VISIBLE ARTIN SUBGROUPS OF RIGHT-ANGLED COXETER GROUPS

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Abstract

We consider an extension of work done by Michael Davis and Tadeusz Januszkiewicz in 1999. Given a right-angled Artin group A, they showed how to construct a right-angled Coxeter group W in which A sits as a subgroup of finite index. We partially classify the RACGs W which admit such an embedding, obtaining complete results when W is hyperbolic, virtually free, or 1-ended. We introduce the notion of visible Artin subgroups, and show how to detect them using the combinatorics of the RACG's defining graph and check index using natural CAT(0)-cube complexes associated to the groups. Finally, we discuss open questions and extension to the case where W is infinite-ended.

To my family, friends, and Jennie, for accepting my constant and unrestrained absurdity.

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Chapter 1

Introduction

To any finite simple graph Γ, one can associate two strikingly different groups with very similar constructions. The first, called the right-angled Coxeter group on Γ, is CAT(0), generated by involutions, and consists only of infinite order elements and 2-torsion. The second, the right-angled Artin group on Γ, also CAT(0), is torsion-free, and interpolates, in a sense, between free groups and free Abelian groups. The two classes of groups belong to the much larger class of graph products of cyclic groups, which is well-studied. Despite the similarities between their definitions, the two classes of groups lend themselves naturally to wildly different applications, both serving as useful, concrete examples for testing new mathematics.

Of course, the groups are themselves objects of considerable interest, and much attention has been paid to the properties that they do share. Davis and Januszkiewicz [14] show that given any right-angled Artin group, there exists a right-angled Coxeter group which contains it as a subgroup of finite index. They provide one of several constructions to this effect, proving as a consequence that every right-angled Artin groups is linear. While it is easy to show that some right-angled Coxeter groups do not admit a finite-index Artin subgroup - and hence that a full converse to this construction is unobtainable - it remains unknown which Coxeter groups do. Recent work by Behrstock, Hagen, and Sisto [4] on thickness and relative hyperbolicity in Coxeter groups provides tremendous insight into this question.

In chapters 2-4, we introduce the central problem of the document - finding circumstances under which right-angled Coxeter groups admit finite-index right-angled Artin subgroups - and present relevant background. In chapter 5, we recall recent machinery developed by Dani and Thomas [12] and Behrstock, Drutu, Hagen, Mosher, and Sisto [3] [4] which is central to the subsequent chapters. We also prove some minor results about cases of the problem rendered nearly trivial by the recent work of the aforementioned authors, proving the following directly:

Theorem 1.0.1 (L.) Let W be a right-angled Coxeter group. If W contains a right-angled Artin subgroup A of finite index, then W is thick.

In chapter 6, we completely answer the question for hyperbolic and virtually-free right-angled Coxeter groups; virtually-free groups are clearly virtually Artin, and it is trivial to show that:

Theorem 1.0.2 (L.) If W is hyperbolic, then W contains a finite-index RAAG if and only if W is also virtually free.

In chapter 7, we present a final critical invariant which, combined of previously mentioned theorems, allows us to narrow the range of groups under consideration to a more manageable subset.

In chapter 8, we introduce the concept of visible subgroups of a graph product of groups, which are generated by non-adjacent Coxeter generator pairs. We define two graph-theoretic conditions on subsets of the opposite graph of Γ called the *star-cycle* and *chain-chord* conditions. We show that:

Theorem 1.0.3 (L.) The visible subgroup A_S of $W(\Gamma)$ defined by a set of nonadjacencies S in Γ is a right-angled Artin group if and only if S satisfies the star-cycle and chain-chord conditions.

We prove several necessary lemmas that arise from these conditions and definitions. We provide an algorithm for exhaustively decomposing the right type of right-angled Coxeter group into a partially-ordered set $\mathfrak{C}$ of subgroups, and show that they inherit all of the properties that we need them to. We present an incorrect solution to the problem, and explain the precise cause of its failure. This failure allows us to answer in the negative an open question related to this topic, and show that the class of Coxeter groups containing finite-index visible Artin subgroups is quite substantially smaller than previously thought:

Theorem 1.0.4 (L.) There exists a finite, simple, CFS graph Γ with no induced cycles of length $n \geq 5$ such that the right-angled Coxeter group $W(\Gamma)$ contains no finite-index visible right-angled Artin subgroup.

In chapter 9, we introduce the idea of a link-satisfied graph, prove that this property is inherited by induced subgraphs, and show that it resolves the issue of the previous chapter. We also show that it is a stronger version of another useful property for right-angled Coxeter groups - that of being long cycle-free.

We then prove the main theorem of the document:

Theorem 1.0.5 (L.) Let $\mathfrak{A}$ be the set of finite, simple, CFS, link-satisfied graphs. If $\Gamma \in \mathfrak{A}$, then the 1-ended right-angled Coxeter group $W(\Gamma)$ contains a visible right-angled Artin subgroup of finite index.

In chapter 10, we present open questions introduced by these results, and directions for future research.

Chapter 2

Coxeter groups and the Davis Complex

In this section, we recall various relevant results about right-angled Coxeter groups (RACGs), associate to each a CAT(0) cube complex, and describe a geometric action of the group on the complex. Most of these results are widely known and can be found in a number of publications, so we will generally favor crediting authors for useful formulations of the theorems over identifying provenance. We will wantonly abuse notation by omitting the phrase 'right-angled,' and will discuss only right-angled Coxeter groups except when specifically noted.

2.1 Coxeter groups and relevant properties

Definition 2.1.1 Let Γ be a finite simple graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. We define the *right-angled Coxeter group* $W = W(\Gamma)$ *on* Γ to be the group generated by the elements of V, with these generators given order 2 and with $v_i v_j = v_j v_i$ whenever $\{v_i, v_j\} \in E$. That is,

$$W(\Gamma) =< v_i \mid v_i^2; [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \in E > .$$

We call Γ the *defining graph* of $W(\Gamma)$. Note that W is finite exactly when Γ is a complete graph and is a free product of cyclic groups exactly when $E(\Gamma)$ is empty. An *induced subgraph of* Γ consists of some subset of v and all edges between these vertices; we call a Coxeter group on an induced subgraph of Γ a *special subgroup of* W. If H is a special subgroup of W and $K = gHg^{-1}$ for some $g \in W$, then we call K a *parabolic subgroup of* W. If the induced subgraph $\Delta \subset \Gamma$ is a clique, then we call $W(\Delta)$ a *spherical subgroup*, with *spherical generating set* $V(\Delta)$.

If Γ is a graph whose largest clique contains n vertices, then we say that the group W is n-dimensional. Notice that if Γ contains an n-clique, it contains an

m-clique for all $m \leq n$. Thus, requiring that a graph Γ be *triangle-free* is exactly the same as demanding that the corresponding right-angled Coxeter group $W(\Gamma)$ be 2-dimensional. This paper will rely heavily on triangle-free graphs for examples and intuition.

In the following proposition, we record some useful, well-known facts about defining graphs of right-angled Coxeter groups.

Proposition 2.1.2 Let Γ be a finite simple graph and let W be the right-angled Coxeter group associated to W. Then:

1. W is hyperbolic if and only if Γ contains no induced cycle of length 4,

2. W is virtually free if and only if Γ decomposes as $\bigcup \Gamma_i$, where each Γ_i is a clique and $\Gamma_i \cap \Gamma_j$ is a clique for all i, j, and

3. W is 1-ended if and only if no complete subgraph Δ of Γ separates Γ; that is, $\Gamma \setminus \Delta$ consists of two or more components.

