MATHEMATICAL THINKING: A SEMIOTIC COORDINATION OF IDEAL-MATERIAL COMPONENTS

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Abstract. In this case study, I provide evidence for, and operationalize (fill the gaps and provide specific analytic methods to replicate) the "dynamic unity of material and ideal components" framework for mathematical cognition (Radford, 2014, p. 268). Mathematical thinking, according to this framework, is a "semiotic coordination" (Radford, 2014, p. 268) of ideal and material components, such as speech, gestures, tactility, rhythm, perception, sensuous imagination, and actions with cultural artifacts. The analytical methods in the present case study involve a frame-by-frame analysis of data collected during an interview with a seventh grade student exploring the shadow of a figurine. In this case study, I identified the following six ideal-material components described by Radford: outer speech, gestures, tactility, rhythm, perception, and actions with cultural artifacts. To illustrate Radford's framework, I identify ideal-material components that co-occur in space and time, discuss ways in which they might be semiotically coordinated in terms of carrying identical, supporting, complementary or additional meaning, and argue that these are the components of learner's mathematical thinking. Lastly, I draw implications for mathematics education and research.

Introduction

What are students thinking with when doing mathematics? According to the traditional cognitivist approach students are thinking with their brains through information processing of mental representations. Post-cognitivist approaches, on the other hand, paint a much broader picture. General post-cognitivist perspectives include views of cognition as situated within the bodily interactions with the physical, cultural and social environment rather than being confined to one's head. Sociocultural perspectives view cognition as situated within the culture and social interactions (Cole, 1998; Lave & Wenger, 1991; Rogoff, 2003; for review see Tenenberg & Knobelsdorf, 2014). Cognition, as situated, is further characterized as embodied (e.g., Davis &

Markman, 2012; Lakoff & Núñez, 2000; Rosch, Thompson & Varela, 1992), or, for example, as embedded/distributed (e.g., Hutchins, 1995), or extended into the agent's environment (e.g., Clark & Chalmers, 1998). Human capacities for cognition within the head are limited, but are increased significantly when embedded, extended, and distributed across resources available outside the limits of the physical brain. Namely, it has been argued that external representations (e.g., written text, tables, drawings) "serve as vehicles for thought" (Kirsh, 2010, p. 445), thus as tools mediating learning and thinking, and as "forms of knowing" (Pérez Echeverría & Scheuer, 2009, p. 2). Gesture (e.g., finger counting), often considered a form of external representations, is seen as a "tool for thinking" (Goldin-Meadow, 1999, p. 428), situating thinking within our bodies. Arguably, perception and action serve a cognitive role that goes far beyond a mere sensory input and motoric output for computations carried out in the mind (Hurley, 1998). Literature on gestures (e.g., Arzarello, Paola, Robutti, & Sabena , 2009; Goldin-Meadow, 1999; Radford, 2009), perception (Hurley, 1998), actions with tools (Rasmussen, Nemirovsky, Olszewski, Dost, & Johnson , 2004) or, for example, interactions with external representations (Andersen, Scheuer, Pérez Echeverría & Teubal, 2009, Kirsh, 2010) help us understand how human cognition spans beyond the limits of the physical brain to include the body and the environment. Understanding mechanisms by which cognition is situated within the bodily interactions with the physical, social and cultural environment is the main problem this paper is trying to address.

Recent studies in the domain of mathematics educational research have made a significant contribution to the post cognitivist approaches to mathematics cognition (e.g., Abrahamson, Lee, Negrete & Gutiérrez ., 2014; Nemirovsky, Rasmussen, Sweeney & Wawro, 2012; Radford, 2014). Nemirovsky et al. (2012), for example, took multimodal approach and, alongside speech, looked into students' bodily activities such as gesture, gaze, posture and pointing and found that students' ideas related to addition and multiplication of complex numbers "are expressed in and constituted by perceptuo-motor activity" (p. 288). Authors, for example, provided evidence that students gestured rotation in the complex plane when discussing multiplication by the imaginary unit *i*. Similarly, Abrahamson et al. (2014) studied how students developed the mathematical idea of proportionality while interacting with a piece of motor-sensor technology enabled to provide an instant feedback on learner's bodily movement. In this study, learners had a task to keep the computer screen green, without knowing that in order to do so they needed to preserve the proportion between their arms. During this kinesthetic activity, learners expressed and 'enacted' the mathematical idea of proportionality. Studies reviewed above provide evidence that mathematical thinking is situated (e.g., offloaded, distributed, constituted) within the body and the environment, including gesture and interactions with cultural artifacts such as a computer screen. Situatedness implies acknowledgment that the meaning depends on the characteristics of the bodily interactions with the environment, and is the key to understanding the difference between the computational and the embodied view of the mind. Mathematical thinking the participant in this study engages with is a form of algebraic thinking, in which she seeks to generalize an algebraic rule that describes the numerical pattern in her function table. Her thinking, as this study will reveal, is situated within her bodily interactions with the function chart on paper as well as on the computer screen.

The present study aims to provide evidence for, and operationalize (fill the gaps and provide specific analytic methods to replicate the studies), Radford's (2014) theoretical framework for mathematical cognition as situated within culture, body, and material world. Radford (2014) refers to mathematical thinking as the "dynamic unity of material and ideal components" (p. 268) and claims that mathematical cognition is a semiotic coordination of ideal and material components such as gestures, interactions with cultural artifacts, speech (inner and outer), tactility, perception, rhythm, sensuous imagination, and actions with cultural artifacts. The main distinctions from Radford's framework are: 1) I collapsed terminology for ideal and material components into ideal-material to emphasize that their separation is artificial; 2) I provided definitions for each of the six Radford's component of thinking; 3) I provided specific analytic methods to replicate; and 4) I demonstrated analysis of semiotic coordination through the analysis of identical, supporting, complementary, additional, mismatched (different or contradicting) information carried by different components. This study contributes to the literature on situated cognition in general, and embodied cognition in particular, by providing replicable analytical methods that help characterize mathematical thinking as a semiotic coordination of components in a variety of sensory modalities.

Radford talks about a "dynamic unity" as a semiotic, thus meaning-making, coordination of ideal and material components. I contribute to Radford's framework by including the following definitions: outer speech (utterances), gestures (eye and body movement), rhythm (regular movement or sound, such as rhythmic tapping), perception (interpreting sensory information, such as visual images, into a coherent understanding of something), actions with cultural artifacts (e.g., drawing, writing, including idiosyncratic and canonical representations, carrying out a standard subtraction algorithm on paper), and tactility (perception through employing the sense of touch). While keeping the original Radford's classification, I also acknowledge that gestures can be rhythmic, actions are a form of gesture, and that tactility is a form of perception. Radford considers gestures, language, and perception as material (2009, 2011) and imagery and inner speech as immaterial (2011) components of thinking It is my interpretation that the distinction Radford makes between ideal and material components lies strictly within the observer, a researcher, in terms of whether a component is visible, thus material (e.g., gesture, outer speech, tactility, rhythm, perception, action) or invisible, thus ideal (e.g., inner speech, imagination). However, I argue, if a visible component, such as a gesture, is to participate in the construction of meaning, and not be an "empty" gesture, then such a gesture also belongs to the world of ideas and therefore is also ideal in nature, and we cannot consider it to be strictly material. Similarly, an invisible component such as imagination could potentially involve simulated actions and therefore be reminiscent of the material word, and thus we cannot call it strictly ideal. The boundaries between the world of ideas and the material world are blurred and, consequently, the differentiation between the material and ideal components of thinking becomes artificial. Many authors, on the other hand, while acknowledging the "phantom of dualism" (Pérez Echeverría & Scheuer, 2009) choose to maintain the distinction between the outer and inner worlds to allow greater precision in methods, or for simplicity in order to support the distinction made by an observer. In this paper I will refer to what Radford calls ideal and material components (2014)—or, at times, as material-ideational components (2011)—as ideal-material components, to further reinforce that their separation is artificial, and thus to highlight the situated and embodied nature of a human mind and activity. At times, Radford also refers to thinking as "an ideal-material form of reflection and action." (Radford, 2014, p. 268)

In what follows, I first pose a research question, then review Radford's framework for mathematical cognition, discuss ways through which I operationalize his framework, and illustrate it through a case study. Finally, I discuss implications for mathematics education and research in mathematics education.

