# Orbits and Centralizers for Algebraic Groups in Small Characteristic and Lie Algebra Representations in Standard Levi Form

A dissertation submitted by

Alex P. Babinski

In partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
*Mathematics*

Tufts University

Medford, Massachusetts
May 2015

Adviser: George McNinch

## Abstract

The purpose of this work is two-fold. First, we will explore what can be said about some particular conjectures concerning centralizers and orbits of algebraic groups when considering a ground field of small characteristic. Second, we attempt to understand non-restricted Lie algebra representations for standard Levi form by generalizing some existing machinery.

Specifically, in Chapter 2 we provide a proof of the existence of Levi decompositions of nilpotent centralizers in classical groups of bad characteristic. Then, in Chapter 3, we provide an initial approach to a conjecture of Steinberg in good characteristic related to understanding the orbits of an algebraic group by that of its faithful representations. This conjecture was previously known (due to Steinberg) in characteristic zero or "sufficiently large", while our approach is valid for certain elements in almost good characteristic and provides a smaller restriction for the analogous case of certain elements in the Lie algebra. Finally, in Chapter 4 we generalize a construction of Jantzen in the special setting of standard Levi form. Here we study an important type of module called a baby Verma module and build its smaller parabolic analogue. It turns out that these both yield the same unique simple quotient.

# Acknowledgements

This document is a confluence of many streams of inspiration and support, some of which I would like to acknowledge here. First and foremost, I would like to thank my adviser George McNinch for innumerable lessons, suggestions, and pieces of advice. This would certainly not have be possible without him, and I am extremely grateful for all of the ideas he shared with me. I would also like to thank my committee members Eric Sommers, Montserrat Teixidor i Bigas, and Richard Weiss for generously taking the time to read and comment on this work.

It would be remiss of me not to mention my appreciation for the entire mathematics department at Tufts University for expanding my mathematical horizons on all fronts through so many rewarding classes and relationships. Thanks, too, to everyone at the Poincaré Institute for an amazing opportunity gaining insights in mathematics education from educators and researchers alike. I am grateful as well for my fellow graduate students for help along the road and for listening to me give talks that unfortunately involved very few pictures.

Thank you to my parents and sister for constant love, encouragement, and decompression, and for giving the means through supposedly unused mathematical genes. I would not be here without you, literally and figuratively. Thanks as well to my indescribably great friends D. Bruce, R. Talbot, and G. Allan for over a decade of illumination, music, and gondolas.

Lastly and essentially, I would like to thank my incomparable wife Megan Creighton. Her love and support is truly remarkable, and there is not enough space here for me to say how much I appreciate everything she does; perhaps I will find a postscript.

# Contents


1 Introduction 2

1.1 Orbits and Centralizers of Nilpotent Elements in Small Characteristic 3

1.1.1 Levi Decomposition of Centralizers . . . . . 3

1.1.2 Conjugacy Under Representations . . . . . 5

1.2 Lie Algebra Representations . . . . . 5

1.2.1 Standard Levi Form . . . . . 8

1.2.2 Categories and Filtrations . . . . . 9

2 Levi Decompositions of Nilpotent Centralizers in Bad Characteristic: Classical Groups 11

2.1 Preliminaries and Good Characteristic . . . . . 11

2.2 Indecomposables . . . . . 13

2.3 The Symplectic Group . . . . . 15

2.4 The Orthogonal Group . . . . . 17

2.5 Examples . . . . . 24

2.5.1 The 210-dimensional Symplectic Group . . . . . 24

2.5.2 The 1653-dimensional Orthogonal Group . . . . . 26

3 Steps Towards a Conjugacy Conjecture of Steinberg in Small Characteristic 29

3.1 Preliminaries and Semisimple Classes . . . . . 29

3.2 Algebraic Groups . . . . . 32

3.3 Lie Algebras . . . . . 34

3.4 Future Work . . . . . 37

**4 Modules in Standard Levi Form** **39**

4.1 Preliminaries and Parabolics . . . . . . . . . . . . . . . . . . . . . . . 39

4.2 Parabolic Baby Verma Modules . . . . . . . . . . . . . . . . . . . . . . 40

4.3 Isomorphic Modules . . . . . . . . . . . . . . . . . . . . . . . . . . . . 43

4.4 Further Parabolic Exploration . . . . . . . . . . . . . . . . . . . . . . . 45

4.5 Categories of Modules . . . . . . . . . . . . . . . . . . . . . . . . . . . 48

**Bibliography** **53**

# Orbits and Centralizers for Algebraic Groups in Small Characteristic and Lie Algebra Representations in Standard Levi Form

Alex P. Babinski

# Chapter 1

# Introduction

We begin this work by investigating questions related to orbits and centralizers for the conjugation action of a reductive algebraic group on elements in its Lie algebra, as well as orbits of elements in the group. In particular, we study the structure of nilpotent centralizers and how to begin to understand conjugacy in the group under different representations, concerning ourselves with how these objects change when we consider our group over a field of smaller characteristic and many traditionally useful constructions are not valid.

Next, we examine the irreducible representations of Lie algebras of algebraic groups in positive characteristic by exploring modules for their reduced enveloping algebras, looking at so-called nonrestricted representations and try to understand them algebraically. We will be interested primarily in the situation of "standard Levi form", when certain standard modules are endowed with unique maximal submodules. Our aim is to expand on and generalize existing machinery to better realize simple modules.

In the following sections of this chapter, we will provide some background and an overview of the results for each of these subjects.

## 1.1 Orbits and Centralizers of Nilpotent Elements in Small Characteristic

For more than 50 years, mathematicians have been studying the structure of orbits and centralizers of nilpotent and unipotent elements in algebraic groups, applying this information to better understand subgroup structures and Lie algebra representations, among other subjects. Major results in this area include Steinberg's connectedness theorem for centralizers of semisimple elements, the Richardson-Lusztig theorem on the finiteness of unipotent orbits, and the Bala-Carter-Pommerening classification of unipotent orbits (see [H95] or [Ca85]).

Over time, much has been learned about these objects, particularly for algebraic groups over fields of "large enough" characteristic, where the situation mirrors that of characteristic zero. In chapters 2 and 3, we are interested in what can be said in the remaining cases — do certain properties still hold, or do certain pathological aspects of smaller characteristic change the structure of what we are looking at? Specifically, does a centralizer of a nilpotent element have a Levi decomposition for classical groups in bad characteristic, and can we distinguish between orbits in small characteristic by looking at the images of elements under representations?

### 1.1.1 Levi Decomposition of Centralizers

Let $H$ be a connected linear algebraic group over an algebraically closed field $k$ with unipotent radical $U$. A *Levi factor* of $H$ is a reductive subgroup $L$ such that $H \simeq U \rtimes L$. We then say that $H$ has a *Levi decomposition*. It is important to note that this is an isomorphism of algebraic groups, so the existence of a subgroup of $H$ isomorphic to $H/U$ is not enough to ensure a Levi decomposition; one must check that the projection map is an isomorphism at the level of the tangent spaces as well. This amounts to showing that the Lie algebras of $U$ and $L$ have an empty intersection. When the characteristic of $k$ is $p > 0$, Levi factors need not exist for an arbitrary group $H$ (see Section 3.2 of [M10]).

Now, consider a reductive algebraic group $G$ over a field $k$ of characteristic $p > 0$, and let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. For a nilpotent element $e \in \mathfrak{g}$, we might ask whether its centralizer in $G$, denoted $C_G(e)$, has a Levi decomposition. When the characteristic of our field is very large, this can be shown to be true by embedding $e$ in an $\mathfrak{sl}_2$ subalgebra, much as one would in characteristic zero. In the cases where $k$ is of *good* characteristic, we can (after some work) use the analogous notion of associated cocharacters and proceed similarly (see Section 5 of [J04]). The question remains: what should we do when the characteristic is *bad*, and associated cocharacters need not exist?

As a natural starting place, we will first consider the classical groups. So, let $G = Sp(V)$ or $O(V)$ for a finite dimensional vector space $V$ over a field $k$ of characteristic 2. A recent book by Liebeck and Seitz ([LS12]) provides a wealth of information about the structure of centralizers in these (and other) situations. In it, they produce a *distinguished normal form* for the action of a nilpotent $e$ on $V$. This involves decomposing $V$ into certain indecomposables for the action of $e$ (dating back to Hesselink in [He79]). In defining this form, they give rise to a certain one-dimensional torus $T$ which acts by particular weights on chosen basis vectors, somewhat taking the place of an associated cocharacter.

Liebeck and Seitz give the precise form of the reductive quotient of the connected centralizer $C_G(e)^0/R_u(C_G(e))$ and show that there exists a closed subgroup $L \leqslant C_G(e)^0$ isomorphic to it as an abstract group. We hope that this $L$ is our Levi subgroup but, unfortunately, existence as an abstract subgroup is not enough to ensure a Levi decomposition in bad characteristic (see Section 3.3 of [M10]). To be sure, we have to check that the projection map is compatible on the level of $\mathrm{Lie}(C_G(e)^0)$. That is, if $\pi : C_G(e)^0 \to C_G(e)^0/R_u(C_G(e))$ is the projection map, we need $d\pi|_{\mathrm{Lie}(L)}$ to be bijective.

When $G = Sp(V)$, we are able to establish uniformly that $L$ is an honest Levi factor infinitesimally as well as on the level of groups through a bit of representation theory (see Proposition 2.3.2). This method works occasionally when $G = O(V)$; though when $C_G(e)^0/R_u(C_G(e))$ contains $SO_{2a_i+1}$ factors, these will not act simply on their natural modules, making this case incompatible with our current approach. To get a full result in the orthogonal group, we are forced to use a much more "hands-on" approach (see Proposition 2.4.5). With this, we are able to combine the two results in Theorem 2.4.6 to state finally that the connected centralizer of a nilpotent element in a classical group in bad characteristic does indeed have a Levi decomposition.

### 1.1.2 Conjugacy Under Representations

Given an algebraic group $G$ with Lie algebra $\mathfrak{g}$, at the outset the question might arise of how to best arrange the elements of $G$ and $\mathfrak{g}$. A natural way to do so is to collect the elements into conjugacy classes for the action of $G$ on itself, or for the adjoint action of $G$ on $\mathfrak{g}$. Then we might wonder: how can we tell these classes apart? Can we parameterize them in some way?

It turns out that the regular class functions for $G$, regular functions which are constant on conjugacy classes, give us a partial answer to these further questions. They can distinguish between classes of *semisimple* elements, though they do not provide enough information to separate classes in general. So, if class functions are not robust enough, perhaps studying all of the faithful representations of $G$ will do the job.

In 1966, Steinberg conjectured that it could be determined whether two elements in a semisimple algebraic group were conjugate by checking that they were conjugate under every rational representation. He later proved this result when the ground field was of characteristic zero or "sufficiently large", greater than roughly four times the Coxeter number. In Chapter 3, we use a result of Lawther to describe and approach for verifying Steinberg's conjecture for certain group elements with exceptional semisimple centralizers when the characteristic is good, with a single exception in type $F_4$. We then extend the proof, using results of McNinch, to orbits of certain elements in the Lie algebra under the adjoint action of $G$, though with a few more restrictions on the characteristic.

## 1.2 Lie Algebra Representations

Early work in the field of modular Lie algebra representation theory is highlighted by papers of Zassenhaus and Curtis from the 1940's and 1950's (see [Z40], [Z54], [Cu53], [Cu60]), though activity receded a bit over the next few decades. More recent developments include the algebraic constructions of Friedlander and Parshall in the late 1980's and early 1990's (see [FP88], [FP90], [FP91]); the fundamental 1994 paper of Andersen, Jantzen, and Soergel ([AJS94]); and later work by Jantzen in the late

1990's and early 2000's (see [J99], [J00], [J04]). Even more recently, the algebro-geometric methods of Bezrukavnikov, Mirković, and Rumynin gave a count for the number irreducible Lie algebra representations in [BMR08]. The goal of chapter 4 is to explore and generalize a previous algebraic approach due to Jantzen.

Let us begin by briefly discussing the mathematical story of Lie algebra representation theory in positive characteristic, largely following the great surveys of Humphreys [H98] and Jantzen [J97]. We will note that realizing the representation theory in general will depend strongly on understanding different types and structures of nilpotent elements, suggesting our work in the previous two chapters. Though we will not use those previous results directly, the interplay between these two fields, Lie algebra representation theory and nilpotent orbit structure, is very evident.

Let $\mathfrak{g}$ be the Lie algebra of a reductive algebraic group $G$ over a field $k$ of characteristic $p > 0$. As is often the case, we want to be able to understand the representations of $\mathfrak{g}$. These correspond, as in characteristic 0, to modules for the universal enveloping algebra $U(\mathfrak{g})$. In contrast to the characteristic 0 situation, a result of Curtis in [Cu53] states that the dimensions of these modules are finite and, moreover, bounded. This is due to the fact that the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is a finitely generated $k$-algebra in characteristic $p$, and $U(\mathfrak{g})$ is a finitely generated $Z(\mathfrak{g})$-module.

Now, $Z(\mathfrak{g})$ contains $x^p - x^{[p]}$ for all $x \in \mathfrak{g}$, where $x^p$ is the $p$th power in the enveloping algebra and $x^{[p]}$ is the $p$th power in $\mathfrak{g}$ (which injects into $U(\mathfrak{g})$). So, given a simple $U(\mathfrak{g})$-module $M$, each $x^p - x^{[p]}$ acts on $M$ as a scalar $\chi(x)^p$ by Schur's lemma, and hence defines a character $\chi$ of $\mathfrak{g}$ called the "$p$-character" of $M$. For any $\chi \in \mathfrak{g}^*$, define

$$U_\chi(\mathfrak{g}) = U(\mathfrak{g})/\langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\rangle.$$

This is called the *reduced* enveloping algebra of $\mathfrak{g}$ for $\chi$. It has dimension $p^{\dim(\mathfrak{g})}$ and a Poincaré-Birkhoff-Witt basis of monomials made up of elements of a basis for $\mathfrak{g}$ with each exponent less than $p$. In a way, we can now partition our search for simple $U(\mathfrak{g})$-modules by characters of $\mathfrak{g}$: simple $U(\mathfrak{g})$-modules with $p$-character $\chi$ correspond to simple modules for $U_\chi(\mathfrak{g})$. Note that $U_\chi(\mathfrak{g}) \simeq U_{\chi'}(\mathfrak{g})$ if $\chi$ and $\chi'$ are in the same $G$-orbit, so we can choose $\chi$ up to conjugacy.

When $\chi = 0$, we call $U_\chi(\mathfrak{g})$ the *restricted* enveloping algebra. Understanding this is the goal in some sense, as simple $U_0(\mathfrak{g})$-modules correspond to an important class of simple rational modules for G. One prospective approach to understanding the restricted representations of $\mathfrak{g}$ has been by so-called "deformation" from the non-restricted setting, moving representations continuously through non-restricted characters until we reach the restricted case (see [FP90]). This, along with interest in the relationship between the structure of the category of modules of $U(\mathfrak{g})$ and the geometry of conjugacy classes in $\mathfrak{g} = \mathrm{Lie}(G)$, compels us to study modules for $U_\chi(\mathfrak{g})$ when $\chi \neq 0$.

Following Jantzen, we will often impose what are referred to as the "Standard Hypotheses" on our reductive group G:

(SH1) The derived subgroup of $G$ is simply connected;

(SH2) The prime $p$ is *good* for $\mathfrak{g}$;

(SH3) There exists a $G$-invariant nondegenerate bilinear form on $\mathfrak{g}$.

Highlighting the third condition, notice that such a form $(\ ,\ )$ gives us an isomorphism of $\mathfrak{g} \simeq \mathfrak{g}^*$ by identifying $x \in \mathfrak{g}$ with $\chi_x(y) = (x, y)$.

Now, fix a maximal torus $T \subseteq G$ and set $\mathfrak{h} = \mathrm{Lie}(T)$. Let $R$ be the root system for G with $R^+$ a system of positive roots, and let $\Delta$ be the set of simple roots. We have a triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

Here $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) is the sum of all $\mathfrak{g}_\alpha$ root spaces with $\alpha > 0$ (respectively $\alpha < 0$), and we call $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ (respectively $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$) the positive (respectively negative) Borel subalgebra containing $\mathfrak{h}$.

We have one last important observation due to Kac and Weisfeiler in [KW71] that allows us to further narrow down our study of reduced enveloping algebras. It is enough to understand $U_\chi(\mathfrak{g})$-modules when $\chi$ is nilpotent, i.e. $\chi(\mathfrak{b}^+) = 0$, since one can essentially find another reductive Lie algebra $\mathfrak{m}$ and nilpotent character $\chi'$ in $\mathfrak{m}^*$ such that the category of modules for $U_\chi(\mathfrak{g})$ is equivalent to the category of modules for $U_{\chi'}(\mathfrak{m})$.

### 1.2.1 Standard Levi Form

Next, we discuss the algebraic constructions that allow us to find simple $U_\chi(\mathfrak{g})$-modules and label them by certain characters of $\mathfrak{g}$. This is the positive characteristic analogue to "Category $\mathcal{O}$" story over $\mathbb{C}$ (see [H91]). We will pay specific attention to the situation when $\chi$ is in so-called "standard Levi form" and later investigate what can be gained by considering parabolic induction, the focus of chapter 4.