We note that one direction of (1) is obvious, as an induced 4-cycle in Γ corresponds to a $D_\infty \times D_\infty$ subgroup of W (ie, a flat); the other direction is due to Moussong [23]. Condition (2) essentially requires that Γ have the large-scale structure of a tree, with cliques functioning as vertices and intersections as edges. Condition (3) is more subtle; it amounts to preventing W from decomposing as an amalgamated product. In the case that W is 2-dimensional, hence triangle-free, this means that Γ has no separating vertices and edges. The proofs of (2) and (3) are due to Mihalik and Tschantz.

The combinatorics of word reduction and identification in right-angled Coxeter groups is greatly enabled by the well-known *deletion and exchange conditions*, proved separately by Bourbaki, Davis, et al. We cite a useful formulation below [9]:

Lemma 2.1.3 *Deletion condition for right-angled Coxeter groups:* Let $W = W(\Gamma)$ be a right-angled Coxeter group and suppose that $w = v_1 v_2 ... v_p$ is not reduced. Then there exists $1 \leq i < j \leq p$ such that $v_i = v_j$ and the vertex v_i is adjacent in Γ to each of the vertices $v_{i+1}, ..., v_{j-1}$.

Lemma 2.1.4 *Exchange condition for right-angled Coxeter groups:* Let W be as above. If two words w and w' represent the same element of W, then w can be transformed into w' by a finite sequence of transpositions of adjacent, commuting letters.

We will be using these conditions heavily in chapters 8 and 9 to prove facts about certain subgroups of right-angled Coxeter groups. It should be noted that they use an intuitive definition of a *reduced word* in a right-angled Coxeter group. A minimal length word $w \in W$ may have many representations given by commuting adjacent vertices; such a word will decompose into a product of sets of commuting letters (belonging to cliques of Γ) $w = w_1 w_2 ... w_k$. If we fix an initial ordering of the vertex set of Γ, we can choose a preferential representation of any such reduced word, in which each w_i has generators arranged by ascending index.

One last critically important property is relative hyperbolicity. It is known how to detect relative hyperbolicity in right-angled Coxeter groups, and even how to detect it graph-theoretically. Caprace first showed how to check W for relative hyperbolicity [5], but his method is rather complicated. We will rely on an alternate formulation developed by Behrstock, Hagen, and Sisto [4], which characterizes the set of Γ such that $W(\Gamma)$ is *not* hyperbolic relative to any collection of subgroups. We will present this in great detail in chapter 4.

2.2 CAT(0) cube complexes

In this section, we provide a brief overview of CAT(0) cube complexes. For an arbitrary right-angled Coxeter group W, one can construct a CAT(0) cube complex X on which W acts nicely. This construction is due to Davis [13], and is now referred to as the *Davis complex* for W. We'll review the construction in the case where W is right-angled; W can be shown to act geometrically on X, and is thus CAT(0) and has many desirable properties.

Definition 2.2.1 Let $I = \{1, ..., n\}$. We call the product $[-1, 1]^n \subset \mathbb{E}^n$ the *standard n-cube*. An n-cube is then some translate of this cube. A *face* of the cube is isometric to a translate of $[-1,1]^m$ for some $m \leq n$. A *vertex* of the cube is an element of the set $\{\pm 1\}^n$. Let X be a union of cubes of possibly different dimensions. If for every pair of cubes $C_i, C_j \subset X$, we have that $C_i \cap C_j = \varnothing$ or $C_i \cap C_j$ is a face of at least one of C_i, C_j, then we say that X is a *cube complex.*

To avoid pathologies, we require that no cube in X be glued to itself along a face, and that no two distinct cubes are glued along more than one (shared) face. These restrictions allow allow for cubulations of products of S^1; for example, S^1 can be viewed as three 1-cubes glued cyclically.

We say that the *dimension* of X is equal to the dimension of its largest cube - that is, the largest n such that $[-1,1]^n \subset X$. We define the *n-skeleton* $X^{(n)}$ of a cube complex X in the usual way, and call the points in $X^{(0)}$ the *vertices* of X. If X has the property that, for any vertex v, at most finitely many edges (1-cubes) in X contain v as an endpoint, then we say that X is *locally finite.* We will only be concerned with finite-dimensional, locally finite cube complexes, and will thus omit these modifiers.

We impose on X the standard metric for cube complexes, where $d_X(x,y)$ is the infimum across all paths γ in X connecting X and y, with $length(\gamma)$ the sum of the Euclidean lengths of its restrictions to the cubes of X which it intersects. With this metric, provided that X is finite-dimensional or locally finite, X is a complete geodesic metric space.

Definition 2.2.2 Let X be a cube complex as above, and let v be a vertex of X. The *link of* v *in* X, denoted $link(v,X)$ or simply $link(v)$, is the intersection with X of an ϵ-sphere centered at v. We say that $link(v)$ is *flag* if whenever $link(v)$ contains the 1-skeleton of an n-simplex, it contains the entire n-simplex. If X is such that $link(v)$ is flag for all $v \in X^{(0)}$, then we say that X is a *flag complex.*

When v is the corner of a 2-cube, $link(v)$ is a 1-cube, and could be visualized as the diagonal of the original cube. When v is a vertex of a 3-cube, $link(v)$ is a filled in triangle. In general, $link(v)$ is a cell-complex with a $(k-1)$-cell for each k-cube in X abutting v.

Gromov introduced a particularly useful tool for determining whether a cube complex X is CAT(0):

Theorem 2.2.3 *Gromov's link condition:* A cube complex X is locally CAT(0) if and only if X is a flag complex.

In particular, X is CAT(0) if and only if it satisfies Gromov's link condition and is simply connected. This formulation, which has been proved by various authors, allows us to determine whether or not X is CAT(0) using the combinatorics of the cube gluings, rather than the curvature of the metric.

Cube complexes have arisen as objects of particular interest in recent decades, as useful spaces for groups to act on. Many spaces, like $\mathbb{E}^n$ or any graph, can be cubulated in an obvious way. For most groups G, it's unclear a priori if there's a natural cube complex that admits a useful G-action. In the remainder of this chapter, we'll discuss how to construct the Davis complex X for a RACG W to act on.

2.3 Building the Davis complex

Given a right-angled Coxeter group W , one can construct the associated Davis complex from the defining graph Γ. Consider the poset of cliques of Γ, ordered by inclusion, and draw a Hasse diagram Δ of the poset (a directed graph with maximal cliques at the "top," terminating at a vertex for the empty clique). The combinatorics of the cliques guarantees that Δ has the structure of the 1-skeleton of a finite cube complex; the proof of this fact is a bit far afield from the purpose of this document.

Each vertex of Δ is now labeled with a set of vertices of Γ (the members of the clique to which the vertex corresponds). We obtain a copy of Δ for each g in the Coxeter group W, and can quotient by the set of maps identifying paired (identically labeled) faces of the complex. The resulting structure, denoted $\Sigma(W)$, is known as the *Davis complex* of W. The Coxeter group W acts on Σ by left multiplication on these vertex sets (eg, $a * \{a, b, c\} = \{1, ab, ac\}$) with fundamental domain Δ. Geometrically, the involutions that generate W act by reflection across labeled reflection walls in Δ. In the case that W is a finite group and Γ is itself an n-clique, Δ is an n-cube and $\Sigma(W)$ is a gluing of 2^n copies of that cube. If W is infinite, any pair of non-commuting generators of W generates a virtually $\mathbb{Z}$ subgroup of W which carves out a translation axis in Σ_W; thus $\Sigma(W)$ is also infinite. Much attention will be paid to these reflections and translations in chapter 9. Davis showed the following:

Theorem 2.3.1 The cube complex Σ, with its standard metric, is CAT(0). The action of W on Σ is geometric, with fundamental domain Δ.