Research Question

The present case study is part of a larger project, which explores how students construct mathematical ideas while engaging in physical situations and how mathematics instruction and curricula might integrate physical situations to enhance learning. I refer to physical situations as instances of the material world in which students can see, touch, smell, hear, taste, and interact with material artifacts and tools. Specifically, in this study, I explore the following research question:

How does a middle school student coordinate gestures, tactility, rhythm, perception, actions with cultural artifacts, and outer speech to reason algebraically while exploring numerical patterns in a function table?

Cognitive Framework for Mathematical Thinking

Radford's framework for mathematical cognition builds on his earlier work on the development of algebraic thinking (e.g., Radford, 2011, 2012), on "the theory of knowledge objectification" according to which "thinking is considered to be a mediated reflection in accordance with the form or mode of the activity of individuals" (Radford, 2008, p. 218), on his work on gestures (e.g., Radford, 2009), and on his study of signs and signification (e.g., Radford, 2000, 2003).

For Radford (2011, 2014), thinking is "an ideal-material form of reflection and action, which occurs not solely in the head but also in and through a sophisticated semiotic [in regards to mathematics] coordination of speech, body, gestures, symbols, and tools" (Radford, 2014, p. 268). In his studies on the development of algebraic thinking, he provides examples of young students who coordinated components such as gestures, utterances, and perception, into a single "dynamic unity" in order to describe what a 50th term in a certain algebraic sequence looks like.

For example, he reported on a student who made a certain horizontal gesture to accompany his utterance “like this.” The gesture and the utterance, in that case, conveyed complementary, yet algebraic, aspects of meanings regarding student’s generalization what a distant term in an algebraic sequence looks like. The fact that the significance of each component in the unity depends on its role within the unity gives the unity a dynamic nature different from being just a collection of items whose significance is predetermined and constant. A component, such as utterance “this”, might mean one thing in one dynamic unity, and something completely different in another, which, for example, could be further determined by a student’s pointing gesture. The dynamic nature reflects the variability in what each individual component signifies depending on a dynamic unity it belongs to. Whether a dynamic unity is a stable or an unstable formation, or whether it emerges, from context or elsewhere, or is constructed does not relate to how Radford describes its dynamic nature. What each component signifies to the learner, thus what meaning it conveys, within a dynamic unity is central to understanding how such component becomes a component of thinking.

While some authors (e.g., Abrahamson et al., 2014; Nemirovsky et al., 2012) make connections between bodily activity, such as gesture, and mathematical thinking, Radford’s framework has greater explanatory power in terms of highlighting semiotic bonds among a variety of ideal-material components—beyond just gestures and thought—students use to think mathematically. Radford’s framework not only helps us provide evidence that mathematical thinking is situated within the socio-cultural and physical interactions with the environment but also explains how students coordinate these interactions in multiple sensory modalities to form a unified meaning. Comparable to Radford’s is the “semiotic bundle model” (Arzarello et al., 2009) according to which gesture is a semiotic resource, thus a sign, whereas “[a] semiotic bundle is a system of signs" (p. 100). Radford's "dynamic unity" framework provides a more holistic approach to mathematical thinking than Arzarello's "semiotic bundle", which only concerns gestures, words (outer speech), and inscriptions (a form of interaction with cultural artifacts) as semiotic mediation tools, while omitting other components Radford identified such as inner speech, tactility, rhythm, perception, sensuous imagination, and actions with cultural artifacts beyond inscriptions.

Development of the Algebraic Thinking

The "dynamic unity" framework for mathematical thinking originated in Radford's earlier work on the development of students' algebraic thinking. Literature on the development of early algebraic reasoning helps us frame Laura's exploration of numerical patterns in function tables. Existing research on young students' capacities to think algebraically have examined their ability to generalize, justify, represent, and reason with mathematical structures and relationships (e.g., Blanton, Levi, Crites, & Dougherty, 2011; Kaput, 2008). Such studies have focused on such topics as students' meaning of the equal sign (e.g., Carpenter, Franke & Levi, 2003), and understanding of proof (e.g., Knuth, Choppin & Bieda., 2009). A few important studies have focused on a functional approach to algebra in the early grades (e.g., Blanton & Kaput, 2011; Carraher, Schliemann, Brizuela, & Earnest, 2006; Carraher, Schliemann, & Schwartz, 2008). When looking for a pattern in a function table young learners might look into how values vary sequentially within a single column (see "recursive patterning" in Blanton & Kaput, 2011; also, "scalar approach" or "scalar relationship" in Vergnaud, 1983, 1988, and in Martínez & Brizuela, 2006), or they might attend to a relationship between the two variables (see "covariational thinking" and "correspondence relationship" in Blanton & Kaput, 2011; also, "functional approach" or "functional relationship" in Vergnaud, 1983, 1988, and in Schliemann, Carraher, &

Brizuela, 2001). In the present study Laura's "scalar approach" is a form of early algebraic thinking.

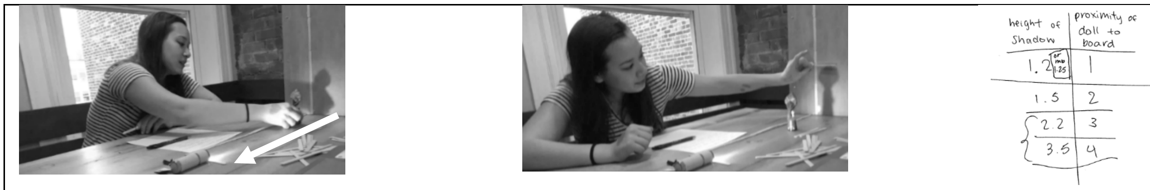
Method

Participant

The participant in this study was a seventh grade student, Laura (pseudonym). Laura learned about my study from another participant who was recruited through emails sent to parents I knew. Participants were told that the purpose of the interview was for the researcher to look into what mathematics questions they might have about a real-world situation and how they might look for answers to those questions. I interviewed Laura outside of her regular school hours in a private home setting.