For any $\lambda \in \mathfrak{h}^*$, we can define a one-dimensional $\mathfrak{h}$-module $k_\lambda$, which is just the field $k$ with $h \in \mathfrak{h}$ acting as $\lambda(h)$. Now, for $\lambda \in \Lambda_\chi$, where we define

$$\Lambda_\chi = \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\},$$

we can extend $k_\lambda$ to a one-dimensional module for $U_\chi(\mathfrak{b}^+)$. Then, define

$$Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}^+)} k_\lambda.$$

This is called a "baby Verma module", suggesting the Verma modules over $\mathbb{C}$, and it has dimension $p^{\dim(\mathfrak{n}^-)}$ as well as a $k$-basis $\{x_{\alpha_1}^{a_1} x_{\alpha_2}^{a_2} \dots x_{\alpha_n}^{a_n} \otimes 1 \mid 0 \leqslant a_i < p, \alpha_i \in R^+\}$. With a theorem due to Rudakov, we can now (almost) label simple modules.

**Theorem 1.2.1.** *([R70]) Every simple $U_\chi(\mathfrak{g})$-module is the quotient of some $Z_\chi(\lambda)$ for $\lambda \in \mathfrak{h}^*$.*

When $\chi = 0$, each of the $Z_\chi(\lambda)$ have a unique simple quotient, though this can break down in the non-restricted case. Let us now focus on a particularly nice class of $\chi \neq 0$ where we can still identify the simple quotients of baby Verma modules.

As in [FP90], we define $\chi \in \mathfrak{g}^*$ to have *standard Levi form* if there exists a subset $I \subseteq \Delta$ such that $\chi(x_{-\alpha}) \neq 0$ for $\alpha \in I$, and $\chi(x_{-\alpha}) = 0$ otherwise. We can think of such a character under the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ as a regular element in the Levi subalgebra of the standard parabolic $\mathfrak{p}_I$.

By an early theorem of Zassenhaus [Z40], attributed even earlier to Whitehouse and Witt, a unipotent Lie algebra (which one can think of as the Lie algebra of a unipotent algebraic group) has only one simple module up to isomorphism for its reduced enveloping algebra. This, along with the $\mathfrak{n}^-$-module isomorphism $Z_\chi(\lambda) \simeq U_\chi(\mathfrak{n}^-)$, gives us the following theorem (see [J97] 10.2, or [FP90]).

**Theorem 1.2.2.** *If $\chi$ has standard Levi form, then each $Z_\chi(\lambda)$ has a unique maximal submodule.*

So, each such $Z_\chi(\lambda)$ has a unique simple quotient, which we can unambiguously call $L_\chi(\lambda)$. Since each simple $U_\chi(\mathfrak{g})$-module is the quotient of a $Z_\chi(\lambda)$, these make up all of the simple representations. Furthermore, two such simple modules $L_\chi(\lambda)$ and $L_\chi(\mu)$ are isomorphic if and only if $\mu \in W_{I\bullet}\lambda$. Here $W_I$ is the Weyl of the root system $R_I$ generated by $I$, and it acts on $\mathfrak{h}^*$ via the "dot action": $w_\bullet\lambda = w(\lambda + \rho) - \rho$, with $\rho$ the unique root such that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple roots $\alpha$.

In the special case of $I = \Delta$, we are looking at a *regular* nilpotent character $\chi$. Premet's proof ([P95b]) of the Kac-Weisfeiler conjecture ([KW71]) states that every simple $U_\chi(\mathfrak{g})$-module has dimension divisible by $p^{\frac{1}{2}\dim \Omega(\chi)}$, where $\Omega(\chi)$ is the $G$-orbit of $\chi$. When $\chi$ is regular, $\dim \Omega(\chi) = 2|R^+| = 2\dim(\mathfrak{n}^-)$. But now, as observed above, each $Z_\chi(\lambda)$ has dimension $p^{\dim(\mathfrak{n}^-)} = p^{\frac{1}{2}\dim \Omega(\chi)}$, hence must be simple. So, in this situation, we have $Z_\chi(\lambda) \simeq L_\chi(\lambda)$.

### 1.2.2 Categories and Filtrations

The simplicity (for lack of a better word) of the regular case helps us unravel the baby Verma modules for more general characters in standard Levi form since, as observed above, such a character corresponds to a regular element in a Levi subalgebra. Complete knowledge of the composition factors of the $Z_\chi(\lambda)$ would provide valuable information about the simple modules $L_\chi(\lambda)$. In [J97], Jantzen begins the process of achieving this by building, piece by piece, a map from each baby Verma module to an appropriate "shifted" module that is dual to it in some sense. In chapter 4, we will attempt to emulate this machinery in the setting of *parabolic* baby Verma modules. It turns out that adapting Jantzen's approach to the parabolic case encounters computational and combinatorial difficulties.

Later, in [J00], Jantzen is able to find filtrations of $Z_\chi(\lambda)$ by considering a local ring $A$ obtained by localizing the polynomial ring $k[T]$ at $\langle T \rangle$ and constructing generalized baby Verma modules over $A$ in a graded category of modules $\mathcal{C}_A$. He builds filtrations of these $A$-forms using maps as mentioned in the previous paragraph, which in turn yield filtrations of the original baby Verma modules over $k$. This gives a formula describing the sum

$$\sum_{i>0}[Z_\chi(\lambda)^i]$$

in the Grothendieck group of $U_\chi(\mathfrak{g})$-modules, where the $i$ index the filtration. This "sum formula" proves useful because, among other things, it leads to a type of linkage principle that says if the multiplicity $[Z_\chi(\lambda)\colon L_\chi(\lambda)] \neq 0$, then $\mu \uparrow \lambda$ for an order relation $\uparrow$. These sorts of results in the parabolic case would be the natural continuation of the material in chapter 4, if the aforementioned issues were to be overcome.

This all itself is a generalization of results from [AJS94] for when $\chi = 0$ (and $I = \varnothing$). In [J00], Jantzen obtains explicit lists of characters (of length $(W\colon W_I)$) in a dot orbit of $\lambda$ such that, given a character $\lambda'$ in a list, the only $\mu$ such that $[Z_\chi(\lambda')\colon L_\chi(\mu)] \neq 0$ are those also in the list ([J00], Lemma 4.12). He then uses these results to explicitly compute character and dimension formulae for the simple modules in specific cases where $I$ is large, and in these cases confirms "Lusztig's Hope" [Lu97].

Furthermore, in [AJS94], the authors note that when $A = F$ is a field, the analogous category $\mathcal{C}_F$ becomes semisimple. Later in chapter 4, we will see that the baby Verma modules over $F$ are simple just as *loc. cit.*. We conjecture, though, that they are not projective, thus we expect not to have $\mathcal{C}_F$ semisimple for $\chi \neq 0$ in standard Levi form.

# Chapter 2

# Levi Decompositions of Nilpotent Centralizers in Bad Characteristic: Classical Groups

## 2.1 Preliminaries and Good Characteristic

Let $G$ be a connected linear algebraic group over an algebraically closed field $k$ with unipotent radical $U$. A *Levi factor* of $G$ is a reductive subgroup $L$ such that $G$ is isomorphic to the semidirect product of $U$ and $L$. We then say that $G$ has a *Levi decomposition*. The key here is that we must have an isomorphism of algebraic groups, so the existence of a subgroup of $G$ isomorphic to the reductive quotient $G/U$ is not quite enough to know that we have a Levi decomposition. We must check that the differential of the projection map is an isomorphism of Lie algebras. So, we need to show that $\mathrm{Lie}(U)$ and $\mathrm{Lie}(L)$ have an empty intersection. When the characteristic of $k$ is $p > 0$, Levi factors need not exist in general (see, for example, [H67]).

To see that the existence of a complement to the unipotent radical isomorphic to the reductive quotient is not enough to ensure a Levi decomposition, consider the example from Section 3.3 of [M10] (which comes from [BT65] Section 3.15). Let $W = k^2$ and $V = S^pW$, the $p$th symmetric power of $W$. Now, consider the subspace of $V$ of all $p$th powers of vectors in $W$, which we will denote $W^{[1]}$. Then the stabilizer $P = \mathrm{stab}(W^{[1]}) \subseteq GL(V)$ is a maximal parabolic subgroup. Let $W'$ be any linear complement to $W^{[1]}$ in $V$, and let $M'$ be the reductive subgroup of $P$ generated by $GL(W) \subseteq GL(V)$ and $GL(W')$, the latter acting trivially on $W^{[1]}$. Now, if $\pi : P \to P/R_u(P)$ is the projection map, then $\pi|_{M'}$ is a purely inseparable isogeny, since $M' \cap R_u(P)$ is trivial and hence $M'$ maps isomorphically as abstract groups to $P/R_u(P) \simeq GL(W^{[1]}) \times GL(V/W^{[1]})$. The usual choice for a Levi factor, $M = GL(W^{[1]}) \times GL(W')$, leaves $W'$ invariant, yet there is no complement to $W^{[1]}$ in $V$ which is stable under $M'$. Therefore, $M$ and $M'$ cannot be conjugate, which is a requirement for all Levi factors of a parabolic subgroup ([H75] Theorem 30.2). Thus $M'$ is not a Levi factor of $P$.

Return now to our general connected algebraic group $G$ over the field $k$. We say that the characteristic $p$ of the field $k$ is *bad* when $p = 2$ and $G$ contains a simple factor not of type $A_n$; $p = 3$ and $G$ contains a simple factor of type $G_2$, $F_4$, or $E_n$; or $p = 5$ and $G$ contains a simple factor of type $E_8$. Otherwise, we say that the characteristic *good*.

Let $\mathfrak{g}$ be the Lie algebra for $G$, and consider $C_G(e)$ the centralizer of a nilpotent element $e \in \mathfrak{g}$ for the adjoint action of $G$. In [J04] Sections 5.10-11, Jantzen proves that, when the characteristic of $G$ is good, $C_G(e)$ has a Levi decomposition. Let us present a rough sketch of the proof.

For a cocharacter $\tau : k^\times \to G$, we can get a grading of $\mathfrak{g}$ given by

$$\mathfrak{g}(i) = \{A \in \mathfrak{g} \mid Ad(\tau(t))(A) = t^i A \text{ for all } t \in k^\times\}.$$

Define $\tau$ to be *associated* to our nilpotent element $e$ if $e \in \mathfrak{g}(2)$, and if there exists a Levi subgroup $H$ of $G$ such that $e$ is distinguished in the Lie algebra of $H$ and the image of $\tau$ contained in the derived group of $H$. These cocharacters are meant to replicate the existence of an $\mathfrak{sl}_2$-subalgebra containing $e$ in smaller characteristic, where such subalgebras may not exist unless the characteristic is sufficiently large. In good characteristic, associated cocharacters exist and are conjugate under the connected centralizer of $e$ ([J04] Lemma 5.3, originally [P95a] Theorem 2.5).

Now, $\tau$ defines a parabolic subgroup $P_\tau$ of elements $g \in G$ such that the limit $\lim\limits_{t\to 0} \tau(t)g\tau(t)^{-1}$ exists. The Lie algebra of $P_\tau$ is the direct sum of $\mathfrak{g}(i)$ with $i \geqslant 0$, and has a Levi decomposition with Levi subalgebra equal to $\mathfrak{g}(0)$. Hence, the unipotent radical of $P_\tau$ has Lie algebra the sum of $\mathfrak{g}(i)$ with $i > 0$.

By intersecting the centralizer of $e$ with this Levi decomposition, we find, with help from the aforementioned characterization as graded pieces of $\mathfrak{g}$, that multiplication induces an isomorphism of varieties between the centralizer and the product of its intersections with the Levi factor and unipotent radical of $P_\tau$, respectively. Furthermore, the intersection with the Levi factor is reductive, and hence $C_G(e)$ has a Levi decomposition.

The challenge in bad characteristic is that associated cocharacters do not exist for all nilpotent elements, so this approach will not work. We will need some other machinery to understand whether or not $C_G(e)$ has a Levi decomposition. To begin, we may restrict our focus to the first examples of algebraic groups in bad characteristic: the classical symplectic and orthogonal groups over a field of characteristic two. Fortunately, in [LS12], Liebeck and Seitz provide ample structure which we may use to answer this question.

## 2.2 Indecomposables

From this point forward, let $G = Sp(V)$ or $O(V)$ for a finite dimensional vector space $V$ over an algebraically closed field $k$ of characteristic $p = 2$, and let $\mathfrak{g} = \text{Lie}(G)$. Then $G$ preserves a nondegenerate symmetric bilinear form $( \, , \, )$ on $V$ and, when $G = O(V)$, a quadratic form $Q$. For a nilpotent element of $e \in \mathfrak{g}$, let $V \downarrow e$ be the restriction of $V$ to the action of $e$. In Chapter 5 of [LS12], Liebeck and Seitz produce $V \downarrow e$ as an orthogonal direct sum of certain indecomposables $V(m)$, $W(m)$, and $W_l(m)$ using a one-dimensional torus $T \subseteq G$, the impetus of which comes from Hesselink in [He79]. For $Sp(V)$, they are defined as follows:

1. $V(m)$ has basis $v_i$ for $i = -(m-1), -(m-3), \ldots, m-3, m-1$, with $(v_i, v_{-i}) = 1$ and $(v_i, v_j) = 0$ for $j \neq -i$. Each $v_i$ is a vector of $T$-weight $i$, and $e$ acts as a single Jordan block by $e.v_i = v_{i+2}$ for $i < m-1$ and $e.v_{m-1} = 0$.

2. $W(m)$ has basis $r_i, s_i$ for $i = -(m-1), -(m-3), \ldots, m-3, m-1$, with $(r_i, s_{-i}) = (s_i, r_{-i}) = 1$ and all other basis inner products zero. Each $r_i, s_i$ is a vector of $T$-weight $i$, and $e$ acts with two totally singular Jordan blocks by $e.r_i = r_{i+2}$ and $e.s_i = s_{i+2}$ for $i < m-1$, and $e.r_{m-1} = e.s_{m-1} = 0$.

3. $W_l(m)$ for $0 < l < \frac{m}{2}$ has basis $v_i, w_j$ for $i = -(2l-1), \ldots, 2l-1, 2l+1, \ldots, 2m-$

$2l-1$ and $j = -(2m-2l-1), \ldots, 2l-3, 2l-1$. Here, the subspace $\langle w_j \rangle$ is totally singular, $\langle v_i \rangle$ has a radical subspace of $\langle v_i \mid i = 2l+1, \ldots, 2m-2l-1 \rangle$ with nondegenerate quotient, and $(v_i, w_{-i}) = 1$ with $(v_i, w_j) = 0$ for $j \neq -i$. Each $v_i$ and $w_j$ is a vector of $T$-weight $i$ and $j$, respectively, and $e$ acts by $e.v_i = v_{i+2}$ and $e.w_j = w_{j+2}$, with $e.v_{2m-2l-1} = e.w_{2l-1} = 0$.

When $G = O(V)$, the indecomposables are:

1. $W(m)$ is defined identically as for $Sp(V)$, with the stipulation that $Q(v_i) = Q(w_i) = 0$ for all $i$.

2. $W_l(m)$ for $\frac{m+1}{2} < l \leqslant m$ has basis $v_i, w_j$ for $i = -(2l-2), \ldots, -2, 0, 2, \ldots, 2m-2l$ and $j = -(2m-2l), \ldots, 2l-2$. Here, the subspaces $\langle v_i \rangle$ and $\langle w_j \rangle$ are totally singular for the bilinear form, $Q(v_0) = 1$, and $Q(v_i) = Q(w_i) = 0$ otherwise. Each $v_i$ and $w_j$ is a vector of $T$-weight $i$ and $j$, respectively, and $e$ acts by $e.v_i = v_{i+2}$ and $e.w_j = w_{j+2}$, with $e.v_{2m-2l-1} = e.w_{2l-1} = 0$.

Using these indecomposables, Seitz and Liebeck give an orthogonal decomposition referred to as *distinguished normal form*:

$$V \downarrow e = \sum_i W(m_i)^{a_i} + \sum_i W_{l_i}(n_i) + \sum_j V(2k_j)^{c_j}$$

Here the sequences $(n_i)$, $(l_i)$, and $(n_i - l_i)$ are strictly decreasing, all $c_j \leqslant 2$, and $k_i > n_i - l_i$ or $k_j < l_i$ for all $i, j$ (when $G = Sp(V)$). Note that the $V(2k_j)^{c_j}$ factors do not appear when $G = O(V)$.

The following lemma will go very far in aiding our pursuit of Levi decompositions, particularly in the symplectic group:

**Lemma 2.2.1.** *Let $C \subseteq GL(V)$, $R$ be the unipotent radical of $C$, and $H \leqslant C$ be a reductive closed subgroup such that $H \simeq C/R$ as abstract groups. Let $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ be a filtration for $C$. If $W = \bigoplus_i V_i/V_{i-1}$ is a faithful module for $H$ on which $R$ acts trivially, then the projection map*

$$\pi : C \to C/R$$

*has the property that $d\pi|_{Lie(H)}$ is a bijection. In particular, $H$ is a Levi factor of $C$.*

*Proof.* First, note that, since $H \simeq C/R$, the bijectivity of $d\pi|_{\mathrm{Lie}(H)}$ follows from injectivity for dimensional reasons. So, we need only prove that $d\pi|_{\mathrm{Lie}(H)}$ is injective.

Let $P \subseteq GL(V)$ be the full stabilizer of $W$. So, $C \subseteq P$ and $P/R_u(P) \simeq GL(W) \times GL(V/W)$. Consider the map from $C$ to $GL(W)$ given by

$$\phi : C \hookrightarrow P \twoheadrightarrow P/R_u(P) \simeq GL(W) \times GL(V/W) \twoheadrightarrow GL(W)$$

composing inclusion, the quotient map, and projection to the first coordinate. The restriction $\phi|_H$ is a closed embedding, as $W$ is a faithful $H$-module. Since the unipotent radical of $C$ acts trivially on $W$, and we have $R \subseteq \ker(\phi)$.

