GEOMETRIC, ALGEBRAIC, AND TOPOLOGICAL METHODS FOR STUDYING OUTER AUTOMORPHISM GROUPS OF FREE GROUPS

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Contents

1 Introduction

Through the study of groups, one can gain insight into the symmetries of various mathematical objects. What if these objects are groups themselves? To study the symmetries of a group G in the broadest sense would amount to studying properties of $Aut(G)$, the group of automorphisms of G. An inner automorphism of G is any automorphism that arises from conjugation. $Inn(G)$, the group of all inner automorphisms, is a normal subgroup of $Aut(G)$, and so when trying to study automorphisms up to conjugation it is natural to look at $Aut(G)/Inn(G)$. This quotient is denoted by $Out(G)$ and is called the outer automorphism group of G.

This thesis aims at giving a brief survey of a few of the algebraic, geometric, and topological techniques used in developing the theory of $Out(F_n)$, the outer automorphism group of the free group on n generators. Much of the study of $Out(F_n)$ has been developed through analogies and connections with mapping class groups and general linear groups. Throughout this paper, we will repeatedly discuss the evolution of methods for $Out(F_n)$ in regards to their analogous counterparts in the setting of mapping class groups and general linear groups.

We will start by introducing the mapping class group and looking into its structure. We will then investigate the topological spaces which model F_n. There will arise a connection with the construction of the extended mapping class group of a surface and the Dehn-Nielsen-Baer theorem when thinking about $Out(F_n)$ topologically.

We'll then look at a representation of $Out(F_n)$ into the general linear group over the integers, $GL_n(\mathbb{Z})$, which emerges from the action of $Out(F_n)$ on $\mathbb{Z}^n$. This representation is analogous to the representation of the mapping class group into the integer points of the symplectic group, $Sp_{2g}(\mathbb{Z})$. Just as for mapping class groups, the kernel of this homomorphism from $Out(F_n) \to GL_n(\mathbb{Z})$, called the Torelli subgroup, is of interest to many. This chapter will discuss how this representation arises and what it looks like in the case of $Out(F_2)$

Almost all of the results in this paper are not a consequence of my own work. I would like this survey to be a resource for mathematicians with varying levels of experience. In this vein, I will use some of the most basic examples and ignore many technicalities. I would like to thank Genevieve Walsh for supervising my thesis and providing insightful, productive advice along the way. I would also like to thank Rylee Lyman, Kim Ruane and Lorenzo Ruffoni for their willingness to discuss these topics and help me attempt to understand many of the seemingly impossible to grasp concepts in this area of research.

2 Model Spaces

In the first section of this chapter we will define the mapping class group and study some simple aspects of its structure via Dehn-twists. Then we will work toward understanding the Dehn-Nielsen-Baer theorem by building intuition for the connection between the mapping class group of a surface S and $\pi_1(S)$. After this, we will study two spaces which have fundamental group isomorphic to the free group, graphs and punctured surfaces. In the spirit of the Dehn-Nielsen Baer theorem, we will see how mapping class groups interact with the fundamental groups of punctured surfaces and how homotopy equivalence groups interact with graphs. In the last section, we explore Stallings' folding techniques for graphs, one of the most useful tools for studying $Out(F_n)$.

2.1 The Mapping Class Group and Dehn Twists

2.1.1 The Definition

For a compact, path connected, orientable surface S of genus g with b boundary components and p punctures, the *mapping class group* of S, denoted by $Mod(S)$, is defined as

$$Mod(S) \coloneqq Homeo^{+}(S)/\sim$$

where $Homeo^{+}(S)$ is the group of orientation preserving homeomorphisms of $S_{g,b,p}$ which restrict to the identity on ∂S and $\sim$ is the equivalence relation on $Homeo^{+}(S)$ induced by isotopy $rel(\partial S)$. If we consider $Homeo^{+}(S)$ with the compact-open topology, the mapping class group can be realized as $\pi_0(Homeo^{+}(S)$. For our purposes, it will be useful to work with the group of isotopy classes of *all* (i.e. even orientation reversing) homeomorphisms of a surface S with no boundary components [1]. This group is called the *extended* mapping class group of S and is denoted $Mod^{\pm}(S)$.

2.1.2 Dehn-Twists

When trying to study a group algebraically, one of the first things we can investigate is whether we can find a generating set for it. Since this generating set encodes the building blocks for the entire group, the algebraic relationships between generators will in turn encode the global structure of the group we have in question. Some of the most fundamental elements of the mapping class group are certain types of "twisting" homeomorphisms called Dehn-twists. These elements were first studied by Max Dehn, a prominent combinatorial group theorist during the early 20th century. For many surfaces, Dehn-twists will in fact generate the mapping class group; thus, it will be useful to spend some time discussing them. Since we are dealing with a symmetry group of a surface, the relationships between these generators will have nice visual interpretations in

[1] Since homeomorphisms that restrict to the identity on ∂S are always orientation preserving, we are only considering surfaces with no boundary components.

