Limit Loads - I

Limit loads on metal structures

plasticity beyond elastic limit

<u>Goal</u> : approx. estimate of limit loads

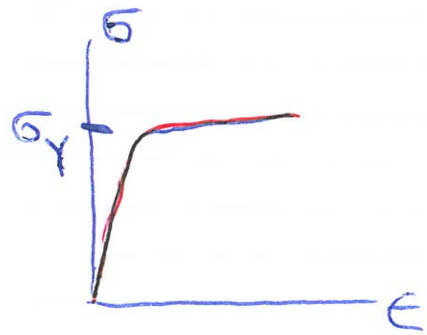
(collapse)

Full analysis (motion of elastic–plastic boundary, as load $\uparrow$) usually cannot be done. Pressurized pipe problem is an exception (1-D problem, dependence on r only)

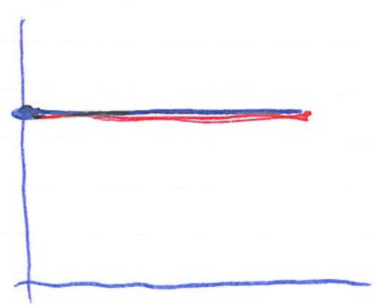
Methodology of limit analysis :

- give up full analysis of elastic-plastic evolution
  Focus on limit load only
- Even this task is difficult
  $\Rightarrow$ establish $\left.\begin{array}{l}\text{upper bound}\\ \text{lower bound}\end{array}\right\}$ for the limit load
- Further simplification :

Rigid-plastic idealization of material behavior



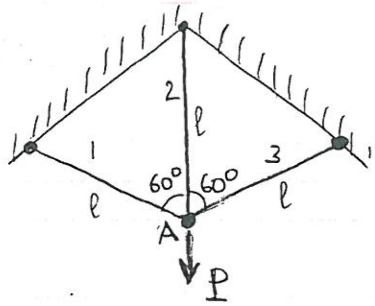
$\Rightarrow$



Limit loads may be substantially higher than the onset of plasticity!

(example : pressurized pipe)

Another example: Truss



- All bars have the same length ℓ, cross-section F

Ball joints connections $\Rightarrow$ uniaxial tension in bars

Find stresses in bars $\sigma_1 = \sigma_3$ and σ_2

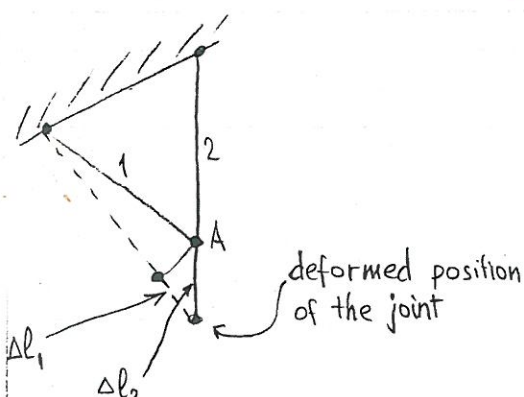
The truss is statically indeterminate (median bar can be removed)

$$2 \cdot \sigma_1 \cdot \cos 60^\circ + \sigma_2 = P/F \quad \text{— equilibrium}$$

or

$$\sigma_1 + \sigma_2 = \frac{P}{F}$$

Additional condition: by compatibility of elongations



$\Delta\ell_1$ and $\Delta\ell_2$ are geometrically interrelated

$\Delta\ell_1 = \Delta\ell_2 \cdot \cos 60^\circ$ (to within small values of higher order)

- same for strains -

$\Rightarrow$ $\sigma_1 = \frac{1}{2}\sigma_2$ in elastic regime

assuming Hooke's law for stress-strain relations

Together with equilibrium:

$$\begin{cases} \sigma_1 = \sigma_3 = \dfrac{1}{3}\dfrac{P}{F} \\ \sigma_2 = \dfrac{2}{3}\dfrac{P}{F} \end{cases}$$

