mid v^+ = w^+\}} e^{\delta_\Gamma \beta_{v^-}(x,p)} d\mu_x(v^-) = 0$ for μ_x-almost every $w \in A$. In other words, the induced measure on $W^{os}(w)$ of A must be trivial for almost every w. Immediate consequences are these absolute continuity (AC) properties:

I. If A has trivial $\widetilde{\mu}_{BM}$-measure but πA has positive measure in Ω, then for μ_x-a.e. $\eta \in \partial_V \Omega$, the set $\{\xi \mid \exists v \in A,\ v^+ = \xi,\ v^- = \eta\}$ has trivial μ_x-measure.

II. Any $\tau \subset \partial_V \Omega$ has trivial μ_x-measure if and only if $\{u \in \Omega \mid u^+ \in \tau\}$ has trivial $\widetilde{\mu}_{BM}$-measure.

We are now prepared to prove the main theorem of the section.

Theorem 6.2.4. *The Bowen–Margulis measure is ergodic for the geodesic flow.*

Proof. By Lemma 6.2.3, it suffices to verify that f^+ is constant almost everywhere for $f \in L^1(\mu_{BM})$ continuous. By Lemma 6.2.2,

(1) f^+ is constant everywhere on $W^{ss}(v)$ for regular v

(2) f^- is constant everywhere on $W^{su}(v)$ for regular v

(3) $f^+ = f^-$ almost everywhere since φ is invertible.

It suffices to prove that f^+ is constant almost everywhere in a neighborhood of a regular vector by transitivity and that $\mu_{BM}(SM_{\text{flat}}) = 0$ (Remark 6.2.5 and Lemma 6.1.1). Extend $f^\pm$ to Γ-invariant functions on the universal cover $S\Omega$ and recall $\widetilde{\mu}_{BM}$ is the Γ-invariant Radon measure on $S\Omega$ which induces the Bowen–Margulis measure.

Suppose $v_h \in S\Omega$ is a regular fixed point for $h \in \Gamma_{\text{hyp}}$. Then by Lemma 5.3.2 there are open neighborhoods $V \ni h^+ = v_h^+$, $U \ni h^- = v_h^-$ in $\partial_V \Omega$ such that for all $\xi \in V, \eta \in U$, the projective line $(\xi\eta)$ is contained in Ω but not in any properly embedded triangle. Let

$$G = \{w \in S\Omega \mid w^+ \in V, w^- \in U\}$$

be our neighborhood of v_h. Note by choice of U, V we have $G \cap S\Omega_{\text{flat}} = \varnothing$: this is fine because $\mu_{BM}(S\Omega_{\text{flat}}) = 0$ (Lemma 6.1.1). Then

$$G' = \{w \in G \mid f^+(w) = f^-(w)\}$$

has full measure in G by (3).

For $\xi \in U$, let $G_\xi = \{\eta \in V \mid \exists u \in G',\ u^- = \xi, u^+ = \eta\}$. Then there exists a regular $w \in G'$ such that G_{w^-} has full $\tilde{\mu}_{BM}$-measure in V. This is because every pair $\xi \in U, \eta \in V$ can be connected by a projective ray contained in Ω, so we can realize the complement $V \setminus G_{w^-} = \{\eta \in V \mid \forall u$ such that $u^- = w^-$ and $u^+ = \eta,\ u \in G \setminus G'\}$. Then $V \setminus G_{w^-}$ has μ_x-measure 0 because $G \setminus G'$ has $\tilde{\mu}_{BM}$-measure 0 by ACI.

By Proposition 5.3.9, μ_x-almost every point in V and hence in G_{w^-} is a proper extremal point. Then by ACII, $v^+ \in G_{w^-}$ and v^+ is a proper extremal point for almost every $v \in G$. By definition of G_{w^-} there is a $u \in G'$ such that $u^+ = v^+$ and $u^- = w^-$. Since $w \in S\Omega_{\text{reg}}$ and v^+ is proper extremal, $W^{ss}(v)$ and $W^{su}(w)$ are defined globally and so there exist t_1, t_2 such that $\varphi^{t_1} u \in W^{ss} v$ and $\varphi^{t_2} u \in W^{su}(w)$.

Then by (1), (2), φ^t-invariance of $f^\pm$, and choice of $G' \ni u, w$,

$$f^+(v) = f^+(\varphi^{t_1} u) = f^+(u) = f^-(u) = f^-(\varphi^{t_2} u) = f^-(w) = f^+(w) \tag{6.2.3}$$

Thus, f^+ is constant $\tilde{\mu}_{BM}$-almost everywhere in a neighborhood of v_h in $S\Omega$. By choosing a fundamental domain for SM containing v_h and taking the intersection of a small neighborhood of v_h with G', we conclude moreover that f^+ on SM is constant μ_{BM}-almost everywhere on a neighborhood of v_h in SM, which suffices by transitivity. □

Remark 6.2.5. It suffices to prove ergodicity in a neighborhood by transitivity (Proposition 2.2.4) and invariance of μ_{BM}. Suppose A is a full measure subset of N, an open neighborhood, such that f^+ is constant on A. Then there exists $v \in N$ with a dense orbit, so we can cover $\varphi \cdot v$ and hence SM by $\varphi \cdot N$. Take a finite subcover $\{\varphi^{t_n} N\}_{n=1}^{k}$ of SM. Then for each $n = 1, \cdots, k$, by invariance of μ_{BM} we have

$$\mu_{BM}(N) = \mu_{BM}(A \cap N) = \mu_{BM}(\varphi^{t_n} A \cap \varphi^{t_n} N) \le \mu_{BM}(\varphi^{t_n} N) = \mu_{BM}(N) \tag{6.2.4}$$

implying $\cup_{n=1}^{k} \varphi^{t_n} A$ is a full measure set in $\cup_{n=1}^{k} \varphi^{t_n} N \supset SM$ on which f^+ is constant.

Corollary 6.2.6. *The μ_{BM}^{x} are unique up to a constant.*

Proof. For any dynamical system, if two measures μ, ν are invariant and equivalent, then their Radon–Nikodym derivative is invariant μ- and ν-almost everywhere. Moreover, if they are ergodic then the Radon–Nikodym derivative must be constant almost everywhere. By definition, Lemma 6.1.2, and Theorem 6.2.4 the μ_{BM}^{x} and μ_{BM}^{y} are invariant, equivalent, and ergodic, so we have the result. □

Thus as noted in the previous section, we can choose a preferred x and normalize so that $\mu_{BM}(SM) = 1$.

Corollary 6.2.7. *The regular set has full measure for the Bowen–Margulis measure.*

Proof. By ergodicity of μ_{BM} for the flow, and φ^t-invariance of SM_{reg}, it suffices to show that $\mu_{BM}(SM_{\text{reg}}) > 0$. Consider $A = \{v \mid v^- \in \partial_V \triangle\}$. Then

$$\mu_{BM}(A) \le \int_{\partial_V \Omega} \int_{\partial_V \triangle} \text{length}_{\Omega}\left((v^- v^+) \cap \pi A\right) d\mu_x(v^-) d\mu_x(v^+) = 0$$

Since there are countably many properly embedded triangles, we can conclude $\mu_{BM}(SM_{\text{reg}}) = 1$. □

Remark 6.2.8. An invariant measure μ for a flow $\varphi^t : X \to X$ where X is a probability space is *mixing* if for all measurable $A, B \subset X$,

$$\mu(\varphi^{-t}(A) \cap B) \to \mu(A)\mu(B) \text{ as } t \to \infty$$

This is equivalent to the following dual definition: for all $f, g \in L^2(X, \mu)$,

$$\int f(\varphi^t(x)) \cdot g(x)\, d\mu(x) \to \int f\, d\mu \cdot \int g\, d\mu \text{ as } t \to \infty$$

A generalization of the Hopf argument originally due to Babillot [2, Theorem 2] for the geodesic flow on rank one manifolds yields mixing of the Bowen–Margulis measure as soon as the length spectrum is dense in $\mathbb{R}^+$, which applies to the Benoist 3-manifolds by Proposition 1.4.7. One is then able to apply the Hopf argument as outlined in Lemma 6.2.2 and Theorem 6.2.4 for weak limits of $f \circ \varphi^t$ in $L^2(\mu_{BM})$. This argument requires working with regular vectors, hence the need for density of the hyperbolic length spectrum.