We now have the commutative diagram

$$\begin{tikzcd}
 & & C/R \arrow[dd] \\
H \arrow[r] \arrow[urr, bend left, "\pi|_H"] \arrow[drr, bend right, "\phi|_H"'] & C \arrow[ur, "\pi"'] \arrow[dr, "\phi"] & \\
 & & GL(W)
\end{tikzcd}$$

where the map $C/R \to GL(W)$ comes from the universal property of the quotient. Since $\phi|_H$ is a closed embedding, so too is $\pi|_H$. Therefore, $d\pi|_{\mathrm{Lie}(H)}$ is injective. □

## 2.3 The Symplectic Group

For the one-dimensional torus $T$ used to define the indecomposables, let $C_G(T, e) = C_G(e) \cap C_G(T)$. Then, using the distinguished normal form, Seitz and Liebeck give the following results in [LS12].

**Theorem 2.3.1.** *Let $G = Sp(V)$ and $e$ be a nilpotent element of Lie$(G)$ with the given distinguished normal form of $V \downarrow e$. Then*

*(i)* $C_G(e)$ *is connected with* $C_G(e) \simeq R_u(C_G(e))C_G(T, e)$*,*

*(ii)* $C_G(e)/R_u(C_G(e)) \simeq \prod_i Sp_{2a_i}$*,*

*(iii)* $\prod_i Sp_{2a_i} \leqslant C_G(T, e)$*.*

So, the reductive quotient of $C_G(e)$ exists as a closed subgroup, but we must check that the projection map is compatible on the level of Lie$(C_G(e))$ in order to have a Levi decomposition. To show this, we will use Lemma 2.2.1 along with the explicit construction of $\prod_i Sp_{2a_i} \leqslant C_G(T,e)$ by Liebeck and Seitz in [LS12] Lemma 5.7. As we will see, the actions defined in *loc. cit.* are fairly natural; the key is that these subgroups can be shown to give the entire reductive quotient.

Also note that we need only check the Levi decomposition of $C_G(T,e)$. By Theorem 2.3.1, the Lie algebra of the proposed Levi factor is contained entirely in $T$-weight zero, and thus it can only intersect the Lie algebra of $R_u(C_G(e))$ within Lie$(C_G(T,e))$. Specifically, since $C_G(T,e)\cap R_u(C_G(e))$ is connected (see [H75] Proposition 28.1) and $C_G(e)\simeq R_u(C_G(e))C_G(T,e)$, we have $C_G(e)/R_u(C_G(e))\simeq C_G(T,e)\cap R_u(C_G(T,e))$.

**Proposition 2.3.2.** *Let $G = Sp(V)$ for $V$ a finite dimensional vector space over a field $k$ of characteristic 2, and $e$ be a nilpotent element of Lie$(G)$ with distinguished normal form $V\downarrow e$ as in Section 2.2. Then $C_G(e)$ has a Levi decomposition with Levi factor $\prod_i Sp_{2a_i}$.*

*Proof.* Let $H = \prod_i Sp_{2a_i}$, and let $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ be a composition series for $C = C_G(T,e)$. Consider a summand $Z = W(m_i)^{a_i}$ of the distinguished normal form of $V\downarrow e$.

(i) Suppose $m_i$ is **even**. Then we have an embedding $Sp_{m_i}\otimes Sp_{2a_i}\leqslant Sp(Z)$. The restrictions of $e$ and $T$ to $Z$ are conjugate to a nilpotent element $e_{m_i}$ acting as a single Jordan block on $\mathfrak{sp}_{m_i}$ and a one-dimensional torus $T'$ acting with the appropriate weights on $Sp_{m_i}$, respectively. These are obviously centralized by $Sp_{2a_i}$, giving the factor $Sp_{2a_i}\leqslant C_G(e_{m_i},T') = C_G(e,T)$. Now, $\ker(e_{m_i}) = k\vec{x}\otimes k^{2a}$ for some $m_i$-dimesnional vector $\vec{x}$. So $1\otimes Sp_{2a_i}$ acts on $\ker(e_{m_i})$ as $Sp_{2a_i}$ naturally acts on $k^{2a_i}$, which is simple and faithful.

(ii) Suppose $m_i$ is **odd**. Then $T$ decomposes $Z$ into $2a_i$-dimensional weight spaces $Z_j$ for weights

$$j = -(m_i-1),\ldots,-2,0,2,\ldots,(m_i-1).$$

$Z_0$ is non-degenerate under the bilinear form with a group $Sp_{2a_i}$ acting on it preserving this form. The action of $Sp_{2a_i}$ is extended to all of the other weight spaces on $Z$ by applying $e$ (which takes $Z_j$ to $Z_{j+2}$), and this action commutes with $T$ and $e$. Let $e_{m_i}$ be the restriction of $e$ to $Z$. Now, we have $Sp_{2a_i} \leqslant C_G(T, e)$ acting simply and faithfully on, in particular, $\ker(e_{m_i}) = Z_{m_i - 1} \simeq k^{2a_i}$.

Putting this all together, we have $H = \prod_i Sp_{2a_i}$ acting simply and on each $\ker(e_{m_j})$, since an $Sp_{2a_k}$ factor acts trivially when $k \neq j$.

Let $M_i = V_i/V_{i-1}$. Since $R = R_u(C_G(T, e))$ is normal, the fixed point set $M_i^R \subseteq M_i$ is a $C$-submodule. By the Borel fixed point theorem ([H75], 21.2), $M_i^R$ is nonempty, thus we must have $M_i^R = M_i$ by simplicity. Now that we know $R$ acts trivially on each $V_i/V_{i-1}$, they become simple $C/R$-modules. Hence, as $H \simeq C/R$, each $\ker(e_{m_i})$ must appear as a composition factor, with $H$ acting faithfully on $\bigoplus_i \ker(e_{m_i})$. We now extend the faithful action of $H$ to all of $\bigoplus_i V_i/V_{i-1}$ by letting it act trivially on the remaining factors. By Lemma 2.2.1, $H$ is a Levi factor of $C_G(T, e)$, proving the claim.

□

## 2.4 The Orthogonal Group

As in the symplectic group, Liebeck and Seitz give an explicit description of the reductive quotient of the centralizer of a nilpotent element in $O(V)$ in [LS12] using distinguished normal form.

**Theorem 2.4.1.** *Let $G = O(V)$ and $e$ be a nilpotent element of Lie$(G)$ with the given distinguished normal form of $V \downarrow e$. Then*

*(i)* $C_G(e)/C_G(e)^0$ *is a 2-group with* $C_G(e) \simeq R_u(C_G(e))C_G(T, e)$,

*(ii)* $C_G(e)^0/R_u(C_G(e)) \simeq \prod_{m_i \text{ even}} Sp_{2a_i} \times \prod_{m_i \text{ odd}} I_{2a_i}$, *where* $I_{2a_i} = SO_{2a_i+1}$ *or* $SO_{2a_i}$ *according to whether or not* $V \downarrow e$ *has a summand of the form* $W_l(n)$ *with* $2(n - l) \leqslant m_i \leqslant 2l - 1$,

*(iii)* $\prod_{m_i \text{ even}} Sp_{2a_i} \times \prod_{m_i \text{ odd}} I_{2a_i} \leqslant C_G(T, e)$.

In the case that the reductive quotient contains only symplectic and even dimensional special orthogonal factors, we can hope to proceed using Lemma 2.2.1 much as we did when $G = Sp(V)$. There arise some issues with this approach, though, when we encounter factors of the form $SO_{2a_i+1}$. For example, the natural module $k^{2a_i+1}$ is not necessarily simple, so when building a filtration which has it as a factor we cannot guarantee that $R$ will act trivially. We will deal with these issues with more "hands-on" methods, but first we record what our original yields in the orthogonal case with a lemma.

**Lemma 2.4.2.** *Let $G = O(V)$ for $V$ a finite dimensional vector space over a field $k$ of characteristic 2, and $e$ be a nilpotent element of Lie$(G)$ with distinguished normal form $V \downarrow e$ such that for each odd $m_i$, $m_i < 2(n_j - l_j)$ or $m_i > 2l_j - 1$ for all $j$. Then $C_G(e)^0$ has a Levi decomposition with Levi factor $\prod_{m_i \text{ even}} Sp_{2a_i} \times \prod_{m_i \text{ odd}} SO_{2a_i}$.*

*Proof.* This follows exactly as in the symplectic case of Proposition 2.3.2. In the situation where $m_i$ is odd, the quadratic form is preserved on the zero weight space (see [LS12] Lemma 5.7), so we get a simple, faithful action of $SO_{2a_i}$ on its natural module $k^{2a_i} \simeq \ker(e_{m_i})$. □

Now we wish to proceed with the case where the reductive quotient contains odd dimensional special orthogonal factors. First, we have a technical lemma that states that the $W(m)^{a_i}$ factors of $V \downarrow e$ for which there are no $W_l(n)$ factors satisfying the inequality in Theorem 2.4.1 part (ii) are $C_G(T, e)$-stable.

**Lemma 2.4.3.** *Let $V \downarrow e = \sum_i W(m_i)^{a_i} + \sum_i W_{l_i}(n_i)$ be the distinguished normal form for $e$ with $G = O(V)$. Consider a factor $W(m_i)^{a_i}$. If $m_i$ is even, then $W(m_i)^{a_i}$ is fixed by $C_G(T, e)$. Furthermore, if $m_i$ is odd and, for each factor $W_{l_j}(n_j)$, we have $m_i > 2l_j - 1$ or $m_i < 2n_j - 2l_j$, then $W(m_i)^{a_i}$ is fixed by $C_G(T, e)$.*

*Proof.* First, suppose a factor $W(m_j)$ has $m_j$ of a different parity than $m_i$. Then for a $t$-weight vector $v \in (W(m_i)^{a_i})_t$ and $g \in C_G(T, e)$, a linear combination of basis vectors for $g.v$ cannot have a summand in $W(m_j)$ since $g$ centralizes $T$. So, suppose we have $W(m_j)$ where $m_i$ and $m_j$ have the same parity and are unique.

Assume $m_i$ is odd, and let $q = \frac{m_i - 1}{2}$. In the zero-weight space, let $r_0 \in (W(m_i)^{a_i})_0$ and $v_0 \in (W(m_j))_0$. For $g \in C_G(T, e)$, suppose that $g.r_0$ has a linear combination with a summand of $v_0$. If $m_i < m_j$, then

$$0 = g.(e^{q+1} r_0) = e^{q+1}(g.r_0)$$

has a $e^{q+1}v_0 = v_{m_i+1} \neq 0$ summand, a contradiction. Similarly, if $m_i > m_j$, then

$$g.r_0 = g.(e^q r_{-(m_i-1)}) = e^q(g.r_{-(m_i-1)}).$$

This implies that $v_0$ is the image of some vector under $e^q$, contradicting the the block size of $W(m_j)$ relative to $e$, since $m_i > m_j$.

The case of $m_i$ even follows similarly, beginning in the 1-weight space. So, $C_G(T,e)$ cannot map a vector in $W(m_i)^{a_i}$ to a linear combination containing vectors from $W(m_j)$ with $m_i \neq m_j$. Now we wish to show that such a linear combination cannot contain vectors from the other factors. Note that the only odd weight spaces occur for $W(m_i)^{a_i}$ when $m_i$ is even, so a similar parity argument as above shows that, in this case, $W(m_i)^{a_i}$ must fixed by $C_G(T,e)$. We now proceed with $m_i$ odd.

Consider a $W_{l_j}(n_j)$ factor with basis as given in Section 2.2, so that the zero-weight space is $(W_{l_j}(n_j))_0 = \{v_0, w_0\}$. By assumption, we have $m_i > 2l_j - 1$ or $m_i < 2n_j - 2l_j$. Once again, let $r_0 \in (W(m_i)^{a_i})_0$ and $g \in C_G(T,e)$. We need to consider a few cases.

Suppose first that $g.r_0$ is a linear combination of basis vectors with a $v_0$ summand, and once again let $q = \frac{m_i-1}{2}$. If $m_i > 2l_j - 1$, then

$$g.r_0 = g.(e^q r_{-(m_i-1)}) = e^q(g.r_{-(m_i-1)})$$

has a $v_0$ summand. Hence $v_0$ is the image of some vector under $e^q$, contradicting the the block size of $W_{l_j}(n_j)$ relative to $e$, since $m_i > 2l_j - 1$. On the other hand, if $m_i < 2n_j - 2l_j$, then

$$e^{q+1}(g.r_0) = g.(e^{q+1}r_0) = g.0 = 0.$$

But $e^{q+1}v_0 = v_{m_i+1} \neq 0$ by block sizes, contradicting $v_0$ as a summand of $r_0$.

Now suppose that $g.r_0$ is a linear combination with a $w_0$ summand. By the definition of $W_{l_j}(n_j)$ in Section 2.2, we have $\frac{n_j+1}{2} < l_j$, which implies $2l_j - 1 > 2n_j - 2l_j + 1$. Now suppose $m_i > 2l_j - 1$. This implies, with the inequality above, that $-(m_i - 1) < -(2n_j - 2l_j)$. Therefore, $w_0$ cannot be the image of any vector under $e^q$. But now,

$$g.r_0 = g.(e^q r_{-(m_i-1)}) = e^q(g.r_{-(m_i-1)})$$

has a $w_0$ summand, a contradiction. Similarly, suppose $m_i < 2n_j - 2l_j$. Note from above that $2n_j - 2l_j < 2l_j - 2$, and hence $m_i < 2l_i - 2$. Therefore, $e^{q+1}(g.r_0)$ has a

$w_{m_i+1} \neq 0$ summand while

$$e^{q+1}(g.r_0) = g.(e^{q+1}r_0) = g.0 = 0$$

for a final contradiction.

Thus, we have shown that a vector in $(W(m_i)^{a_i})_0$ must be mapped by $g \in C_G(T, e)$ to a linear combination of basis vectors with summands neither in $W(m_j)$ for $m_i \neq m_j$ nor in $W_{l_j}(n_j)$ with $m_i > 2l_j - 1$ or $m_i < 2n_j - 2l_j$. Since $g$ must centralize $e$, this applies in all weight spaces. Therefore, given our hypotheses, each $W(m_i)^{a_i}$ is $C_G(T, e)$-stable.

□

We will now need a bit of homological algebra concerning the natural module. Let $H = SO_{2n+1}$ with maximal torus $T_H$ contained in a Borel subgroup $B_H$. Thus we have determined a set of simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n \in X^*(T_H)$. Considering the first fundamental dominant weight $\varpi_1$, Section II.2.18 of [J03] tells us that the standard module $H^0(\varpi_1) \simeq N^*$, where $N$ is a vector space of dimension $2n + 1$ over $k$ with quadratic form $Q$ on which $SO_{2n+1}$ acts as the natural module. Therefore, by Section II.2.13 of *loc. cit.*, $N$ must be the Weyl module $V(\varpi_1)$. In the case $\mathrm{char}(k) = p \neq 2$, this means that $N$ is simple. Though, in our situation of $p = 2$, we have a $2n$-dimensional simple quotient $L = N/kf$ for a fixed point $f$, so that $L = L(\varpi_1)$. This actually gives the natural highest weight representation for $Sp_n$ (see the discussion following Theorem 3.2 in [H05], or exercise 6 of [Sp98] 7.4.7).

**Lemma 2.4.4.** *Consider $N$ the natural module for $H = SO_{2n+1}$, with $L = N/kf$ the simple quotient for a fixed point $f$. Let $W$ be a finite dimensional, trivial $H$-module, and let*

$$0 \to W \to E \to L \to 0$$

*be a short exact sequence of $H$-modules. Then either there is a $H$-module isomorphism $E \simeq L \oplus W$ or $E \simeq N \oplus W_1$ for some subspace $W_1 \subset W$ of codimension one.*

*Proof.* First note that it follows from [J03] Proposition II.2.14 that, for a trivial module $k$, the space of extensions $\mathrm{Ext}^1_H(L, k)$ is one dimensional. Suppose that it is spanned by a nonzero extension class $\beta$. Then classes $\gamma \in \mathrm{Ext}^1_H(L, k)$ correspond to isomorphism classes of extensions of $H$-modules

$$0 \to k \to E_\gamma \to L \to 0,$$

with $E_0 \simeq L \oplus k$ and $E_{t\beta} \simeq E_\beta \simeq N$ as $H$-modules for all $0 \neq t \in k$.

Now, the short exact sequence

$$0 \to W \to E \to L \to 0$$

determines a class $\delta$ in

$$\mathrm{Ext}^1_H(L, W) \simeq W \otimes_k \mathrm{Ext}^1_H(L, k) = W \otimes_k k\beta,$$

hence we may write $\delta = w \otimes \beta$ for some $w \in W$. In the case that $w = 0$, we have $\delta = 0$ and $E \simeq L \oplus W$.

If $w \neq 0$, we can extend it to a $k$-basis $\{w, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{d-1}\}$ for $W$. Letting

$$W_1 = \sum_{i=1}^{d-1} k\tilde{w}_i,$$

we have $W = kw \oplus W_1$ and an isomorphism:

$$\phi : \mathrm{Ext}^1_H(L, W) \xrightarrow{\sim} \mathrm{Ext}^1_H(L, kw) \oplus \mathrm{Ext}^1_H(L, W_1)$$

Under this isomorphism we have $\phi(\delta) = (\beta, 0)$, and hence $E \simeq E_\beta \oplus W_1 \simeq N \oplus W_1$. $\square$

We are now ready to consider Levi decompositions in the orthogonal group in earnest, improving upon Lemma 2.4.2.