Data Collection

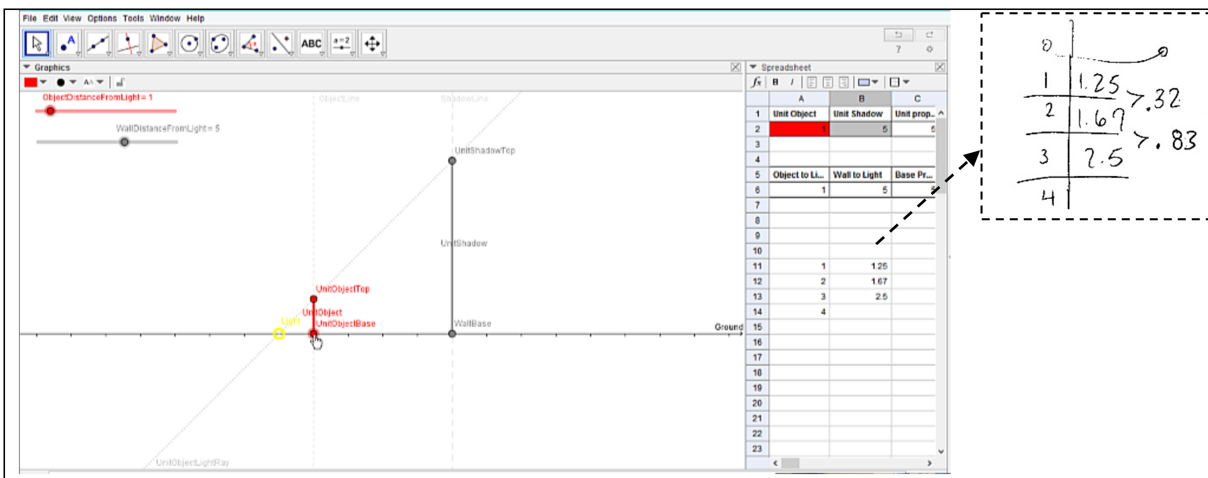
The primary data collection instrument used in the study was a clinical interview (Ginsburg, 1997),which aims to understand students' thinking but also progress they make during the interview. At the beginning of an hour-long interview Laura was presented with a physical situation involving a desk, flashlight, a figurine, a pretend wall, and measuring sticks (see Figure 1). Laura's task in this interview was to make predictions about shadow height as the figurine moved between the pretend wall and the flashlight. To do so, she chose to record the results of measurements in a function table. She decided on the variables and placed them in two columns: height of the shadow and proximity of the object to the wall (see Figure 1, top right). She moved the figurine one measuring stick at a time away from the wall, measured the shadow, and recorded the data as pairs of "Height of shadow" (her left column) and "Proximity of doll to board" (the right column in her function table). She assumed that the wall is at a constant 90-degree angle to the desk and that the flashlight is set on the desk.



[14:38] Laura: From five Popsicle sticks away [from the wall], on zero inches height [of flashlight], [shadow] will be one… and 90--ish degrees off the wall [against the desk], it [shadow] will be one and a half Popsicle sticks. […] So, the doll is two Popsicle sticks away from the wall-board, and three Popsicle sticks from the light.

Figure 1. Laura collecting and recording measurement data.

Following the physical exploration, the interview incorporated a computer-based simulation using *Geogebra* software (see Figure 2, top left). Laura copied the measurements from the computer screen into a new function table (see Figure 2, top right) and proceeded to look for a numerical pattern that would explain the shadow growth numerically.



[00:27:10.11] Interviewer: See if we can find out a way to predict, for example [step] four.

(ACTIVITY) Laura copying the function table from the spreadsheet portion of the Geogebra simulation onto her paper.

Figure 2. Shadow heights based on computer-generated measurements

My intention was to incorporate software-generated measurements of changes in shadow height in an idealized condition in which the height of the light is restricted to zero. Such computerized measurements comply with the indirect proportion model in Figure 3. If the light source is, for example, 6 units away from the wall, and the doll is, for example, 2 units away from the light, the shadow according to the formula in Figure 3 will be 6 divided by 2, thus 3 units in height.

shadow(when doll is x units away from the light) = distance-wall-to-light / x.

Figure 3. Object's shadow is indirectly proportional to its distance from the light source

Data Sources

Data collected during the interview included a video recording made with a stand-alone video camera, two video recordings generated by *Camtasia Studio* (one capturing Laura's face and another capturing the screen), an audio stream made with a stand-alone voice recorder, and scanned versions of her written work. Raw video data was transcribed, and the transcript was synchronized with the video and audio recording into a single movie file. For the analysis, I coordinated video data with Laura's written work.

Data Selection

To answer my research question, I selected for analysis two short episodes (< 5 minutes) in which I observed a number of different ideal-material components from the hour-long interview, in which Laura explores numerical patterns in a function table. In the first episode, Laura looked for an additive difference in shadow height and did not find a numerical pattern in the data, whereas in the second, she found the pattern through a multiplicative comparison of differences between shadow heights. During these two episodes, Laura looked for a numerical pattern of shadow change based on the function table from the computer-generated measurements. In both episodes she employed the "scalar approach" (Vergnaud, 1983, 1988) and sequentially compared pairs of values in the column for shadow height, top to bottom, at first in search for additive (Episode 1), and later for multiplicative (Episode 2) differences. This approach is different than the "functional approach" (Vergnaud, 1983, 1988) modeled by the formula in Figure 3.

Episode 1 is a short, one-minute-long episode in which Laura analyzed the table in search of a numerical pattern of change in shadow height (right column of her table). The real-life situation that Laura was exploring in this episode had the distance between the wall and the light source set to five Popsicle sticks (in Figure 3, distance-wall-to-light = 5). Such parameters resulted in two decimal places, and some rounding, for the height of the shadow (according to the formula provided in Figure 3), both of which might have prevented Laura from looking beyond the additive differences between the pairs of values in the shadow height column and discovering a multiplicative pattern.

In Episode 2, which was about three-minutes long, Laura also looked for a numerical pattern, but this time she used a function table that reflected the distance of six Popsicle sticks between the wall and the light source. This distance resulted in shadow height expressed as a whole number or a number with only one decimal space, which made the multiplicative pattern easier for Laura to identify (in Figure 3, distance-wall-to-light = 6).

For the purpose of analysis of distinct clusters of mathematical thinking, each of the two episodes has been divided into five individual frames. Frames are chosen so that each frame contains one or more ideal-material components co-occurring in space and time. Arzarello et al., 2009 refer to an analysis of co-timed semiotic resources as *Synchronic Analysis* as opposed to the *Diachronic analysis,* which takes into consideration semiotic resources that could be related to student's previous activities. In my analysis I only observed co-timed semiotic resources and therefore the analysis I performed would be classified as synchronic in Arzarello's terminology.

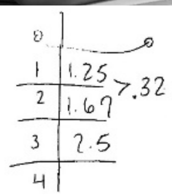
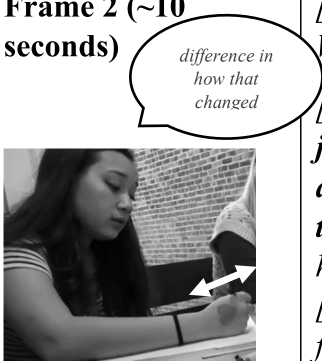
Analytical Methods