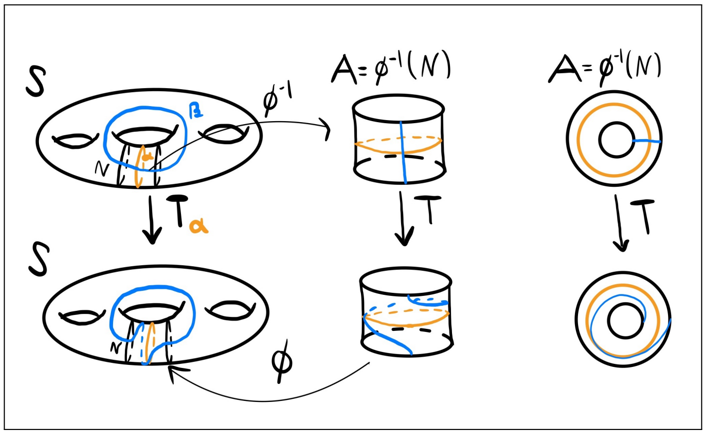

FIGURE [2A]

terms of whats happening topologically on the surface. Let's begin with the definition, which we will pull directly from [FM12].

Consider the annulus $A := S^1 \times [0,1]$. To orient A we embed it in the (θ, r)-plane via the map $(\theta, t) \mapsto (\theta, t+1)$ and take the standard orientation of the plane. Let $T : A \to A$ be the twist map given by $(\theta, t) \mapsto (\theta + 2\pi t, t)$. Let S be an arbitrary (oriented) surface and let α be a simple closed curve in S. Let N be a regular neighborhood of α and choose an orientations preserving homeomorphism $\phi : A \to N$. We obtain a homeomorphism $T_\alpha : S \to S$ called a *Dehn-twist about* α, as follows:

$$T_\alpha = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\ \text{x} & \text{if } x \in S \setminus N \end{cases} \tag{1}$$

See [2A] for an illustration of T_α with an intersecting curve β to emphasize the twisting effect. Since elements in the mapping class group are unique up to isotopy, Dehn-twists about simple closed curves on a surface will be unique in the mapping class group up to the isotopy class of curve we choose. Further, if a simple closed curve in S is isotopic to a point, the Dehn-twist about that curve will be isotopic to the identity. This is because isotopy classes of simple closed curves which intersect that curve will get sent to themselves under the Dehn-twist.

Dehn-twists provide us with a useful way of immediately getting our feet wet with subgroups of the mapping class group. Note that we can take powers T_α^n by $T^n : A \to A$ where $(\theta, t) \mapsto (\theta + 2n\pi t, t)$. We see that $T_\alpha^0 = Id_{Mod(S)}$ and $T_\alpha^{-1} \circ T_\alpha = Id_{Mod(S)}$. Since there are no other relations, T_α generates an infinite cyclic subgroup of the mapping class group.

What if we have two non-trivial simple closed curves α and γ in different isotopy classes and we want to study the structure of the subgroup $\langle T_\alpha, T_\gamma \rangle \leqslant Mod(S)$? Well, as we have mentioned previously, we can investigate the relationship between these Dehn-twists via the topology of S; specifically, via the

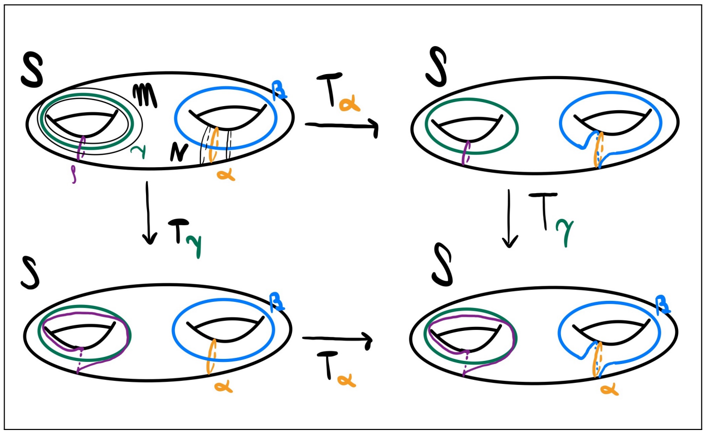

FIGURE [2B]

intersection number of the isotopy classes of the curves α and γ. If α and γ can be realized disjointly in S, then we can pick disjoint neighborhoods N and M of α and γ respectively which represent the (important part of the) support of T_α and T_γ. Thus, in terms of the effect of these two Dehn-twists on our surface, the regions in which they are doing something non-trivial have no connection with eachother. It follows that the order in which we perform them does not matter; i.e. they commute with eachother. See [2B].

Thus, when alpha and gamma don't intersect, $\langle T_\alpha, T_\gamma \rangle \cong \mathbb{Z}^2$, and we see that there arises flat structured subgroups within the mapping class group. It is shown in [FM12] that, depending on the intersection number, subgroups generated by two Dehn-twists are isomorphic to either $\mathbb{Z}$, $\mathbb{Z}^2$, braid groups on two generators, or F_2.