Beyond the elastic limit:

yield limit in tension

σ_2 reaches σ_Y when $\frac{P}{F} = \frac{3}{2}\sigma_Y$

It stays const $(= \sigma_Y)$ under further increase of P

In lateral rods, 1 & 3 : from equilibrium

$$\sigma_1 + \sigma_2 = P/F$$

$$\Rightarrow \sigma_1 = \frac{P}{F} - \sigma_Y$$

At

$\sigma_1 < \sigma_Y$ (means $P/F < 2\sigma_Y$) : elastic-plastic regime

When P/F reaches $2\sigma_Y$, then $\sigma_1 = \sigma_Y$

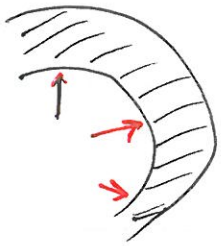
the system is fully plasticized

$\rightarrow$ limit load

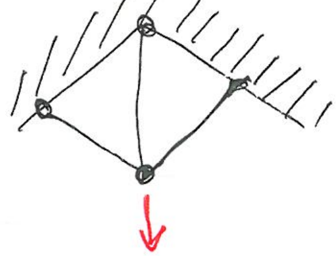
Thus: first plastic def. at $P/F = 3/2\,\sigma_Y$ $\leftarrow$ not a failure! elastic "core" remains

limit load at $P/F = 2\sigma_Y$

Summary : in some simple cases :



pressurized pipe



truss

can trace the evolution (as applied load $\uparrow$)

elastic regime $\rightarrow$ elastic-plastic $\rightarrow$ fully plastic (collapse) Limit load

$\Uparrow$

Simple geometries

Insufficient for eng-g needs

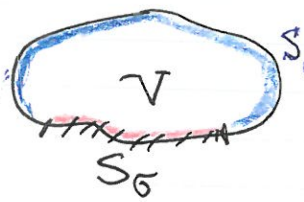
More complex geometries ?

Approximate estimates of Limit loads — upper & lower boun.

means : collapse

use stress / displacement fields that do not have to be exact,
but should satisfy certain minimal requirements :

A) Statically admissible stress field $\sigma_{ij} = \sigma_{ij}(\underset{\sim}{x})$:



- Satisfies eq-m, $\sigma_{ij,j} = 0$ in V
- Agrees with b.c. in stresses (if any) on S_σ

B) Kinematically admissible displ. field $u_i(\underset{\sim}{x})$:

- incompressible, $u_{i,i}$ $(= \epsilon_{ii}) = 0$ in V
- Agrees with b.c. in displac. (if any) on S_u

Lower Bound theorem

Suppose we construct some stress field $\bar{\sigma}_{ij}$ - not necessarily the actual one, may be fictititous - such that

- it's statically admissible (in particular, agrees with appl. loads on boundary)
- it is below yield everywhere

Then:

The loading conditions (with which $\bar{\sigma}_{ij}$ agrees) are below limit load

(physically: we are able to match b.c. with $\bar{\sigma}$ below yield)

How to use it: try to find the highest $t|_{\text{prescribed on boundary}}$ for which $\bar{\sigma}_{ij}$ can still be constructed

(to get the highest lower bound)

Upper Bound theorem

Suppose we are able to construct some displac. field $\bar{u}(x)$

— not necessarily the actual one, may be fictitious —

such that

- It's kinematically admissible
- Rate of work of appl. loads on $\bar{u}$ on S_σ

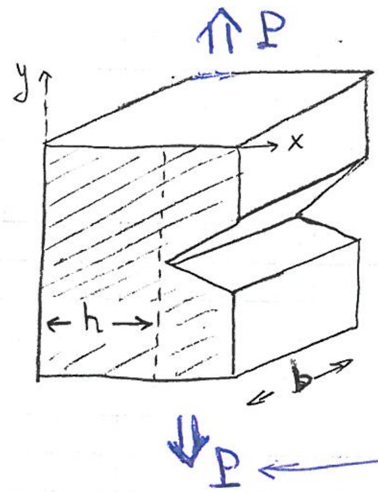
$>$ Rate of plastic dissipation inside the body

Then: loads on S_σ are above the limit load

How to use: try to find the lowest t for which such $\bar{u}$ can be constructed

(The lowest upper bound)

Edge - Cracked Plate in Tension



total axial force
(no details given)

Lower Bound Estimate.

Assume the stress state:

1. Equilibrium-satisfied (uniform in each part)
2. Agrees with b.c. on S_σ:

on average only!