**Proposition 2.4.5.** *Let $G = O(V)$ for $V$ a finite dimensional vector space over a field $k$ of characteristic 2, and $e$ be a nilpotent element of $Lie(G)$ with distinguished normal form $V \downarrow e$ as in Section 2.2. Then $C_G(e)^0$ has a Levi decomposition with Levi factor $\prod_{m_i\ even} Sp_{2a_i} \times \prod_{m_i\ odd} I_{2a_i}$, where $I_{2a_i} = SO_{2a_i+1}$ or $SO_{2a_i}$ according to whether or not $V \downarrow e$ has a summand of the form $W_l(n)$ with $2(n-l) \leqslant m_i \leqslant 2l-1$.*

*Proof.* Recall from Theorem 2.4.1 that $C_G(e)^0/R_u(C_G(e)) \simeq \prod_{m_i\ even} Sp_{2a_i} \times \prod_{m_i\ odd} I_{2a_i}$ and there exists a subgroup $M \leqslant C_G(T, e)$ such that $M \simeq C_G(e)^0/R_u(C_G(e))$. If

$R = R_u(C_G(e))$, we know that $M$ and $R$ have trivial intersection *as abstract groups*, which we will write as $M(k) \cap R(k) = 1$. As mentioned in Section 2.1, we must see that these have trivial intersection in the Lie algebra.

Let $\Xi = \{m_i \mid m_i > 2l_j - 1 \text{ or } m_i < 2n_j - 2l_j \text{ for each } W_{l_i}(n_i)\}$. Consider a factor $W(m_i)^{a_i}$ such that $m_i \in \Xi$ and recall that, by Lemma 2.4.3, $W(m_i)^{a_i}$ is stable under $C_G(T, e)$. For each weight $t$, the subgroup $M$ acts on $(W(m_i)^{a_i})_t$ as $Sp_{2a_i}$ or $SO_{2a_i}$ depending on whether $m_i$ is even or odd, respectively. Let us look at the odd case. So, the restriction of the action of $M$ to $(W(m_i)^{a_i})_t$ is precisely $SO_{2a_i} \simeq SO((W(m_i)^{a_i})_t)$. Since we know that $M(k) \cap R(k) = 1$ and $R$ preserves the quadratic form, we must have $R$ acting trivially on $(W(m_i)^{a_i})_t$. Thus, Lie$(R)$ acts trivially on $(W(m_i)^{a_i})_t$ and intersects the Lie$(SO_{2a_i})$ factor of Lie$(M)$ trivially. We have a similarly trivial intersection in the case of $Sp_{2a_i}$ with $m_i$ even.

Since $\sum_{m_i \in \Xi} W(m_i)^{a_i}$ is stabilized by $C_G(T, e)$, so too must be

$$\Big(\sum_{m_i \in \Xi} W(m_i)^{a_i}\Big)^{\perp} = \sum_{m_i \notin \Xi} W(m_i)^{a_i} + \sum_i W_{l_i(n_i)}.$$

This is where we look next. Consider $W(m_i)^{a_i}$ with $m_i \notin \Xi$, and let $W_{l_j}(n_j)$ be a factor such that $2n_j - 2l_j \leqslant m_i \leqslant 2l_j - 1$. Then the action of $SO_{2a_i+1}$ on the zero-weight space is defined as follows. Let $W_{l_j}(n_j)$ have the basis as given in Section 2.2 with zero weight vectors $\{v_0, w_0\}$, and recall that $Q(v_0) = 1$. Then the stabilizer of $\langle v_0 \rangle$ gives an $SO_{2a_i+1}$ subgroup for the restrcition of the quadratic form on $V$ that also stabilizes the $2a_i + 1$-dimensional $\langle v_0 \rangle^{\perp} \cap (W(m_i)^{a_i} + W_{l_j}(n_j))_0$, which is generated by $v_0$ and the $2a_i$ basis vectors in $(W(m_i)^{a_i})_0$. We then let this $SO_{2a_i+1}$ act on the other weight spaces of $W(m_i)^{a_i} + W_{l_j}(n_j)$ by surjecting along the map given by $e$ to the positive weight spaces and pulling back along the injective map given by $e$ to the negative weight spaces. Lastly, we let $SO_{2a_i+1}$ act trivially on all other weight spaces, as well as other factors of $V \downarrow e$. This choice we have made of a specific $W_{l_j}(n_j)$ leads potentially non-conjugate Levi factors. For an example of the setup given here, see Section 2.5.2.

We must take some care here since, since though we have a natural module for $SO_{2a_i+1}$, we need to ensure that this is a $C_G(T, e)$-module. As a module for $SO_{2a_i+1}$, the zero weight space $V_0$ is an extension of the natural module $N_{2a_i+1}$ we have just identified by the sum of several trivial modules. Then we have the short exact sequence

$$0 \to W \to V_0 \xrightarrow{\pi} N_{2a_i+1} \to 0$$

where $W$ is a finite dimension trivial module for $SO_{2a_i+1}$. Consider $V_0^R$, the $R = R_u(C_G(T,e))$ fixed points on $V_0$. By the Borel fixed point theorem ([H75], 21.2), we have $V_0^R \neq \varnothing$, and thus we have three possibilities. First, we could have $V_0^R \subseteq \ker(\pi)$. In this case, we have $V_0^R$ contained in the sum of trivial modules, so we might as well start again considering $V_0/V_0^R$. Next, we could have $V_0^R = V_0$. If this is true, we are done, since then $R$ acts trivially on the entire zero weight space and thus so must $\mathrm{Lie}(R)$, ensuring that it intersects $\mathrm{Lie}(SO_{2a_i+1})$ trivially as desired. Our final possibility is that $V_0^R = \pi^{-1}(kv_0)$, where $v_0$ is again the fixed point in the natural module for $SO_{2a_i+1}$.

When $V_0^R$ is the preimage of the line generated by the fixed point, the situation is more interesting. Note that in this case $SO_{2a_i+1}$ acts trivially on $V_0^R$. Then the short exact sequence above gives way to

$$0 \to V_0^R \to V_0 \xrightarrow{\pi'} L \to 0$$

where $L = N_{2a_i+1}/\langle kv_0\rangle$ is the $2a_i$-dimensional simple quotient of $N_{2a_i+1}$. By Lemma 2.4.4, such extensions by trivial modules must be split, hence $V_0 \simeq L \oplus V_0^R$ or $V_0 \simeq N_{2a_i+1} \oplus F_1$ as $SO_{2a_i+1}$-modules, where $F_1$ is a codimension one subspace of $V_0^R$. The former case is impossible, though, as $V_0$ contains $N_{2a_i+1}$ as a quotient whereas $L \oplus V_0^R$ does not. Hence we must have $V_0 \simeq N_{2a_i+1} \oplus F_1$.

Now, $F_1 \subseteq V_0^R = \pi^{-1}(kv_0)$ is acted on trivially by $M$ and $R$, and since these generate $C_G(T,e)$ as an abstract group, $F_1$ is a trivial $C_G(T,e)$-submodule. Therefore we have $V_0/F \simeq N_{2a_i+1}$ as a $C_G(T,e)$-module where $SO_{2a_i+1}$ acts naturally. Thus, since $R$ must respect the quadratic form and intersects $SO_{2a_i+1}$ trivially as an abstract group, it must act trivially on the natural module as before. Therefore, $\mathrm{Lie}(R)$ intersects $\mathrm{Lie}(SO_{2a_i+1})$ trivially, as desired.

With this, we have successfully shown that $\mathrm{Lie}(R)$ intersects all factors of $\mathrm{Lie}(M)$ trivially, and therefore must intersect $\mathrm{Lie}(M)$ itself trivially. Thus, our abstract subgroup $M \simeq C_G(e)^0/R_u(C_G(e))$ is in fact a Levi factor. □

Finally, by combining Propositions 2.3.2 and 2.4.5, we have the unified result:

**Theorem 2.4.6.** *Let $G = Sp(V)$ or $O(V)$ for $V$ a finite dimensional vector space over a field $k$ of characteristic 2, and $e$ be a nilpotent element of Lie$(G)$. Then $C_G(e)^0$ has a Levi decomposition.*

*In particular, it has Levi factor*

*(i)* $\prod_i Sp_{2a_i}$ *if $G = Sp(V)$; or*

*(ii)* $\prod_{m_i \text{ even}} Sp_{2a_i} \times \prod_{m_i \text{ odd}} I_{2a_i}$ *if $G = O(V)$, where $I_{2a_i} = SO_{2a_i+1}$ or $SO_{2a_i}$ according to whether or not $V \downarrow e$ has a summand of the form $W_l(n)$ with $2(n-l) \leqslant m_i \leqslant 2l-1$.*

*Remark* 2.4.7. In the case that $G$ is an exceptional group, Liebeck and Seitz prove that we have a semidirect product $C_G(e) = R_u(C_G(e))C_G(T,e)$ in [LS12] Theorem 9.1 part (iv), so long as $p \neq 2$ and $(G,p,e) \neq (E_8, 3, (A_7)_3)$ or $(G_2, 3, (\tilde{A}_1)_3)$. In type $E_8$, outside of the excluded case, $C_G(T,e)$ is in fact seen to be reductive, and hence this gives a Levi decomposition. Though, dealing with the rest of the exceptional groups seems much more difficult in general.

## 2.5 Examples

We will conclude this chapter with some examples of cases that have been covered. As we have seen, the nontrivial pieces of the reductive quotient come from the factors of $V \downarrow e$ of the form $W(m)$, with $W_l(n)$ factors also playing a part in the orthogonal group. As each $W(m)$ has dimension $2m$, in order to see more general situations it will be convenient to consider larger vector spaces and hence quite large classical groups.

### 2.5.1 The 210-dimensional Symplectic Group

Let $V$ be a 20-dimensional vector space over a field $k$ of characteristic 2 and consider $G = Sp(V)$, which has dimension $\frac{20}{2}(20+1) = 210$. Let $e \in \mathfrak{sp}(V)$ be a nilpotent element such that

$$V \downarrow e = W(3)^2 \oplus W(4).$$

Now, we turn our attention to the centralizer $C_G(e)$. By [LS12] Lemma 5.4, it has dimension

$$\dim(C_G(e)) = \sum_{i=1}^{6}(it_i - \chi(t_i)) = 66$$

where $t_1, t_2 = 4$ and $t_j = 3$ for $3 \leqslant j \leqslant 6$ are the Jordan block sizes of $e$, and $\chi$ is a function given by taking the maximum of Hesselink's $\chi_W$ functions over the indecomposables (see [He79]). In this case, each $\chi(t_i) = 0$. We also know that $C_G(e)$ is connected by Theorem 2.3.1.

We expect to have a Levi factor of the form $Sp_2 \times Sp_4$. Let us see how to construct each piece. First, consider the $W(4)$ factor of $V \downarrow e$. This has basis (divided into weight spaces)

$$\{r_{-3}, s_{-3}\}, \{r_{-1}, s_{-1}\}, \{r_1, s_1\}, \{r_3, s_3\}$$

where the vectors pair such that $(r_i, s_{-i}) = (s_i, r_{-i}) = 1$ and all other pairings are zero. Now, we have a natural, faithful action of $Sp_2$ on the top weight space $\{r_3, s_3\} = \ker(e|_{W(4)})$ which centralizes $e$ and $T$.

Next, consider the $W(3)^2$ factor of $V \downarrow e$. Here we have a basis

$$\{x_{-2}, y_{-2}, u_{-2}, v_{-2}\}, \{x_0, y_0, u_0, v_0\}, \{x_2, y_2, u_2, v_2\}$$

with nonzero pairings $(u_i, v_{-i}) = (v_i, u_{-i}) = (x_i, y_{-i}) = (y_i, x_{-i}) = 1$. The zero weight space is clearly nondegenerate under the bilinear form and has a natural $Sp_4$ action preserving this form as well as centralizing $e$ and $T$. We can then extend this action to the top weight space $(W(3)^2)_2 = \ker(e|_{W(3)^2})$.

Now, $\ker(e|_{W(4)}) = \{r_3, s_3\}$ and $\ker(e|_{W(3)^2}) = \{x_0, y_0, u_0, v_0\}$ are simple submodules of $V$ for the action of $C_G(T, e)$, hence they appear as composition factors in the composition series $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$. Suppose

$$\ker(e|_{W(4)}) = V_k/V_{k-1} \text{ and } \ker(e|_{W(3)^2}) = V_l/V_{l-1}$$

for $1 \leqslant k, l \leqslant n$. Then $Sp_2 \times Sp_4$ acts faithfully on $(V_k/V_{k-1}) \oplus (V_l/V_{l-1})$, and we can extend this action trivially to the rest of the composition factors giving a faithful action of $Sp_2 \times Sp_4$ on $\bigoplus_i V_i/V_{i-1}$. Since $R_u(C_G(T, e))$ acts trivially on each of the simple composition factors, Lemma 2.2.1 allows us to conclude that $Sp_2 \times Sp_4$ is a Levi factor of $C_G(T, e)$.

### 2.5.2 The 1653-dimensional Orthogonal Group

Let $V$ be an 58-dimensional vector space over a field $k$ of characteristic 2 and consider $G = O(V)$, which has dimension $\frac{58(58-1)}{2} = 1653$. Let $e \in \mathfrak{o}(V)$ be a nilpotent element such that

$$V \downarrow e = W(13) \oplus W(5) \oplus W_6(7) \oplus W_4(4).$$

Again by [LS12] Lemma 5.4, the centralizer of $e$ has dimension

$$\dim(C_G(e)) = \sum_{i=1}^{2}(it_i - \chi(t_i)) = 161$$

where the Jordan block sizes are $t_1, t_2 = 13$; $t_3, t_4 = 7$; $t_5, t_6 = 5$; and $t_7, t_8 = 4$. This time we have $\chi(13) = 7$, $\chi(7) = 6$, and $\chi(5) = \chi(4) = 4$. By [LS12] Theorem 5.12, the component group of $C_G(e)$ is a 2-group, specifically $C_G(e)/C_G(e)^0 \simeq \mathbb{Z}_2$.

We hope that $C_G(e)^0$ has a Levi factor of the form $SO_2 \times SO_3$. Starting with $W(13)$, we see that $13 > 2(6) - 1$ when considering $W_6(7)$ (our "largest" piece of $V \downarrow e$ of type $W_l(n)$). Now we can build a factor of $SO_2$ acting first on the $C_G(T, e)$ submodule $(W(13))_0$. Here $SO_2$ centralizes $e$ and $T$ while preserving the bilinear and quadratic forms on its natural two-dimensional module, and thus the unipotent radical acts trivially.

On the other hand, $2 = 2(7-6) \leqslant 5 \leqslant 2(6) - 1 = 11$ and $0 = 2(4-4) \leqslant 5 \leqslant$

$2(4) - 1 = 7$, so we must consider $W(5) \oplus W_6(7) \oplus W_4(4)$ together. This has basis:

| $W(5)$ : | $W_6(7)$ : | $W_4(4)$ : |
|---|---|---|
| | $\{w_{-10}\}$ | |
| | $\{w_{-8}\}$ | |
| | $\{w_{-6}\}$ | $\{x_{-6}\}$ |
| $\{u_{-4}, v_{-4}\}$ | $\{w_{-4}\}$ | $\{x_{-4}\}$ |
| $\{u_{-2}, v_{-2}\}$ | $\{w_{-2}, z_{-2}\}$ | $\{x_{-2}\}$ |
| $\{u_0, v_0\}$ | $\{w_0, z_0\}$ | $\{x_0, y_0\}$ |
| $\{u_2, v_2\}$ | $\{w_2, z_2\}$ | $\{y_2\}$ |
| $\{u_4, v_4\}$ | $\{z_4\}$ | $\{y_4\}$ |
| | $\{z_6\}$ | $\{y_6\}$ |
| | $\{z_8\}$ | |
| | $\{z_{10}\}$ | |

The nonzero pairings are $(w_i, z_{-i}) = (x_i, y_{-i}) = (u_0, v_{-i}) = (v_i, u_{-i}) = 1$, and only $w_0$ and $x_0$ are isotropic with respect to the quadratic form. In [LS12] Lemma 5.10, the $SO_3$ action which we hope to be part of our Levi factor is defined as follows.

Consider part of the zero weight space $Z_0 = (W(5) \oplus W_6(7))_0 = \{u_0, v_0, w_0, z_0\}$. This has an action of $O_4$ with the stabilizer of $\langle w_0 \rangle$ giving the action of $SO_3 \times \mathbb{Z}_2$ on $\langle w_0 \rangle^\perp$ and $Z_0/\langle w_0 \rangle$. We can extend this to the other weight spaces by the action of $e$; moving up in weight surjectively from $Z_0/\langle w_0 \rangle$ in the the zero weight space, or pulling back to the negative weight spaces from $\langle w_0 \rangle^\perp$ along the injective map given by $e$. Finally, let $SO_3$ act trivially on all other weight spaces. The action of $SO_3$ is then on all of $W(5) \oplus W_6(7) \oplus W_4(4)$, centralizes $e$ and $T$, and preserves both the bilinear and quadratic forms.

In the proof of Proposition 2.4.5, we showed that we could pick $w_0$ for our fixed point and, with some work, get the 3-dimensional natural module we wanted in a $C_G(T, e)$-filtration. Thus, with the unipotent radical acting trivially, we get a trivial intersection with $SO_3$ in the Lie algebra. So finally we can say that $SO_2 \times SO_3$ is a Levi factor.

Note that we made a choice in letting $w_0$ be our fixed point rather than similarly acceptable $x_0 \in W_4(4)$, or any other anisotropic vector. Choosing a different fixed point amounts to choosing a different, often nonconjugate, Levi factor. For instance, the submodule structure of $V$ for the action of the $SO_3$ given by fixed point $w_0$ involves four copies of the natural module and one copy of the two-dimensional simple module, whereas the action of the $SO_3$ given by fixed point $x_0$ involves three copies of the natural module and two copies of the simple. So, although both of these choices give a Levi factor for $C_G(T, e)$, they must be nonconjugate due to having different submodule structures on $V$.