At this point the reader will most likely notice that the key to understanding Dehn-twists lies in an understanding of the structure of isotopy classes of simple closed curves in S. Actually, understanding isotopy classes of simple closed curves in S can give one a good understanding of the entire mapping class group.

Theorem 2.1 (Dehn-Lickorish)**.** *Let S be a surface of genus $g \geq 0$ with no boundary components nor punctures, then $Mod(S)$ is generated by finitely many Dehn-twists about non-separating simple closed curves.*

This is yet another piece of evidence for the hypothesis that the combinatorial data coming from subsets of simple closed curves in a surface encodes practically all one needs to know topologically about the surface. Let's look at an example of a surface where the mapping class group is generated by Dehn-twists.

Example 2.1. $Mod(T^2)$

In [FM12] it is shown that $Mod(T^2) \cong SL_2(\mathbb{Z})$ by looking at how the mapping class group of the torus acts on its first homology. Another way to see that

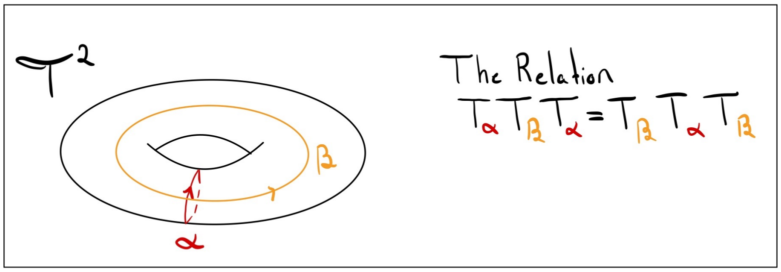

FIGURE [2C]

$Mod(T^2) \cong SL_2(\mathbb{Z})$ is through Dehn-twists. The two Dehn-twists that generate $Mod(T^2)$ are about the homotopy classes of the curves α and β shown in figure [2F]. Since the homotopy classes of these curves intersect once, their Dehn-twists will have a braid relation, and this braid relation corresponds exactly to the relation between the two generators $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ of $SL_2(\mathbb{Z})$.

2.2 The Dehn-Nielsen-Baer Theorem

In 2.1.2 we saw that isotopy classes of simple closed curves played a big role in studying the structure of the mapping class group. We will now start exploring a connection between the extended mapping class group of a surface and its fundamental group. For a surface S, a continuous map $f : (S, x_0) \to (S, f(x_0))$ induces a homomorphism $f_* : \pi_1(S, x_0) \to \pi_1(S, f(x_0))$. If f is a homotopy equivalence, then f_* will be an isomorphism. Thus, for a surface S, any $\phi \in Homeo^{\pm}(S)$ will induce an isomorphism $\phi_* : \pi_1(S, x_0) \to \pi_1(S, \phi(x_0))$. As we have hinted at previously, we would like elements of $Mod^{\pm}(S)$ to induce *automorphisms* of the fundamental group of our surface. The first thing to take into account is that two homotopic maps will induce the same homomorphism of fundamental groups. More specifically, if we have two continuous maps $\phi : (S, x_0) \to (S, \phi(x_0))$ and $\psi : (S, x_0) \to (S, \phi(x_0))$ where $\phi(x_0) = \psi(x_0)$ and $\phi \sim \psi \ rel(x_0)$, then $phi_* = \psi_*$. We would like to not have to deal with all of these hindrances in terms of basepoints: i.e. we would like to take an arbitrary element in $Mod^{\pm}(S)$ and get a unique outer automorphism of $\pi_1(S, x_0)$ for a fixed x_0. The way to do this is as follows.

For $\phi \in Mod^{\pm}(S)$ choose a path γ from x_0 to $\phi(x_0)$. Then, define $\phi_* : \pi_1(S, x_0) \to \pi_1(S, x_0)$ by

$$[\ell] \mapsto [\gamma \phi(\ell) \gamma^{-1}].$$

Note that we made a choice for the path from x_0 to $\phi(x_0)$. Say we choose another path β from x_0 to $\phi(x_0)$. Then we will get a different automorphism $\phi_* : \pi_1(S, x_0) \to \pi_1(S, x_0)$ defined by

$$[\ell] \mapsto [\beta \phi(\ell) \beta^{-1}].$$

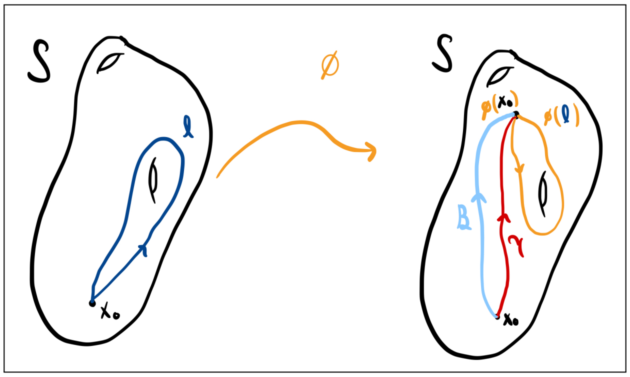

FIGURE [2D]