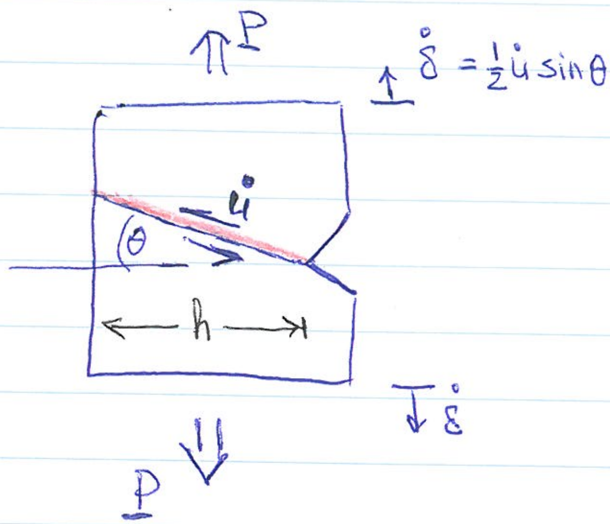
$$\sigma \cdot bh = P \quad \Rightarrow \quad \sigma = \frac{P}{bh}$$

If $\sigma < \sigma_Y$ then $P < P_{limit}$

The best (highest) lower bound corresponds to $\sigma \to \sigma_Y$

$$P_{l.b.} = \sigma_Y \, bh$$

Upper bound estimate: assumes certain kinematics of collapse



Rigid blocks, sliding one past another (relative speed $\dot u$)

$\dot\delta = \frac{1}{2}\dot u \sin\theta$

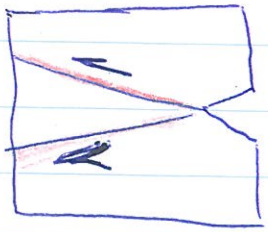
$$\underbrace{2P\dot\delta}_{\text{Rate of work of applied load}} > \underbrace{\tau_Y \cdot \underbrace{\frac{h}{\cos\theta} \cdot b}_{\text{area of sliding}}\,\dot u}_{\text{rate of work done inside body}}$$

τ_Y: shear stress along sliding line

or $$P > \tau_Y \frac{bh}{\sin\theta\cos\theta}$$

Any such P is above the limit load

Note: alternative kinematics



yields same result (doubles both left & right-hand parts)

For the best (lowest) upper bound :

choose θ to minimize $\frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta}$

$\underline{\text{Min}} = 2 \quad (\text{at } \theta = 45^\circ)$

$$\Rightarrow \quad P_{u.b.} = 2\tau_Y bh$$

Combining l.b. & u.b. : $\quad \sigma_Y < \frac{P_{lim}}{bh} < 2\tau_Y$

are interrelated
either by Tresca condition
or by Mises cond.

<u>Tresca material</u> : $\quad \sigma_Y = 2\tau_Y \Rightarrow$ upper & lower bounds coincide $\Rightarrow$ <u>exact</u> <u>result</u> !

Such luck is rare

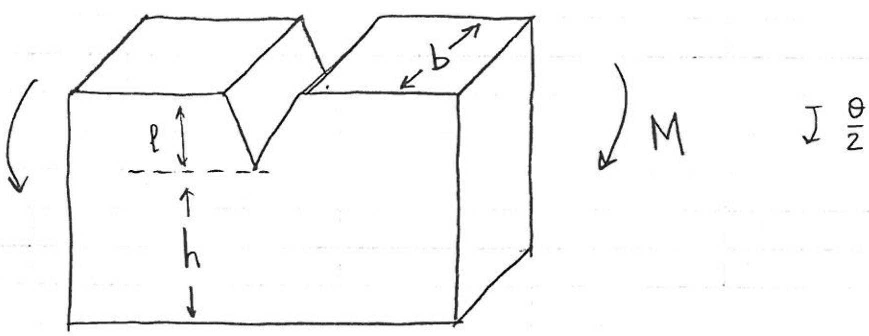
<u>Mises material</u> : $\quad \sigma_Y = \sqrt{3}\,\tau_Y$

$$1 < \frac{P_{lim}}{bh\,\sigma_Y} < \frac{2}{\sqrt{3}}$$

differ by 15% only

Midpoint would guarantee ~~error~~ < 7.5%

Plate with Sharp Notch in Bending

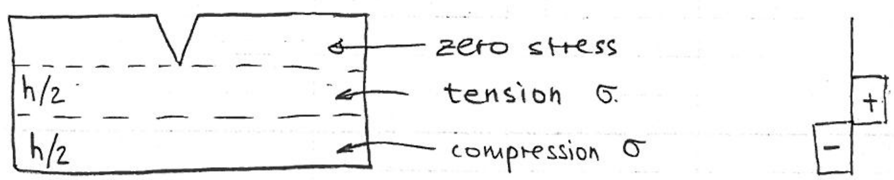


M – appl. moment

θ – relative rotation of ends at remote points

Limit Moment = ?