6.3 A measure of maximal entropy

In this section, we prove that μ_{BM} has entropy $\delta_\Gamma \geq h_{top}(\varphi)$ and conclude that μ_{BM} is a measure of maximal entropy.

We will need some definitions from entropy theory. Let $\mathscr{A}$ be a finite measurable partition of SM. Then the entropy of a partition $\mathscr{A} = \{A_1, \ldots, A_m\}$ with respect to a measure μ is

$$H_\mu(\mathscr{A}) = \sum_{i=1}^{m} -\mu(A_i)\log(\mu(A_i)).$$

A refinement of a partition need only satisfy that all elements of the refinement are subsets of elements from the original partition. The least common refinement of two partitions $\mathscr{A}, \mathscr{B}$ is then $\mathscr{A} \vee \mathscr{B} := \{A \cap B \mid A \in \mathscr{A},\ B \in \mathscr{B}$. The φ-measure-theoretical entropy of μ with respect to $\mathscr{A}$, also know as the Kolmogorov-Sinai entropy, is

$$h_\mu(\varphi, \mathscr{A}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(A_\varphi^{(n)})$$

where

$$A_\varphi^{(n)} = \bigvee_{i=0}^{n-1} \varphi^{-i}\mathscr{A} = \mathscr{A} \vee \varphi^{-1}\mathscr{A} \vee \cdots \vee \varphi^{-(n-1)}\mathscr{A}$$

is the partition consisting of all intersections $\bigcap_{i=0}^{n-1} \varphi^{-i}\mathscr{A}_{j_i}$ over all possible $\{j_1, \ldots, j_{n-1}\} \subset \{1, \ldots, m\}$. Then the measure-theoretic entropy of the pair (φ, μ) is given by

$$h_\mu = h_\mu(\varphi) = \sup_{\mathscr{A}} h_\mu(\varphi, \mathscr{A})$$

By Theorem 3.2.5, φ^t is entropy-expansive with expansivity constant $\epsilon = \mathrm{diam}(M)/3$. By [11, Theorem

3.5], a consequence of h-expansivity is that

$$h_\mu = h_\mu(\varphi, \mathscr{A}) \qquad \text{for } \operatorname{diam}(\mathscr{A}) < \epsilon.$$

By the variational principle, we have that

$$h_{top}(\varphi) = \sup_{\mu\, \varphi^t\text{-inv}} h_\mu(\varphi),$$

and a measure μ which realizes the supremum is a measure of maximal entropy.

We first need a mildly technical lemma to prove the main result of this section.

Lemma 6.3.1. *There exists some $a > 0$ such that*

$$\mu_{BM}(\alpha) \le e^{-\delta_\Gamma n} a$$

for all $\alpha \in \mathscr{A}_\varphi^{(n)}$.

Proof. Let $\mathscr{A}$ be a partition with diameter less than ϵ, the h-expansivity constant for h_{BM}. Note that for all $v \in \alpha \in \mathscr{A}_\varphi^{(n)}$,

$$\alpha \subset \bigcap_{k=0}^{n-1} \varphi^{-k} B(\varphi^k v, \epsilon).$$

Let $v, w \in \alpha$ be regular vectors Then $d_\Omega(\ell_v(t), \ell_w(t)) \le \epsilon$ for all $t \in [0, n]$. Choose $p \in \Omega$ to be the reference point for μ_{BM}. Since ϵ is sufficiently small, we can lift v, w to $\tilde{v}, \tilde{w} \in \Omega$ such that $\tilde{v}, \tilde{w}, p$ are all in the same fundamental domain with diameter $\operatorname{diam}(M)$. Denote by $c_{\xi,x}(t)$ the projective line through any two $\xi \in \partial\Omega$ and x parameterized at unit Hilbert speed such that $c_{\xi,x}(n) = x$. Choose $\xi = w^-$ and $x = \ell_v(n)$. Then

$$\begin{aligned} d_\Omega(c_{\xi,x}(0), p) &\le d_\Omega(c_{\xi,x}(0), \pi\tilde{v}) + d_\Omega(\pi\tilde{v}, p) \\ &\le d_\Omega(c_{\xi,x}(0), \pi\tilde{w}) + d_\Omega(\pi\tilde{w}, \pi\tilde{v}) + d_\Omega(\pi\tilde{v}, p) \\ &\le \epsilon + \epsilon + \operatorname{diam}(M) = 2\epsilon + \operatorname{diam}(M). \end{aligned}$$

Note that $d_\Omega(c_{\xi,x}(0), \ell_w(0)) < d_\Omega(c_{\xi,x}(n), \ell_w(n)) \le \epsilon$ because $\xi = w^-$ and w is a regular vector. Also, clearly

$w^+ \in \mathcal{O}_\epsilon(\xi, x)$. Recall the projection $p_\infty(v) = (v^-, v^+) \in \partial\Omega^2$. The above inequalities imply that

$$p_\infty(\tilde{\alpha}) \subset \bigcup_{\eta \in \mathcal{O}_{2\epsilon+\operatorname{diam} M}(x,p)} \{\eta\} \times \mathcal{O}_\epsilon(\eta, x).$$

Now, for all $\eta \in \mathcal{O}_{2\epsilon+\operatorname{diam} M}(x, p)$, choose $q \in c_{\eta,x} \cap B_\Omega(p, 2\epsilon + \operatorname{diam} M)$. Then

$$\begin{aligned} d_\Omega(x, \pi\tilde{v}) &\leq d_\Omega(p, x) + d_\Omega(p, \pi\tilde{v}) \\ &\leq d_\Omega(q, x) + d_\Omega(p, q) + d_\Omega(p, \pi\tilde{v}) \end{aligned}$$

implying

$$d_\Omega(q, x) \geq d_\Omega(x, \pi\tilde{v}) - d_\Omega(p, q) - d_\Omega(p, \pi\tilde{v}) \geq n - 2\epsilon - 2\operatorname{diam} M$$

because $q \in B_\Omega(p, 2\epsilon + \operatorname{diam} M)$ and $p, \pi\tilde{v} \in M$.

Now, since $-d_\Omega(p, q) \leq \beta_\eta(p, q)$ for all p, q and for μ_{BM}-almost every η (for which the Busemann function is well-defined),

$$\begin{aligned} \mu_p(\mathcal{O}_\epsilon(\eta, x)) &= e^{-\delta_\Gamma \beta_\eta(p,q)} \mu_q(\mathcal{O}_\epsilon(\eta, x)) && \text{transformation rule} \\ &\leq e^{\delta_\Gamma d_\Omega(p,q)} \mu_q(\mathcal{O}_\epsilon(\eta, x)) \\ &\leq e^{\delta_\Gamma(2\epsilon+\operatorname{diam} M)} b e^{-\delta_\Gamma d_\Omega(q,x)} && \text{Lemma 5.3.11} \\ &\leq e^{\delta_\Gamma(2\epsilon+\operatorname{diam} M)} b e^{-\delta_\Gamma(n-2\epsilon-2\operatorname{diam} M)} = \bar{b} e^{-\delta_\Gamma n}. \end{aligned}$$