Note that in $Aut(\pi_1(S, x_0))$, ϕ_* and ϕ_* differ by conjugation (see [2D]):

$$\begin{aligned}[\gamma\beta^{-1}]\phi_*(\ell)[\beta\gamma^{-1}] &= [\gamma\beta^{-1}][\beta\phi(\ell)\beta^{-1}][\beta\gamma^{-1}] \\ &= [(\gamma\beta^{-1})(\beta\phi(\ell)\beta^{-1})(\beta\gamma^{-1})] \\ &= [\gamma(\beta^{-1}\beta)\phi(\ell)(\beta^{-1}\beta)\gamma^{-1}] \\ &= [\gamma C_{x_0}\phi(\ell)C_{x_0}\gamma^{-1}] \\ &= [\gamma\phi(\ell)\gamma^{-1}] = \phi_*(\ell).\end{aligned}$$

Thus, for each element in $Mod^{\pm}(S)$ we get a unique element in $Out(\pi_1(S, x_0))$. Since we did this for an arbitrary $x_0 \in S$, and the fundamental group of S at any choice of basepoint will be isomorphic to the fundamental group of S at any other choice of basepoint, we can surpress the basepoints in our notation and say we have a well defined map $\Phi : Mod^{\pm}(S) \to Out(\pi_1(S))$.We easily verify that, for $\phi, \psi \in Homeo(S)$, by properties of induced homomorphisms on fundamental groups,

$$\Phi(\psi \circ \phi) = (\psi \circ \phi)_* = \psi_* \circ \phi_* = \Phi(\psi) \circ \Psi(\phi).$$

And so $\Phi : Mod^{\pm}(S) \to Out(\pi_1(S))$ is a homomorphism.

We are now ready to state the main theorem of this section.

Theorem 2.2 (Dehn-Nielsen-Baer)**.** *Let S_g be a surface of genus $g > 0$ with no punctures nor boundary components. The homomorphism*

$$\Phi : Mod^{\pm}(S) \to Out(\pi_1(S))$$

is an isomorphism.

The injectivity of this homomorphism follows from the fact that S is a $K(\pi_1(S), 1)$ space, and so there is a one-to-one correspondence between free homotopy classes of unbased maps $S \to S$ and conjugacy classes of homomorphisms $\pi_1(S) \to \pi_1(S)$. Proving surjectivity for $g \geq 2$ can be done by looking at actions of the fundamental group on the universal cover of S. See [Hat02] and [FM12] for thorough explanations of the injectivity and surjectivity of this homomorphism.

This theorem is great in many ways, but for us it is especially nice because it gives us our first glimpse into a connection between mapping class groups and outer automorphism groups. We now know that, if we were to be given a group G, and all we knew about it was, say, its presentation, then if we can find a surface of genus $g > 0$ with no punctures nor boundary components that has fundamental group isomorphic to G, we can conduct a complete study of $Out(G)$ by studying the mapping class group of the surface.

2.3 Model Spaces for F_n

As we discussed previously, we would like to use the ideas behind Dehn-Nielsen-Baer theorem to guide our journey when trying to model $Out(F_n)$ as a group associated to a topological space. A good place to start is to look into some spaces which have free groups as their fundamental groups.

2.3.1 Punctured Surfaces

The first types of spaces we will look into when thinking about this process are punctured surfaces, because there is already a well-studied group associated to them that we have discussed at-length: the mapping class group! One might ask how we know that the fundamental groups of punctured surfaces are free groups. This comes from the fact that a surface S of genus g with p punctures and no boundary components is homotopy equivalent to R_{2g+p-1}, the wedge of $2g+p-1$ circles. Since $\pi_1(R_{2g+p-1}) \cong F_{2g+p-1}$ and the fundamental group is an invariant under homotopy equivalence, $\pi_1(S) \cong F_{2g+p-1}$.

Does it still hold that we can model the whole of the outer automorphism group of the fundamental group of a punctured surface via its extended mapping class group? The answer is no. The reasoning comes from how elements in the mapping class group interact with the punctures of a surface. We know that any element of the mapping class group of a punctured surface S is a homeomorphism, and homeomorphisms of S will at most permute the set of punctures. Elements in the fundamental group of S are, by definition, homotopy classes of loops in S, and we know that there will be loops in S, which are unique up to homotopy, that bound the punctures. Thus, their corresponding elements in the fundamental group can only be sent to conjugacy classes of themselves or conjugacy classes of loops which bound the other punctures under an element of the mapping class group. Now, if we look at the outer automorphism group of the fundamental group of S, it will have elements which send those representatives to representatives of loops which are not in the conjugacy class of a curve which bounds a puncture of the surface. Let's look at an example.