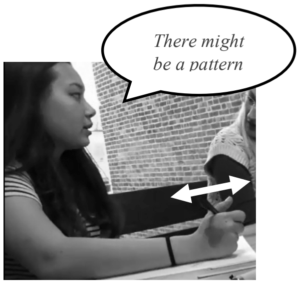
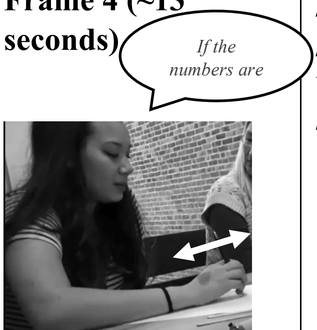
Analysis of the video data from the two episodes involved a frame-by-frame dissection of Laura's behavior. I searched for six types of ideal-material components as listed in Radford's framework (which I defined earlier): outer speech, gestures, rhythm, perception, actions with cultural artifacts (e.g., drawing, writing, carrying out a standard subtraction algorithm on paper), and tactility (perception through employing the sense of touch). I organized and presented data in a table. I then analyzed these ideal-material components and interpreted what they signified to Laura; the ways in which she coordinated them in space, time, and semiotics; what kind of information they were conveying to her; and what role they served within the dynamic unity of each frame. In my analysis, I will point out ways in which outer speech, gestures, rhythm, perception, actions with cultural artifacts, and/or tactility were semiotically coordinated into a dynamic unity in regards to mathematics. In order to determine if two or more components co-occurring in time and space are also semiotically coordinated I will demonstrate that co-occurrence was not incidental (which is also possible) and that it did involve cognition. To do so, I will argue that co-occurring components I observed carried identical, supporting, complementary or additional mathematical information (I found no evidence of other types, such as when components are carrying mismatched, different or contradicting information). At the same time, to contribute to the main argument of this study, I will interpret mathematical thinking that is taking place and argue that it happened through a semiotic coordination of ideal-material components. I will consider evidence of different ideal-material components carrying identical, supporting, complementary, or additional meaning as evidence of their semiotic coordination. Analysis of semiotic coordination in terms of whether co-occurring components confirm, support, complement, extend or contradict one another in meaning appears to be lacking, or at best is implicit, in Arzarello's at al. (2009) synchronic or diachronic analysis and is my unique contribution to Radford's framework. In summary, my analytic method operationalizes the "dynamic unity" framework through an analysis of six types of ideal-material components described above by providing the following specific analytical methods to replicate: 1) data is analyzed frame-by-frame; 2) frames are selected based on components co-occurring in space and time; 3) for each frame interpretations are made to determine the significance (conveyed meaning) of each component based on the information they contribute (identical, supporting, complementary, or additional); and 4) mathematical thinking is discussed as a semiotic coordination of components in the frame.

Results

In this section, I first present data from the two episodes in a tabular format, frame by frame (Episode 1 is about the additive scalar relations, and episode 2 is about multiplicative scalar relations). The table helps reveal the links between the 1) video and written data, 2) transcripts and descriptions, 3) evidence of ideal-material components found in the data, and 4) my interpretations about the student's mathematical thinking. This, in turn, addresses parts of the research question by 1) listing the specific gestures, tactility, rhythm, perception, actions with cultural artifacts, and outer speech Laura is using, and 2) listing aspects of Laura's mathematical thinking.

Figure 4 is a frame-by-frame summary of Episode 1 in which I present video data and written work (column 1); transcripts and descriptions (column 2); description of Laura's gestures, outer speech, perception, and actions with cultural artifacts I observed in the data (column 3); and my interpretations about Laura's mathematical thinking (column 4). Following the structure of Figure 4, I present, in Figure 5, a frame-by-frame summary of Episode 2, which contains evidence of rhythm and tactility, in addition to the four ideal-material components I observed in Episode 1.

Images from video / Scanned written work	Transcript / Description	Ideal-Material Components	Mathematical Thinking
Frame 1 (~6 seconds)  	*[28:14] Laura: [lifts the pencil, pauses, looks at the function table for five seconds]* *[28:19] Laura: [writes 0.32, continues to look at the table]*	Gestures (lifted and lowered the pencil, wrote .32). Actions with cultural artifacts (wrote alongside her table the additive difference of .32). Perception (looked for five seconds at the table before writing .32).	L(aura) carried out a calculation (she wrote .32).
Frame 2 (~10 seconds) 	*[28:20] Interviewer:* ***What are you doing?*** *[28:21] Laura**: I am just trying to find the difference in how that** [shadow heights]* ***changed.*** *[still looking at the function table she pauses for 2 seconds, then lifts the pencil and gestures with it vertically. During this gesture, she raises her eyebrows briefly]*	Outer speech. Gestures (made a repeated vertical motion with the pencil, and raised eyebrows). Perception (looked for two seconds at the table).	L looked for a difference in change in vertical values (utterance “to find the difference in how that changed” followed by a vertical hand gesture). L was possibly surprised (eyebrow lift). A possible, although there is no further evidence for this interpretation, is that she realized that the additive difference might not be constant.

Frame 3 (~3 seconds)  	*[28:30] Interviewer:* ***And why do you think that matters to find the difference?*** *[28:32] Laura:* ***Because,*** *[makes vertical gesture with her fingers, then looks up at the interviewer]* ***there might be a pattern.***	<u>Outer speech</u>. <u>Gestures</u> (made a repeated vertical motion with her fingers, and looked up at the interviewer. Direct eye contact with the interviewer. Maintained vertical body posture with a slight turn towards the interviewer).	L signified a vertical pattern (vertical finger gesture simultaneous with the utterance “Because there might be a pattern”). L might be thinking that a pattern might reveal itself as a numerical relationship between values in a column of her table.
Frame 4 (~13 seconds)    	*[28:41] Laura: [gestures vertically with pencil]* ***If the numbers are …*** *[24:48] Laura: [pauses and gazes up in the air, lifts eyebrows, then looks at the interviewer]* ***… mmm-multiples of each other, maybe …*** *[28:51] … [makes a horizontal hand gesture]* ***or related in some way.***	<u>Outer speech</u>. <u>Gestures</u> (made a repeated up-down (vertical) motion with the pencil along the right column of the table, gazed in the air, lifted eyebrows, made a repeated left-right (horizontal) gesture above the table). Rhythm (repeated up-down gesture)	L signified a vertical pattern (utterance “if the numbers are” … preceded by a vertical hand gesture). L signified a multiplicative pattern (utterance “mmm-multiples of each other” preceded by a gaze in the air and eyebrow lift). L signified a horizontal pattern (utterance “or related in some way” accompanied by a horizontal hand gesture).

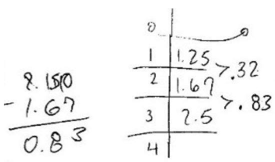
Frame 5 (~13 seconds)  	*[29:04] Laura: [manually computed .83]* ***OK. I didn't find an obvious pattern***.	<u>Actions with cultural artifacts</u> (calculated using standard algorithm for subtraction, and wrote the additive difference of .83 in the table, a cultural artifact). <u>Outer speech</u>. <u>Perception</u> (looked at the table during interaction with it).	L computed the next additive difference (manually performed a subtraction algorithm and wrote .83 in the table). L signified that she did not find the pattern (utterance).

Figure 4. Episode 1: Semiotic coordination of gesture, outer speech, perception, and actions with cultural artifacts

In frame 1, Laura's perception of and interaction with the function table co-occur in space and time. To demonstrate that this co-occurrence is not incidental I will now argue that it involves semiotic coordination. Laura is focusing on the differences between the numbers in the right column, thus is taking a "scalar approach" (Vergnaud, 1983, 1988), as she performs a calculation and denotes the additive difference of 0.32 alongside and in-between the two numbers 1.25 and 1.67 in the table (should be 0.42). She looks at the table for five seconds before writing 0.32. Looking is not incidental but is accompanied with perceiving the additive difference between the two numbers. What makes this coordination of action and perception semiotic, rather than just a co-occurrence in space and time, is that her perceived meaning of the relationship between the three numbers is identical to the meaning signified through her writing: the number 0.32 stands for the difference between the two numbers 1.25 and 1.67, which she later confirms with her utterance in Frame 2. In other words, she perceives number 0.32 as a difference between 1.25 and 1.67, and she writes 0.32 to denote the difference between the two numbers. Her perception and her writing thus both share the same identical information, which is that 0.32 is the difference between 1.25 and 1.67. Frame 1 is an example of how Laura semiotically coordinated perception and action with a cultural artifact to think mathematically. To show that these components are semiotically coordinated I argued that they conveyed identical meaning to one another. The meaning is mathematical as it relates to the difference between two numbers.