Note that $p_\infty(\tilde{\alpha}) \subset \mathcal{O}_{2\epsilon+\operatorname{diam} M}(x, p) \times \mathcal{O}_\epsilon(\eta, x)$ and $\frac{d\tilde{\mu}_p}{d\mu_p \otimes d\mu_p}(v^-, v^+) = e^{2\delta_\Gamma \langle v^-, v^+ \rangle_p}$ is bounded above by some $C > 0$ since $2\langle v^-, v^+ \rangle_p = \beta_{v^-}(\pi v, p) + \beta_{v^+}(\pi v, p) \leq 2d_\Omega(\pi v, p) \leq \operatorname{diam} M$ by choice of p in the same fundamental domain as πv. Then

$$\begin{aligned}
\mu_{BM}(\alpha) &\leq \int_{\alpha} \text{length}_{\Omega}(\pi\alpha \cap \ell_v)\, d\bar{\mu}_p(v) \\
&\leq \int_{\alpha} \operatorname{diam} SM\, d\bar{\mu}_p(v) \\
&\leq C \operatorname{diam} SM\, \mu_p(\mathscr{O}_{2\epsilon+\operatorname{diam} M}(x,p))\mu_p(\mathscr{O}_{\epsilon}(\eta,x)) \\
&\leq C \operatorname{diam} SM\, \mu_p(\partial_V \Omega)\mu_p(\mathscr{O}_{\epsilon}(\eta,x)) \\
&\leq a e^{-\delta_\Gamma n}.
\end{aligned}$$

□

Theorem 6.3.2. *The Bowen–Margulis measure is a measure of maximal entropy.*

Proof. By definitions and Lemma 6.3.1,

$$\begin{aligned}
H_{\mu_{BM}}(\mathscr{A}_{\varphi}^{(n)}) &= \sum_{\alpha \in \mathscr{A}_{\varphi}^{(n)}} -\mu_{BM}(\alpha)\log \mu_{BM}(\alpha) \\
&\geq \sum_{\alpha \in \mathscr{A}_{\varphi}^{(n)}} -\mu_{BM}(\alpha)\log(e^{-\delta_\Gamma n} a) \\
&= (\delta_\Gamma n - \log a) \sum_{\alpha \in \mathscr{A}_{\varphi}^{(n)}} \mu_{BM}(\alpha) \\
&= \delta_\Gamma n - \log a
\end{aligned}$$

because $\mathscr{A}_{\varphi}^{(n)}$ is a partition and we normalized $\mu_{BM}(SM) = 1$. Then by Theorem 5.1.4,

$$\begin{aligned}
h_{top}(\varphi) \geq h_{BM}(\varphi) &= h_{\mu_{BM}}(\varphi, \mathscr{A}) \\
&= \lim_{n\to\infty} \frac{1}{n} H_{\mu_{BM}}(\mathscr{A}_{\varphi}^{(n)}) \\
&\geq \lim_{n\to\infty} \frac{1}{n}\left(\delta_\Gamma n - \log a\right) = \delta_\Gamma = h_{top}(\varphi).
\end{aligned}$$

□

6.4 Future directions

An obvious project of interest is proving that the Bowen–Margulis measure is the unique measure of maximal entropy for the geodesic flow. Even more enticing would be existence and uniqueness of equilibrium states for all sufficiently nice potential functions, especially one which would recover an SRB measure as an equilibrium state. Knieper's program for rank one manifolds suffices for uniqueness of the measure of maximal entropy, but both more promising and more general are recent results of Climenhaga and Thompson on sufficient conditions for existence and uniqueness of equilibrium states for continuous, nonuniformly hyperbolic flows [13]. Notably, Climenhaga and Thompson develop a nonuniform specification property which is currently being applied to the framework of geodesic flows of rank one manifolds.

Given Knieper's success for rank one manifolds and the potential application of [13], we are confident of the following conjecture:

Conjecture 6.4.1. *The Bowen–Margulis measure is the unique measure of maximal entropy for the geodesic flow of the Benoist 3-manifolds.*

Appendix A

Hilbert geometries

A.1 Classical results

Fact A.1.1. *If $g \in PGL(n+1,\mathbb{R})$ acts on any convex Ω with attractor g^+ and repellor g^-, then the translation length of g is $\frac{1}{2}\log\frac{\lambda_0}{\lambda_n}$ where λ_0 is the largest eigenvalue of g in modulus and λ_n the smallest.*

Proof. To preserve Ω, the axis of g must be contained in $\overline{\Omega}$. This axis is the projective line from g^- to g^+, and this is where g will realize its minimum translation length. The action of g restricted to this projective line is isometric to the projective action of $\begin{pmatrix} \lambda_0 & \\ & \lambda_n \end{pmatrix}$ on $\mathbb{RP}^1 = \{\text{lines through } 0 \text{ in } \mathbb{R}^2\}/\text{homothety}$, where g^+ is sent to the x-axis and g^- to the y-axis under a projective transformation. Using a cross ratios of four lines arguments allows us to compute immediately that the translation length of g in this nice chart is $\frac{1}{2}\log\frac{\lambda_0}{\lambda_n}$. □

Fact A.1.2 (Benzecri [10] as cited in [9])**.** *Let Ω be a properly convex domain in $\mathbb{RP}^n$. Let $G_n = \mathrm{PSL}(n+1,\mathbb{R})$, $X_{m,0}$ the space of projective domains in $\mathbb{RP}^n$ containing a marked point x_0, and X_m the union of the $X_{m,0}$. Then*

(a) The group G_m acts properly and cocompactly on $X_{m,0}$.

(b) The divisible domain Ω has closed G_m-orbit in X_m.

Corollary A.1.3 ([10] as cited in [9])**.** *For indecomposable properly convex open $\Omega \subset \mathbb{RP}^n$,*

(a) Every nonstrictly convex Ω contains a properly embedded triangle.

(b) $\partial\Omega$ does not contain a side of dimension $n-1$.

(c) Every side of dimension $n-2$ in $\partial\Omega$ admits a unique supporting hyperplane to Ω.

(d) The boundary does not contain any extremal angular *points (unless* Ω *has dimension two).*

Definition A.1.4. A point p in $\partial\Omega$ for an n-dimensional Ω is *angular* if p can be realized as a vertex of a closed n-simplex containing $\overline{\Omega}$. We will say an n-simplex S *supports* Ω (at $p \in \partial\Omega$) if $\Omega \subset S$ and $\partial S \cap \partial\Omega \neq \varnothing$ ($\partial S \cap \partial\Omega \supset \{p\}$). So in particular, if p is the vertex of a supporting n-simplex to Ω then p is an angular boundary point of Ω.

Example A.1.5. A proper, strictly convex $\Omega \subset \mathbb{RP}^2$ with C^1-boundary has only C^1-extremal points in $\partial\Omega$. Each boundary point admits a unique tangent hyperplane, and thus cannot be the vertex of a triangle (2-simplex) which also contains Ω. Thus, there are no angular points for a strictly convex Ω with C^1-boundary (this argument clearly generalizes to any dimension).

In contrast, the teardrop in $\mathbb{RP}^2$ has a non-C^1-boundary point p which admits nonunique supporting hyperplanes. Taking two such hyperplanes intersecting at p and a third hyperplane outside of Ω crossing these two hyperplanes, we form the boundary of a triangle which contains $\overline{\Omega}$ such that p is a vertex of the triangle. Note that it is possible to enclose Ω in this triangle because Ω is properly convex. This makes p an angular point, because Ω is of dimension two so the closed triangle with vertex p is of highest possible dimension.

As a final remark, the vertices of triangles in Benoist's examples are not angular points. They are vertices of supporting 2-simplices, but not supporting 3-simplices, because Ω is divisible and indecomposable, so Corollary A.1.3 applies. The exception in two dimensions amounts to the fact that the triangle is indecomposable, but the tetrehedra can be decomposed into a nontrivial projective sum of two projective lines. For more on this, the curious reader is directed to Benzecri's thesis, Benoist's automorphismes de cones convexes, and Marquis' thesis.