Example 2.2. $S_{1,3}$

Let $S_{1,3}$ be the genus one surface with three punctures. The fundamental group of $S_{1,3}$ is isomorphic to F_4 and is generated by the loops x_1, x_2, x_3, and x_4 shown in [2D]. Consider the conjugacy class of $\phi \in Aut(F_4)$ where

$$\phi : F_4 \to F_4$$
$$x_1 \mapsto x_1 x_3$$
$$x_2 \mapsto x_4$$
$$x_3 \mapsto x_2$$
$$x_4 \mapsto x_1.$$

Note that, as x_1 bounds a puncture, any element in $Mod^{\pm}(S_{1,3})$ will map x_1 to a conjugacy class of either x_1, x_2, or $x_3 x_4 x_3^{-1} x_4^{-1}$. However, note that the image of x_1 is not in any of these conjugacy classes, and so it is impossible to represent this element by an element in $Mod^{\pm}(S_{1,3})$.

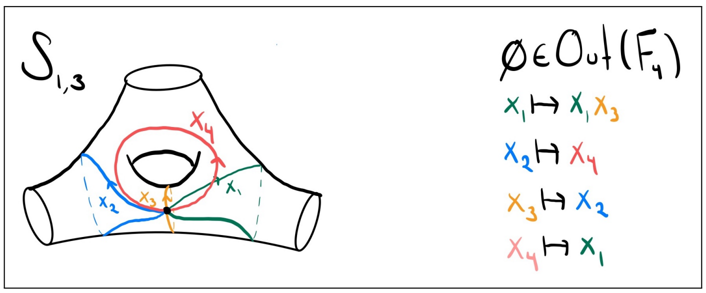

FIGURE [2E]

While in general for a punctured surface S, $Mod^{\pm}(S)$ will not be isomorphic to $Out(\pi_1(S))$, it will inject into a subset of $Out(\pi_1(S))$, namely:

Theorem 2.3 (Dehn-Nielsen-Baer for Punctured Surfaces)**.** *Let $S_{g,p}$ be a hyperbolic surface with genus g and p punctures and let $Out^{\star}(\pi_1(S_{g,p}))$ be the subgroup of $Out(\pi_1(S_{g,p}))$ which preserves conjugacy classes of simple closed curves surrounding individual punctures. Then the natural map*

$$\Phi : Mod^{\pm}(S_{g,p}) \to Out^{\star}(\pi_1(S_{g,p}))$$

is an isomorphism.

There is one punctured surface for which the extended mapping class group surjects onto the outer automorphism group of its fundamental group. Consider $S_{1,1}$, the once punctured torus. $\pi_1(S_{1,1})$ is isomorphic to F_2 and is generated by the loops x_1 and x_2 shown in [2E]. A special thing that happens with F_2 is that any element in $Aut(F_2)$ fixes the conjugacy class of commutator $[x_1, x_2]$, which is exacly the homotopy class of the curve on $S_{1,1}$ which bounds the puncture.

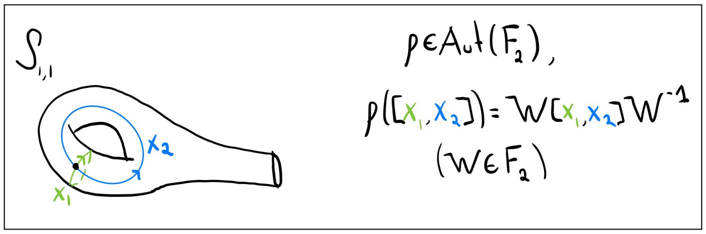

FIGURE [2F]

2.3.2 Graphs

We know that for a connected graph Γ, $\pi_1(\Gamma) \cong F_n$ for some n. If we constructed a homeomorphism group of a graph, would that be able to model the outer automorphism group of its fundamental group in the same way the extended mapping class group does for fundamental groups of surfaces without punctures or boundary components?

Restricting ourselves to homeomorphisms turns out to be too rigid of a viewpoint if we want to model outer automorphism groups of fundamental groups of connected graphs. For a connected graph Γ with $\pi_1(\Gamma) \cong F_n$ there will be many elements in $Out(F_n)$ which will not be able to be modeled by a homeomorphism of Γ. Take R_4 for example. Note that the petals x_1, x_2, x_3, and x_4 shown in [2G] correspond to the generators of $\pi_1(R_4) \cong F_4$. Consider the conjugacy classes of $\phi, \varphi \in Aut(F_4)$ where

$$\begin{aligned} \phi : F_4 &\to F_4 \\ x_1 &\mapsto x_1 \\ x_2 &\mapsto x_4 \\ x_3 &\mapsto x_2 \\ x_4 &\mapsto x_3; \end{aligned}$$

$$\begin{aligned} \varphi : F_4 &\to F_4 \\ x_1 &\mapsto x_1 x_2 \\ x_2 &\mapsto x_4 \\ x_3 &\mapsto x_2 \\ x_4 &\mapsto x_3. \end{aligned}$$

If we wanted to model these maps topologically as maps from $R_4 \to R_4$, we would build the maps from where the loops of the generators of $\pi_1(R_4)$ are sent under the automorphism. Note that if we follow this procedure, the map $\tilde{\phi} : R_4 \to R_4$ will indeed be a homeomorphism, but the map $\tilde{\varphi} : R_4 \to R_4$ will not. See [2G]. As a consequence we might want to consider different types of maps when attempting to model $Out(\pi_1(\Gamma))$.