Thus W is CAT(0), and has many desirable qualities. We'll revisit this complex and its utility repeatedly in later chapters. We add as a final note that one can less rigorously and more efficiently see the Davis complex for $W(\Gamma)$ by 'cubifying' the Cayley graph $Cay(\Gamma)$ - that is, adding $(n+1)$-cubes wherever the 1-skeleton of an n-simplex appears; a simplex is then the link of a corner of a cube. This more intuitive process requires some additional familiarity with the group and complex.

Chapter 3

Artin groups and embeddings into RACGs

In this chapter, we review definitions and properties of right-angled Artin groups, a family of groups analogous in many ways to right-angled Coxeter groups. We recall the standard complex associated to the groups, and present the motivating result for this document, a canonical embedding map from an arbitrary RAAG into a RACG and the associated map on complexes. Finally, we discuss some recent results identifying surface subgroups of graph products of groups.

3.1 Artin groups, the Salvetti complex, and special cube complexes

We begin by collecting some well-known results about Artin groups and associated cube complexes. While these results have been widely published, we'll follow the conventions for notation set by Charney [6]. Let Γ be a finite simple graph as before with vertex set $V(\Gamma) = \{v_1, ..., v_n\}$ and edge set $E = E(\Gamma)$. Define the *right-angled Artin group* $A = A(\Gamma)$ associated to Γ by $A = \langle v_1, ..., v_n | [v_i, v_j] \iff \{i, j\} \in E(\Gamma) \rangle$. Note that this definition differs from that of a right-angled Coxeter group only in that the generators are infinite order; the group is thus torsion free, and is finite only when Γ is the empty graph and A is the trivial group.

In the previous chapter, we introduced the deletion and exchange conditions for right-angled Coxeter groups. A classical result of Servatius [25] serves as a useful analogue to these combinatorial tools; we note that, as before, this is only applicable to *right-angled* Artin groups:

Theorem 3.1.1 Let $A = A(\Gamma)$ be a right-angled Artin group. Then any element of A can be represented by a reduced word which does not contain a subword of the form $a_i^p x a_i^{-p}$ where all letters of the word x commute with the generator a_i. Also, any two reduced representatives of the same word w are related by a finite number of commutation relations; no insertions or deletions of trivial pairs of generators are needed.

To an arbitrary RAAG A is associated its *Salvetti complex* $S = S_A$, a cube complex with 1-skeleton a wedge of n circles with labels corresponding to the generators of A. For each 2-clique (edge) $\{v_i, v_j\}$ in Γ, glue in a 2-torus with boundary $v_i v_j v_i^{-1} v_j^{-1}$. For each 3-clique Δ in Γ, glue in a 3-torus with faces the tori corresponding to the edges of Δ. For each k, glue a k-torus in this way for every k-clique in Γ. By construction, $\Pi_1(S_A) = A$.

Wise introduced the notion of a *special cube complex.* We'll briefly introduce the relevant definitions and important consequences; a strictly better treatment of the material can be found in a paper by Haglund and Wise [16].

Definition 3.1.2 Let $C = [-1, 1]^n$ be an n-cube as in Definition 2.2.1. A *midcube* M in C is the subset obtained by setting one of the coordinates equal to 0. We say that an edge $e \in C^{(1)}$ is *dual* to M if it is perpendicular to it.

Definition 3.1.3 Let X be a nonpositively curved cube complex. Define the relation $\sim$ on the edges of X by $e \sim f$ if and only if e and f are opposite edges of some cube C in X. It is well-known that $\sim$ is an equivalence relation. For the class $[e]$, define the *hyperplane* of X dual to $[e]$ to be the collection of midcubes of cubes in X which intersect representatives of $[e]$.

A special cube complex is characterized by its hyperplanes satisfying several restrictive conditions, which we record below:

Definition 3.1.4 Let X be a nonpositively curved cube complex. A midcube of X extends to a unique immersed *hyperplane* of X. The cube complex X is called *special* if the following conditions are satisfied by the hyperplanes of X:

1. Each immersed hyperplane embeds in X.

2. Each hyperplane is 2-sided (consistent direction of dual 1-cubes.)

3. No hyperplane self-osculates (is dual to two distinct 1-cubes with the same initial or terminal vertex.)

4. No two hyperplanes interosculate (cross, and also have dual 1-cubes sharing a 0-cube but not belonging to a common 2-cube).

These rather severe restrictions on the behavior of the hyperplanes of X do more than avoid pathologies - they guarantee the following:

Theorem 3.1.5 The Salvetti complex S_A for any RAAG A is special. Further, a nonpositively curved cube complex X is special if and only if there is a local isometry $X \to S$ for some Salvetti complex S.

Thus, for any special cube complex X, $\pi_1(X)$ embeds into a right-angled Artin group. The theory of special cube complex is immensely helpful in understanding the behavior of these complexes.

3.2 The Davis-Januszkiewicz embedding

Davis and Januszkiewicz [14] show that right-angled Artin groups are commensurable with right-angled Coxeter groups by exhibiting, for an arbitrary RAAG $A(\Gamma)$, an embedding $A(\Gamma) \to W(\Gamma')$ into a right-angled Coxeter group $W(\Gamma')$. Given the Artin group's defining graph Γ, they construct the defining graph for Γ' for W as follows:

Suppose that Γ has n vertices $\{v_1, ..., v_n\}$, and let Δ be an n-clique with vertices labeled $\{v_{1'}, ..., v_{n'}\}$. Define $V(\Gamma') = V(\Gamma) \cup V(\Delta)$ and $E(\Gamma') = E(\Gamma) \cup E(\Delta) \cup \{\{v_i, v_{j'}\} \mid i \neq j\}$. That is, Γ' contains a copy of the original graph Γ (its "upper level"), a copy of the clique Δ (its "lower level"), and almost all edges connecting vertices in the upper level to those in the lower level.

The n omitted edges $\{v_i, v_{i'}\}$ for $i = 1, ..., n$ yield n infinite-order elements $v_i v_{i'}$ which generate an Artin subgroup A' of the Coxeter group $W(\Gamma')$ isomorphic to the original group A. In this document, we'll refer to such generators as *visible generators* and the subgroup as a *visible subgroup* of $W(\Gamma')$.

Using the cube complexes associated to the groups, Davis and Januszkiewicz show the following:

Theorem 3.2.1 The subgroup $A' \leq W'$ is a right-angled Artin group, and is a subgroup of index 2^n in W'.

Thus, for each RAAG A, we have a RACG W in which to embed A. It should be noted that not all virtually-RAAG W have this structure, even after replacing W with a suitable substitute to remove graph-theoretic pathologies like central torsion. Rather, this is a very narrow subset of the Coxeter groups which admit finite index RAAG subgroups.

3.3 Surface subgroups of Artin and Coxeter groups

Given a RACG $W(\Gamma)$, it is easy to see that an immersed (chord-free) n-cycle C_n for $n \geq 5$ in Γ corresponds to a hyperbolic surface subgroup H embedded in W. A priori, it is more difficult to graphically detect surface subgroups of RAAGs. As we will see later, such a subgroup in W presents an obstruction to the existence of a large Artin subgroup that is very difficult to overcome. It is therefore of interest to collect recent results on detecting surface subgroups of right-angled Coxeter and Artin subgroups.

In a series of papers, Kim [18] [19] investigates this question in the more general setting of graph products of arbitrary groups (in particular, of cyclic or finite groups). He defines the class $\mathfrak{S}$ of groups that contain hyperbolic surface subgroups, and proves the following remarkable result:

Theorem 3.3.1 Let $\mathfrak{S}$ be as above, and let $W(\Gamma)$ and $A(\Gamma)$ be the right-angled Coxeter and Artin groups on Γ. Then

1. $W(\Gamma) \in \mathfrak{S}$ if and only if the graph product of arbitrary nontrivial groups over Γ is in $\mathfrak{S}$

2. $A(\Gamma) \in \mathfrak{S}$ if and only if the graph product of some cyclic groups over Γ is in $\mathfrak{S}$

In particular, we see that $W(\Gamma) \in \mathfrak{S} \iff A(\Gamma) \in \mathfrak{S}$. In a sense, this result shows that many surface subgroups exist as a consequence of the graph product structure on W (or A), rather than the algebra of the vertex groups. A reasonable question to ask is whether every surface subgroup of $W(\Gamma)$ arises in this way; are there nonobvious surface subgroups of some right-angled Coxeter groups?