Proof of Corollary A.1.3 in dimension 2. In fact, what we really prove here is a small subcase of Benoist's dichotomy 1.3.2: In dimension two, a divisible, properly convex, projective domain Ω has C^1-boundary if and only if it is strictly convex, because the only divisible Ω which is either not strictly convex or has non-C^1 boundary is a triangle. See Figure A.1.1 and apply Fact A.1.2(b) to conclude if Ω is divisible then there exists a $g_\infty \in \mathrm{PSL}(3,\mathbb{R})$ such that $g_\infty\Omega = \triangle$ and thus Ω must be a projective triangle. □

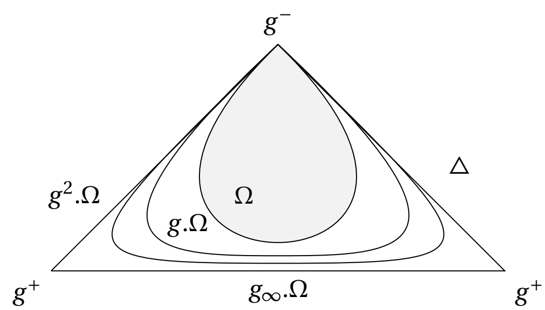


Figure A.1.1: If Ω is not C^1, we can construct a supporting triangle to Ω and realize the triangle $\triangle$ as $\lim_{n\to\infty} g^n\Omega$ where $g \in \mathrm{Stab}\,\triangle$ such that $g^+ > 1 > g^-$ are the eigenvalues for the eigenvectors as pictures.

A.2 Benoist's 3-manifolds

In this section, we summarize some of Benoist's results in [9] on the geometry of a nonstrictly convex, divisible, open subset of $\mathbb{RP}^3$. Brief explanations of ideas in the proof will be provided at times.

Proposition A.2.1 ([9, Proposition 1.3])**.** *There exist discrete Zariski dense sub-groups* $\Gamma < \mathrm{SL}(4, R)$ *which divide open properly-convex but not strictly convex* $\Omega \subset \mathbb{RP}^3$.

Corollary A.2.2 ([9, Corollary 2.4])**.** *Let* $\Gamma_0 < \mathrm{SL}(4\mathbb{R})$ *be a discrete* $\mathbb{Z}^2$ *subgroup preserving an open properly convex* Ω *in* $\mathbb{RP}^3$. *Then the eigenvalues of* Γ_0 *are real.*

Definition A.2.3. A projective transformation $g \in \mathrm{SL}(4,\mathbb{R})$ is *unipotent* if there exists an integer m such that $(g-1)^m = 0$.

Lemma A.2.4 ([9, Lemma 2.8])**.** *Let* $\Gamma < \mathrm{SL}(4,\mathbb{R})$ *be a discrete subgroup which divides an open properly convex* $\Omega \subset \mathbb{RP}^3$. *Then* $\Gamma \setminus \{1\}$ *does not contain any unipotent elements.*

Proposition A.2.5 ([9, Proposition 3.1])**.** *Let* $\Gamma < \mathrm{SL}(4,\mathbb{R})$ *be a discrete torsion-free subgroup which divides an open, properly convex, indecomposable* $\Omega \subset \mathbb{RP}^3$. *Let* $\triangle_1 \neq \triangle_2$ *be two properly embedded triangles in* Ω. *Then for any other* $\triangle \in \mathcal{T}$,

(a) The stabilizer $\Gamma_\triangle$ *contains an abelian subgroup of index 2.*

(b) Every line segment $s \subset \partial\Omega$ *which intersects* $\partial\triangle$ *is included in* $\partial\triangle$.

(c) The closures in $\partial\Omega$ *are disjoint:* $\overline{\triangle_1} \cap \overline{\triangle_2} = \varnothing$.

(d) The stabilizers are disjoint: $\Gamma_{\triangle_1} \cap \Gamma_{\triangle_2} = \{1\}$.

Proof. (a) Every $\gamma \in \Gamma_\triangle$ preserves the three vertices of $\triangle$. By the torsion-free assumption, γ cannot be a cyclic permutation on the vertex set, else γ would fix a point inside of $\triangle \subset \Omega$. Thus, γ might not fix the vertices pointwise but at worst, γ^2 fixes the vertices of $\triangle$. Then γ^2 also stabilizes the projective lines between the vertices and the unique supporting hyperplanes to these lines as an element of $\mathrm{Aut}(\Omega)$ (uniqueness of the supporting hyperplanes is a corollary of a fact of Benzecri's, 2.1 as cited in [9]). In projective space these hyperplanes intersect at a unique point outside of Ω which γ^2 must also fix. Then for all $\gamma \in \Gamma_\triangle$, γ^2 fixes four points in $\mathbb{RP}^3$, and any subgroup of $\mathrm{PGL}(4,\mathbb{R})$ fixing four points is abelian.

(b) If a line segment $s \subset \partial\Omega$ intersects a side I of $\partial\triangle$, then the unique supporting hyperplane to I also contains s. Then the convex closure of I and s must be included in $\partial\Omega$. That convex closure can be at most one dimensional (Corollary 2.2(a) [9]), implying $s \subset I$.

(c) Should two triangles $\triangle_1$ and $\triangle_2$ be such that $\overline{\triangle_1} \cap \overline{\triangle_2} \neq \varnothing$, then the boundaries must also intersect. By part (b), the sides cannot intersect transversally. By Benzecri, if two triangles share the same edge I, then the convex hull of lines $\ell_1 \subset \partial\triangle_1$ and $\ell_2 \subset \partial\triangle_2$ meeting at I must be contained in $\partial\Omega$, implying the lines ℓ_1 and ℓ_2 coincide by the dimension argument.

(d) If $\gamma \in \Gamma_{\triangle_1} \cap \Gamma_{\triangle_2}$, then γ^2 fixes the three vertices of $\triangle_1$ and the three vertices of $\triangle_2$. If $\triangle_1 \neq \triangle_2$, by part (c) these are 6 distinct points and hence γ^2 is the identity and, by the torsion free assumption, γ is the identity.

□

Proposition A.2.6 ([9, Proposition 3.2])**.** *Let $\Gamma < \mathrm{SL}(4,\mathbb{R})$ be a discrete torsion-free subgroup which divides an open, properly convex, indecomposable $\Omega \subset \mathbb{RP}^3$.*

(a) $\mathscr{L} := \bigcup_{\triangle \in \mathscr{T}} \triangle$ is closed in Ω.

(b) Every $\triangle \in \mathscr{T}$ is isolated in $\mathscr{L}$.

(c) The subset $\Gamma / \mathscr{L}$ is a finite union of disjoint tori and Klein bottles.

Proof. (c) Kneser-Haken: for a compact, orientable 3-manifold M, every compact, incompressible family of surfaces which are pairwise disjoint and nonparellel is finite, with cardinality bounded by some constant depending only on M [3]. Two surfaces in a 3-manifold are parallel if they bound a submanifold isomorphic to a surface cross an interval. Should this be the case for two tori in M, then their stabilizers would coincide, contradicting Proposition A.2.5(d).

□

Proposition A.2.7 ([9, Proposition 3.8])**.** *Let* $\Gamma < \mathrm{SL}(4,\mathbb{R})$ *be a discrete torsion-free subgroup which divides an open, properly convex, indecomposable* $\Omega \subset \mathbb{RP}^3$.

(a) For every nontrivial segment $s \subset \partial\Omega$*, there exists a triangle* $\triangle \in \mathscr{T}$ *such that* $s \subset \triangle$.

(b) A point $x \in \partial\Omega$ *admits a unique supporting hyperplane if and only if* $x \notin V$.

Fixing the vertex set of a properly embedded triangle completely characterizes automorphisms which stabilize that triangle. A Corollary of this observation is the following:

Lemma A.2.8. *For all* $g \in \mathrm{Aut}(\Omega)$,

$$g\triangle_1 = \triangle_2 \iff g^{-1}\,\mathrm{Stab}(\triangle_2)g = \mathrm{Stab}(\triangle_1)$$

Proof. If $g\triangle_1 = \triangle_2$, then g maps the vertex set $\{v_1^i\}_{i=1,2,3}$ of $\triangle_1$ to the vertex set $\{v_2^i\}_{i=1,2,3}$ of $\triangle_2$. So g simply permutes the indices of the vertex sets by some σ_g (note that the association of g to $\sigma_g \in S_3$ is a homomorphism). Then for all $h \in \mathrm{Stab}(\triangle_2)$, h fixes pointwise $\{v_2^i\}_{i=1,2,3}$ by the torsion free assumption and for all $i = 1,2,3$,

$$g^{-1}hg(v_1^i) = g^{-1}h(v_2^{\sigma_g(i)}) = g^{-1}v_2^{\sigma_g(i)} = v_1^i$$

so $g^{-1}hg \in \mathrm{Stab}(\triangle_1)$.