Let $H.E.(\Gamma)$ denote the group of homotopy equivalences of Γ. In a similar vein as when we examined the connection between the mapping class group

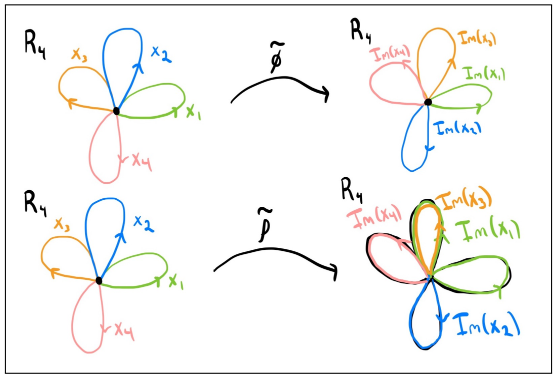

FIGURE [2G]

of a surface and the outer automorphism group of its fundamental group, any $f \in H.E.(\Gamma)$ will induce an isomorphism $f_* : \pi_1(\Gamma, x_0) \to \pi_1(\Gamma, f(x_0))$. To get rid of the basepoint issue we choose a path from x_0 to $f(x_0)$ and follow the exact same reasoning as in 2.2 to get a well-defined homomorphism from $\Psi : H.E.(\Gamma)/Homotopy \to Out(\pi_1(\Gamma))$.

Theorem 2.4. *Let Γ be a connected graph. Then the homomorphism*

$$\Psi : H.E.(\Gamma)/Homotopy \to Out(\pi_1(\Gamma))$$

is an isomorphism.

While $\tilde{\varphi}$ is not a homeomorphism, it *is* a homotopy equivalence from $R_4 \to R_4$. Indeed, we obtain its homotopy inverse by constructing a map $\tilde{\varphi}^{-1} : R_4 \to R_4$ from $\varphi^{-1} \in Aut(F_4)$ where

$$\varphi^{-1} : F_4 \to F_4$$
$$x_1 \mapsto x_1 {x_4}^{-1}$$
$$x_2 \mapsto x_3$$
$$x_3 \mapsto x_4$$
$$x_4 \mapsto x_2.$$

Now we know that, if we are given a free group F_n, we can study its outer-automorphism group by finding a connected graph Γ for which $\pi_1(\Gamma) \cong F_n$ and looking at $H.E.(\Gamma)/Homotopy$. Note that there are *many* graphs Γ we can pick with $\pi_1(\Gamma) \cong F_n$ and hence many different homotopy equivalences which represent an element in $Out(F_n)$.

2.4 Stallings' Folding Techniques and their Ramifications

We are now at a point where we can consider connected graphs Γ satisfactory spaces to model F_n. With this in our tool belt, we will be able to investigate graphs further and see if there are any methods in their study which will in turn help us to understand $Out(F_n)$. Thurston's classification theorem for elements in the mapping class group gives a way of uniquely representing homeomorphisms of a surface S. There is an analogue classification process for elements of $Out(F_n)$, developed by Bestvina, using certain homotopy equivalences called train track maps. This train track technology associates to every element of $Out(F_n)$ a map of graphs that in a sense respects paths of train tracks. The creation of train track maps relied on the technique of folding for graphs created by Stallings. Further, folding allows one to topologically prove that Nielsen's set of automorphisms generates $Aut(F_n)$ and gives a method to check whether a map $F_n \to F_n$ is an automorphism.

2.4.1 Folding

Suppose e_1 and e_2 are edges of a graph Γ with the same initial vertices. Form a new graph $\Gamma^{'}$ where e_1 is identified with e_2 and the terminal vertices of e_1 and e_2 are also identified. The resulting quotient map $\phi : \Gamma \to \Gamma^{'}$ is called a *fold*. When considering what folding does on the level of fundamental groups, there are a few different cases. If e_1 and e_2 have different terminal vertices, then $\pi_1(\Gamma^{'})$ is isomorphic to $\pi_1(\Gamma)$. If e_1 and e_2 have the same terminal vertex, then the rank $\pi_1(\Gamma^{'})$ will be one less than the rank of $\pi_1(\Gamma)$. See [2H].