<u>Lower bound</u>. Assume the stress field:



– equil. eq-ns : satisfied (uniform stress in each part, $\sigma_{ij,j} = 0$)

– b.c. in stresses : stress distribution must balance M

$$2 \cdot \underbrace{\sigma \frac{h}{2}}_{\substack{\text{force} \\ \text{per unit thick}}} \cdot \underbrace{\frac{h}{4}}_{\text{arm}} \cdot b = M$$

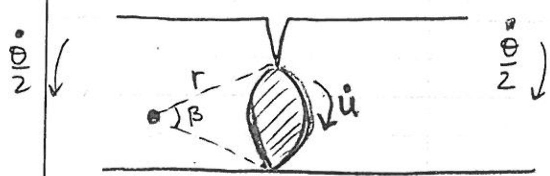
for any $\sigma < \sigma_Y$ M is below M_{Limit}

For the best (highest) bound, let $\sigma \to \sigma_Y$

$$M_{l.b.} = \frac{1}{4} \sigma_Y \cdot h^2 b$$

Upper bound

Assumed kinematics of collapse:



Rigid body rotation around rigid core along two circular arcs (kinematically, cannot be non-circular)

$$M\dot\theta > \tau_Y \underbrace{r\beta}_{\text{arc length}} b \cdot 2 \cdot \underbrace{\frac{r\dot\theta}{2}}_{\dot u}$$

[rate of external work] [rate of internal dissipation]

r and β are not independent : $2r\sin\frac{\beta}{2} = h$ — eliminate r

$$M > \frac{1}{4}\tau_Y bh^2 \left(\frac{\beta}{\sin^2 \beta/2}\right)$$

minimum = 2.76 at $\beta = 134°$

$$\Rightarrow \quad M_{u.b.} = .69\ \tau_Y bh^2$$

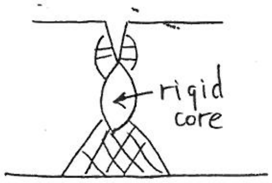
whereas $M_{\ell.b.} = \frac{1}{4}\underbrace{\sigma_Y}\, bh^2$