To this end, Kim introduces an operation on finite graphs called *co-contraction* and shows that if a graph Γ co-contracts to Γ', the right-angled Artin group $A(\Gamma)$ contains $A(\Gamma')$ as a subgroup. This operation allows the ready identification of a large family of right-angled Artin groups with hyperbolic surface subgroups and whose defining graphs lack induced cycles of length $n \geq 5$, answering the preceding question in the affirmative. This theme will be revisited later in this document, in the context of Artin subgroups of right-angled Coxeter groups.

For now, we introduce a small bit of terminology suggested by Kim:

Definition 3.3.2 Let Γ be a finite simple graph. An induced n-cycle, for $n \geq 5$, in Γ is called a *long cycle*. If Γ contains no such induced cycles, we say that Γ is *long cycle-free*.

As we will see later, many graphs with long cycles immediately fail to be virtually right-angled Artin, for reasons discovered quite recently [8] [10].

Chapter 4

Relatively hyperbolic groups

The class of relatively hyperbolic groups is generally defined in order to enlarge the class of hyperbolic groups to include classical groups that are not strictly hyperbolic but that do enjoy many properties reminiscent of hyperbolic groups - for example, finite volume cusped hyperbolic 3-manifolds. Such a group acts properly discontinuously by isometries on $\mathbb{H}^3$ which finite volume quotient rather than co-compact quotient. In particular these groups contain $\mathbb{Z} \times \mathbb{Z}$ subgroups (from the toroidal cusps) so are not hyperbolic.

The class contains many more examples of interest to various problems, and can be defined rigorously in terms of an action on a δ-hyperbolic space. However, since we will not need a formal definition of this class for this work, we will only provide this intuitive description before giving some of the basic properties and facts we will need about the class. In the very next chapter, we will give a somewhat surprising definition of relative hyperbolicity for right-angled Coxeter groups, defining the class of relatively hyperbolic RACGs to be precisely those which are *not* thick. This useful and intriguing result is due to Behrstock, Hagen, and Sisto [4].

In this short chapter, we recall relevant results about relative hyperbolicity, in particular in the setting of right-angled Coxeter groups. We briefly present known results and definitions connecting relative hyperbolicity to the defining graph of a right-angled Coxeter group, and to subgroup distortion and quasiconvexity.

4.1 Relative hyperbolicity for Coxeter and Artin groups

We first note that relative hyperbolicity is well-understood in the context of right-angled Artin groups with connected defining graph. Behrstock and Charney show the following [2]:

Theorem 4.1.1 Except for $\mathbb{Z}$, freely indecomposible right-angled Artin groups are not relatively hyperbolic with respect to any collection of subgroups, or even subsets.

This feel-good result has the benefit of also being readily believable, as right-angled Artin groups with connected defining graphs of cardinality > 1 have networks of flats. Charney and Crisp prove more extensive results about weak relative hyperbolicity [6], but those results aren't necessary for the scope of this document.

Relative hyperbolicity in right-angled Coxeter groups, while now well-understood, is a much more recently developed field of study. Caprace provides an extremely elegant but relatively complex categorization of relative hyperbolicity in general Coxeter systems in the context of buildings with isolated subspaces [5]; unfortunately, connecting that result as stated to the rest of the machinery used in the paper requires a significant diversion. We will instead defer the statement of which right-angled Coxeter groups are relatively hyperbolic with respect to which subgroups to the next chapter, and present it in relation to strong algebraic thickness.

4.2 Subgroup distortion and relatively hyperbolic inheritance

A natural question when dealing with a relatively hyperbolic group G is the following: under which circumstances is a subgroup H of G also hyperbolic relative to some collection of its subgroups? Further, can the group H be seen as inheriting, in some sense, the property of the larger group G?

Various formulations of relative hyperbolicity are presented by Gromov, Osin, and Bowditch; as expected, these definitions have variable utility depending on the application at hand. Hruska shows the equivalence of these definitions of relative hyperbolicity for countable groups, and answers the questions above in this setting [17]. We recall two of his results below, after the following necessary definitions.

Definition 4.2.1 Let (G, S) be a finitely generated group with (H, T) a finitely generated subgroup of G. Then the *distortion* of (H, T) in (G, S) is the function

$\Delta_H^G(n) := \max\{|h|_T \mid h \in H \text{ and } |h|_S \le n\}$

Subgroup distortion is unique up to the usual equivalence relation on functions $\simeq$. A subgroup H is *undistorted* if and only if $\Delta_H^G \simeq n$.

The following definition requires a bit more background knowledge of metric spaces, but is most easily applicable to our setting:

Definition 4.2.2 A subgroup H of a countable, relatively hyperbolic group G is *relatively quasiconvex* if, for any compact metric space M on which G acts as a geometrically-finite convergence group, the induced convergence action of H on the limit set $\Lambda\mathrm{H} \subseteq M$ is geometrically finite.

Hruska proves [17] the following two results, which we have simplified (and weakened) to better suit our purposes:

Theorem 4.2.3 Let G be a finitely generated relatively hyperbolic group and let H be a finitely generated subgroup. If H is undistorted in G, then H is relatively quasiconvex.

Theorem 4.2.4 Suppose G is hyperbolic relative to to some collection of subgroups $\{H_i\}$. If a subgroup $K \le G$ is relatively quasiconvex in G (relative the collection $\{H_i\}$), then K is hyperbolic relative to $\{K \cap H_i\}$.

Among other things, these theorems show that if a subgroup H of G is quasi-isometrically embedded, it inherits the relatively hyperbolic structure of G. In particular, the results hold if G is a relatively hyperbolic right-angled Coxeter group and H is a nicely-embedded Artin subgroup. We will need this property to prove a result later in the document. Further reading on the topic of relative hyperbolicity and quasiconvexity [17], though strongly recommended for personal enjoyment, is not strictly prerequisite for the remainder of this paper.

Chapter 5

Thickness in RACGs

In recent years, much attention has been paid to (strong) algebraic thickness of certain classes of groups, particularly graph products. Distinct from the metric notion of geometric thickness, this quasi-isometry invariant provides useful data about groups of interest, including an upper bound on divergence and, for right-angled Coxeter groups, a new characterization of relative hyperbolicity. In this short chapter, we'll introduce relevant tools and results which we'll be using for the remainder of this paper. We'll largely follow the notation of Behrstock, Hagen, Sisto, et al [4].

5.1 Divergence in metric spaces and groups

Let $0 < \delta < 1$ and $\gamma \geq 0$, and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(r) = \delta r - \gamma$. For a geodesic metric space (M, d), let $a, b, c \in M$ such that $d(c, \{a, b\}) = r > 0$, define $div_f(a, b; c) = \inf\{l(P)\}$, where P ranges over all geodesic paths from a to b avoiding the $f(r)$ ball about c, and $l(p)$ is the path length. If the set of such paths is empty, we define $div_f(a, b; c) = \infty$. Then the *divergence function* $Div_f^M(s) : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $Div_f^M(s) = sup\{div_f(a, b; c) : d(a, b) \leq s\}$.

Given $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we say that $f \preceq g$ if, for some $K \geq 1$, $f(s) \geq Kg(Ks + K) + Ks + K$ for all $s \in \mathbb{R}_+$, and we say that $f \sim g$ if $f \preceq g$ and $g \preceq f$. If there exists some function g as above such that $Div_f^M(s) \preceq g(s)$, then we say that (M, d) has divergence of order at most g (e.g., linear).