# Chapter 3

# Steps Towards a Conjugacy Conjecture of Steinberg in Small Characteristic

## 3.1 Preliminaries and Semisimple Classes

Let $G$ be a semisimple algebraic group over an algebraically closed field $k$. Recall the definition of *good* characteristic from Section 2.1. We say furthermore that $p$ is *very good* if it is good and $p$ does not divide $n-1$ whenever $G$ contains a simple factor of type $A_n$. We will also need slight modifications of these definitions. We will say that the characteristic is *L-good* (resp. *very L-good*) if $p$ is good (resp. very good) and, if $p = 7$, $G$ contains no simple factor of type $F_4$.

If $k[G]$ is the usual algebra of regular functions on $G$, then define the class functions as

$$C[G] = \{f \in k[G] \mid f(ghg^{-1}) = f(h) \text{ for all } g, h \in G\}.$$

So, these are the regular functions on $G$ which are constant on orbits. Then $C[G]$ is just $k[G]^G$, the invariants of $k[G]$ under the action induced by conjugation with $G$. We will see that, in good characteristic, the class functions on $G$ are enough to separate semisimple orbits.

Fix a maximal torus $T$ of $G$, and let $W = N_G(T)/T$ be the corresponding Weyl group. Following Humphreys' exposition in [H95] chapter 3 (see also [S65] Section

6), the semisimple classes correspond to the $W$-orbits in $T$, since any semisimple element is conjugate to an element of the maximal torus and any two toral elements are conjugate under $N_G(T)$. If $k[T]^W$ is the algebra of regular function on $T$ which are constant on $W$-orbits, then it turns out that the restriction from $k[G]$ to $k[T]$ induces an isomorphism these two algebras $C[G]$ and $k[T]^W$. This gives [H95] Theorem 3.2, due to Steinberg:

**Theorem 3.1.1.** *There is a $k$-algebra isomorphism $C[G] \simeq k[T]^W$. Furthermore, $C[G]$ has a $k$-basis of the characters of irreducible rational representations of $G$.*

Moreover, the same approach works for the Lie algebra, with class functions replaced with $Ad(G)$-invariant polynomials on $\mathfrak{g}$. Now, $T/W$ has an affine quotient variety structure with affine algebra $k[T]^W$. This variety corresponds to the semisimple classes by the discussion above, and hence $C[G]$ separates semisimple classes by the isomorphism of Theorem 3.1.1. Thus, we get [H95] Theorem 3.4 (alternately, [SS70] Corollary II.3.3) and [SS70] 3.17:

**Theorem 3.1.2.** *(i) If $s, t \in G$ are semisimple and not conjugate, then there exists a class function $f$ such that $f(s) \neq f(t)$.*

*(ii) If $h, k \in \mathfrak{g}$ are semisimple and not conjugate, then there exists $f$ an $Ad(G)$-invariant polynomials on $\mathfrak{g}$ such that $f(h) \neq f(k)$.*

When $G = SL_n$ and $W = S_n$, one can think of the class functions as symmetric polynomials in $n$ variables $T_1, T_2, \ldots, T_n$. Then each class function is a polynomial in the elementary symmetric functions in the variables $T_i$, with $W$ acting by permutation on the indices. The value of $f(g)$ for $f \in C[G]$ and semisimple $g \in G$ is just the symmetric polynomial $f$ evaluated at the eigenvalues of $g$.

Unfortunately, class functions are in general not enough to distinguish orbits; they must take same value on the closure of a class, whereas the semisimple classes are the only closed orbits. In [S78] Theorem 3, though, Steinberg proves that two elements of an algebraic group over a field of large characteristic are conjugate in the group if and only if they are conjugate under each rational representation of $G$. Thus, understanding the orbits under each representation is sufficient to determine the orbits in the group.

**Theorem 3.1.3.** *Suppose that $G$ is a semisimple algebraic group over a field $k$ of characteristic $p = 0$ or $p > 4h$, where $h$ is the Coxeter number of $G$. Two elements $a, a' \in G$ are conjugate in $G$ if and only if $f(a)$ and $f(a')$ are conjugate in $GL(V)$ for each irreducible rational representation $(f, V)$ of $G$.*

This was orginally conjectured in [S66]. Gauger proved in [G77] the analogous result for Lie algebras over fields of characteristic zero. As mentioned, a year later in [S78], Steinberg presented a somewhat simpler proof of the conjecture in the Lie algebra which was adaptable to the original setting of groups. Steinberg's proof involves the use of $\mathfrak{sl}_2$-triples as well as exponentiation, and so is valid only in characteristic zero or when the the characteristic is "sufficiently large", roughly four times the Coxeter number.

One might initially hope to replace the $\mathfrak{sl}_2$-triples in Steinberg's approach with the associate cocharacters mentioned last chapter in Section 2.1, as this tactic is often successful in replicating similar arguments in smaller characteristic. The problem in doing this is that associated cocharacters do not necessarily behave well under homomorphisms. Specifically, if $\phi : G \to H$ is a homomorphism of reductive algebraic groups and $X \in \mathfrak{g}$ is nilpotent with $\tau : k^\times \to G$ a cocharacter associated to $X$, then $\phi \circ \tau$ need not be associated to $d\phi(X)$ (see [J04] Section 5.12). So, we are not guaranteed that a cocharacter associated to a nilpotent element will stay associated under a representation, and we must find a different solution.

In the next section, we are going to verify the validity of Theorem 3.1.3 for good primes smaller than four times the Coxeter number in the event that the centralizers of the semisimple parts of $a, a' \in G$ can distinguish between unipotent orbits in the group by way of its adjoint representation. One such fundamental situation is when the centralizer is the product of exceptional groups, where a result of Lawther from [La95] verifies this unipotent class split in this way. The goal here is to display an approach which, while only valid when this centralizer can split unipotent orbits in its adjoint representation, we hope in the future can be applied in general using other representations.

## 3.2 Algebraic Groups

When $G$ is a classical group, we can usually recover the conjugacy classes of unipotent elements by examining their Jordan block sizes on the natural module, though this fails when we are examining "very even" orbits in type $D_n$ (see [J04] Proposition 1.12). Unfortunately, when $G$ is simple and exceptional, there is no such geometrically "natural" module to consider, though we can look at the adjoint representation on the Lie algebra. In most cases there is a smaller representation for each exceptional group, though we will be utilizing the adjoint representation (see remark 3.2.4).

Through much calculation and working with cases, Lawther proves that in fact the Jordan block sizes for the adjoint action are enough to determine the conjugacy classes in exceptional groups. The following is contained in [La95] Theorem 2.

**Theorem 3.2.1.** *Let $G$ be a simple exceptional algebraic group over a field $k$ whose characteristic is L-good, and let $\mathfrak{g}$ be its Lie algebra. Let $u, u' \in G$ be unipotent. If $u$ and $u'$ have the same Jordan block sizes for their action on $\mathfrak{g}$, then they are conjugate in $G$.*

In the classical case, it seems that the adjoint representation may not be able to determine the conjugacy of unipotent elements in classical groups, as this might cause issues even in type $A_n$. Sometimes, though, this representation will be sufficient, as is evident by the previous theorem of Lawther for exceptional groups.

With all of this in mind, define a semisimple algebraic group $G$ to be *adjoint adequate* when it is true that if $u$ and $u'$ are conjugate under the adjoint representation of $G$, then $u$ and $u'$ are conjugate in $G$. So, we have the following lemma restating Lawther's result in these terms and adapting it to semisimple groups.

**Lemma 3.2.2.** *Suppose that $G$ is a semisimple group over a field $k$ of L-good characteristic. If each simple factor of $G$ is of exceptional type, then $G$ is adjoint adequate.*

*Proof.* Suppose that $u, u' \in G$ are unipotent elements such that $u$ and $u'$ are conjugate under the adjoint representation of $G$. It suffices to prove the lemma for $G$ adjoint, as conjugacy is preserved under isogeny. So, suppose $G = G_1 \times G_2 \times \dots G_n$ with each $G_i$ simple and exceptional, and let $u = u_1 + \dots + u_n$ and $u' = u'_1 + \dots + u'_n$ with $u_i, u'_i \in G_i$. Now, we know that $Ad(u)$ and $Ad(u')$ are conjugate in the adjoint representation. Therefore $Ad(u_i)$ and $Ad(u_i')$ are conjugate for each $i$ and, since each $G_i$ is exceptional, we know by Lawther's Theorem 3.2.1 that $u_i$ and $u_i'$ are conjugate in $G_i$. Thus, $u$ and $u'$ are conjugate in $G$. □

So, by assumption we know Theorem 3.1.3 for unipotent elements in adjoint adequate groups over fields of L-good characteristic, and this version of Lawther's result from [La95] gives us a class of semisimple groups which satisfy the hypothesis. Since we have the result for semisimple elements in good characteristic from Theorem 3.1.2, the work now is to combine both of these cases with a Jordan decomposition to obtain the result for arbitrary elements, with some adjoint adequacy restriction. We initially follow Steinberg's approach in [S78] Theorem 3.

**Theorem 3.2.3.** *Suppose that $G$ is a semisimple algebraic group over a field $k$ of good characteristic, and consider two elements $a, a' \in G$ with Jordan decompositions $a = us$ and $a' = u's'$. Suppose that the centralizer $G_s$ is adjoint adequate. Then, $a$ and $a'$ are conjugate in $G$ if and only if $f(a)$ and $f(a')$ are conjugate in $GL(V)$ for each irreducible rational representation $(f, V)$ of $G$.*

*Proof.* The "only if" direction is clear; it remains to prove the "if" direction.

Once again, we may assume that $G$ is adjoint. Consider $a, a' \in G$ and suppose that $f(a)$ and $f(a')$ are conjugate in $GL(V)$ for each irreducible rational representation $(f, V)$ of $G$. Recall that $a = us$ and $a' = u's'$ are the Jordan decompositions of $a$ and $a'$, respectively, with $u, u'$ unipotent. The semisimple parts $f(s)$ and $f(s')$ are conjugate in $GL(V)$, hence $s$ and $s'$ are conjugate in $G$ by the Theorem 3.1.2 part (i). So we may assume $s = s'$, and now $u, u' \in G_s$, the centralizer of $s$ in $G$.

Moreover, $f(u)$ is conjugate to $f(u')$ in $GL(V)_{f(s)}$ for each rational representation. Consider the adjoint representation $(Ad, \mathfrak{g})$. We wish to conclude that $u$ and $u'$ are conjugate in $G_s$, but the previous lemma only allows us to say that they are conjugate in $G$ given the appropriate hypotheses. We need to show that an element of $GL(\mathfrak{g})_{Ad(s)}$ restricts to an element of $GL(\mathfrak{g}_s) = GL(\text{Lie}(G_s))$.

So, let $h \in GL(\mathfrak{g})_{Ad(s)}$ such that $h$ conjugates $Ad(u)$ to $Ad(u')$. Then, for $Y \in \mathfrak{g}_s$,

$$Ad(s)(h(Y)) = h(Ad(s)(Y)) = h(Y),$$

hence the image of $h|_{\mathfrak{g}_s}$ is contained in $\mathfrak{g}_s$. On the other hand, assume $h(Z) \in \mathfrak{g}_s$ for

$Z \in \mathfrak{g}$. Then
$$h(Ad(s)(Z)) = Ad(s)(h(Z)) = h(Z),$$
thus $Ad(s)(Z) = Z$ as $h$ is invertible. So, $Z \in \mathfrak{g}_s$. Now we've shown that $h$ restricts to an endomorphism of $\mathfrak{g}_s$, and therefore $h|_{\mathfrak{g}_s} \in GL(\mathfrak{g}_s)$.

To summarize, we have $u, u' \in G_s$ such that $Ad(u)$ is conjugate to $Ad(u')$ in $GL(\mathfrak{g}_s)$. Since we have assumed $G_s$ is adjoint adequate, we may now conclude that $u$ and $u'$ are conjugate in $G_s$. Thus, $a = us$ and $a' = u's$ are conjugate, completing the proof.

□

Unfortunately, there does not seem to be a satisfactory assumption we can place on $G$ to ensure $G_s$ is adjoint adequate. The hope is that this can be avoided and that a similar argument may work with any rational representation (since we have access to all of them by hypothesis). In particular, we might use the natural modules for simply connected classical groups of types $A_n$, $B_n$, and $C_n$, or use the spin representations to take care of simple factors of type $D_n$.

*Remark* 3.2.4. The L-good condition restricting the appearance of simple factors of type $F_4$ when $p = 7$ comes from Theorem 2 in [La95]. There, we see that there are unipotent elements which may be conjugate under the adjoint representation with partition $[7^7, 1^3]$ despite being in separate $B_3$ and $C_3$ orbits in the group (in Bala-Carter notation), given by partitions $[7^3, 1^5]$ and $[7^2, 6^2]$, respectively. In *loc. cit.*, Lawther also verifies that a certain "natural module" $V$ also splits orbits in the group without this condition. It does not seem obvious here why an element of $GL(V)_{f(s)}$ should restrict to an endomorphism of the natural module for $G_s$, which is the crux of the difficulty in attempting to use the usual natural modules for classical groups.

## 3.3 Lie Algebras

As mentioned above, we have the analogous conjecture from [S66] for Lie algebras, proved by Steinberg for sufficiently large characteristic in [S78] Theorem 2.

**Theorem 3.3.1.** *Suppose that $G$ is a semisimple algebraic group over a field $k$ of characteristic $p = 0$ or $p > 4h$, where $h$ is the Coxeter number of $G$. Let $\mathfrak{g}$ be the*

*Lie algebra of $G$. Two elements $A, A' \in \mathfrak{g}$ are conjugate in $G$ if and only if $f(A)$ and $f(A')$ are conjugate in $GL(V)$ for each irreducible rational representation $(f, V)$ of $G$.*

The difficulty here, in small characteristic, is that it is not obvious how Lawther's result for the adjoint representation of unipotent elements can be transported to consider nilpotent elements in the Lie algebra. Fortunately, using a Springer isomorphism, good $A_1$ subgroups, and sub-principal homomorphisms, McNinch obtains Lawther's result in in [M02b] when our nilpotent element is *p-nilpotent*, as well as in general for classical groups. We say that an element $X \in \mathfrak{g}$ is $p$-nilpotent if $X^{[p]} = 0$.

We call the characteristic of $k$ *very M-good* for $G$ if $p$ is very good and greater than the Coxeter number for each simple factor of type $F_4$ or $E_n$. This ensures that every nilpotent element of $G$ is $p$-nilpotent when restricted to exceptional factors not of type $G_2$ ([M02a], Corollary 4.4). If $X \in \mathfrak{g}$ is nilpotent, then by [M02b] Proposition 3 there is a $G$-equivariant isomorphism $\theta : \mathcal{N} \to \mathcal{U}$ such that $\theta(X^{[p]}) = (\theta(X))^{[p]}$. In the simple exceptional cases, the requirement that $X$ be $p$-nilpotent allows for the existence of the good $A_1$ subgroup (see [Sz00]) and sub-principal homomorphism (see [M03]) containing $\theta(X)$ needed in the proof of [M02b] Theorem 10.

Finally, note that very M-good implies very L-good.

**Lemma 3.3.2.** *Let $G$ be an adjoint adequate algebraic group over a field $k$ of very M-good characteristic. Suppose that $X, X'$ are nilpotent elements of $\mathfrak{g} = Lie(G)$. If $X$ and $X'$ are conjugate under the adjoint representation of $G$, then $X$ and $X'$ are conjugate in $G$.*

*Proof.* Once again, by [M02b] Proposition 3 there exists a $G$-equivariant isomorphism $\theta : \mathcal{N} \to \mathcal{U}$ such that $\theta(Y^{[p]}) = (\theta(Y))^{[p]}$ for all $Y \in \mathfrak{g}$. Let $u = \theta(X)$ and $u' = \theta(X')$.

Now, by *loc. cit.* Theorem 10, Theorem 24, and Theorem 30, the partitions of $Ad(u)$ and $Ad(u')$ are the same as the partitions of $Ad(X)$ and $Ad(X')$, respectively. Thus, $u$ and $u'$ are conjugate under the adjoint representation and, by adjoint adequacy, $u$ and $u'$ are conjugate in $G$. Since $\theta$ was $G$-equivariant, we must have $X$ and $X'$ conjugate in $G$.

□

We are now able to to prove a result for Lie algebras mirroring our proof for group elements (again following Steingberg in [S78] Theorem 2 initially).

**Theorem 3.3.3.** *Suppose that $G$ is a semisimple algebraic group over a field $k$ of good characteristic. Let $\mathfrak{g}$ be the Lie algebra of $G$ and consider two elements $A, A' \in \mathfrak{g}$ with Jordan decompositions $A = X + S$ and $A' = X' + S'$. Suppose that the centralizer $G_S$ is adjoint adequate and has very M-good characteristic. Then, $A$ and $A'$ are conjugate in $G$ if and only if $f(A)$ and $f(A')$ are conjugate in $GL(V)$ for each irreducible rational representation $(f, V)$ of $G$.*

*Proof.* Let $A = X + S$ and $A' = X' + S'$ be the Jordan decompositions with $X, X'$ nilpotent and $S, S'$ semisimple. The semisimple case is known from Theorem 3.1.2 part (ii), so since $f(S)$ is conjugate to $f(S')$ in $GL(V)$, we know that $S$ is conjugate to $S'$ in $G$. Thus we may assume $S = S'$ and we have $X, X' \in \mathfrak{g}_S$, the centralizer of $S$ in $\mathfrak{g}$.

Considering the adjoint representation, we now know that $ad(X)$ is conjugate to $ad(X')$ in $GL(\mathfrak{g})_{ad(S)}$. As before, we want to show that an element of $GL(\mathfrak{g})_{ad(S)}$ restricts to an element of $GL(\mathfrak{g}_S)$.