$= 2\tau_Y$ for Tresca mat'l

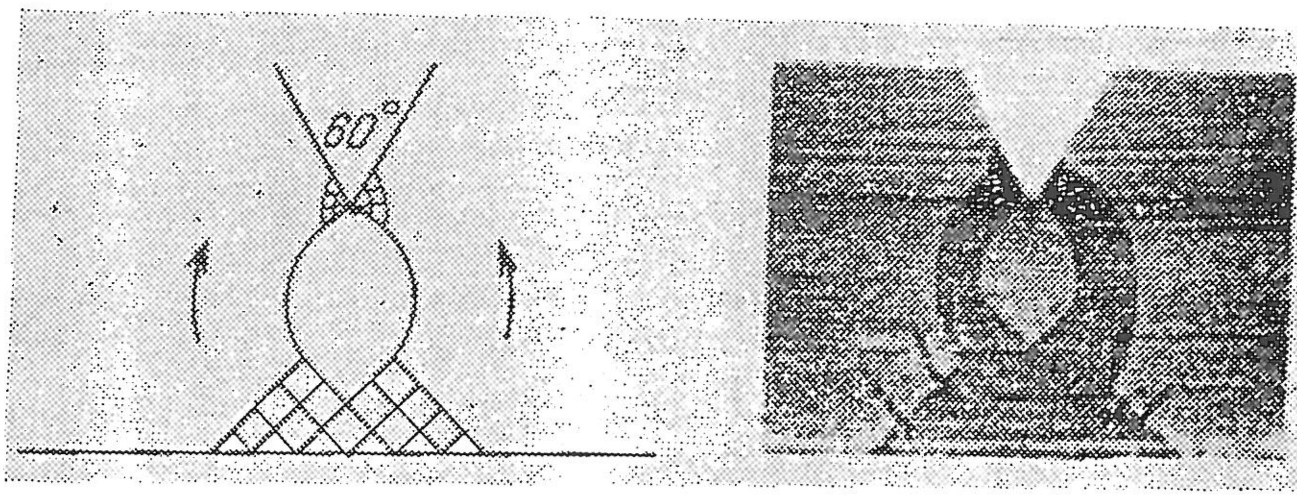
$= \sqrt{3}\,\tau_Y$ — Mises —

$$\left.\begin{matrix} .5 & \text{(Tresca)} \\ .43 & \text{(Mises)} \end{matrix}\right\} < \frac{M_{limit}}{\tau_Y bh^2} < .69$$

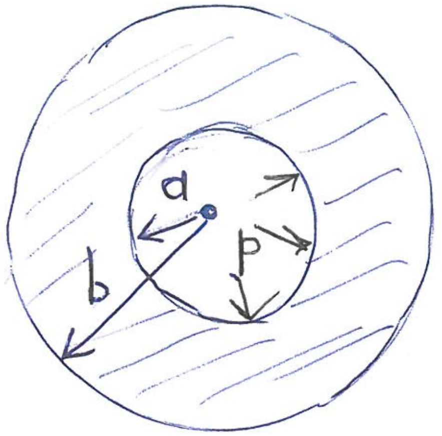
Actual kinematics:



our model: reflects the basic feature (rotation about rigid core) but there are substantial differences


60°

Circular Pipe – limit pressure



We know the exact result: $p_{lim} = 2\tau_Y \ln\frac{b}{a}$

What can we get from the bounds?

Lower B. construct stress field (as simple as possible)
that – is equilibrial
– agrees with b.c.

Start with σ_{rr}

Should be $= 0$ at $r = b$
$-p$ at $r = a$ } b.c.

Assume: linear f-n

$$\sigma_{rr} = -p\frac{b}{a-b}\left(\frac{r}{b}-1\right)$$ – satisfies b.c.

What about $\sigma_{\theta\theta}$? from equilibrium eq. (cannot be violated!):

$$\frac{d\sigma_{rr}}{dr} = -\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}$$

$$\Downarrow$$

$$\sigma_{\theta\theta} = -p\frac{b}{a-b}\left(2\frac{r}{b}-1\right)$$

Note: This field is obviously wrong:
for example, $\sigma_{\theta\theta} < 0$ (??) But it's OK for our purpose

Has to be below yield everywhere. Taking Tresca condition

$$|\sigma_{\theta\theta}-\sigma_{rr}| = p\frac{r}{b-a} < 2\tau_Y$$ (has max. at $r=b$) nonsense – but OK for our purpose

Ensuring that we are below yield, even at $r=b$:

$\Rightarrow$ $p_{lower\ bound} = \tau_Y \cdot 2\left(1-\frac{a}{b}\right) = 1$ for $\frac{b}{a}=2$ (exact: $2\ln 2 \approx 1.39$)

explore the lower bound generated by assuming

$$\sigma_{rr} = Ar^2 + B$$

By matching b.c. at $(r=a,\ r=b)$ find A, B:

$$\sigma_{rr} = -p\,\frac{b^2 - r^2}{b^2 - a^2}$$

Equilibrium equation then yields

$$\sigma_{\theta\theta} = p\,\frac{3r^2 - b^2}{b^2 - a^2}$$

Note: Stresses do not look physically realistic: $\sigma_{\theta\theta}$ may be negative
But this is OK for our purposes

Below yield everywhere:

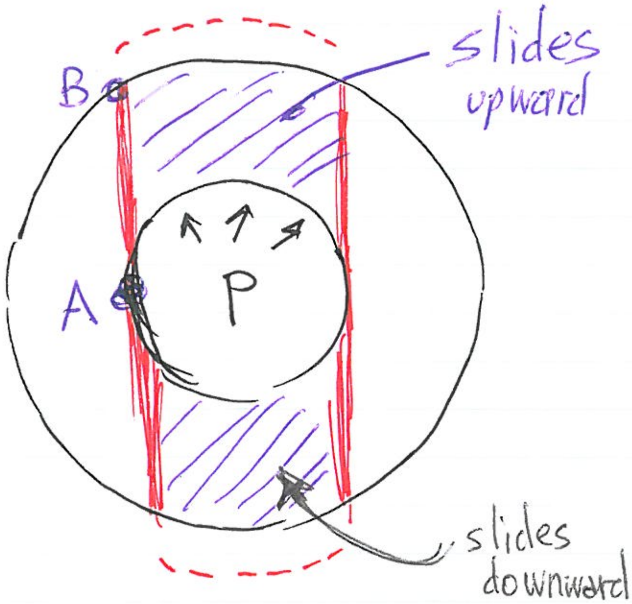
$$\underbrace{|\sigma_{\theta\theta} - \sigma_{rr}|}_{\frac{2p}{b^2-a^2} r^2} < 2\tau_Y$$

$\frac{2p}{b^2-a^2} r^2$ – max at outer b. $r=b$
(another nonsense)

$$p_{l.b.} = \underbrace{\left(1 - \frac{a^2}{b^2}\right)}_{= \frac{3}{4} \text{ at } b=2a} \tau_Y$$

$= \frac{3}{4}$ at $b=2a$ $\Rightarrow$ estimate got worse

Circular pipe : Upper Bound



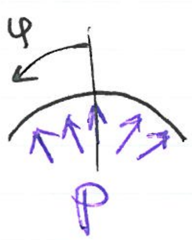
Assumed kinematics of collapse

If :

Rate of work of p > rate of plast. dissipation on 4 lines AB

Then : p is above limit

Work of pressure (per unit thickness)



rate of sliding

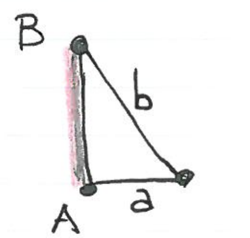
$$2 \cdot \int_{-\pi/2}^{\pi/2} (p\cos\varphi \cdot a\, d\varphi)\, \dot{\delta} = 4pa\dot{\delta}$$

(top & bottom)

Dissipated work :

Along AB : yield stress τ_Y

$$4 \times \tau_Y \sqrt{b^2 - a^2}\, \dot{\delta}$$



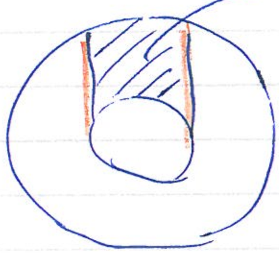
$AB = \sqrt{b^2 - a^2}$

Then $4pa\dot{\delta} > 4\tau_Y \sqrt{b^2-a^2}\,\dot{\delta} \implies \dfrac{p_{u.b.}}{\tau_Y} = \sqrt{\dfrac{b^2}{a^2} - 1}$

$\sqrt{3}$ at $b = 2a$ (exact: 1.39)

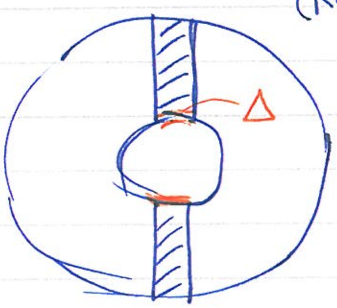
Other Attempts:

(1)



– both work of pressure & dissipation decrease by factor of 2 → same result

(2) Try a very thin sliding zone:



(Rate of) Work of pressure: $\approx 2 \cdot p \cdot \Delta \cdot \dot{\delta}$

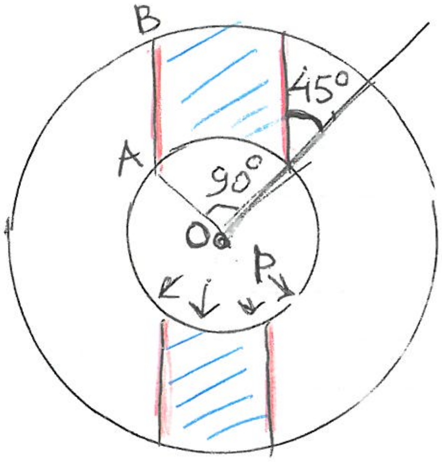
(Rate of) plastic diss. inside: $\approx 4 \cdot \tau_Y \cdot \dot{\delta} \cdot (b-a)$