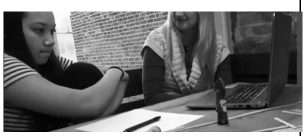
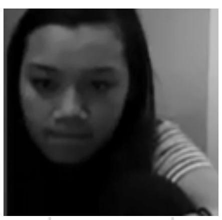
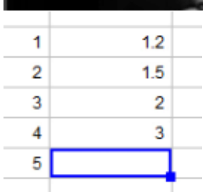
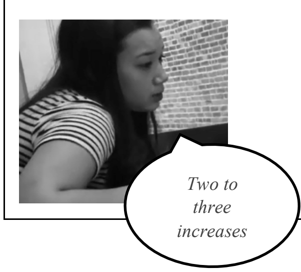
In Frame 2, Laura continued to look at the table “to find the difference in how that [shadow heights] changed” while making a vertical gesture above the table. This gesture for her provided complementary (feature-enhancing) meaning to her utterance in terms of vertical spatial orientation for the values in the right column of the table, which she additively compared to find the difference. Laura’s utterance and gesture, by virtue of carrying complementary mathematical meaning for her, are thus semiotically coordinated. Arguably, Laura’s perception of the table as the organization of numbers with a possible pattern in the difference in values in the right column is identical to the information signified simultaneously by her gesture and utterance. Vertical spatial orientation of Laura’s gesture, and her utterance, together signify her “scalar approach”, thus finding the additive difference between the two subsequent values in the shadow heights denoted in the right column. In Frame 2, Laura thinks mathematically through a semiotic coordination of outer speech, perception, and vertical hand gesture. To show that these components are semiotically coordinated I argued that they conveyed complementary, and in certain instances identical, meaning to one another. The meaning is mathematical as it reflects a “scalar approach” to finding a pattern in a function table.

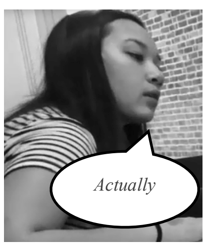
In Frame 3, Laura simultaneously gestured vertically above the table while saying, "Because there might be a pattern." The utterance and the gesture occurred simultaneously, and therefore they were coordinated in space and time. Each component carried complementary semiotic information: the utterance signified a possibility of a numerical pattern within the data recorded in the table, whereas her gesture signified that the numerical pattern could possibly be present in the vertical spatial orientation within the table. The significance of a component, according to Radford, is determined by its role within the dynamic unity. The role of the vertical gesture within the unity that included the utterance was to provide a spatial orientation for the possible numerical pattern in the table. The vertical gesture alone might mean something entirely different to Laura in a different situation (the dynamic notion of the unity). Accompanied by the utterance that there might be a pattern, the significance of the vertical gesture is a possibility for a vertical pattern in the table. Similarly, in a different situation, the utterance alone might mean something else to Laura and not the vertical spatial orientation of a possible pattern. Together, these two components complement one another in meaning, and therefore create a semiotic unity in regards to mathematics as Laura talks about a numerical pattern in her table. This single unit of meaning, which was a result of a semiotic, temporal, and spatial coordination of the two ideal-material components, represents a dynamic unity. In Frame 3, Laura's mathematical thinking occurred through a semiotic coordination of gesture and utterance in which components complemented one another in meaning. The meaning is mathematical as it relates to finding a pattern in a function table while comparing numerical entities vertically within a column.

In Frame 3 Laura was thinking that there might be a vertical pattern (within a column), but in Frame 4, she made a shift in her thinking and considered the possibility of a pattern as a horizontal relationship in the data, that is, as a relationship between the two columns (functional approach). Her horizontal gesture and utterance "or related in some way" carry complementary information and together create a dynamic unity of meaning that there might be a horizontal pattern in numbers in the table. The significance of the gesture is determined by its role in the dynamic unity. The gesture provides the horizontal spatial orientation for such a pattern. The significance of the utterance is also determined by its role in the unity. The utterance signifies that the numbers could be related in some way to create such a pattern. Just like in Frame 3, both the utterance and the hand gesture are ideal-material components of Laura's mathematical thinking.

In Frame 5, Laura coordinated the action of physically carrying out the subtraction algorithm to find the difference between 1.67 and 2.5; writing the differences .83 alongside the table; her perception of the inscriptions; and outer speech ("I didn't find an obvious pattern"). All four components co-occur in space and time but they also relate to one another semiotically. Physical execution of the subtraction algorithm to find the difference, and writing that difference alongside the table, both carry identical meaning for Laura, in terms of the difference between 1.67 and 2.5 being .83. Perception provides spatial information to Laura on where her inscriptions should go on the paper when she copies the difference .83 from where she carried out the algorithm to the place alongside the table and between the two numbers being subtracted. For Laura, I interpret, her utterance signifies additional meaning as it suggests that finding the difference between 1.67 and 2.5 did not help her find a pattern to continue the previously found difference of .32. Frame 5 is an example of Laura's mathematical thinking though a coordination of actions with cultural artifacts (carrying an algorithm and writing), perception and outer speech, in which the semiotics is reflected in components in the dynamic unity carrying identical or additional meaning.

In summary, the analysis of Episode 1 reveals that Laura's mathematical thinking consisted of spatial, temporal, and semiotic coordination of gesture, perception, actions with cultural artifacts, and/or outer speech. I now move to the summary and analysis of Episode 2.

Images from video / Scanned written work	Transcript / Description	Ideal-Material Components	Mathematical Thinking
Frame 6 (~17 seconds)   	*[37:09] Interviewer:* ***Ok, so. Do that one*** *[move red doll (see Figure 5) four units away from gray board, (step 4)****] and pause at five*** *[when the red line is five units away from gray (step 5)]****. And then, do not move it*** *[to step 5]****. Just try to predict.***	<u>Gestures</u> (briefly raised her eyebrows, sighed, lifted her knees up and wrapped her arms around them, bit her lip, stared at the computer screen for a total of seventeen seconds, with an increasingly intense laser focus, and almost complete physical stillness for about 6 last seconds, stillness accompanied with frequent blinking and movement of her eyes)*. <u>Perception</u> (looked at the screen for 17 seconds, eye movement, blinking).	Prompted by the interviewer to predict step 5 Laura focused, with great intensity, on the computer screen for an extended period of 17 seconds. I interpret the intensity and duration of perception as an intense thinking episode. What comes next, in frame 7, when Laura verbalizes a pattern she finds, supports this interpretation.
Frame 7 (~11 seconds) 	*[37:37] Laura: [still looks at the function table on the computer screen but lets go of her knees and leans over the table] [Step]* ***two*** *[corresponds to shadow of 1.5]* ***to***	<u>Gestures</u> (changed her body position, gazed up in the air, raised both eyebrows). <u>Outer speech</u> (ended the sentence with a	L calculated the difference (looked at the screen, said "increases by," gazed in the air, and said "a third"). L questioned whether