So, let $h \in GL(\mathfrak{g})_{ad(S)}$. Then for $Y \in \mathfrak{g}_S$, we have

$$ad(S)(h(Y)) = h(ad(S)(Y)) = h(0) = 0.$$

Thus the image of $h|_{\mathfrak{g}_S}$ is contained in $\mathfrak{g}_S$.

Alternately, if $h(Z) \in \mathfrak{g}_S$ for $Z \in \mathfrak{g}$, then

$$h(ad(S)(Z)) = ad(S)(h(Z)) = 0.$$

Since $h$ is invertible, $ad(S)(Z) = 0$ and $Z \in \mathfrak{g}_S$.

Once again, we have shown that $h \in GL(\mathfrak{g})_{ad(S)}$ restricts to an automorphism on $\mathfrak{g}_S$, thus $h|_{\mathfrak{g}_S} \in GL(\mathfrak{g}_S)$. So, we have $X, X' \in \mathfrak{g}_S$ such that $ad(X)$ is conjugate to $ad(X')$ in $GL(\mathfrak{g}_S)$. Therefore, by Lemma 3.3.2, $X$ is conjugate to $X'$ in $G_S$. Hence, $A$ is conjugate to $A'$.

□

It may be possible to achieve a more universal restriction than $G_S$ having very M-good characteristic. To ensure very good characteristic descends to the centralizer, it suffices to assume that $G$ is "strongly standard" in the sense of [M05] and subsequent papers along these lines (see Remark 3 in *loc. cit.*). In this case we have Springer isomorphisms as above, but we will still need $p > h$ in $G_S$ to ensure $p$-nilpotentcy for the application of results from [M02a].

*Remark* 3.3.4. Of course, if we know that the exceptional parts of $X$ and $X'$ are $p$-nilpotent to begin with, the restriction to characteristics greater than the Coxeter number is unnecessary. One hopes that Lawther's result for unipotent partitions in exceptional groups always has an analogue for nilpotent elements in good characteristic, rather than just in the $p$-nilpotent case, though a non-computational approach remains unclear.

## 3.4 Future Work

Our work here presents ample opportunity for future explorations. First, as discussed above, we hope that a similar argument is admissible for representations beyond the adjoint. We possibly need the spin representations to deal with unipotent elements in groups of type $D_m$, and the natural modules for other simply connected classical groups. Then we may only need top level restrictions relative to the group $G$, such as $G$ being strongly standard for the Lie algebra analogue.

One would imagine that there is a more uniform solution to the results of [La95] and subsequently Lemma 3.2.2. A closer examination of associated cocharacters might yield similar conclusions, but for now we use Lawther's computational approach. Additionally, we could attempt to modify the argument in the proof of Theorem 3.2.3 to utilize the representation $V_d$ from [La95] rather than the adjoint representation, thus removing the L-good restriction for components of type $F_4$ when $p = 7$ (see remark 3.2.4).

We might also expect that the restriction to $p$-nilpotent elements built into the M-good characteristic condition of Lemma 3.3.2 and Theorem 3.3.3 could be avoided. Such a hypothesis does not seem intrinsic to the result but, as mentioned above, was used in a fundamental way in the proof of [M02b] Theorem 10. A case-checking approach to the nilpotent elements similar to that of [La95] should work, though an alternative uniform approach with Springer isomorphisms might be less computational.

Lastly, in the future we plan to investigate the setting of bad characteristic, where the situation is surely more delicate. In this case we cannot hope for a correspondence between unipotent and nilpotent elements, though we might still expect the bad characteristic analogue to Theorem 3.2.3 to be true. As noted in [H95] section 3.7, a complete proof would imply a more direct approach to proving the Richardson-Lusztig finiteness theorem (see Theorem 3.11 in *loc. cit.*).

# Chapter 4

# Modules in Standard Levi Form

## 4.1 Preliminaries and Parabolics

We start by considering the Lie algebra $\mathfrak{g}$ of an algebraic group $G$ over a field $k$ of characteristic $p > 0$. Assume the "standard hypotheses" for $\mathfrak{g}$ from Section 1.2.Now, for $\chi \in \mathfrak{g}^*$, let

$$U_\chi(\mathfrak{g}) = U(\mathfrak{g})/\langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\rangle$$

be the reduced enveloping algebra for $\mathfrak{g}$, where $U(\mathfrak{g})$ is the usual enveloping algebra of $\mathfrak{g}$. Fix a maximal torus $T \subset G$ and $X = X(T)$ its group of characters. Let $R^+ \subset R$ be a system of positive roots with set of simple positive roots $\Delta$ and Weyl group $W$. Now, let $\mathfrak{h} = \mathrm{Lie}(T)$ and let $\mathfrak{n}^+$(respectively, $\mathfrak{n}^-$) be the sum of the root spaces $\mathfrak{g}_\alpha$ with $\alpha > 0$ (respectively, $\alpha < 0$). Denote $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ the standard positive Borel subalgebra.

In our quest to understand $U_\chi$-modules, a well known result of Kac and Weisfeiler in [KW71] says that we need only consider those for conjugacy classes of nilpotent $\chi$. So, in fact, it will suffice to work with those characters such that $\chi(\mathfrak{b}^+) = 0$. We will be concerned with a special type of nilpotent $\chi$, investigated first by Friedlander and Parshall in [FP90], said to be in *standard Levi form*. We define $\chi \in \mathfrak{g}^*$ to have standard Levi form if there exists a subset $I \subseteq \Delta$ such that $\chi(x_{-\alpha}) \neq 0$ for $\alpha \in I$, and $\chi(x_{-\alpha}) = 0$ otherwise.

Let $J \subseteq \Delta$ be another subset of simple roots such that $I \cap J = \varnothing$, and let $P = P_J$ be the standard parabolic subgroup defined by $J$ containing our maximal torus $T$.

Then $\mathfrak{b}^+ \subseteq \mathfrak{p} = \mathrm{Lie}(P)$. The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ decomposes as the direct sum $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{z}$, where $\mathfrak{l}' = \mathrm{Lie}([L, L])$ and $\mathfrak{z} = \mathrm{Lie}(Z(L))$. Let $T'$ be a maximal torus in $[L, L]$, and define $\mathfrak{h}' = \mathrm{Lie}(T')$. Then $R_J \subseteq X^*(T')$ and $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}$.

If $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$) is the nilradical of $\mathfrak{p}$ (resp. $\mathfrak{p}^-$), then $\mathfrak{g}$ has a triangular decomposition of sorts:

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+ = \mathfrak{u}^- \oplus \mathfrak{l}' \oplus \mathfrak{z} \oplus \mathfrak{u}^+$$

Finally, following [J97] Section 11.3, we first have a grading of $U_\chi(\mathfrak{g})$ by $\mathbb{Z}R$ due to the Poincaré-Birkhoff-Witt basis, where each $x_\alpha^p$ is contained in $U_\chi(\mathfrak{g})_{p\alpha}$ for $x_\alpha \in \mathfrak{g}_\alpha$ and $h^p - h^{[p]} - \chi(h)^p$ has degree zero. In standard Levi form, we then get a coarser grading of $U_\chi(\mathfrak{g})$ by $\mathbb{Z}R/\mathbb{Z}I$.

## 4.2 Parabolic Baby Verma Modules

Here we will define the central objects of our study in standard Levi form. For each $\lambda \in \mathfrak{h}^*$, let $k_\lambda$ be the one-dimensional $\mathfrak{h}$-module such that each $h \in \mathfrak{h}$ acts as $\lambda(h)$ acting on $k$. We wish to extend this to a module for $\mathfrak{p}$, much as in section 1.2.1, by simply letting $\mathfrak{n}^+$ and $\bigoplus_{\alpha \in R_J^+} \mathfrak{g}_{-\alpha}$ act trivially. Here we must consider, though, that if $\lambda$ is to vanish on root vectors $x_\alpha$ and $x_{-\alpha}$ for $\alpha \in R_J$, we must have $\lambda([x_\alpha, x_{-\alpha}]) = 0$ on $\mathfrak{h}'$. So, in a sense, by increasing the size of $J$ we are restricted to fewer characters $\lambda \in \mathfrak{z}^*$. Now let

$$\Lambda_{\chi,J} = \{\lambda \in \mathfrak{h}^* \mid \lambda(\mathfrak{h}') = 0 \text{ and } \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p = 0 \text{ for all } h \in \mathfrak{h}\}.$$

Note that $\chi(h) = 0$ for $h \in \mathfrak{h}$ since we are assuming that $\chi$ is nilpotent. Then for $\lambda \in \Lambda_{\chi,J}$ we can we may define the $\mathfrak{p}$-module $k_\lambda$ and further extend it to a $U_\chi(\mathfrak{p})$-module.

We are now ready to define:

$$Z_{\chi,J}(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} k_\lambda$$

We call this a parabolic baby Verma module. Since $U_\chi(\mathfrak{g})$ is free over $U_\chi(\mathfrak{p})$, we know $Z_{\chi,J}(\lambda)$ has basis $\{u_{-\alpha_1}^{a_1} u_{-\alpha_2}^{a_2} \dots u_{-\alpha_l}^{a_l} \otimes 1 \mid \alpha_i \in R^+ \backslash R_J \text{ and } 0 \leqslant a_i < p\}$ and write $1 \otimes 1$ for the generator. Since $\dim(Z_{\chi,J}(\lambda)) = p^{\dim(\mathfrak{u}^-)}$, these modules are smaller in general than the usual baby Verma modules $Z_\chi(\lambda)$ defined in Section 1.2.1. Furthermore, as $\mathfrak{u}^-$-modules, $Z_{\chi,J}(\lambda) \simeq U_\chi(\mathfrak{u}^-)$.

These objects were first explored for standard Levi form in [FP90], though simply inducing from a Borel subalgebra (when $J = \varnothing$) soon attracted the most interest, as this provides a bit more "flexibility" in constructions. We will see some of the challenges that arise when dealing with a parabolic subalgebra that is not a Borel in the next few sections.

As we mentioned in Theorem 1.2.2, in standard Levi form each $Z_\chi(\lambda) = Z_{\chi,\varnothing}(\lambda)$ has a unique maximal submodule with simple quotient denoted $L_\chi(\lambda)$. This is also true of parabolic baby Verma modules and, in fact, we get the *same* simple quotient. To prove this, we first need to know about representations when $\mathfrak{g}$ is unipotent; that is, when $\mathfrak{g}$ is the Lie algebra of a unipotent algebraic group. More directly, this means that for all $g \in \mathfrak{g}$ there exists $r > 0$ such that $g^{[p^r]} = 0$. The following Lemma is contained in [J97] Proposition 3.2 and Theorem 3.3, due to Zassenhaus in [Z40].

**Lemma 4.2.1.** *Let $\chi \in \mathfrak{g}^*$. If $\mathfrak{g}$ is unipotent, then $U_\chi(\mathfrak{g})$ has only one simple module up to isomorphism.*

*Proof.* First, we consider $\chi = 0$. In $U_0(\mathfrak{g})$, $x^p = x^{[p]}$ and hence $x^{p^r} = x^{[p^r]} = 0$. Therefore, $U_0(\mathfrak{g})\mathfrak{g}$ is a nilpotent ideal which annihilates the trivial $\mathfrak{g}$-module $k$ and thus is contained in the radical of $U_0(\mathfrak{g})$. Since $k \simeq U_0(\mathfrak{g})/(U_0(\mathfrak{g})\mathfrak{g})$, the trivial $\mathfrak{g}$-module can be the only simple $U_0(\mathfrak{g})$-module up to isomorphism.

Now, return to general $\chi \in \mathfrak{g}^*$. If $M$ and $N$ are simple $U_\chi(\mathfrak{g})$-modules, then $M^* \otimes N \simeq \mathrm{Hom}_k(M, N)$ is a $U_0(\mathfrak{g})$-module. By above we have the $\mathfrak{g}$-submodule $k \subseteq \mathrm{Hom}_k(M, N)$, and hence $\mathrm{Hom}_k(M, N) \neq 0$. Thus, by an application of Schur's lemma, we must have $M \simeq N$. $\square$

Recall that, if $M$ and $L$ are modules with $L$ simple, we denote the multiplicity of $L$ as a composition factor of $M$ by $[M : L]$.

Now we are ready to prove the parabolic analogue to Theorem 1.2.2, which said that, when $J = \varnothing$, the modules $Z_\chi(\lambda)$ had unique simple quotient $L_\chi(\lambda)$. It turns out that our smaller parabolic baby Verma modules will have the *same* simple quotient. We approach the first part of the following theorem in a fashion similar to [J97] Proposition 10.2.

**Theorem 4.2.2.** *Let $\chi \in \mathfrak{g}^*$ have standard Levi form for $I \subseteq \Delta$, and let $J \subseteq \Delta$ such that $I \cap J = \varnothing$. For $\lambda \in \Lambda_{\chi,J}$, $Z_{\chi,J}(\lambda)$ has a unique maximal submodule, and this submodule gives a unique simple quotient isomorphic to $L_\chi(\lambda)$. Furthermore, we have $[Z_{\chi,J}(\lambda)\colon L_\chi(\lambda)] = 1$.*

*Proof.* Recall that we denote by $\mathfrak{u}^-$ the nilradical of $\mathfrak{p}^-$, the negative parabolic subalgebra defined by $J$. Since $\chi$ is nonzero only on simple negative root spaces, looking at weights we can conclude that $\chi([\mathfrak{u}^-, \mathfrak{u}^-]) = \chi(\mathfrak{u}^{-[p]}) = 0$. So, $\chi$ defines a one dimensional $\mathfrak{u}^-$-module $k_\chi$ which we can consider as a $U_\chi(\mathfrak{u}^-)$-module. By Lemma 4.2.1, since $\mathfrak{u}^-$ is unipotent, this is the unique such module up to isomorphism. Now, as an $\mathfrak{u}^-$-module, $U_\chi(\mathfrak{u}^-)$ has a simple head since it is the projective cover of $k_\chi$. On the other hand, we know that $Z_{\chi,J}(\lambda) \simeq U_\chi(\mathfrak{u}^-)$ as $\mathfrak{u}^-$-modules. Thus, $Z_{\chi,J}(\lambda)$ has a simple head (and unique maximal submodule) as a $U_\chi(\mathfrak{g})$-module.

Now, consider both $U_\chi(\mathfrak{g})$-modules $Z_\chi(\lambda)$ and $Z_{\chi,J}(\lambda)$ with unique simple quotients $L_\chi(\lambda)$ and $L_{\chi,J}(\lambda)$, respectively. Consider the map $\psi : Z_\chi(\lambda) \to Z_{\chi,J}(\lambda)$ given by $\psi(1 \otimes 1) = 1 \otimes 1$. Since by definition $u.\psi(1 \otimes 1) = 0$ for $u \in \mathfrak{n}^+$, this is a homomorphism which is moreover seen to be surjective by examining bases. Let $N$ be the unique maximal submodule of $Z_\chi(\lambda)$ and let $\pi : Z_\chi(\lambda) \to L_\chi(\lambda)$ be the projection map. By unique maximality we have $\ker(\psi) \subseteq N = \ker(\pi)$. Therefore, by the universal property of quotients, there exists a unique homomorphism

$$\tilde{\pi} : Z_\chi(\lambda)/\ker(\psi) \simeq Z_{\chi,J}(\lambda) \to L_\chi(\lambda)$$

such that $\tilde{\pi} \circ \psi = \pi$.

Since $\psi$ and $\pi$ are both nonzero, we must have $\tilde{\pi}$ nonzero. Thus, the image of $\tilde{\pi}$ must be isomorphic to $L_\chi(\lambda)$ by simplicity, and we have $Z_{\chi,J}(\lambda)/\ker(\tilde{\pi}) \simeq L_\chi(\lambda)$. This is a simple quotient of $Z_{\chi,J}(\lambda)$, hence we must have $L_\chi(\lambda) \simeq L_{\chi,J}(\lambda)$ by uniqueness.

By the discussion in section 2.8 of [J00], we know that $[Z_\chi(\lambda)\colon L_\chi(\lambda)] = 1$. Moreover, since we have just displayed $Z_{\chi,J}(\lambda)$ as a quotient of $Z_\chi(\lambda)$, each composition factor of $Z_{\chi,J}(\lambda)$ must appear as a composition factor of $Z_\chi(\lambda)$. Thus we must have $[Z_{\chi,J}(\lambda)\colon L_\chi(\lambda)] = 1$, proving the final assertion. □

Combining this with Theorem 1.2.1, we can use the parabolic baby Verma modules to find *all* of the simple $U_\chi(\mathfrak{g})$-modules as unique simple quotients just as we did with the larger $Z_\chi(\lambda)$. It also allows us to use some of the other results from the $J = \varnothing$ case in the parabolic setting, as we will see soon.

## 4.3 Isomorphic Modules

Given that we have now explicitly constructed modules with which we can identify simple modules in our nonrestricted enveloping algebra, we may wish to know when such modules (and simple quotients) are isomorphic. We will explore such an idea in this section, with some restriction on our choice of subset of simple roots $J \subseteq \Delta$.

Considering a simple root space $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$, let $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ be the toral element with $\alpha(h_\alpha) = 2$. Now, for any element $w \in W$, the "dot action" on $\mathfrak{h}^*$ is defined as $w_\bullet\lambda = w(\lambda + \rho) - \rho$, where $\rho \in \mathfrak{h}^*$ is the element such that $\rho(h_\alpha) = 1$ for all $\alpha \in \Delta$ (see, for example, [J97] section 9.2). We will write $W_I$ for the subgroup of $W$ generated by the simple reflections $s_\alpha$ for $\alpha \in I$.