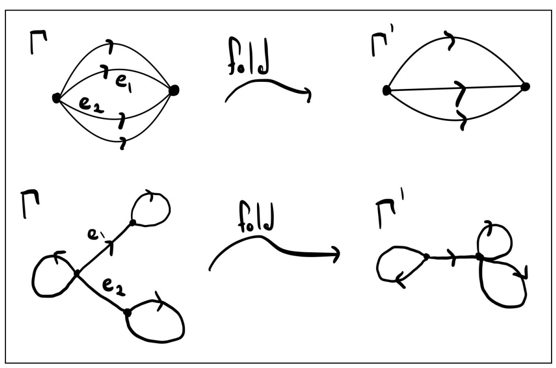

FIGURE [2H]

If one has a map $F_n \to F_n$, where F_n is generated by $x_1, x_2, ..., x_n$, an easy way to check if it is an automorphism is through folding. Identify the image of each generator with a petal of $R_n^{'}$, where, if a generator gets sent to a word of length m in the generators, subdivide the petal its image is identified with into m edges. Then the ith edge is identified with the ith generator in the image. This graph has an obvious graph morphism to R_n, the rose where each petal is identified with $x_1, x_2, ..., x_n$. If this graph morphism can be factored through a sequence of folds such that the graph for which we can perform no more folds is homeomorphic to R_n, then the initial map $F_n \to F_n$ is an automorphism. Take, for example, the map φ where

$$\varphi : F_4 \to F_4$$
$$x_1 \mapsto x_1 x_2$$
$$x_2 \mapsto x_4$$
$$x_3 \mapsto x_2$$
$$x_4 \mapsto x_3.$$

In [2I] we see that the graph morphism $R_4' \to R_4$ can be factored through a sequence of folds such that the graph in which we can perform no more folds is homeomorphic to R_4.

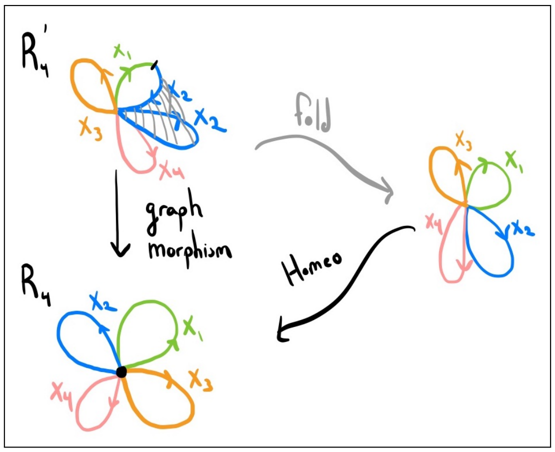

FIGURE [2I]

3 Linear Representations

We will see that for both mapping class groups and outer automorphism groups of free groups there are well defined homomorphisms $Mod(S) \to GL_n(\mathbb{Z})$ and $Out(F_n) \to GL_n(\mathbb{Z})$. From these maps we'll have a direct way of algebraically comparing mapping class groups and outer automorphism groups of free groups with automorphism groups of free abelian groups. Knowing how similar these groups are will also give us a way of measuring how "far away" surface groups and free groups are from being linear.

3.1 The Linear Representation of $Out(F_n)$ and the Torelli Subgroup

3.1.1 In the Realm of Mapping Class Groups

Let $\phi \in Mod(S)$. In 2.1.3 we saw that ϕ induces an automorphism ϕ_* of $\pi_1(S)$ that is unique up to conjugation. Since any automorphism of $\pi_1(S)$

will fix the commutator subgroup, we know ϕ_* induces an automorphism of $\pi_1(S)/\langle\langle[\pi_1(S), \pi_1(S)]\rangle\rangle$. From [Hatcher], we know that $\pi_1(S)/\langle\langle[\pi_1(S), \pi_1(S)]\rangle\rangle$ is isomorphic to $H_1(S, \mathbb{Z})$, and so we get that ϕ induces an automorphism of $H_1(S, \mathbb{Z})$ that is unique up to conjugation. This shows that we get a well defined map from $Mod(S) \to Aut(H_1(S, \mathbb{Z})) \cong GL_n(\mathbb{Z})$. The value of studying the action of the mapping class group on homology is that for many surfaces the mapping class group will be a group that is complex and hard to understand, while $GL_n(\mathbb{Z})$ is better understood. The action on homology is able to retain a decent amount of information and let's us gain algebraic information about the mapping class group through $GL_n(\mathbb{Z})$.