$$p_{u.b.} = \frac{2(b-a)}{\Delta} \tau_Y \quad \text{– large}$$

Δ ← small

much worse

Try to improve the upper bound – by more realistic kinematics of failure



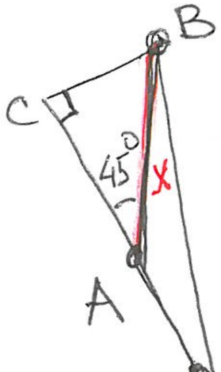
known : yield starts at inner boundary

at 45°

Assume: this is where vertical sliding occurs

$$\text{Rate of work of } p = \underbrace{2}_{\text{top \& bottom}} \cdot \int_{-\pi/4}^{\pi/4} p \cdot a \cos\varphi \, d\varphi \cdot \dot{\delta} = 2 \cdot \sqrt{2}\, p a \dot{\delta}$$

Rate of dissipated work:



Find length of AB $= x$

$$\underbrace{CO^2}_{\left(a+\frac{x}{\sqrt{2}}\right)^2} + \underbrace{BC^2}_{\left(\frac{x}{\sqrt{2}}\right)^2} = \underbrace{BO^2}_{b^2} \quad \Rightarrow \quad x = \sqrt{b^2 - \tfrac{1}{2}a^2} - \frac{a}{\sqrt{2}}$$

$$\text{Dissipation rate} = \underbrace{4}_{\text{4 lines}} \cdot \tau_Y \cdot \left(\sqrt{b^2 - \frac{a^2}{2}} - \frac{a}{\sqrt{2}}\right) \dot{\delta}$$

Equating $\Rightarrow$ $$p_{u.b.} = \underbrace{\left[\sqrt{2\left(\frac{b}{a}\right)^2 - 1} - 1\right]}_{1.65 \text{ for } b/a = 2} \tau_Y$$

Slightly better than $\sqrt{3}$

(exact: 1.39)

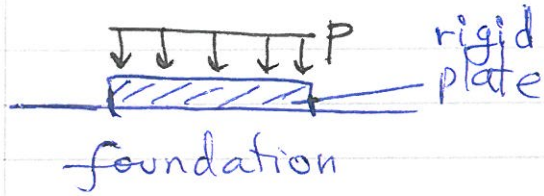
Thus : the best we can get from our attempts :

$$2\left(1-\frac{a}{b}\right) < \frac{p_{lim}}{\tau_Y} < \sqrt{2\left(\frac{b}{a}\right)^2-1}$$

exact solution : $2\ln\frac{b}{a}$

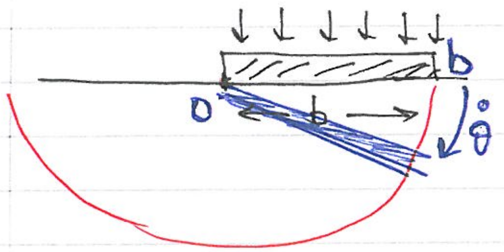
At $\frac{b}{a}=2$: $\left[\,1 \quad \underset{\text{exact}}{(1.39)} \quad 1.65\,\right]$

Indentation problem : ultimate bearing capacity of foundation

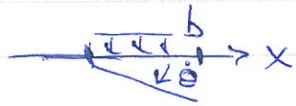


[problems involving punches, foundations]

Upper bound : Assume circular rotation (at plastic collapse)



Rate of work of applied load :



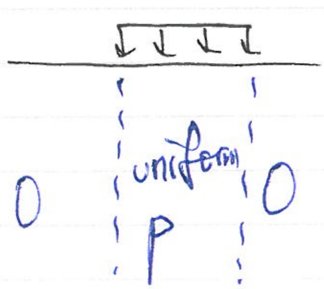
$$\int_0^b p \cdot \underbrace{x\dot{\theta}}_{\text{linear velocity}} \cdot dx = p\dot{\theta}\frac{b^2}{2}$$

Rate of plastic dissip: $\tau_Y \cdot \underbrace{\pi b}_{\text{half-circle}} \cdot b\dot{\theta}$

Equating :

$$p_{u.b.} = 2\pi\tau_Y .$$

Easy lowerbound : uniform uniaxial compression :



$$p = \sigma_Y = 2\tau_Y$$ (assuming Tsesca)

$$\Rightarrow \boxed{1 < \frac{p_{lim}}{2\tau_Y} < \pi}$$