Given two subsets of simple roots $I_1, I_2 \subseteq \Delta$, we will say that $I_1$ and $I_2$ are *non-adjacent* if they are disjoint and $\alpha_1 + \alpha_2$ is not a root for all $\alpha_1 \in I_1$ and $\alpha_2 \in I_2$. This implies that $\langle \alpha, \beta^\vee \rangle = 0$ (see [H80] Lemmas 9.4 and 10.1) and, since $\alpha$ and $\beta$ generate a root subspace of type $A_1 \times A_1$, we have $\alpha(h_\beta) = 0$. In what follows, we will require $I$ and $J$ to be non-adjacent, as interaction between their respective root systems turns out to be problematic. This restriction will be important if we wish to study any sort of "linkage" between parabolic baby Verma modules, as evident in the proof of the following lemma.

**Lemma 4.3.1.** *Let $\chi \in \mathfrak{g}^*$ have standard Levi form for $I \subseteq \Delta$, and let $J \subseteq \Delta$ such that $I \cap J = \varnothing$. Consider $\lambda \in \Lambda_{\chi,J}$. For $\alpha \in I$, if $I$ and $J$ are non-adjacent, then $s_{\alpha\bullet}\lambda \in \Lambda_{\chi,J}$.*

*Proof.* First we will check that $s_{\alpha\bullet}\lambda(\mathfrak{h}') = 0$. It suffices to consider $h_\beta$ for $\beta \in J$. Now,

$$s_{\alpha\bullet}\lambda(h_\beta) = (s_\alpha(\lambda) - \alpha)(h_\beta) = (\lambda - \lambda(h_\alpha)\alpha - \alpha)(h_\beta),$$

and since $\lambda(h_\beta) = 0$ by assumption, we have

$$s_{\alpha\bullet}\lambda(h_\beta) = -\alpha(h_\beta)(\lambda(h_\alpha) + 1).$$

By non-adjacency, we know that $\alpha(h_\beta) = 0$, and therefore $s_{\alpha\bullet}\lambda(h_\beta) = 0$.

Now, for any $\gamma \in \Delta$, the calculations above show that

$$s_{\alpha\bullet}\lambda(h_\gamma) = \lambda(h_\gamma) - (\lambda(h_\alpha) + 1)\alpha(h_\gamma).$$

Therefore, using that $\alpha(h_\gamma)^p = \alpha(h_\gamma^p) = \alpha(h_\gamma)$ and $\chi$ vanishes on $\mathfrak{h}$, we have

$$\begin{aligned}
(s_{\alpha\bullet}\lambda(h_\gamma))^p - s_{\alpha\bullet}\lambda(h_\gamma) &= [\lambda(h_\gamma) - (\lambda(h_\alpha) + 1)\alpha(h_\gamma)]^p - [\lambda(h_\gamma) - (\lambda(h_\alpha) + 1)\alpha(h_\gamma)] \\
&= \lambda(h_\gamma)^p - (\lambda(h_\alpha)^p + 1)\alpha(h_\gamma) - \lambda(h_\gamma) + (\lambda(h_\alpha) + 1)\alpha(h_\gamma) \\
&= [\lambda(h_\gamma)^p - \lambda(h_\gamma)] - [\lambda(h_\alpha)^p - \lambda(h_\alpha)]\alpha(h_\gamma) \\
&= \chi(h_\gamma) - \chi(h_\alpha)\alpha(h_\gamma) \\
&= 0
\end{aligned}$$

Hence, $s_{\alpha\bullet}\lambda \in \Lambda_{\chi,J}$ as desired. □

*Remark* 4.3.2. We see in the proof of Lemma 4.3.1 that, were $I$ and $J$ to be adjacent, we could encounter $\alpha \in I$ and $\beta \in J$ such that $\alpha + \beta \in R$, and thus $\alpha(h_\beta) \neq 0$. In this case, if we were to want $s_{\alpha\bullet}\lambda \in \Lambda_{\chi,J}$, we would have to ensure $\lambda(h_\alpha) = -1$. In some sense, this simply builds the restriction of non-adjacency into the what we may choose for $\lambda$.

Now that we know when we can define $Z_{\chi,J}(s_{\alpha\bullet}\lambda)$ for $\alpha \in I$, we will see that the resulting modules are isomorphic.

**Lemma 4.3.3.** *Let $\chi \in \mathfrak{g}^*$ have standard Levi form for $I \subseteq \Delta$, and let $J \subseteq \Delta$ such that $I$ and $J$ are non-adjacent. If $\lambda \in \Lambda_{\chi,J}$ and $\alpha \in I$, then $Z_{\chi,J}(\lambda) \simeq Z_{\chi,J}(s_{\alpha\bullet}\lambda)$.*

*Proof.* Consider $s_\alpha$ for any $\alpha \in I$. Proceeding as in [J97] Section 6.9, we see that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ generate an $\mathfrak{sl}_2$ subalgebra, so we may chose $x_\alpha$ and $x_{-\alpha}$ such that $h_\alpha = [x_\alpha, x_{-\alpha}]$ satisfies $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha, x_{-\alpha}] = -2x_{-\alpha}$, and $h_\alpha^{[p]} - h_\alpha = 0$.

Now, since $\lambda \in \Lambda_{\chi,J}$ and $\chi(h_\alpha) = 0$, we have $\lambda(h_\alpha)^p - \lambda(h_\alpha) = 0$, and hence $\lambda(h_\alpha) = a$ for some integer $0 \leqslant a < p$. Let $v_\lambda = 1 \otimes 1 \in Z_{\chi,J}(\lambda)$. Then the vector $x_{-\alpha}^{a+1}.v_\lambda$ can be seen to be annihilated by all of $\mathfrak{n}^+$. For $\beta \in R_J^+$, $x_{-\beta}$ commutes with $x_{-\alpha}$ due to the non-adjacency of $I$ and $J$, hence $x_{-\beta}(x_{-\alpha}^{a+1}.v_\lambda) = 0$ as well. Furthermore, $\mathfrak{h}$ acts on $x_{-\alpha}^{a+1}.v_\lambda$ as $s_{\alpha\bullet}\lambda = \lambda - (a+1)\alpha$. Therefore, we have a homomorphism of the induced modules

$$\phi_\alpha : Z_{\chi,J}(s_{\alpha\bullet}\lambda) \to Z_{\chi,J}(\lambda)$$

given by $\phi_\alpha(v_{s_\alpha \bullet \lambda}) = x_{-\alpha}^{a+1}.v_\lambda$. Since

$$x_{-\alpha}^{p-(a+1)}.(x_{-\alpha}^{a+1}.v_\lambda) = x_{-\alpha}^{p}.v_\lambda = \chi(x_{-\alpha})^p v_\lambda,$$

the image contains a multiple of our basis vector, hence the map is surjective. Moreover, as both modules have the same dimension, $\phi_\alpha$ must be an isomorphism. Thus, $Z_{\chi,J}(\lambda) \simeq Z_{\chi,J}(s_\alpha \bullet \lambda)$ for all $\alpha \in I$. □

Now we are in position to prove a isomorphism principle similar to [J97] Proposition 10.8 when $J = \varnothing$.

**Proposition 4.3.4.** *Let $\chi \in \mathfrak{g}^*$ have standard Levi form for $I \subseteq \Delta$, and let $J \subseteq \Delta$ such that $I$ and $J$ are non-adjacent. Then,*

$$Z_{\chi,J}(\lambda) \simeq Z_{\chi,J}(\mu) \iff L_\chi(\lambda) \simeq L_\chi(\mu) \iff \mu \in W_I \bullet \lambda$$

*for $\lambda, \mu \in \Lambda_{\chi,J}$.*

*Proof.* The right-hand equivalence is precisely [J97] Proposition 10.8. Also, if $Z_{\chi,J}(\lambda) \simeq Z_{\chi,J}(\mu)$, then clearly $L_\chi(\lambda) \simeq L_\chi(\mu)$. So what is left to prove is the "if" implication of the left-hand equivalence. Suppose then that $L_\chi(\lambda) \simeq L_\chi(\mu)$. In this case, the right-hand equivalence implies $\mu \in W_I \bullet \lambda$, so let $\mu = w \bullet \lambda$ for $w \in W_I$. If $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_l}$ is a reduced decomposition in $W_I$ for $w$, then repeated application of Lemma 4.3.3 gives an isomorphism $Z_{\chi,J}(\lambda) \simeq Z_{\chi,J}(w \bullet \lambda) = Z_{\chi,J}(\mu)$. □

## 4.4 Further Parabolic Exploration

One might hope to follow a similar path to that of Jantzen started in [J97] and continued in [J00], where homomorphisms are built from simple reflections in $W_I$ (much as in Lemma 4.3.3) which, when composed, give a map from a chosen $Z_{\chi,J}(\lambda)$ to another module that has $L_\chi(\lambda)$ as its socle. Unfortunately, a similar construction has several difficulties in the case of nonempty $J$.

To see this, define

$$w\mathfrak{p} = \bigoplus_{\alpha > 0} \mathfrak{g}_{w\alpha} \oplus \bigoplus_{\alpha \in J} \mathfrak{g}_{-w\alpha} \oplus \mathfrak{h}$$

as the parabolic subalgebra "shifted" by $w$. Now let $w\mathfrak{u}^+ = \bigoplus_{\alpha\in R^+\backslash R_J} \mathfrak{g}_{w\alpha}$, and let $w\mathfrak{l}$ be the Levi subalgebra of $w\mathfrak{p}$ with center denoted $w\mathfrak{z}$. Finally, let $w\mathfrak{l}'$ be the derived subalgebra of $w\mathfrak{l}$ and let $w\mathfrak{h}'$ denote its maximal torus.

As above, each $\lambda \in (w\mathfrak{h}')^*$ defines a one-dimensional $w\mathfrak{p}$-module $k_\lambda$ where $h \in \mathfrak{h}$ acts as $\lambda(h)$, and with $\bigoplus\limits_{\alpha>0} \mathfrak{g}_{w\alpha}$ and $\bigoplus\limits_{\alpha\in R_J^+} \mathfrak{g}_{-w\alpha}$ acting trivially. But now, to consider $k_\lambda$ as a $U_\chi(w\mathfrak{p})$-module, we need $\chi(w\mathfrak{p}) = 0$. This is only satisfied by shifting with elements in the following subset of $W$:

$$W^{I,J} = \{w \in W \mid w^{-1}(\alpha) \in R^+\backslash R_J \text{ for all } \alpha \in I\}$$

When $J = \varnothing$, this is the subset $W^I$ in [J97] Section 11.12. In that case, $W^I$ has a description as minimal length coset representatives for $W/W_I$, where $W_I$ is the Weyl group generated by simple reflections for the roots in $I$ (see [H90] Section 1.10). It seems, though, that $W^{I,J}$ does not seem to have a similar characterization.

We are now ready to generalize the construction in the previous section and define the *shifted* parabolic baby Verma modules. For $\lambda \in \Lambda_{\chi,wJ}$ and $w \in W^{I,J}$, we now know that $k_\lambda$ defines a $U_\chi(w\mathfrak{p})$-module, since $\chi(w\mathfrak{p}) = 0$. Furthermore, $U_\chi(\mathfrak{g})$ is free over $U_\chi(w\mathfrak{p})$ of rank $\dim(U_\chi(w\mathfrak{u}^-)) = p^{\dim(\mathfrak{u}^-)} = p^{|R^+\backslash R_J|}$. So define:

$$Z^w_{\chi,J}(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(w\mathfrak{p})} k_\lambda$$

Once again, we will write $1 \otimes 1$ for the generator of this module. When $w = 1$, we have our usual parabolic baby Verma modules from the previous section, and will omit the superscript to simply write $Z_{\chi,J}(\lambda)$.

Given two elements of $W^{I,J}$ that differ by a simple reflection, we want to define a homomorphism from one to the other much as we did in Lemma 4.3.3. In this way we hope to obtain a parabolic analogue to the contents of [J97] Section 11.12 and [J00] Lemma 3.4.

**Conjecture 4.4.1.** *Let $\lambda \in \Lambda_{\chi,J}$, $w \in W^{I,J}$, and suppose $\alpha$ is a simple root such that $ws_\alpha \in W^{I,J}$. Then there exists a homomorphism*

$$\phi_\alpha : Z^w_{\chi,J}(\lambda) \to Z^{ws_\alpha}_{\chi,J}(\lambda - (p-1)w\alpha).$$

Here is where constructions that mirror those of Jantzen encounter difficulties, largely due to the necessity of keeping track of $\mathfrak{p}$ after it is shifted by elements of $W^{I,J}$.

First, note that if $\alpha \in J$ then $w\mathfrak{p} = ws_\alpha\mathfrak{p}$, so in this case the map $1 \otimes 1 \mapsto 1 \otimes 1$ gives an isomorphism. Thus, suppose that $\alpha \in \Delta \backslash J$. We may attempt to define the map $\phi_\alpha$ by

$$\phi_\alpha(1 \otimes 1) = x_{w\alpha}^{p-1} \otimes 1,$$

where $x_{w\alpha}$ is the generator for the root space $\mathfrak{g}_{w\alpha}$. In order for this to be a homomorphism of induced modules, we need to see that $x_{w\beta}(x_{w\alpha}^{p-1} \otimes 1) = 0$ for all $\beta \in R^+ \cup R_J^-$. It turns out that this is not always possible.

Consider, for example, $\mathfrak{g} = \mathfrak{sl}_6$ with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, and let $I = \{\alpha_1\}$ and $J = \{\alpha_3\}$. Suppose that we wish to define the map

$$\phi_{\alpha_2} : Z_{\chi,J}(\lambda) \to Z_{\chi,J}^{s_{\alpha_2}}(s_{\alpha_2} \bullet \lambda)$$

given by $\phi_{\alpha_2}(1 \otimes 1) = x_{\alpha_2}^{p-1} \otimes 1$. Now, for $x_{-\alpha_3} \in U_\chi(\mathfrak{p})$, we need to look at

$$x_{-\alpha_3}(x_{\alpha_2}^{p-1} \otimes 1) = x_{\alpha_2}^{p-1} x_{-\alpha_3} \otimes 1.$$

This is nonzero, though, since $x_{s_{\alpha_2}(-\alpha_3)} = x_{-(\alpha_2+\alpha_3)} \notin \mathfrak{p}$ implies $x_{-\alpha_3} \notin s_{\alpha_2}\mathfrak{p}$. Therefore, this cannot be a homomorphism.

One possible solution to this problem is to build more information about $J$ into the $\phi_\alpha$ maps; that is, to make the image of the generator involve root vectors for the shifted parabolic rather than simply using $x_\alpha^{p-1} \otimes 1$. The issue then becomes the computation necessary to verify that this is a homomorphism, which greatly increase in rank from the $\mathfrak{sl}_2$ calculations of the original approach of Jantzen with $J = \varnothing$.

As mentioned above, the eventual goal in creating these maps would be to compose them so as to build a homomorphism from $Z_{\chi,J}(\lambda)$ to a module which contains the unique simple quotient as its socle. In this way, through a bit of category theory, we hope to construct a filtration of $Z_{\chi,J}(\lambda)$ much as in [J00] Proposition 3.10. In the case of parabolic baby Verma modules, this filtration should be shorter, though at the same time contain the simple quotient as its first factor.

The "target" module in the parabolic case seems to be $Z_{\chi,J}^{w^{I,J}}(w^{I,J} \bullet \lambda)$ where $w^{I,J} = w_I w_0 w_J$ for $w_0, w_I$, and $w_J$ the longest words in $W_0, W_I$, and $W_J$, respectively. Another combinatorial challenge then arises. For each intermediate composition of maps

$$\phi_w : Z_{\chi,J}(\lambda) \to Z_{\chi,J}^w(w \bullet \lambda)$$

between $Z_{\chi,J}(\lambda)$ and $Z_{\chi,J}^{w^{I,J}}(w^{I,J}{}_\bullet\lambda)$, we must verify that $w \in W^{I,J}$. In the case of $J = \varnothing$, this is due to the shortest length coset representative description of $W^I$ and results concerning more general reflection groups. Unfortunately, with parabolic baby Verma modules it turns out that it is not always possible to ensure this. The question is then: given a subset of simple root $I$, what restrictions are there on a second non-adjacent subset $J$ such that there is a reduced decomposition $w_I w_0 w_J = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_n}$ with $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_i} \in W^{I,J}$ for all $1 \leqslant i \leqslant n$?

## 4.5 Categories of Modules

As mentioned in the previous section as well as section 1.2.2, the filtration of $Z_\chi(\lambda)$ in [J00] is obtained by considering more general categories of modules than those for our usual reduced enveloping algebra $U_\chi(\mathfrak{g})$. We will now investigate some properties of these categories. For the remainder of this chapter, the baby Verma modules will be induced over a Borel subalgebra (so that $J = \varnothing$).

Let $U = U(\mathfrak{g})/F_\chi$, where $F_\chi$ is the ideal

$$< x^p - x^{[p]} - \chi(x)^p \mid x = x_\alpha \text{ for } \alpha \in R > .$$

Note that this is similar to the $U_\chi(\mathfrak{g})$ that we have been working with, but without quotienting by elements $h^p - h^{[p]} - \chi(h)^p$ for $h \in \mathfrak{h}$. By the Poincaré–Birkhoff–Witt basis theorem, we have an isomorphism of vector spaces:

$$U_\chi(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U_\chi(\mathfrak{n}^+) \simeq U$$

Let $U^0$ be the image of $U(\mathfrak{h})$ in $U$. Now, let $A$ be any commutative $k$-algebra with identity, and consider $\pi : U^0 \to A$ a $k$-algebra homomorphism. There is a $\mathbb{Z}R/\mathbb{Z}I$-grading on $U$ with each $\mathfrak{g}_\alpha$ contained in degree $\alpha + \mathbb{Z}I$.