3.1.2 The Linear Representation of $Out(F_n)$

Let $\phi \in Aut(F_n)$, where F_n is generated by $x_1, x_2, ..., x_n$. Note that for any $a, b \in F_n$, $\phi([a, b]) = [\phi(a), \phi(b)]$. Thus ϕ induces an automorphism

$$\phi_* : F_n/\langle\langle[F_n, F_n]\rangle\rangle \to F_n/\langle\langle[F_n, F_n]\rangle\rangle, \text{ where}$$
$$\phi_*([W]) = [\phi(W)].$$

Let $\varphi \in Inn(F_n)$, i.e. $\varphi(a) = WaW^{-1}$ for some $W \in F_n$. The corresponding element $\varphi_* \in Aut(F_n/\langle\langle[F_n, F_n]\rangle\rangle)$ takes $[W] \mapsto [WaW^{-1}] = [W][a][W]^{-1} = [a]$ and so is the identity in $Aut(F_n/\langle\langle[F_n, F_n]\rangle\rangle)$. Since any element in the inner automorphism group is mapped to the identity, this map factors through $Out(F_n)$, and so we see we have a map

$$\Phi : Out(F_n) \to Aut(F_n/\langle\langle[F_n, F_n]\rangle\rangle) \cong Aut(\mathbb{Z}^n) \cong GL_n(\mathbb{Z}).$$

Note that for $\phi, \varphi \in Out(F_n)$

$$\Phi(\phi \circ \varphi) = (\phi \circ \varphi)_* = \phi_* \circ \varphi_*$$

and so Φ is a homomorphism.

One of the first and most accessible cases of this homomorphism is $\Phi : Out(F_2) \to GL_2(\mathbb{Z})$. It was shown in 2.3.1 that the only punctured surface for which its mapping class group surjects onto the outer automorphism group of its fundamental group is the once punctured torus $S_{1,1}$. It turns out that $Mod^{\pm}(S_{1,1}) \cong Out(\pi_1(S_{1,1}))$, and so $Mod^{\pm}(S_{1,1}) \cong Out(F_2)$. In Example 2.1 we discussed how $Mod(T^2) \cong SL_2(\mathbb{Z})$. Due to the special nature of $S_{1,1}$, its mapping class group is also isomorphic to $SL_2(\mathbb{Z})$. There is a natural homomorphism $Mod^{\pm}(S) \to \mathbb{Z}/2\mathbb{Z}$ which records whether an element in the mapping class group is orientation preserving or reversing. Note that the kernel of this homomorphism is $Mod(S)$, and thus $Mod(S)$ is an index 2 subgroup of $Mod^{\pm}(S)$. This implies that $Mod^{\pm}(S_{1,1}) \cong GL_2(\mathbb{Z})$, which, further, shows that $Out(F_2) \cong GL_2(\mathbb{Z})$. What if we wanted to prove that $Out(F_2) \cong GL_2(\mathbb{Z})$ algebraically? One way we could do it is to show that $Aut(F_n)$ surjects onto $GL_2(\mathbb{Z})$, and then show that the kernel of this homomorphism is exactly $Inn(F_2)$. Let Ψ: $Aut(F_2) \to GL_2(\mathbb{Z})$ induced from the correspondence discussed in the previous paragraph. Nielsen provided a nice set of generators for $Aut(F_n)$ which are given in [MKS04]. For $Aut(F_2)$ these reduce down to the set α, β, σ defined by

$$
\begin{aligned}
\alpha : F_2 &\to F_2 \\
x_1 &\mapsto x_2 \\
x_2 &\mapsto x_1; \\
\beta : F_2 &\to F_2 \\
x_1 &\mapsto x_1^{-1} \\
x_2 &\mapsto x_2; \\
\sigma : F_2 &\to F_2; \\
x_1 &\mapsto x_1 x_2 \\
x_2 &\mapsto x_2
\end{aligned}
$$

Which map to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ under Ψ respectively. Note that these three matrices correspond exactly to the elementary row operation matrices that generate $GL_2(\mathbb{Z})$.

Showing that $ker(\Psi) \cong Inn(F_2)$ algebraically is much harder and a proof is given in [Nie24].

4 Possible Future Directions

There are two different directions I would like to continue with this research. I will list them in order of presendence below:

1. From what I can tell, there is only one algebraic proof of the isomorphism between $Out(F_2)$ and $GL_2(\mathbb{Z})$. This proof [Nie24] is also somewhat long and uses complicated techniques. My goal is to provide a simpler algebraic proof of this isomorphism. In doing this I would also like to study the kernel of $\Phi : Out(F_n) \to GL_n(\mathbb{Z})$ and determine if the types of automorphisms in the kernel differ as one increases n.

2. In 2.1.2 we saw how, through Dehn-twists, simple closed curves play a huge role in the study of mapping class groups. A structure that we did not get to touch on in this paper is the complex of curves, a simplicial complex that encodes the combinatorial data of simple closed curves on a surface. Via this geometric structure, we can study the structure of the mapping class group in many ways. For example, the curve complex encodes free abelian subgroups in a way that is concretely related to how we saw them generated in 2.1.2. There is an analogue to the complex of curves for free groups called the free factor complex. In a similar vein, this is a simplicial complex which can encode a lot of information about outer automorphism groups of free groups. I would like to study the Cauhen-Mcaulay complex for right angled Artin groups and see whether theorems for right angled Artin groups can be proved using the action of their outer automorphism groups on this complex.

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