As divergence is a quasi-isometry invariant, this gives a well-defined notion of the divergence of a finitely generated group G; it follows that the divergence of G (equivalently, its Cayley graph) is finite if and only if G is 1-ended. This is essentially the reason why this paper restricts its attention to 1-ended groups.

It is known that the divergence of a right-angled Artin group $A(\Gamma)$ is linear if and only if Γ is a join (and hence A is a direct product), and is quadratic otherwise [2].

Thus, for a right-angled Coxeter group to contain a finite index RAAG, it must have at most quadratic divergence. For two-dimensional right-angled Coxeter groups, this subclass was characterized by Dani and Thomas [12]; a modification of their machinery was more recently used by Levcovitz (2017) to generalize the result to right-angled Coxeter groups of arbitrary dimension [21]. For the sake of convenience, we'll present both results using the latter version of the machinery.

Definition 5.1.1 Let Γ be a finite simple graph and $W(\Gamma)$ the right-angled Coxeter group defined by Γ. Define the *4-cycle graph* Γ^4 associated to Γ as follows:

- The vertices of Γ^4 are the induced 4-cycles of Γ. For a vertex v in Γ^4, let C_v be the 4-cycle in Γ associated to v.
- For $v, w \in \Gamma^4$, $\{v, w\} \in E(\Gamma^4)$ if and only if C_v and C_w share a pair of alternating vertices - that is, there exists an edge in Γ_{opp} that is a missing chord to both C_v and C_w.

Note that even if Γ is connected, Γ^4 may fail to be so (and may even be empty, in the case that Γ contains no induced 4-cycles). For a component Δ of Γ^4, let $\mathrm{supp}(\Delta)$ be the set of vertices of Γ involved in the 4-cycles associated to the vertices of Γ^4; that is, $supp(\Delta) = \{v \in V(\Gamma) \mid v \in C_{v'} \text{ and } v' \in V(\Gamma^4)\}$. If some component of Γ^4 has *full support* - that is, its support is the entire vertex set of Γ - then we'll pick one such component and denote it Γ^4_0, and say that Γ is a *CFS graph*. We say that the group $W(\Gamma)$ is CFS if and only if Γ is. Then the following is true [12] [21]:

Theorem 5.1.2 Let $W = W(\Gamma)$ be a right-angled Coxeter group. Then W has linear or quadratic divergence if and only if Γ is CFS.

Visually, CFS graphs contain a collection of squares with the property that you can move from one to the next (adjacent in the above sense) and travel anywhere in the graph's vertex set. It is clear that for Γ to be CFS, it must at least be connected, and also true that no complete subgraph of Γ must separate Γ. This more precise condition is equivalent to requiring that $W(\Gamma)$ be 1-ended, which is not surprising, as the result guarantees polynomial divergence.

5.2 Thickness and its implications

In this section, we'll briefly recall the notion of strong algebraic thickness as developed in several recent papers [4] [3]. No familiarity with the related concept of metric thickness is assumed, needed, or provided here.

Definition 5.2.1 Let G be a finitely generated group. We say that G is *thick of order 0* if G has linear divergence. For $n \geq 1$, G is *thick of order at most* n if there exists a finite collection $\mathfrak{H}$ of subgroups of G such that the following hold:

1. Each $H \in \mathfrak{H}$ is strongly algebraically thick of order at most $n-1$.
2. $\langle \cup_{H \in \mathfrak{H}} H \rangle$ has finite index in G.
3. There exists C ≥ 0 such that, for all $H, H' \in \mathfrak{H}$, there is a sequence $H = H_1, ..., H_k = H'$ with each $H_i \in \mathfrak{H}$ such that, for all $i \leq k$, the intersection $H_i \cap H_{i+1}$ is infinite, and the C-neighborhood of $H_i \cap H_{i+1}$ is path-connected with respect to some fixed word metric on G.
4. For all $H \in \mathfrak{H}$, any two points in H can be connected in the C-neighborhood of H by a (C, C)-quasigeodesic.

If such G is strongly algebraically thick of order at most n and is not strongly algebraically thick of order at most $n-1$, we say that G is *strongly algebraically thick of order* n.

As mentioned earlier, strong algebraic thickness is not equivalent to metric thickness when G is viewed as a metric space; it is true, however that algebraic thickness is an upper bound on metric thickness [4]. As the rest of this document will focus solely on strong algebraic thickness, we will abuse notation and refer to groups as simply "thick," meaning strongly algebraically thick.

Intuitively, a space is thick if it can be be decomposed into a union of subspaces of lower thickness orders which are sufficiently networked (have sufficient intersections and are nicely embedded). As the base level, thick of order 0, consists exactly of the finitely generated groups with linear divergence, one might hope that a deeper connection between thickness and divergence exists. Sadly, it's not true in general that a group is thick of order n if and only if its divergence is polynomial of degree $n + 1$, even under generous hypotheses. In a collection of original and compiled results about low-thickness order right-angled Coxeter groups, Levcovitz [22] shows the following:

Theorem 5.2.2 *Let $W = W(\Gamma)$ be a right-angled Coxeter group. Then W is thick of order 1 if and only if W has quadratic divergence.*

As the result is automatic for groups thick of order 0, this means that the sets of Coxeter groups that are thick of orders 0 and 1 exactly correspond to those with linear and quadratic divergence, respectively. Thus, these are precisely the groups this paper is concerned with. A natural question is whether this set of groups is exactly those that contain finite-index right-angled Artin subgroups; this is false, and this question will be revisited in a later chapter.

5.3 Thickness in RACGs

Behrstock, Drutu, and Mosher show that thickness is a quasi-isometry invariant. While the above formulation of thickness was for finitely generated groups, this invariant is of particular interest in the case that the thick group G is also a right-angled Coxeter group. They show the following [4]:

Theorem 5.3.1 Let $W = W(\Gamma)$ be a right-angled Coxeter group. Then exactly one of the following is true:

- W is thick, or
- W is hyperbolic relative to some collection of thick subgroups (in particular, is simply relatively hyperbolic).

The relatively hyperbolic right-angled Coxeter groups had previously been characterized by Caprace [5]; that characterization is rather complex and is not easily computable for large graphs. The machinery of algebraic thickness allows one to identify precisely when the Coxeter group associated to a graph is *not* relatively hyperbolic, and is advantageous in that it is computable in polynomial time [4]. Therefore, in this paper, we will use relative hyperbolicity almost exclusively in the context of thickness, and could take failure to be thick as a crude definition of a relatively hyperbolic right-angled Coxeter group. This allows us to use some techniques of relative hyperbolicity to prove things about thick right-angled Coxeter groups.

To this end, we recall Behrstock, Hagen, and Sisto's characterization [4] of the class of *thick graphs*; a graph is defined to be thick if the right-angled Coxeter group associated to it is strongly algebraically thick.

Theorem 5.3.2 The set of thick graphs is the smallest collection $\mathfrak{T}$ of graphs satisfying the following conditions:

- $K_{2,2} \in \mathfrak{T}$; that is, the graph consisting of a single 4-cycle is in $\mathfrak{T}$
- If $\Gamma \in \mathfrak{T}$ and Λ is an induced subgraph of Γ of diameter greater than 1, then the graph obtained by adding an external vertex v to Γ and adding all edges between $w \in \Lambda$ and v is also in $\mathfrak{T}$. This operation is called *coning off* Λ.
- If Γ_1 and $\Gamma_2 \in \mathfrak{T}$, and each Γ_i contains a subgraph isomorphic to some graph Γ which is not a clique, then the graph obtained by gluing Γ_1 to Γ_2 along the copies of Γ is also in $\mathfrak{T}$. Further, if any collection of edges joining vertices in $\Gamma_1 \setminus \Gamma$ to vertices in $\Gamma_2 \setminus \Gamma$ is added, then the resulting graph is also in $\mathfrak{T}$. This operation is called *taking the generalized union* of Γ_1 and Γ_2.