  	*[step]* ***three*** *[corresponds to shadow of 2]* ***increases*** *[gazes up in the air for a brief moment]* ***by a third, and it does the same*** *[some nodding]* ***as it did*** *[raises eyebrows]* ***before*** *[in the previous "jump" from step 1 at 1.2 to step 2 at 1.5]****?***	question). <u>Perception</u> (looked at the screen for about 10 seconds in total).	the same fraction would apply to the preceding difference (said "does the same as it did," raised both eyebrows, and said "before?").
Frame 8 (~45 seconds)    	*[37:50] Laura: [tilts head to the left] Actually* *[37:52] Laura: [takes a measuring stick, then points repeatedly up and down the screen]* ***point five*** *[the difference between 1.5 and 2]****, which is a third of one point five.*** *[...]* *[38:07] Laura: [repeatedly taps the screen with the Popsicle stick]* ***Increases by point five, and point five is a third of one point five, so increases by a third*** *[taps the screen]****.*** *[38: 33] Laura:* ***And then increases by*** *[gazes in the air]* ***a***	<u>Gestures</u> (tilted her head, gazed in the air). <u>Tactility</u> (held and touched the screen with a measuring stick). <u>Actions with cultural artifacts</u> (pointing and vertical motion with a Popsicle stick, tapping the table on the screen). <u>Outer speech</u>. <u>Rhythm</u> (co-timed tapping the screen and saying "third" and "half"). <u>Perception</u> (looked at the screen on and off).	L changed her mind (said "Actually" and tilted her head). L confirmed that the additive difference between 1.5 and 2 is a third of 1.5 (- said "point five, which is a third of one point five" and made a vertical gesture with the Popsicle stick; - said "Increases by point five, and point five is a third of one point five, so increases by a third" and proceeded to tap on the screen with a Popsicle stick simultaneously with saying "a third"). L computed the next additive difference

	half *[taps at the screen with the Popsicle stick]**.**		(said "And then increases by," gazed in the air, said "a half," and proceeded to tap on the screen).
Frame 9 (~29 seconds)   	*[38:56] Laura:* ***So, it*** *[shadow of 2 in step 3 to shadow of 3 in step 4**]** **increases by one** [looks down], **and one** [points at the screen**]** **is half of two** [taps the screen, then looks away**]**.* ***So, half of two is one, so one plus two is three. But then*** *[tilts her head**]**,* ***it has increased by point five*** *[gestures vertically with the Popsicle stick**]**,* ***which is a third*** *[looks away**]*** ***of one point five. So, I just, I assumed that one point two to one point five is, it is zero point three, which is a fourth of*** *[1.2]**...***	Tactility (held a measuring stick, perceived the dragging of the stick vertically along the screen, perceived tapping the screen). Gestures (pointing at the screen, tapping on the screen, vertical gesture with the Popsicle stick, looking at the screen, looking away from the screen, head tilt). Outer speech. Perception.	L restated her pattern (utterance "So it increases by one, and one is half of two […]" while pointing, tapping, and gesturing vertically with the Popsicle stick. She alternated looking at the screen and into space).
Frame 10 (~13 seconds) 	*[39:59] Laura: [copies down the function table onto the paper. Then, writes the additive differences. Then writes down and circles the fraction.]* ***So, I would assume this*** *[fraction between steps four and five]* ***would be one*** *[writes*	Actions with cultural artifacts (wrote table, denoted additive differences and fractions) Outer speech. (coordinated witting numbers five and six with saying it).	L made a prediction for the value in the fifth step (utterance "…so this would probably be five five, no six six…" writing).

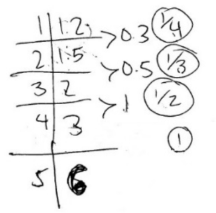
	*and circles one* **] so this** *[value in the right column in step five]* ***would probably be five five*** *[writes five at the same time]* **, no six six** *[simultaneously writes six over five multiple times]* **.**	Perception.	

Figure 5. Episode 2: Semiotic coordination of tactility, rhythm, gesture, outer speech, perception, and actions with cultural artifacts.
* Some of the gestures are discussed as components of framing or epistemic affect (see future research).

The summary of Episode 2 in Figure 5 reveals dynamic unities that are of similar nature, but also the ones that are of an entirely different nature than the dynamic unities discussed in Episode 1. Frame 6 in particular is qualitatively very different from any other frame in either of the two episodes. The lack of outer speech and writing in this frame leaves us with seemingly little or no evidence to base our interpretations of Laura's mathematical thinking. However, her perception of and interaction with the function table during 17 seconds of staring at it, eye movement and blinking, are components of an intense thinking episode. Laura's culminating utterance "two to three increases by a third […]" in which she describes the pattern she discovered, supports this interpretation. Perception of and interaction with the table, which is a form of action with a cultural artifact, co-occur and form a dynamic unity through shared meaning of the relationships in the table.

Frame 7 presents another thinking session that lasted about 10 seconds, with Laura looking on and off at the function table on the computer screen while verbally describing the pattern she found: "[Step] two [corresponds to shadow of 1.5] to [step] three [corresponds to shadow of 2] increases [gazes up in the air for a brief moment] by a third, and it does the same [some nodding] as it did [raises eyebrows] before [in the previous "jump" from step 1 at 1.2 to step 2 at 1.5]?" She perceived and described in words the multiplicative difference between shadow heights of 1.5 and 2 from the right column on the screen as an increase by a third. There is no reason to doubt that for her the perceived relationship between the two values in the right column 1.5 and 2 was identical to what she described in words as "increases by a third". She then experimented in words and through perception if the same relationship also described the multiplicative difference between 1.2 and 1.5 in the previous step. Through this shared (identical) meaning of the multiplicative difference Laura coordinated the perception of the table with the utterance to think mathematically about the multiplicative pattern she is discovering.

Frame 8 brings yet another aspect of dynamic unities, in which a head tilting gesture is synchronized with the utterance "Actually." Building on Radford's idea that the significance of a component rests within the context of the dynamic unity it belongs to, I argue that the head tilt for Laura was not a mere change of the physical position of her head but also a signifier of a shift in her thinking. Previously, she questioned whether the increase was always by a third, "by a third as it did before?" which after saying "Actually" she then changed to the statement "increase by a third and then increase by a half." The head tilt and the utterance "Actually" were simultaneous, yet sandwiched between those two different thoughts. With this, I argue that tilting of her head and her utterance "Actually" both carried identical information to signify a shift in thinking. Therefore, the particular shift in thinking was a semiotic coordination of gesture and utterance.

Another novelty that Episode 2 brings is tactility and rhythm. In particular, Frame 8 exemplifies a variety of ideal-material components: outer speech, gesture, and perception were coordinated with tactility (holding a Popsicle stick and tapping the screen with it) as well as rhythm (saying "third" or "half" while tapping the screen). I claim that all ideal-material components of this unity supported one another in what they signified, which is best described by her long utterance: "Actually, point five, which is a third of one point five. […] [repeatedly taps the screen with the Popsicle stick] increases by point five, and point five is a third of one point five, so increases by a third [taps the screen]. And then increases by [gazes in the air] a half [taps the screen with a Popsicle stick]." The role of tactility in this dynamic unity was not just having something in her hand for no particular reason at all, but rather, to use the Popsicle stick as an extension of her hand to point and tap loudly, and perhaps increase her sensation of touch, which might be a form of interaction with a function table on a computer screen. The role of rhythm (tapping synchronized with utterance), on the other hand, was to support the utterance by putting emphasis on the relevant parts of the pattern ("half" and "third"). I claim that both the tactility and the rhythm supported the utterance and therefore were part of the same semiotic unity.