Conversely, if $g^{-1}\,\mathrm{Stab}(\triangle_2)g = \mathrm{Stab}(\triangle_1)$, then for all $i = 1,2,3$ and all $h \in \mathrm{Stab}(\triangle_2)$,

$$g^{-1}hg(v_1^i) = v_1^i \iff hg(v_1^i) = g(v_1^i) \iff g(v_1^i) \in \{v_2^i\}_{i=1,2,3}$$

so $g\triangle_1 = \triangle_2$. □

A.3 Volume is not invariant for the geodesic flow

Let (Ω, d_Ω) be a Hilbert geometry on a properly convex Ω. We show in this section that a necessary condition for the volume to be invariant under the line flow is that Ω is actually the ellipse, so (Ω, d_Ω) is hyperbolic space.

First, a volume is any measure μ with a positive, L^1 Lebesgue density. Recall:

Theorem A.3.1 (Livsic [31])**.** *Suppose φ^t is an Anosov flow. Then there exists an invariant μ such that $\mu = f\,d\lambda$ for positive L^1 function f if and only if $|\det d_y\varphi^t| = 1$ at every t-periodic point y.*

Let Γ be the group that divides Ω. Each t-periodic orbit y corresponds to a homotopy class of $\pi_1 M$ and consequently some $\gamma_y \in \Gamma$. Then

$$\begin{aligned} |\det d_y\varphi^t| &= \text{form}(\text{eigenvalue of } \gamma_y) \\ &= 2(n+1)\log(\lambda_0\lambda_{p+1})/\log(\lambda_0/\lambda_{p+1}) \end{aligned}$$

If $\det d_y\varphi^t = 1$, then $2(n+1)\log(\lambda_0\lambda_{p+1}) = \log(\lambda_0/\lambda_{p+1})$ for all $\gamma \in \Gamma$, so the group is not Zariski dense by Benoist's condition for Zariski density of PGL-subgroups.

Theorem A.3.2 (Benoist [7])**.** *Let Γ be a discrete subgroup of $\mathrm{PGL}(n+1,\mathbb{R})$ which divides an open properly convex cone C in $\mathbb{R}^{n+1} \setminus \{0\}$. If C is neither reducible nor symmetric, then the group Γ is Zariski dense in $\mathrm{PGL}(n+1,\mathbb{R})$.*

Then Ω must be reducible or symmetric. We don't care about reducible up to the reduced components, so C must be strictly convex and totally symmetric, ie- hyperbolic space.

Note that this all depended on strict convexity - for Livsic's theorem and to exclude the case of simplicial complexes for symmetric-ness. In the appendix we prove that volume is not ergodic for the geodesic flow in the case of a simplex anyways, which makes it a less promising measure.

A.4 Singular vectors

There are many vectors which remain in the hyperbolic part of $S\Omega$ but do not behave like hyperbolic points. These points, we will see, are in some sense asymptotically flat.

Lemma A.4.1. *Let $v \in SM_{sing}$ be such that any lift $\tilde{v} \in S\Omega$ has positive endpoint $v^+ = \lim_{t\to+\infty} \ell_{\tilde{v}}(t)$ in the boundary of some properly embedded triangle $\triangle$ which projects to a torus or Klein bottle $\mathbb{T}$. Then v is positively asymptotic to $S\mathbb{T}$ under the flow, ie;*

$$d(\varphi^t v, S\mathbb{T}) \to 0 \text{ as } t \to +\infty$$

The analogous result holds for v^-.

Proof. First, the result is immediate for $v \in S\mathbb{T}$, so consider $v \in SM \setminus \cup_{\mathbb{T}\subset M} S\mathbb{T}$. The proof breaks into two cases: (1) when v^+ is in the side of $\triangle$ and (2) when v^+ is a vertex of $\triangle$.

(1) Suppose v^+ is in the side of some properly embedded $\triangle$. Let v_0 be the vertex of $\triangle$ opposite v^+. Consider the hyperplane slicing Ω through the three points v^-, v^+, and v_0, and denote $\mathscr{S}$ by the intersection of this slice with Ω. By Proposition , since v^+ is not a vertex, there is a unique hyperplane supporting Ω at v^+. So there is a unique projective line supporting $\mathscr{S}$ at v^+.

    In this slice, which contains $\ell_{\tilde{v}}$, we can now construct the horosphere at v^+ passing through the footpoint of $\tilde{v}$. Note that since $\mathscr{S}$ is transverse to $\triangle$ by construction, there is a unique vector $\tilde{w} \in S\triangle$ tangent to this horosphere: w is uniquely determined by $w^+ = v^+, w^- = v_0$, and the footpoint of w is on the horosphere at v^+ through the footpoint of v. So by Benoist's construction of horospheres, $d(\varphi^t \tilde{w}, \varphi^t \tilde{v}) \to 0$ as $t \to +\infty$ in the universal cover and also in the quotient SM.

(2) Suppose v^+ is a vertex of $\triangle$. By Corollary A.1.3, the three sides s_1, s_2, s_3 of $\partial\triangle$ each have unique supporting hyperplanes to Ω. Suppose $\{v^+\} = s_1 \cap s_2$, so s_3 is opposite v^+. In projective space, the three supporting hyperplanes intersect at a unique point v_*. This point v_* must be outside of $\overline{\Omega}$, else there would be a line in $\partial\Omega$ supported by two hyperplanes h_i, h_j - for example, by convexity $(v^+ v_*)$ would be in $\partial\Omega$, and h_1, h_2 would be two hyperplanes tangent to this line.

    So consider the hyperplane through v^+, v^-, and v_*, and take $\mathscr{S}$ to be the slice of Ω contained in this hyperplane. Again, $\ell_{\tilde{v}}$ is in $\mathscr{S}$, and the line (v^+, v_*) is the unique projective line supporting $\mathscr{S}$ at v^+ because v^+ is not angular by Corollary A.1.3. Letting $v_0 = \partial\mathscr{S} \cap s_3$, the side of $\triangle$ opposite v^+, we can again choose $\tilde{w}$ uniquely in the same way we did before and the same argument shows that $d(\varphi^t \tilde{w}, \varphi^t \tilde{v}) \to 0$ as $t \to +\infty$ in the universal cover and also in the quotient SM.

□

Appendix B

Regularity of nonstrictly convex Hilbert geometries

B.1 Duality

Definition B.1.1. The *dual of a cone* $\mathscr{C}$ in $\mathbb{R}^{n+1}$ is the collection of positive linear functionals on C. This notion is clearly a projective invariant, so the notion of duality applies to our properly convex Ω in $\mathbb{RP}^n$. The dual of Ω is denoted Ω^*. We think of positive linear functionals $\varphi : \Omega \to \mathbb{R}$ as column vectors also in $\mathbb{RP}^n$, so for $p \in \Omega$ we define $\varphi(p) := \varphi^T p$.

The dual of a group $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$ is denoted ${}^t\Gamma$, and ${}^t\Gamma$ is the image of the dual map

$$\begin{aligned} \partial : \Gamma &\to {}^t\Gamma \\ g &\mapsto (g^{-1})^T \end{aligned}$$

where g^T is a the matrix transpose. Note that Γ and ${}^t\Gamma$ are isospectral.