Notice that this somewhat technical definition is not actually very restrictive. One can obtain new thick graphs by gluing together other thick graphs along non-cliques, adding new vertices and many edges. The final point, in particular, allows great freedom in constructing and recognizing thick right-angled Coxeter groups; as noted previously, it can be algorithmically determined in polynomial time whether or not a given graph is thick. Behrstock, Hagen, and Sisto provide an intuitive method for manually checking a graph's thickness, but it's a bit too lengthy and technical to review here.

In closing, we note that the process for constructing large thick graphs is strikingly similar to that for CFS graphs outlined earlier in this chapter. If you restrict the class of thick graphs that you're considering to those whose associated Coxeter groups have at most quadratic divergence, then the graphs can be recognized as decomposing into networks of subgraphs which are thick of a lower order (that is, order 0; these subgraphs are joins). If a graph fails to be thick, its associated group must be relatively hyperbolic; one would expect it to be quite difficult to embed a right-angled Artin group in such a group as a subgroup of finite index.

5.4 Thickness in RAAGs

In this section, we briefly recall relevant results about thickness in the setting of right-angled Artin groups. As an edge in the defining graph corresponds to a flat in the associated RAAG, connected graphs yield RAAGs consisting of networks of flats glued together in a sensible way. Behrstock and Charney prove the following desirable result [2]:

Theorem 5.4.1 Let Γ be a finite, simple, connected graph with diameter at least 1 and let $A(\Gamma)$ be the right-angled Artin group associated to Γ. If Γ is a join, then $A(\Gamma)$ is thick of order 0; otherwise, $A(\Gamma)$ is thick of order 1.

It is well-known that, except for $\mathbb{Z}$, right-angled Artin groups are 1-ended if and only if their defining graph is connected. Thus, this is a complete characterization of thickness in 1-ended RAAGs. It follows that the divergence of $A(\Gamma)$ is linear if and only such a Γ is a join, and is quadratic otherwise. This result was independendently proven by Abrams, Brady, Dani, Duchin, and Young [1].

5.5 A first look at thickness in virtually RAAG Coxeter groups

Here, we state the first minor result of this paper and prove it in a deliberately inelegant way. The result follows more directly from extremely recent results in thickness and divergence, but is perhaps more illustrative if proven more naively. As stated, the theorem raises several reasonable questions, which will be addressed presently.

Theorem 5.5.1 *Let $W = W(\Gamma)$ be a right-angled Coxeter group which contains a finite index right-angled Artin subgroup A. Further suppose that W is (virtually) freely indecomposible and not virtually $\mathbb{Z}$. Then W is thick.*

Proof: We proceed by the contrapositive. Suppose that such a W is not thick; by an earlier theorem, W is thus hyperbolic relative to some collection $\{H_i\}$ of thick subgroups. Suppose that $A \leq W$ is a finite index RAAG subgroup. We want to show that A splits as a free product.

Since A is finite index in W , it is undistorted and thus also relatively quasi-convex, and hence inherits the relatively hyperbolic structure of W . A is thus hyperbolic relative to some collection of subgroups $\{A \cap H_i\}$. We know that, except for $\mathbb{Z}$, freely indecomposible Artin groups are not relatively hyperbolic with respect to any collection of subgroups [2], so we obtain a free splitting $A = A_1 * A_2$. As A is freely decomposible, we then have the desired result. □

First, we note that by appealing to other machinery, we can actually say quite a bit more about such a W. By virtue of containing A, it's actually thick of low order (0 or 1, depending on whether it is a direct product). It is rather unexpected that the converse of the theorem, with this modification, is false. That is, there exist right-angled Coxeter groups which are thick of order 1 which admit no finite index RAAG subgroups, even under generous assumptions. This was first pointed out by Behrstock in late 2016 via a counterexample in the 2-dimensional case; Dani later provided a distinct class of counterexamples with different utility. In general, the problem lies in the Morse boundary of the groups; this will be discussed in a later chapter. This failing is a cause of great consternation, and adds a great deal of convolution to the results in this thesis.

Second, one might wonder why W is supposed to be virtually freely indecomposible. This restriction is not just to control the ends of W; it also sets aside the easiest case of the problem for the next chapter. Similarly, if W is virtually $\mathbb{Z}$, then it is readily identifiable - its defining graph is *almost complete*, or missing exactly one edge.

Chapter 6

Artin subgroups of hyperbolic and virtually-free RACGs

In this chapter, we collect results that allow us to completely answer the central question of identifying finite index RAAGs for Coxeter groups that are either hyperbolic or virtually-free. These results almost immediately follow from others in the literature, and are explicitly stated here for the sake of completeness.

6.1 The geometry of virtually-free RACGs

Since free groups are right-angled Artin groups on empty (edge-free) graphs, any right-angled Coxeter group that is virtually-free is virtually Artin. Gordon, Long, and Reid [15] completely characterize virtually-free Coxeter groups by their defining graphs. In this section, we'll recall that result and its connection to surface subgroups of Coxeter groups, which are of critical interest to this document. Notice that their results do not require the Coxeter groups to be right-angled, and are thus more general than we actually require.

Definition 6.1.1 Let $\mathfrak{F}$ be the set of all finite Coxeter groups (in the right-angled case, those whose defining graphs are cliques), including the trivial group. Define $\mathfrak{G}$ to be the smallest set of Coxeter groups satisfying the following:

1. $\mathfrak{F} \subset \mathfrak{G}$, and
2. if $G_1, G_2 \in \mathfrak{G}$ and $G_0 \in \mathfrak{F}$, and $G = G_1 *_{G_0} G_2$, where the inclusion of G_0 into the G_i is as a special subgroup, then $G \in \mathfrak{G}$.

A right-angled Coxeter group in $\mathfrak{G}$ thus has a defining graph with a very specific and recognizable structure: large scale, such a Γ is a sort of tree of cliques glued together along subcliques. In the above paper, Gordon, Long, and Reid show the following:

Theorem 6.1.2 Let W be a Coxeter group. The following are equivalent:

1. W is virtually free,
2. $W \in \mathfrak{G}$,
3. W does not contain a surface group.

Thus, $\mathfrak{G}$ is the set of virtually free Coxeter groups; by restricting to appropriate edge labels, we obtain for free the (less objectively useful) result for the right-angled case. If the defining graph of an unknown Coxeter group W is given in a particularly combative way, this property can be checked either algorithmically or by sketching the Hasse diagram associated to W .

In Theorem 6.1.2, the authors do not require the Coxeter group W to be hyperbolic; in the case that it is, we get as an immediate corollary that it is either virtually free or contains a *hyperbolic* surface subgroup, as it contains no flats.

6.2 Hyperbolic RACGs and their subgroups

Hyperbolic groups in general, and the subclass of hyperbolic Coxeter groups more specifically, have attracted a great deal of attention for a variety of reasons. While much can be said about this rich class of groups, they can be easily and concisely described if we restrict our attention to the case where a Coxeter group $W(\Gamma)$ is right-angled. Moussong [23] shows the following:

Theorem 6.2.1 Let Γ be a finite simple graph and $W = W(\Gamma)$ the right-angled Coxeter group on Γ. Then W is hyperbolic if and only if Γ contains no induced cycles of length exactly 4.

In the terminology introduced in Chapter 2, this surprising result states that the hyperbolicity of a Coxeter group can be determined by the presence or absence of a *visible* flat.