In Frame 9 Laura semiotically coordinated gestures, tactility, perception, and outer speech. Her long utterance not only co-occurs in space and time with tapping on the screen, head tilt, and dragging the popsicle stick up and down the screen but these gestures also support her utterance. Namely, taping supports the utterance by adding emphasis to the numbers she is verbalizing "half of two [taps the screen, then looks away];" her head tilt is part of shifting her thinking to the next pair of numbers "But then [tilts her head]" whereas dragging the Popsicle stick vertically along the screen supports that she is looking for a vertical pattern between values in the table.

In Frame 10 Laura semiotically coordinated outer speech, perception, and actions with a cultural artifact in order to make a prediction for shadow of the figurine five steps away from the light. Just like in Episode 1, Laura's mathematical thinking follows the "scalar approach" (Vergnaud, 1983, 1988) as she first additively compares subsequent numbers in the right column by denoting their differences alongside the table (0.3, 0.5, 1), then determines that each difference is a fraction of the top number, and finally that fractions follow an increasing pattern $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, 1. She then adds 3 to itself and writes 6 as her prediction for shadow in step 5 while saying "six" twice. Her utterance and writings for Laura convey complementary and at times identical information. Frame 10 is an example of a dynamic unity in which Laura semiotically coordinated actions with cultural artifact, perception and outer speech.

This in-depth analysis of multiple frames within two short episodes in which I looked for six different types of components of mathematical thinking—gestures, perception, outer speech, tactility, rhythm, and actions with cultural artifacts—reveals that Laura's mathematical thinking is a semiotic coordination of two or more of these components. It is important to state that unobservable components such as inner speech or sensuous imagination, which Radford also includes among the components of mathematical thinking, were not accounted for in my analysis, but that should not exclude the possibility that Laura also coordinated them into the dynamic unities that I observed and discussed.

Summary and Discussion

In this study, I pointed out moments in which Laura's gesture, perception, outer speech, tactility, rhythm, and actions with cultural artifacts were coordinated in time and space and with respect to what they signify in finding a numerical pattern. Her thinking was mathematical in the sense that she was exploring shadow heights within the right column of a function table to find a general pattern. Laura found a recursive pattern by using the “scalar approach” which is a form of an early algebraic thinking.

Radford’s framework views mathematical thinking as a “semiotic coordination” (Radford, 2014, p. 268) of ideal and material components. The main question one might ask is on what grounds components such as gestures, interactions with cultural artifacts, or, for example, rhythm, could count as components of mathematical thinking. To answer this question I demonstrated in a frame-by-frame analysis of ideal-material components that certain components not incidentally co-occurred in space and time but participated in the construction of mathematical meaning for the student by providing identical (e.g., Frame 1), supporting (Frames 8 and 9) complementary (e.g., Frame 3) or additional information (Frame 5) to one another. Arguably, if ideal-material components such as spoken words or gestures convey meaning, and meaning is an aspect of cognition, then such components are also components of Laura’s cognition. Similarly, the meaning conveyed by semiotic coordination of ideal-material components are components of Laura’s cognition, in this case, her mathematical thinking. Ideal-material components that carry mathematical meaning for the student must also be considered components of that student’s mathematical thinking.

As Radford’s (2014) study finds, students semiotically coordinate a variety of components to think mathematically. On a smaller scale, the present study contributes to Radford’s framework by filling in gaps by providing a definition of each component and collapsing the ideal vs. material terminology into ideal-material. More significantly, the present study offers a specific methodology to replicate Radford’s study by describing the analytic methods for determining how each of the components participates in the overall meaning if a dynamic unity. This determination is based on whether a component is carrying identical, supporting, complementary, additional, or mismatched (different or contradicting) information to other components participating in the same dynamic unity.

In particular, my analysis reveals moments in which Laura's gesture signified certain mathematical meaning for her as part of a dynamic unity. For example, analysis of Frame 3 reveals that the student's vertical gesture conveyed complementary information to her utterance "because there might be a pattern". The significance of that gesture was defined by its complementary role within the unity with the utterance. The vertical gesture signified that the pattern might reveal itself as a relationship between subsequent numbers within a column of the function table (known as the "scalar approach" or "scalar relationship," see Vergnaud, 1983, 1988; also Martínez & Brizuela, 2006). Similarly her horizontal gesture in Frame 4 complemented her utterance "or related in some way" and signified that a pattern might reveal itself as a relationship between numbers within a row of a function table (known as the "functional approach" or "functional relationship," see Vergnaud, 1983, 1988; also Schliemann et al., 2001). Literature on gestures helps highlight their ties to cognition. For example, Goldin-Meadow (1999) refers to gestures as a "tool for thinking" (p. 428) with a purpose in expressing and shaping our thoughts, whereas McNeil (2008) views "gesture as active participant in speaking and thinking" (p. 7). Goldin-Meadow (1999) further argues that gestures do not always convey identical information to the one conveyed by the speech they accompany; gestures can also convey additional/complementary or different/mismatched information, as well as occur in the absence of speech altogether. Inspired by Goldin-Meadow (1999) in my analysis I looked for the moments when each of the ideal-material components signified identical, supporting, complementary, or additional meaning (I found no evidence of mismatched information).

My analysis also reveals instances in which student's interaction with a function table played a role in a dynamic unity of student's mathematical thinking. Laura coordinated such interaction with perception (Frames 1 and 6), perception, utterance and writing (Frames 5 and 10), perception and gesture (Frames 2, 3, and 4), as well as perception, tactility and rhythm (Frame 8). To better understand the ties between cognition and, as Radford put it, a student's actions with cultural artifacts, such as Laura's interactions with function tables, I now briefly review Kirsh (2010), who provided detailed accounts regarding how external representations (e.g., written text, tables, drawings) "serve as vehicles for thought" (p. 445), "changing the domain and range of cognition" (p. 442). It is the interaction with external representations that, according to Kirsh, allows us to go beyond what we could accomplish just by thinking in our head alone, in terms of complexity, speed, and accuracy. In other words, human capacities for cognition within the head are limited, but are far more superior when they involve interactions with external representations. Gains in efficiency (speed, accuracy) and efficacy (performing at a high level of difficulty) when we interact with external representations, as compared to engaging in strictly internal cognitive processes, could be, as identified by Kirsh, because we can "rearrange" and "reformulate" external representations without altering their physical persistence. Rearranging and reformulating representations without altering physical persistence, according to the author, presents a challenge when simulated in the mind due to our limited cognitive abilities, such as limited working memory. Kirsh helps us begin to understand how our cognition expands beyond the head, and embeds and distributes onto the environment to include interactions with external representations, which highlight the embodied/embedded/extended aspects of cognition. On the other hand, Laura's interactions with a function table involves culturally established ways of organizing independent and dependent variables in two columns, and taking a scalar (vertical) or a functional (horizontal) approach when looking for a numerical pattern or relationship between variables, which are the cultural aspects of Laura's mathematical cognition. The scope of this paper is to look at mathematical thinking as a semiotic coordination of various ideal-material components, rather than to provide detailed accounts of socio-cultural or embodied aspects of cognition. Laura's interaction with the function table contributed identical (Frames 1, 5, 6, 8), additional (Frame 5), or complementary (Frame 2,3,10) information to the ones conveyed by other components.