From these definitions, it is clear that if any Γ preserves Ω, then ${}^t\Gamma$ preseves Ω^*. We need to check that $(\partial(g).\varphi)(gv) = \varphi(v)$ for $g \in \Gamma$ and $\varphi \in \Omega^*$. Since φ is a point in $\mathbb{RP}^n$, $\partial(g)\varphi = (g^{-1})^T\varphi$ is another point in $\mathbb{RP}^n$ and acting as a linear functional, by definition,

$$\big((g^{-1})^T\varphi\big)(gv) = \big((g^{-1})^T\varphi\big)^T gv = \big(\varphi^T g^{-1}\big) gv = \varphi^T v = \varphi(v)$$

Then for any g which preserves Ω, we have $\partial(g)$ preserves Ω^*.

Lemma B.1.2 (2.8 of Benoist [8])**.** *Γ divides Ω if and only if ${}^t\Gamma$ divides Ω^*.*

Proof. The proof uses properties of the cohomological dimension of Γ. Since we can assume Γ to be torsion free and Ω/Γ is compact, then $\mathrm{cd}(\Gamma) = \dim\Omega$ and hence $\mathrm{cd}({}^t\Gamma) = \dim\Omega^*$. As shown above, ${}^t\Gamma <$ $\mathrm{Aut}(\Omega^*)$ and hence preserves the Hilbert metric on Ω^*. Then ${}^t\Gamma$ acts properly on Ω^*, implying $\Omega^* / {}^t\Gamma$ is compact. □

B.2 Hölder regularity and β-convexity

The long-term goal of this section was to make some generalization of Guichard's [23, Theorem 11] to the nonstrictly convex case in three dimensions:

Conjecture B.2.1. *Let* Ω *be a properly convex divisible domain in* $\mathbb{RP}^3$*, divisible by* $\Gamma < \mathrm{PSL}(4,\mathbb{R})$*. Then*

$$\alpha_\Omega = \alpha_\Gamma \text{ and } \beta_\Omega = \beta_\Gamma.$$

At a minimum, Lemma B.3.2 should generalize letting $1 \leq \alpha \leq \beta \leq \infty$.

Some consequences of this section on the dynamics would be nonzero Lyapunov exponents almost everywhere, which could generalize Crampon's result in [19].

B.2.1 Guichard's numbers for a group

For a hyperbolic projective transformation g, with norms of eigenvalues $\lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1} > \lambda_n$, let $\ell_i(g) := \log \lambda_i$. The Guichard-$\alpha$ and Guichard-β for a projective transformation g are then

$$\alpha_g := \frac{\ell_0 - \ell_n}{\ell_0 - \ell_{n-1}}$$
$$\beta_g := \frac{\ell_0 - \ell_n}{\ell_0 - \ell_1}$$

Then for any subgroup $G < \mathrm{PSL}(n+1,\mathbb{R})$ we can define

$$\alpha_G := \inf_{g \in G \text{ hyp}} \alpha_G \qquad \beta_G := \sup_{g \in G \text{ hyp}}$$

We can check easily that $\alpha_{g^{-1}}^{-1} + \beta_g^{-1} = 1$, and that $\alpha_G^{-1} + \beta_G^{-1} = 1$. The norms of the eigenvalues for g^{-1}

are $\lambda_0^{-1} < \lambda_1^{-1} \leq \cdots \leq \lambda_{n-1}^{-1} < \lambda_n^{-1}$. So $\ell_i(g^{-1}) = -\ell_{n-i}(g)$ and

$$\alpha_{g^{-1}} = \frac{\ell_0(g^{-1}) - \ell_n(g^{-1})}{\ell_0(g^{-1}) - \ell_{n-1}(g^{-1})} = \frac{\ell_0(g) - \ell_n(g)}{\ell_1(g) - \ell_n(g)}$$

Then letting $\ell_i := \ell_i(g)$,

$$\alpha_{g^{-1}}^{-1} + \beta_g^{-1} = \frac{\ell_1 - \ell_n}{\ell_0 - \ell_n} + \frac{\ell_0 - \ell_1}{\ell_0 - \ell_n} = 1$$

This extends to the group level.

Fact B.2.2. *For all* $G < \mathrm{PSL}(n+1, \mathbb{R})$,

$$\alpha_G^{-1} + \beta_G^{-1} = 1$$

Proof.

$$\begin{aligned} \frac{1}{\alpha_G} + \frac{1}{\beta_G} &= \frac{1}{\inf_G \alpha_{g^{-1}}} + \frac{1}{\sup_G \beta_g} \\ &= \sup \alpha_{g^{-1}}^{-1} + \inf \beta_g^{-1} \\ &= \sup(1 - \beta_g^{-1}) + \inf \beta_g^{-1} \\ &= 1 - \inf \beta_{g^{-1}} + \inf \beta_g^{-1} = 1 \end{aligned}$$

□

B.3 Regularity of $\partial\Omega$

Definition B.3.1. A curve f is α-Hölder on a neighborhood U if there is a $C > 0$ such that for all $x, y \in U$,

$$|f(x) - f(y)| \leq C\|x - y\|^{\alpha}.$$

Consider $\partial\Omega$ as locally the graph of a curve in $\mathbb{R}^n$. Then define

$$\alpha_\Omega := \sup\{\alpha \leq 2 \mid \partial\Omega \text{ is } \alpha\text{-Hölder}\},$$

$$\beta_\Omega := \inf\{\beta \geq 2 \mid \partial\Omega \text{ is } \beta\text{-convex}\}.$$

Lemma B.3.2 ([23, Lemma 4]). *If* $1 < \alpha \le 2 \le \beta < \infty$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *then*

$$\partial\Omega \text{ is } \alpha\text{-Hölder} \iff \partial\Omega^* \text{ is } \beta\text{-convex}$$

Proof. The pieces of the proof amount to the following:

- $\Omega \subset \Omega' \implies \Omega'^* \subset \Omega^*$
- the dual of $y > x^\beta$ is $y^* > x^{*\alpha}$ locally for functions

□

At no point in this proof is strict convexity demanded, but it is implied by the strict bounds on α and β. We will need to work around this in the nonstrictly convex case but this is not major.

Corollary B.3.3 (5.3 of Benoist [8]). *For* Ω *a strictly convex proper domain in* $\mathbb{RP}^n$, *divided by a discrete* $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$,

$$1 < \alpha_\Omega \le \alpha_\Gamma \le 2 \le \beta_\Gamma \le \beta_\Omega < \infty$$

The key to the proof of this Corollary is

Proposition B.3.4 (5.1 of Benoist [8]). *For* Ω *as in Corollary 5.3 of [8],*

- *Every* $g \in \Gamma$ *is biproximal and stabilizes a unique geodesic for the Hilbert metric on* Ω.
- *There are no homotopically trivial closed geodesics on* M.
- *Every homotopy class* $[g]$ *in* $\pi_1(M)$ *is represented by a unique closed geodesic on* M *of length* $\ell_0(g) - \ell_n(g)$.

In the nonstrictly convex Benoist 3-manifolds, it is very possible to find group elements which are not biproximal. These projective transformations necessarily stabilize properly embedded triangles in Ω, and their connection to the regularity of $\partial\Omega$ is explored in the next section. What's more, aside from these few exceptions, the hyperbolic group elements impose similar structure on $\partial\Omega$ as they did in the strictly convex case. An analogous Proposition to Benoist's arises and makes itself useful in our context.

B.3.1 Regularity of a fixed point at infinity

These computations follow from Benoist's outline in [8] and Lukyanenko's work in his master's thesis [32].

Let g be a hyperbolic group element with attracting fixed point $g^+ \in \partial\Omega$. Since all hyperbolic group elements are biproximal, we can assume the eigenvalues of g are $\lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1} > \lambda_n$ with associated eigenvectors $v_0, v_1, \ldots, v_n$.

Choose a chart for Ω such that $v_0 = g^+$ is sent to the origin, $v_n = g^-$ is sent to the y-axis at infinity, and v_k for some $0 < k < n$ is sent to the x-axis at infinity. Graph the curve of $\partial\Omega$ intersecting this hyperplane, and g must stabilize that curve.