With minimal legwork, this result directly characterizes the whether a hyperbolic Coxeter group is virtually RAAG. For the sake of completeness and ease of reference, we state the theorem and proof here.

Theorem 6.2.2 Let W be a hyperbolic right-angled Coxeter group. Then W contains a right-angled Artin subgroup A of finite index if and only if W is virtually free, and A is a free group.

Proof: Suppose the hyperbolic group W is virtually free. Then it is by definition virtually a free group - this subgroup is the desired RAAG A. Suppose W is not virtually free, but contains a RAAG A of finite index. Since W contains no $\mathbb{Z} \times \mathbb{Z}$ subgroups, A cannot contain such a subgroup either. Thus the defining graph for A has no edges and A is free. But since W is not virtually free, the result follows by contrapositive. □

This completely answers the question for hyperbolic groups; the rest of this document will focus on the more interesting case, where W is not hyperbolic.

Chapter 7

The Morse boundary

It is known that the visual boundary of a CAT(0) group is not a quasi-isometry invariant [11]. Charney and Sultan recently introduced a new kind of quasi-isometric invariant boundary for CAT(0) groups to help remedy this problem. The Morse, or contracting, boundary has several emerging applications to the study of geometric group theory, and right-angled Coxeter groups in particular. In this chapter, we briefly introduce basic definitions and consequences of the Morse boundary, and show its relevance to the central problem of this paper.

7.1 Preliminary definitions

While the more classical definition of the visual boundary can provide great insights into the properties of hyperbolic groups, many Coxeter groups (and all that we're concerned with) are not hyperbolic. Croke and Kleiner show that visual boundaries are not in general quasi-isometry invariants for CAT(0) groups [11].

Charney and Sultan [8] show the equivalence of five properties of rays in a CAT(0) space and define the Morse boundary to be the set of rays satisfying any of these conditions. As a deep understanding of the proofs of their theorems is not required for our purposes, we will present only one useful and intuitive property and use it in our discussion. For the remainder of this section, we let X be a CAT(0) space. The authors define the following:

Definition 7.1.1 For a constant D, we say that a geodesic γ is *D-contracting* in X if, for all x, y, $d_X(x,y) < d_X(x, \pi_\gamma(x)) \implies d_X(\pi_\gamma(x), \pi_\gamma(y)) < D$. If γ is D-contracting for some D, we say that γ is *contracting.*

It can be shown by various means that the property of being contracting is shared across an equivalence class of rays in the visual boundary. Thus, for a complete space

X, one can define the new invariant as follows:

Definition 7.1.2 Let ∂X be the visual boundary. Then the *Morse boundary* of X is the subset $\partial_C X = \{\alpha(\infty) \in \partial X \mid \alpha \text{ is contracting}\}$.

For A CAT(0) space X, Charney and Sultan show that $\partial_C X$ is preserved under quasi-isometry, and is thus a useful notion for CAT(0) groups. They use it to distinguish quasi-isometry classes of previously indistinguishable right-angled Coxeter groups. We note a particularly crucial consequence of the definition here:

Corollary 7.1.3 Let G be a CAT(0) group and let $H \leq G$ be a hyperbolic subgroup such that, for all $h \in H$, if $\mid h \mid = \infty$, then h is rank 1 in G. Then $\partial_C H \subset \partial_C G$. In particular, if W is a right-angled Coxeter group and S is a special surface subgroup with rank 1 infinite order elements, then $\partial_C W$ contains some subset isomorphic to S^1.

A yet-unpublished conjecture of Dani and Charney gives graph-theoretic conditions for the above special surface group to fail to "appear" in the Morse boundary; they are quite restrictive, and their lemma is not needed for the proofs in this document. One might take as good intuition that special surface subgroups of right-angled Coxeter groups have Morse boundaries which usually show up in the Morse boundary of the Coxeter group; such a subgroup would need to be very carefully constructed to fail to appear.

Cordes and Hume [10] show that any right-angled Artin group A has Morse boundary quasi-isomorphic to the direct limit of a Cantor set, and that consequently:

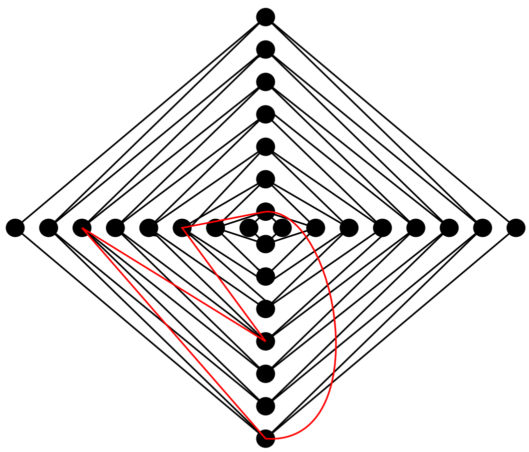
Theorem 7.1.4 If A is a right-angled Artin group, then $\partial_C A$ is totally disconnected.

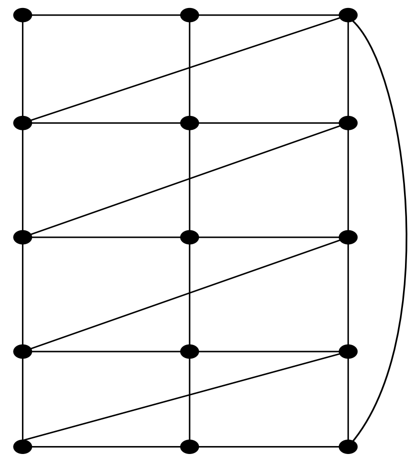
Combining Theorem 7.1.4 with Corollary 7.1.3, we immediately obtain the following lemma:

Lemma 7.1.5 Let W be a right-angled Coxeter group. Then if W contains a long cycle, then W does not contain a right-angled Artin subgroup A of finite index.

7.2 A complication

Prior to the exposition in the previous section, it was perhaps reasonable to guess that the virtually-Artin Coxeter groups were precisely the set of thick ones. We owe a considerable debt to Behrstock for furnishing the following counterexample (left, below); a distinct class of counterexamples based on the same principle (right, below) were furnished by Dani.





We note a few things about these graphs. First, unlike elsewhere in this paper, the red edges in the left example *are* edges in the graph, and not marked nonedges; they are colored for emphasis. Both graphs are CFS, a property which depends entirely on the presence of 4-cycles and is not disrupted by the red or curved edges. Consequently, the associated right-angled Coxeter groups are thick (of order 1), and are even 2-dimensional, as the graphs were constructed to be triangle-free.

However, both graphs contain induced cycles of length 5, and could be generalized, by adding more height and width or by stacking more rows, to examples containing arbitrary long cycles. By Theorems 7.1.3 and 7.1.5, we know that the groups have Morse boundaries which are not totally disconnected, and therefore that the groups cannot be virtually Artin groups.

It now seems reasonable, but perhaps a bit too optimistic, to guess that the presence of such a long cycle is the only obstruction to a CFS defining graph yielding a virtually-Artin Coxeter group. While Coxeter groups can contain many exotic hyperbolic surface subgroups, as discussed in Chapter 2, it is unknown whether they are all detectable in the Morse boundary. If we restrict our investigation to Artin subgroups which are also *visible*, we obtain the surprising result that the class of groups is quite substantially smaller than those that avoid this pathology. We will discuss this in detail in the remainder of this document.

Chapter 8

Geometry and graph theory

Given an arbitrary right-angled Coxeter group W, one would like to have some intuition for where to find reasonable candidate Artin subgroups; such subgroups should at least be 'large' and nicely embedded. The basic purpose of this chapter is to define new properties and machinery that will allow us to address the case where W is neither hyperbolic nor virtually free. In the process, we'll answer in the negative an open question about subgroups of Coxeter groups. When possible, techniques are presented geometrically and graph-theoretically, but most could be restated in more algebraic terms.