We will define a category $\mathcal{C}_A$ of certain $X/\mathbb{Z}I$-graded $U \otimes A$-modules as in [J00] section 3. These will be modules $M$ such that the following conditions hold:

1. $U_\mu \cdot M_\nu \subseteq M_{\mu+\nu}$ for all $\mu \in \mathbb{Z}R/\mathbb{Z}I$ and $\nu \in X/\mathbb{Z}I$.

2. $A \cdot M_\nu \subseteq M_\nu$ for all $\nu \in X/\mathbb{Z}I$.

3. $M$ is finitely generated over $A$.

4. The action of $\mathfrak{h}$ is diagonalizable on each $U^0$-submodule $M_\nu$, with $M_\nu = \bigotimes_\lambda (M_\nu)^\lambda$ over all $k$-linear maps $\lambda : \mathfrak{h} \to A$. Furthermore, we require $\lambda = \pi + d(\xi + \xi')$ for some $\xi' \in \mathbb{Z}I$ when $(M_{\xi+\mathbb{Z}I})^\lambda \neq 0$.

This is a generalization of the category $\mathcal{C}_A$ introduced in [AJS94] for the important case of $\chi = 0$ (that is, when $I = \varnothing$). For any $I \subseteq \Delta$, define

$$W^I = \{w \in W \mid w^{-1}(\alpha) > 0 \text{ for all } \alpha \in I\},$$

which is simply our $W^{I,J}$ from the previous section when $J = \varnothing$.

Define now, for all $\lambda \in X$ and $w \in W^I$, the modules

$$Z_A^w(\lambda) = U \oplus_{U^0 U_\chi(w\mathfrak{n}^+)} A_\lambda,$$

where $A_\lambda$ is the $(w\mathfrak{b}^+) \otimes A$-module $A$ such that $h \in \mathfrak{h}$ acts as multiplication by $\pi(h) + d\lambda(h)$ and $w\mathfrak{n}^+$ acts trivially. This induces an action of $U(w\mathfrak{b}^+)$ factoring through $U^0 U_\chi(w\mathfrak{n}^+)$. In [J00], Jantzen uses a filtration of these modules (with $w = 1$) to obtain a filtration of the usual "baby Verma" modules $Z_\chi(\lambda)$ in the category $\mathcal{C}_k$.

The filtration in question comes from a map seen in [J00] section 3.10:

$$\phi_A : Z_A(\lambda) \to Z_A^{w_I w_0}(\lambda - (p-1)(\rho - w_I w_0 \rho))$$

For the sake of brevity, let $w^I = w_I w_0$ and write

$$Z_A^{w^I}(\lambda^{w^I}) = Z_A^{w_I w_0}(\lambda - (p-1)(\rho - w_I w_0 \rho)).$$

As observed by Jantzen, when we tensor with the field of fractions of $A$, which we will denote by $F$, the map $\phi_F$ becomes an isomorphism. This should imply, as in Lemma 3.11 from *loc. cit.* with scalars extended to $F$, that each $Z_F(\lambda)$ is simple. To prove this, we will work more generally.

The following three results are valid for any discrete valuation ring $A$ with residue field $k = A/\langle t\rangle$ and field of fractions $F$. Considering $U$ any $A$-algebra which is free of finite rank as an $A$-module, we get a $k$-algebra $U_k$ and an $F$-algebra $U_F$ by extension of scalars.

**Lemma 4.5.1.** *Suppose $N$ is a $U_A$-module which is free of finite rank as an $A$-module, and consider a $U_F$-submodule $E \subseteq N \otimes_A F$. Then there exists a $U$-submodule $X \subseteq N$ such that $E = F.X$ and the $U$-module $M/X$ is a free $A$-module.*

*Proof.* Consider the module $X = E \cap N$. Then we have $E = F.X$, so what is left is to show that $N/X$ is free as an $A$-module. So, consider $(n+X) \in N/X$ and let $a \in A$. Then, if $a(n+X) = an + X = 0$, we have $an = x \in X$. Thus, $n = \frac{1}{a}x \in E$ implies $n \in E \cap N = X$, so we must have had $n + X = 0$. What we have just shown is that $N/X$ is torsion free as an $A$-module and therefore, since it is finitely generated and $A$ is a discreet valuation ring, we know that $N/X$ is a free. $\square$

**Lemma 4.5.2.** *Let $M$ and $N$ be $U$-modules which are free of finite rank as $A$-modules. Then the natural map*

$$\mathrm{Hom}_U(M,N)/(t.\,\mathrm{Hom}_U(M,N)) \to \mathrm{Hom}_{U_k}(M_k,N_k)$$

*is injective, where $M_k$ and $N_k$ are the $U_k$-modules $M/tM$ and $N/tN$, respectively.*

*Proof.* Consider the short exact sequence of $U$-modules

$$0 \to N \xrightarrow{\times t} N \xrightarrow{\pi} N/tN = N_k \to 0.$$

Then we can apply the left exact functor $\mathrm{Hom}_U(M,-)$ to get the exact sequence

$$0 \to \mathrm{Hom}_U(M,N) \xrightarrow{\times t} \mathrm{Hom}_U(M,N) \xrightarrow{\pi'} \mathrm{Hom}_U(M,N_k) \simeq \mathrm{Hom}_{U_k}(M_k,N_k).$$

Since $t.\,\mathrm{Hom}_U(M,N) = \ker(\pi')$, we have injectivity of the natural map above. $\square$

**Proposition 4.5.3.** *Let $\phi : M \to N$ be a $U$-module homomorphism, and denote the maps induced by extension of scalars $\phi_k : M_k \to N_k$ and $\phi_F : M_F \to N_F$. Suppose*

*(i) $M_k$ has a unique maximal submodule and $N_k$ has a simple socle $L$;*

*(ii) $\phi_k(M_k) = L$;*

*(iii) $[N_k : L] = 1$;*

*(iv) $\phi_F$ is an isomorphism.*

*Then $N_F$ is a simple $U_F$-module.*

*Proof.* Consider a simple $U_F$-submodule $E \subseteq N_F$. Then by Lemma 4.5.1, there exists a $U$-submodule $X \subseteq N$ with $E = F.X$ and $N/X$ a free $A$-module of finite rank. Given the short exact sequence

$$0 \to X \to N \xrightarrow{\psi} N/X \to 0,$$

we can once again apply the left exact functor $\text{Hom}_U(M, -)$ to get

$$0 \to \text{Hom}_U(M, X) \to \text{Hom}_U(M, N) \xrightarrow{\psi'} \text{Hom}_U(M, N/X),$$

noting that we have $\phi \in \text{Hom}_U(M, N)$. Now, the simple socle $L$ appears as a composition factor for $N_k$ with multiplicity one and $X_k$ contains $L$ since $X_k \subseteq N_k$ is a $U_k$-submodule, hence $L$ is not a composition factor of $N_k/X_k$. Therefore, we must have $\text{Hom}_{U_k}(M_k, (N/X)_k)) = 0$ since $M_k$ certainly contains $L$. Now, Lemma 4.5.2 and Nakayama's Lemma give us that $Hom_U(M, N/X) = 0$. Thus $\phi \in \ker(\psi')$, and so $\phi$ is in the image of $\text{Hom}_U(M, X)$. This means that $\phi(M) \subseteq X \subseteq N$, and hence $\phi_F(M_F) \subseteq E$. Now, since $\phi_F$ is an isomorphism and $E$ was chosen to be simple, we must have $N_F = \phi_F(M_F) = E$. Therefore, $N_F$ is simple.

□

Now, we are able to prove our previously expected result.

**Theorem 4.5.4.** *For $\lambda \in X$, the module $Z_F(\lambda)$ is a simple $U_F$-module.*

*Proof.* By [J00] section 3.10, we have a nonzero homomorphism of $U$-modules

$$\phi : Z_A(\lambda) \to Z_A^{w^I}(\lambda^{w^I})$$

such that extension of scalars gives an isomorphism $\phi_F$ of $U_F$-modules. Over $k$, we know that $Z_\chi(\lambda)$ has a unique maximal submodule as usual; $Z_\chi^{w^I}(\lambda^{w^I})$ has a simple socle $L_\chi(\lambda)$ by [J97] Lemma 11.13; and the image of $\phi_k$ is equal to $L_\chi(\lambda)$ by [J00] Lemma 3.11.

By [J00] 2.8(4), we have $[Z_\chi(\lambda) : L_\chi(\lambda)] = 1$. Now, looking at [J97] section 11.16, we see that $Z_\chi^{w^I}(\lambda^{w^I}) \simeq {}^\tau(Z_\chi(\lambda)^*)$ for $\tau$ an automorphism of $\mathfrak{g}$. Thus, the equivalences in section 11.17 of *loc. cit.* yield:

$$1 = [Z_\chi(\lambda) : L_\chi(\lambda)] = [(Z_\chi(\lambda)^*)^* : L_\chi(\lambda)] = [{}^\tau(Z_\chi(\lambda)^*) : L_\chi(\lambda)] = [Z_\chi^{w^I}(\lambda^{w^I}) : L_\chi(\lambda)]$$

Thus, we may apply Proposition 4.5.3 to conclude $Z_F^{w^I}(\lambda^{w^I})$ is simple. Finally, since $\phi_F$ is an isomorphism, this gives $Z_F(\lambda)$ as a simple $U_F$-module.

□

*Remark* 4.5.5. In fact, the conclusion that $[Z_\chi^{w^I}(\lambda^{w^I}) : L_\chi(\lambda)] = 1$ is true due to general arguments in more abstract situations. See, for example, [Se77] section 15.2 Theorem 32 or [CR81] Proposition 16.16.

In the case of $\chi = 0$, this result is observed in [AJS94] (see the remark in section 6.3). There they go on to observe that, in fact, $Z_F(\lambda) = Q_F(\lambda)$ where $Q_F(\lambda)$ is the projective cover, so that $\mathcal{C}_F$ is a semisimple category. We conjecture that this is *not* the case when $I \neq \varnothing$ for $\chi$ in standard Levi form.

**Conjecture 4.5.6.** *For $\lambda \in X$, let $Q_F(\lambda)$ be the projective cover of $Z_F(\lambda)$ in the category $\mathcal{C}_F$. Then $Q_F(\lambda)$ has a filtration of length $|W_{I\bullet}\lambda|$ where all of the composition factors are isomorphic to $Z_F(\lambda)$.*

If this is true, then our category $\mathcal{C}_F$ is not semisimple in general (when $\chi \neq 0$). In fact, this should be observable directly in the specific case of $\chi$ regular. The intuition for Conjecture 4.5.6 comes from viewing results such as [J00] Proposition 2.9 over $F$, where the modules $Z_F(\lambda)$ are simple as proven in Theorem 4.5.4. In general, though, it takes a bit of care to work with the projectives rigorously.

# Bibliography

[AJS94] H.H. Andersen, J.C. Jantzen, and W. Soergel, Representations of quantum groups at a $p$-th root of unity and of semisimple groups in characteristic $p$: independence of $p$, Astérisque 220 (1994).

[BMR08] R. Bezrukavnikov, I. Mirković, and D.A. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, Ann. of Math. (2) 167 (2008), 945–991.

[BT65] A. Borel, J. Tits, Groupes réductifs, Publ. Math. Inst. Hautes Études Sci. 27 (1965), 55–151.

[Ca85] R.W. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters, Wiley-Interscience, New York, NY, 1985.

[Cu53] C.W. Curtis, Noncommutative extensions of Hilbert rings, Proc. Amer. Math. Soc. 4 (1953), 945–955.

[Cu60] C.W. Curtis, Representations of Lie algebras of classical type with applications to linear groups, J. Math. Mech. 9 (1960), 307–326.

[CR81] C.W. Curtis and I. Reiner, Methods in Representation Theory – with Applications to Finite Groups and Orders, vol. 1, Wiley-Interscience, New York, NY, 1981.

[FP88] E.M. Friedlander and B.J. Parshall, Modular representation theory of Lie algebras, Amer. J. Math. 110 (1988), 1055–1094.

[FP90] E.M. Friedlander and B.J. Parshall, Deformations of Lie algebra representations, Amer. J. Math. 112 (1990), 375–395.

[FP91] E.M. Friedlander and B.J. Parshall, Induction, deformation, and specialization of Lie algebra representations, Math. Ann. 290 (1991), 473–489.

[G77] M. Gauger, Conjugacy in a semisimple Lie algebra is determined by similarity under fundamental representations, J. Algebra 48 (1977), 382–389.

[He79] W.H. Hesselink, Nilpotency in classical groups over a field of characteristic 2, Math. Z. 166 (1979), 165–181.

[H67] J.E. Humphreys, Existence of Levi factors in certain algebraic groups, Pacific J. Math. 23 (1967), 543-546.

[H75] J.E. Humphreys, Linear Algebraic Groups, Grad. Texts in Math., vol. 21, Springer-Verlag, New York, NY, 1975.

[H80] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, 3rd ed., Grad. Texts in Math., vol. 9, Springer-Verlag, New York, NY, 1980.

[H90] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge Univ. Press, Cambridge, 1990.

[H91] J.E. Humphreys, Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$, Grad. Studies in Math., vol. 94, Amer. Math. Soc., Providence, RI, 2008.

[H95] J.E. Humphreys, Conjugacy Classes in Semisimple Algebraic Groups, Math. Surveys and Mono., vol. 43, Amer. Math. Soc., Providence, RI, 1995.

[H98] J.E. Humphreys, Modular representations of simple Lie algebras, Bull. Amer. Math. Soc. (N.S.) 35 (1998) 105–122.

[H05] J.E. Humphreys, Modular Representations of Finite Groups of Lie Type, London Math. Soc. Lect. Note Series, vol. 326, Camb. Univ. Press, Cambridge, UK, 2005.

[J97] J.C. Jantzen, Representations of Lie algebras in prime characteristic, in: A. Broer (Ed.), Proceedings of the NATO ASI, Representations Theories and Algebraic Geometry, Montréal, 1997, pp. 185–235.

[J99] J.C. Jantzen, Subregular nilpotent representations of Lie algebras in prime characteristic, Rep. Theory 3 (1999), 153–222.

[J00] J.C. Jantzen, Modular representations of reductive Lie algebras, J. of Pure and Appl. Alg. 152 (2000), 133–185.

[J03] J.C. Jantzen, Representations of Algebraic Groups, 2nd ed., Math. Surveys and Monographs, vol. 107, Amer. Math. Soc., Providence, RI, 2003.

[J04] J.C. Jantzen, Nilpotent orbits in representation theory, in: Lie Theory: Lie Algebras and Representation Theory, Progress in Mathematics, vol. 228, Birkhäuser, Boston, 2004.

[KW71] V.G. Kac and B.Yu. Weisfeiler, Irreducible representations of Lie $p$-algebras, Functional Anal. Appl. 5 (1971), 111–117.

[La95] R. Lawther, Jordan block sizes of unipotent elements in exceptional algebraic groups, Comm. Algebra 23(11) (1995), 4125–4156.

[LS12] M.W. Liebeck and G.M. Seitz, Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras, Math. Surveys and Monographs, vol. 180, Amer. Math. Soc., Providence, RI, 2012.

[Lu97] G. Lusztig, Periodic $W$-graphs, Rep. Theory 1 (1997), 207-279.

[M02a] G.J. McNinch, Abelian unipotent subgroups of reductive groups, J. Pure Appl. Algebra 167 (2002), 269–300.

[M02b] G.J. McNinch, Adjoint Jordan blocks, arXiv:math/0207001.

[M03] G.J. McNinch, Sub-principal homomorphisms in good characteristic, Math. Z. 244 (2003), 433–455.

[M05] G.J. McNinch, Optimal $SL(2)$ Homomorphisms, Comment. Math. Helv., 80 (2005), 391–426.

[M10] G.J. McNinch, Levi decompositions of a linear algebraic group, Trans. Groups 15 (2010), 937–964.

[P95a] A.A. Premet, An analogue of the Jacobson-Morozov theorem for Lie algebras of reductive groups of good characteristic, Trans. of the Amer. Math. Soc. 347(8) (1995), 2961–2988.

[P95b] A.A. Premet, Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture, Invent. Math. 121 (1995), 79-117.

[R70] A.N. Rudakov, On representations of classical semisimple Lie algebras of characteristic $p$, Mat. USSR-Izv. 4 (1970), 741–749.

[Sz00] G.M. Seitz, Unipotent elements, tilting modules, and saturation, Invent. Math. 141 (2000), 467–502.

[Se77] J.P. Serre, Linear Representations of Finite Groups, trans. by L.L. Scott, Grad. Texts in Math., vol. 42, Springer-Verlag, New York, NY, 1977.

[Sp98] T.A. Springer, Linear Algebraic Groups, 2nd ed., Progress in Math., vol. 9, Birkhäuser, Boston, MA, 1998.

[SS70] T.A. Springer and R. Steinberg, Conjugacy classes, in: Seminar on Algebraic Groups and Related Finite Groups, Lect. Notes in Math., vol. 131, Springer, Berlin, 1970.

[S65] R. Steinberg, Regular elements of semisimple algebraic groups, Publ. Math. I.H.E.S 25 (1965), 49–80.

[S66] R. Steinberg, Classes of elements in semisimple algebraic groups, in: Proceedings, I.C.M., Moscow, 1966.

[S78] R. Steinberg, Conjugacy in semisimple algebraic groups, J. Algebra 55 (1978), 348–350.

[Z40] H. Zassenhaus, Darstellungstheorie nilpotent Lie-Ringe bei Charakteristik $p > 0$, J. reine angew. Math. 182 (1940), 150–155.

[Z54] H. Zassenhaus, The representations of Lie algebras of prime characteristic, Proc. Glasgow Math. Assoc. 21 (1954), 1–36.