In this local chart for g, a point $p = (x, y)$ is sent by g^m to $((\lambda_k/\lambda_0)^m x, (\lambda_n/\lambda_0)^n y)$. Let K be the intersection of $\partial\Omega$ with this chart in local coordinates for v_0, v_k, k_n. Then

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{m}\log d_E(g^m p, g^+) &= \lim_{m\to\infty}\frac{1}{m}\log\sqrt{\frac{\lambda_k^{2m}}{\lambda_0^{2m}}x^2+\frac{\lambda_n^{2m}}{\lambda_0^{2m}}y^2}\\ &= \lim_{m\to\infty}\frac{1}{2m}\log\big(\lambda_0^{-2m}(\lambda_k^{2m}x^2+\lambda_n^{2m}y^2)\big)\\ &= \lim_{m\to\infty}\frac{1}{2m}\big(\log(\lambda_0^{-2m})+\log(\lambda_k^{2m}x^2+\lambda_n^{2m}y^2)\big) = \ell_k-\ell_0\end{aligned}$$

$$\lim_{m\to\infty}\frac{1}{m}\log d_E(g^m p, T_{g^+}K) = \lim_{m\to\infty}\frac{1}{m}\log\big((\lambda_n^m/\lambda_0^m)y\big) = \ell_n-\ell_0$$

Oversetting limits and applying L'Hopital's rule, we uncover the Hölder regularity of the invariant curve $K \subset \partial\Omega$ in these local coordinates for g.

$$\lim_{m\to\infty} \frac{\log d_E(g^m p, T_{g^+}K)}{\log d_E(g^m p, g^+)} = \frac{\ell_n - \ell_0}{\ell_k - \ell_0} =: \alpha_k$$
$$1 = \lim_{m\to\infty} \frac{\log d_E(g^m p, T_{g^+}K)}{\log\left(d_E(g^m p, g^+)^{\alpha_k}\right)} = \lim_{m\to\infty} \frac{d_E(g^m p, T_{g^+}K)}{d_E(g^m p, g^+)^{\alpha_k}}$$

For $\alpha < \alpha_k$ the limit will diverge, so the Hölder regularity at g^+ is at most α_k.

Note that for $k = n-1$, $\alpha_{n-1} = \alpha_g$. Also, λ_{n-1} is the smallest eigenvalue for g other than the repellor point g^- in $\partial\Omega$. So any point in the (v_0, v_{n-1}, v_n)-chart for $\partial\Omega$ tends to g^+ along an α_g-regular curve, but no other curve crossing g^+ in $\partial\Omega$ has this regularity.

Rather, a dense open set of points will be attracted to the (v_0, v_1, v_n) hyperplane crossing Ω, and these points tend to g^+ with regularity $\alpha_1 = \beta_g$.

We can now conclude that the Hölder regularity at g^+ is at most α_g and the β-convexity at g^+ is at least β_g.

B.3.2 Tranversal curves to C^0 vertices of triangles

First, we'll need a proposition which generalizes one of Benoist's.

Proposition B.3.5. *For the Benoist 3-manifolds [9],*

1. *For every $g \in \Gamma$, at least one of g or g^{-1} is proximal.*
2. *If g is hyperbolic then g is biproximal.*
3. *There are no homotopoically trivial closed geodesics in M.*
4. *Every hyperbolic homotopy class $[g] \in \pi_1 M$ is represented by a unique closed geodesic on M of length $\ell_0(g) - \ell_n(g)$.*

Proof. 1. If neither g nor g^{-1} are proximal, then g has eigenvalues $\lambda, \lambda, 1/\lambda, 1/\lambda$ associated to eigenvectors v_0, v_1, v_2, v_3. Then the $(v_0 v_1 v_2)$-triangle and the $(v_1 v_2 v_3)$-triangle must both be properly embedded in Ω, which forces the entire $(v_0 v_1 v_2 v_3)$-simplex to be properly embedded in Ω. Thus in dimension three, Ω must be the simplex, which is excluded from the Benoist 3-manifolds as a reducible divisible domain.

□

Note that for the same reasons, g cannot have 3 redundant eigenvalues.

An important geometric observation for our next computation is that for g to stabilize a properly embedded triangle in Ω, three eigenvectors for g must be in $\partial\Omega$ and the fourth must be associated to an intermediary eigenvalue isolated from the extreme eigenvalues. Let v_ℓ be the eigenvector of g outside of $\partial\Omega$, and note that $v_\ell = v_1$ or v_2, the eigenvectors associated to the middle eigenvalues λ_1 and λ_2. Let $\bar{v_\ell}$ be the other eigenvector.

Choose a local chart for g which sends $v_0 = g^+$ to the origin, v_1 to the x-axis, v_2 to the y-axis, and $v_3 = g^-$ to the z-axis. By Benoist's geometric characterization, Ω admits a unique supporting hyperplane along the projective line $(v_0 v_3)$, which in our local chart is associated to the $v_3 v_\ell$-plane (this is the xz-plane if $v_\ell = v_1$ and the yz-plane if $v_\ell = v_2$).

Then we can compute again for K the curve associated to $\partial\Omega$ in this chart and $p \in K$,

$$\lim_{m\to\infty} \frac{1}{m} \log d_E(g^m p, T_{g^+} K) = \log \bar{\lambda_\ell} - \log \lambda_0$$

$$\lim_{m\to\infty} \frac{1}{m} \log d_E(g^m p, g^+) = \frac{1}{2m} \log\left((\lambda_1/\lambda_0)^2 m x^2 + (\lambda_2/\lambda_0)^{2m} y^2 + (\lambda_3/\lambda_0)^{2m} z^2\right)$$

$$= \log \max\{\lambda_1, \lambda_2\} - \log \lambda_0$$

Appendix C

Dynamical systems

In this Appendix we prove basic facts from ergodic theory for general dynamical systems.

C.1 Basic Ergodic Theory

Fact C.1.1. *If Γ is acting on a compact topological space (X,μ) with Borel probability measure μ by measurable μ-preserving functions, then Γ is ergodic if and only if every Γ-invariant function $f : X \to \mathbb{R}$ is constant μ-a.e.*

Proof. Suppose Γ is ergodic for μ and f is a Γ-invariant integrable function which we assume to be continuous and compactly supported, so that $f(X)$ is compact. For each ϵ, define $A_\epsilon = \{x \in X \mid f(x) \geq \epsilon\}$. If $\epsilon = \sup f(x) < \infty$, then $\mu(A_\epsilon) = 0$. Then let $-\infty \leq \epsilon_0 = \inf\{\epsilon \mid \mu(A_\epsilon) = 0\}$. By ergodicity, since A_{ϵ_0} is Γ-invariant A_{ϵ_0} must have full μ-measure by choice of ϵ_0. Moreover, if $B = \cup_{n \geq k} A_{\epsilon_0 + 1/n}$, then $\mu(B) \leq \sum \mu(A_{\epsilon+1/n}) = 0$ and on the full measure set $A_\epsilon \setminus B$, $f = \epsilon_0$. Since continuous functions are dense in L^1, it follows that every Γ-invariant function is constant almost everywhere.