Radford's framework prompts us to make claims about thinking as being "made up of" ideal-material components (Radford, 2014, p. 268). The way a student interacts with cultural artifacts is, for example, a component of his or her thinking. This is not to say that all thinking is visible and that we as researchers have no place in making hypothesis about children's concepts, methods, and strategies beyond what is directly observable during clinical interviews. However, accounting for cognition as a semiotic coordination of gestures, outer speech, and interactions with cultural artifacts, has a potential to broaden our understanding of a child's ways of thinking to include the thinking that occurs in multiple sensory-motor modalities. Traditional cognitivist approach would consider, for example, a gesture solely as an input or an output to cognitive processing, whereas from a post-cognitivist perspective, a gesture itself could contribute meaning and therefore itself be a component of cognition. The key distinction from earlier cognitivist accounts is that actions, gestures, outer speech, perception, and even tactility and rhythm are modalities of thinking itself, rather than inputs to or outputs from computations done in the mind.

Thinking vs. communication

Reflecting on the examples of Laura's components of thinking, which are not limited to the head alone, one might ask what makes each of these a form of mathematical thinking and not just an act of communication. For example, in Frame 2, prompted by my question: "What are you doing?" Laura said, "I am just trying to find the difference in how that [shadow heights] changed". Her utterance might appear strictly communicational rather than a component of thinking. In Radford's view, making a distinction between thinking and communication would mean that one is taking a cognitivist (i.e., computational) view of the mind:

> Although it might be argued that the teacher and the student are merely communicating ideas, I would retort that this division between thinking and communicating makes sense only within the context of a conception of the mind as a private space within us, where ideas are created, computed and only then communicated. This computational view of the mind has a long history in our Western idealist and rationalist philosophical traditions. (p. 267-268)

Radford's above argument is clear; if we consider gesture or outer speech as forms of communication and not as forms of thinking, then we are taking a cognitivist perspective, which confines thinking solely to the head and communication to the outside of the head. However, arguing that a gesture is a component of thinking just because differentiating between communication and thinking would make us cognitivists is far from convincing. Instead, the key to the argument about what makes a component a part of thinking is what that component means or signifies to the learner within its dynamic unity. When we communicate we convey meaning, meaning in turn is an aspect of cognition, thus the thinking itself. Communication is, I so far argued, a form of thinking. Thinking, on the other hand, is viewed as a form of communication with oneself (Sfard, 2012). Making a distinction between communication and thinking is therefore artificial.

Conclusion

In the present case study, I carried out a frame-by-frame analysis of an interview with a seventh grader exploring shadow. I analyzed video and written data in order to identify her gestures, perception, outer speech, tactility, rhythm, and actions with cultural artifacts. I then interpreted how she coordinated those components in space, time, and semiotically in order to think mathematically when looking for a pattern in a function table.

The main problem this paper aims to address is understanding of the mechanisms by which cognition is situated within the bodily interactions with the physical, social and cultural environment. Semiotic coordination of ideal-material components described in this paper is one such mechanism. I further characterized semiotic coordination by the kinds of information (identical, supporting, complementary or additional) an ideal-material component contributes to the dynamic unity. Although this paper provides evidence that such mechanism exists it does not address *why* students coordinated components in a particular way. For example, in this paper I do not focus on *why* Laura's vertical gesture and her utterance in Frame 2 conveyed complementary information and not identical, supplementary or additional. This might be a direction for future research.

This study contributes to the literature on situated mathematical cognition in general, and embodied mathematical cognition in particular, by providing replicable analytical methods that help characterize mathematical thinking as a semiotic coordination of components in a variety of sensory modalities: 1) frame-by-frame organization of data 2) frame selection based on components co-occurring in space and time; 3) interpretation of the significance (conveyed meaning) of each component in a frame based on the information they contribute (additional, complementary, etc.); and 4) discussion of the mathematical thinking as a semiotic coordination of components in the frame.

The main distinctions from Radford's framework are: 1) terminology for ideal and material components was collapsed into ideal-material to emphasize that their separation is artificial; 2) definitions were provided for each of the six Radford's component of thinking; 3) specific analytic methods to replicate were provided (described above); and 4) coordination was characterized as semiotic by ways in which different components carried identical, supporting, complementary, additional, or mismatched information (embedded in the analytic methods).

My analysis has the following limitations: 1) it fails to account for "inaccessible" components of student thinking such as inner speech and visualizations, 2) it might fail to determine semantics of an observable component (e.g., isolated gesture) unless there is another observable component available in the analysis (e.g., outer speech), 3) it does not take into account whether a component is intentional or involuntary, and 4) it is limited to components that co-occur in time and space and does not take into a consideration components that were established during previous activity (see diachronic analysis in Arzarello, 2009).

The explanatory power of Radford's framework when it comes to looking at student mathematical thinking comes from accounting for a variety of forms that thinking can simultaneously take. If we were to ignore them, some aspects of student thinking would remain hidden to us. Also, if we were to look at those components in isolation from one another, we might miss what a component signifies within a dynamic unity. For example, the significance of Laura's vertical gesture in Frame 3 is unclear to the observer in isolation from the utterance that accompanies it. On the other hand, if we were to look at the utterance in isolation from the gesture, then the vertical spatial orientation of the possible pattern in the function table would remain hidden to us.

As a way of concluding the study, I will now discuss the relevance of Radford's framework to understanding mathematical cognition. First, this framework aligns with the post-cognitivist approach by showing ways in which the body plays an important role in cognition, namely, through perception, gesture, tactility, rhythm, outer speech, and interaction with cultural artifacts. Next, the framework shows that mathematical thinking could simultaneously take multiple forms, from a vertical hand gesture, through tapping and rhythm, to a complex utterance or an interaction with a cultural artifact such as a function table. This, in turn, sheds light on just how complex and powerful our cognitive apparatus is in terms of distributing and extending cognition to resources outside the boundaries of our own physical brains.

Finally, when it comes to mathematics education and research, Radford's framework has a few important implications: 1) looking into students' mathematical thinking should account for outer or embodied forms of thinking; 2) those components are not to be considered mere evidence of, but rather components of, thinking; and 3) components of thinking should not be looked at in isolation from other components because their role in a semiotic unity determines their significance.

Directions for future research

My analysis also revealed ideal-material components that co-occurred in time and space with other components of Laura's mathematical thinking, although they did not convey identical, supporting, complementary, additional, or mismatched information. Namely, in Frame 1, Laura lifted the pencil before she wrote 0.32. I interpret this gesture as framing of her mathematical activity as "I am now in a thinking mode." During Frame 3, Laura also maintained a vertical body position, remained slightly turned towards the interviewer, and maintained direct eye contact with the interviewer. I interpret these gestures as part of how she is framing her mathematical activity as, for example, "I am confident in my understanding of what I am supposed to be doing, and that is to find a pattern." In Frame 6, Laura lifts both eyebrows and sighs, which could signify something similar to taking a deep breath before we start working on something that takes a lot of concentration or effort; physical stillness and arms wrapped around the knees could signify something like "I am in my cocoon isolating myself from external stimulus;" whereas biting the lip could signify "I am in my intense thinking mode." These are only some interpretations of how these gestures could frame Laura's cognitive processes. These examples, arguably, are pointing in the direction of Laura's epistemological framing (Hutchison & Hammer, 2010; Russ et al., 2012) rather than her mathematical thinking. Epistemological framing co-occurs in time and space with mathematical thinking and therefore has the potential to become another layer in Radford's framework.

On the other hand, in Frame 2 Laura also lifts an eyebrow, a gesture that can be interpreted as an element of surprise (or possibly being confused, or annoyed) that the additive difference between the numbers in the right column might not be constant. The eyebrow lift is not just an incidental gesture, but also one that is synced up in time with the emotion of surprise, confusion or being annoyed with her mathematical activity. Epistemic affect (see Jaber, 2015), just like the epistemological framing, might possibly be another layer to be added to Radford's framework.

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