If every Γ-invariant function is constant a.e., then consider for any Γ-invariant set A the indicator function I_A, which is Γ-invariant and hence constant μ-almost everywhere. Then either $\mu(A) = \int I_A \, d\mu = 0$ or $\mu(X \setminus A) = \int I_{X \setminus A} \, d\mu = 0$. □

Fact C.1.2. *If (X,μ) is a probability space, then a unique measure of maximal entropy is ergodic.*

Proof. See [26, Lemma 4.1.10 and Corollary 4.3.17]. Any nonergodic measure μ can be written as a linear combination of nontrivial singular measures, ie- there exists an $\alpha \in (0,1)$ such that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ and $\mu_1 \perp \mu_2$. Without loss of generality, we can take $h(\mu_1) \leq h(\mu_2)$. Then

$$h(\mu) = h(\alpha\mu_1 + (1-\alpha)\mu_2) = \alpha h(\mu_1) + (1-\alpha)h(\mu_2) = \alpha(h(\mu_1) - h(\mu_2)) + h(\mu_2) \leq h(\mu_2)$$

Thus, a nonergodic measure cannot be a unique measure of maximal entropy. □

C.2 Ergodicity is generic for hyperbolic geodesic flows

Let $\mathcal{M}$ denote the space of φ^t-invariant probability measures. Denote also $\mathcal{P}$ for the set of periodic orbits in SM, and $\mathcal{M}_{\mathcal{P}}$ the space of φ^t-invariant probability measures supported on orbits in $\mathcal{P}$. When we refer to orbits in $\mathcal{P}$, we will choose a representative point in SM. In this section, we prove that ergodicity and full support are generic properties of invariant measures, as Coudene and Schapira do for geodesic flows of rank one manifolds in [16]. The set of Dirac probability measures supported on periodic orbits of φ^t is dense in the set of all invariant ergodic Borel probability measures [16].

Lemma C.2.1. *Let φ^t be a continuous flow on SM, a countable separable metric space. If φ^t is a transitive flow which satisfies the Anosov Closing Lemma 2.3.1, then $\mathcal{M}_{\mathcal{P}}$ is a dense G_δ in $\mathcal{M}$ with the weak-$*$ topology.*

Proof. First, since $\mathcal{M}$ is convex and linear, it suffices to consider $\mu \in \mathcal{M}$ ergodic. Suppose also that μ is not supported on $\mathcal{P}$.

Let $\epsilon > 0$. Let $f \in C(SM)$ and let $\delta(\frac{\epsilon}{2}) > 0$ be determined by the continuity of f. We can chose a μ-generic point x which has dense orbit in $\operatorname{supp}\mu$. (This follows from Poincaré Recurrence, cf. [26, Proposition 4.1.18]) In particular, we can choose $x \in \operatorname{supp}(\mu)$ because the compliment is μ-null. By the Birkhoff Ergodic Theorem, for all T greater than a sufficiently large $T_{\frac{\epsilon}{2}}$,

$$\left|\frac{1}{T}\int_0^T f(\varphi^t x)dt - \int f d\mu\right| < \frac{\epsilon}{2}$$

Since $\varphi \cdot x$ will accumulate infinitely often on $x \in \operatorname{supp}(\mu)$, we can apply the Anosov Closing Lemma (Theorem 2.3.1) to the orbit segment with $d(\varphi^T(x), x) < \delta'$ and let $T - \delta > T_{\frac{\epsilon}{2}}$. Choose δ' sufficiently small that the resultant $y \in \mathcal{P}$ with orbit length $S \in]T-\delta, T+\delta[$ will δ-shadow $\varphi \cdot x$ for time $T' = \min\{T, S\}$. Define an invariant measure supported on $\varphi \cdot y$ by

$$\nu := \frac{1}{T'}\int_0^{T'} \delta_{\varphi^t y} dt$$

Then by choice of sufficiently small $\delta(\frac{\epsilon}{2})$,

$$\left|\frac{1}{T'}\int_0^{T'} f(\varphi^t x)\,dt - \int f\,d\nu\right| = \left|\frac{1}{T'}\int_0^{T'} f(\varphi^t x)\,dt - \frac{1}{T'}\int_0^{T'} f(\varphi^t y)\,dt\right|$$
$$\leq \frac{1}{T'}\int_0^{T'} |f(\varphi^t x) - f(\varphi^t y)|\,dt < \frac{\epsilon}{2}$$

And by choice of $T' \geq T - \delta > T(\frac{\epsilon}{2})$ determined by convergence of Birkhoff averages, we can conclude $\nu \in \mathcal{M}_{\mathcal{P}}$ and $\mu \in \mathcal{M}$ are ϵ-close in the weak$*$ topology:

$$\left|\int f\,d\nu - \int f\,d\mu\right| = \left|\int f\,d\nu - \frac{1}{T'}\int_0^{T'} f(\varphi^t x)\,dt + \frac{1}{T'}\int_0^{T'} f(\varphi^t x)\,dt - \int f\,d\mu\right|$$
$$\leq \left|\frac{1}{T'}\int_0^{T'} f(\varphi^t x)\,dt - \int f\,d\nu\right| + \left|\frac{1}{T'}\int_0^{T'} f(\varphi^t x)\,dt - \int f\,d\mu\right| < \epsilon$$

□

We conclude that ergodic invariant measures are dense in $\mathcal{M}$. We can show furthermore that ergodic measures form a G_δ. This is well known, and the proof is included here.

Lemma C.2.2. *Suppose $\varphi^t : SM \to SM$ satisfies the hypotheses of Lemma C.2.1. Then ergodic invariant measures form a dense G_δ in $\mathcal{M}$.*

Proof. Take a countable basis of open balls in $C(SM)$, a separable metric space. Choose centers f_i. For each f_i, construct sets $U_{n,m} \subset M$ as follows:

$$U_{n,m} = \left\{\mu \in \mathcal{M} : \frac{1}{T}\int_0^T f_n \circ \varphi^t\,dt \to \int f_n\,d\mu + \epsilon \text{ for some } \epsilon \in \left(-\frac{1}{m}, \frac{1}{m}\right)\right\}$$

Each $U_{n,m}$ is open in the weak-$*$ topology. The ergodic measures are those for which Birkhoff averages converge for all continuous functions. By continuity and choice of the f_i, we can write all ergodic invariant measures as the countable intersection of the open $U_{n,m}$, thus forming a G_δ set. □

If we lose the ergodicity condition and ask for full support, the same properties hold:

Lemma C.2.3. *Assume again the hypotheses of C.2.1. Then invariant measures of full support form a dense G_δ in $\mathcal{M}$.*

Proof. First, consider some measure μ such that $\operatorname{supp}\mu \neq SM$. By density of periodic points, a consequence of Anosov Closing and transitivity, we can construct a probability measure ν which has full support. The periodic points are countable (correspond to the group), and we can choose weights c_n summing to 1, giving us an invariant probability measure of full support:

$$\nu = \sum_{n \in \mathbb{N}} c_n \delta_{p_n}$$

where δ_{p_n} is the Dirac mass on the periodic orbit p_n.

Now we approximate μ with the following sequence of invariant, fully supported probability measures:

$$\mu_k = \left(1 - \frac{1}{k}\right)\mu + \frac{1}{k}\nu$$

And lastly, we show the invariant, fully supported probability measures are a G_δ.

Consider the countable collection $\mathcal{O} := \{O_{p,m} : p \text{ periodic orbit}, \ m \in \mathbb{N}\}$ where $O_{p,m} = B(p, \frac{1}{m})$. If O is open and invariant, then by density of periodic points we can write O as a union of elements of a subcollection of $\mathcal{O}$. Re-index the collection as $\mathcal{O} = \{O_n\}_{n \in \mathbb{N}}$.

Now we construct open subsets of $\mathcal{M}$ in the weak$*$ topology as follows: for each $O_n \in \mathcal{O}$, define

$$V_n := \{\nu \in \mathcal{M} : \nu(O_n) > 0\}$$

Clearly the complement of each V_n is closed since any weak limit of measures in V_n also gives 0 measure to O_n.

If an invariant measure is full supported, then it is in the intersection of all the V_n. If an invariant measure is *not* fully supported, then $\mu(O) = 0$ for some open and invariant O, and $\mu \notin V_n$ for some n. Hence, the set of fully supported invariant measures is equal to $\cap_{n \in \mathbb{N}} V_n$, a G_δ set.

□

Theorem C.2.4. *The set of ergodic probability measures of full support invariant by the geodesic flow of a Benoist 3-manifold form a G_δ-dense subset of the set of invariant probability measures.*

Proof. Since the intersection of two dense G_δ sets is still a dense G_δ and the geodesic flow satisfies transitivity (Proposition 2.2.4) and Anosov Closing (Theorem 2.3.1), the theorem is an immediate corollary of Lemmas C.2.2 and C.2.3. □

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