8.1 Visible subgroups of right-angled Coxeter groups

Let $W(\Gamma)$ be as usual and let $\{v_1, ..., v_n\} = V(\Gamma)$. Let Γ_{opp} be the opposite graph of Γ, so that $\Gamma \cup \Gamma_{opp}$ is an n-clique. Denote by $e_{i,j}$ the edge in Γ_{opp} joining v_i and v_j. To be clear, when the notation $e_{i,j}$ is used, v_i and v_j are specifically *not* adjacent in Γ.

Definition 8.1.1 Let $S \subseteq E(\Gamma_{opp})$ be some subcollection of nonedges of Γ. Define the subgroup $A_S \leq W$ by $A_S = \langle S \rangle :=< v_i v_j \mid e_{i,j} \in S >$. We call A_S a *visible subgroup* of W.

By construction, a visible subgroup of a right-angled Coxeter group W is generated by infinite order elements of W corresponding to missing edges in the defining graph Γ of W. Notice that $(v_i v_j)^{-1} = v_j v_i$, so we can freely replace $e_{i,j}$ with $e_{j,i}$ in S; this relabeling does not change A_S. We will wantonly abuse notation and refer to the generators of A_S, as well as the nonedges to which they correspond, as $e_{i,j}$.

Consider two elements $e_{i,j}$, $e_{k,l} \in$ the selected set S. If there is a non-chordal 4-cycle $\{v_i, v_k, v_j, v_l\} \subset \Gamma$, then we say that $e_{i,j}$ and $e_{k,l}$ are the *missing chords* to that 4-cycle. We will need the following:

Lemma 8.1.2 The elements v_iv_j and v_kv_l commute in $W = W(\Gamma)$ if and only if there exists a 4-cycle $\{v_i, v_k, v_j, v_l\}$ in Γ.

Proof: Suppose that the elements in question commute in the right-angled Coxeter group W. As a generator of a Coxeter group is its own inverse, this implies that $v_iv_jv_kv_lv_jv_iv_lv_k = 1$. The deletion condition for right-angled Coxeter groups (Theorem 2.1.3) implies that this complete reduction in word length is only possible by shuffling each generator to adjacency with its inverse. This means that the generator v_j of W commutes with v_k and v_l, and that two of the desired edges exist. We are then allowed to perform an initial reduction of the above commutator, yielding the shortened word $v_iv_kv_lv_iv_lv_k$. The same argument shows that v_i commutes with v_k and v_l, and guarantees that the final two edges in the 4-cycle were initially part of the defining graph Γ.

Conversely, suppose that such a 4-cycle exists. Then the commutator in question can be directly reduced to the identity by grouping and canceling the v_j letters, then the v_i letters, yielding the word $v_kv_lv_lv_k$, which is trivial. □

8.2 Visible Artin subgroups

The A_S constructed in Definition 8.1.1 is a subgroup with infinite order generators which might pairwise commute; a priori, it is not clear if this subgroup is a right-angled Artin group. It turns out that visible subgroups are often not even torsion-free. In this section, we will present necessary and sufficient conditions for a chosen set S of nonedges of Γ to generate an Artin group. These conditions are stated in terms of the local combinatorics of Γ, but could be alternatively presented as restrictions on hyperplanes in the Davis complex $\Sigma(W)$ of W.

Lemma 8.2.1 Let e_{i_1,j_1}, e_{i_2,j_2}, ..., $e_{i_m,j_m} \subseteq$ S be m chosen nonedges in S. Suppose that $v_{i_{k+1}} = v_{j_k}$ for all $k = 1, ..., m-1$, and suppose that $v_{j_m} \in star(v_{i_1})$. Then (A_S, S) is not a right-angled Artin system.

Proof: Without loss of generality, we may assume that no subsequence of the chosen edges satisfies the hypothesis of the theorem, else we could run the proof on that subsequence. Thus, at no point does the path marked by the chosen nonedges come within distance 1 of self-intersecting, and we may conclude that all v in the string occur at most twice (in incident nonedges). We note that this set of nonedges has the property that no two visible generators vw associated to distinct members are inverses in the right-angled Coxeter group W.

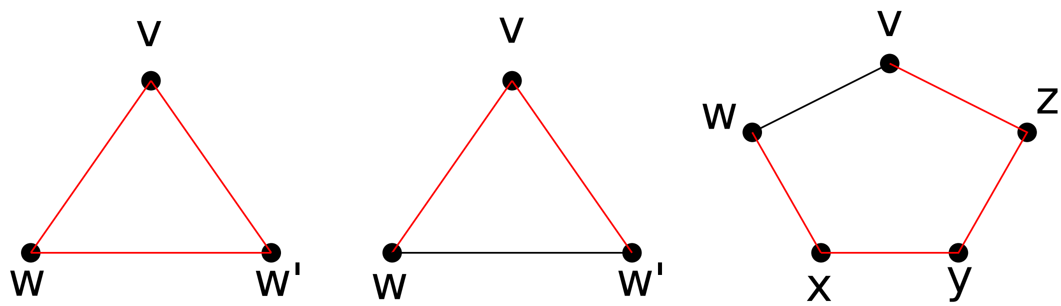
First, suppose that $v_{j_m} = v_{i_1}$. Then this subset of S corresponds to a cycle in Γ_{opp}. If we relabel the order-2 Coxeter generators using the hypothesis of the lemma, the word in W corresponding to this path is $v_{i_1}v_{j_1}v_{j_1}v_{j_2}v_{j_2}...v_{i_1}$, which collapses internally and is trivial in W , and thus in A_S.

Next, suppose that $v_{j_m} \neq v_{i_1}$. As above, the concatenation of these generators yields a word in W that collapses, yielding the element $v_{i_1}v_{j_m}$. Since $v_{j_m} \in star(v_{i_1})$ by hypothesis, we have that $(v_{i_1}v_{j_m})^2 = 1$, and thus the entire string of generators squared is trivial in $A_S \leq W$.

In either case, we have produced a trivial word w in the visible generators of A_S. By Theorem 3.1.1, if A_S is a right-angled Artin group on our generators, this trivial word must be reducible by transposing commuting generators and canceling inverses. Since we noted above that no two letters in w are inverses of each other, the group A_S is therefore not a right-angled Artin group on S. □

The first paragraph of the preceding proof is just a special case of the second; whether the terminal vertex is in the link of the initial vertex or the two vertices are the same, the same multiplication yields a relation in either case. We introduce the following notation:

Definition 8.2.2 Let $S' \subseteq S$ be a chosen set of nonedges of Γ satisfying the conditions of Lemma 8.2.1. We call the subgraph of Γ_{opp} corresponding to this set of edges a *star-cycle.* If S contains no such subset S', we say that the chosen set S satisfies the *star-cycle condition*, and that A_S does, as well.



Above, we present three examples of star-cycles possible in finite simple graphs. The black edges are edges in the graph Γ, and the red edges are nonedges whose generator-products are chosen as generators of a visible subgroup. In the first graph, a true cycle, we note that the product $(vw)(ww')(w'v)$ is trivial. The selected subgroup is thus not any 3-generator (or visible) right-angled Artin group; it is, in fact, isomorphic to a free group on 2 generators. In this case, a correct (and allowable) visible generating set is any two of the three marked edges.

In the second, which is not a cycle, but is a *star-cycle*, the word $((wv)(w'v)^{-1})^2 = (ww')^2$ is trivial, as w and w' commute in $W(\Gamma)$. The final graph illustrates a generalization of the second; the successive product $(wx)(xy)(yz)(zv)$ collapses to (wv), which is again trivial when squared. We see that any visible subgroup of $W(\Gamma)$ whose generators exhibit these combinatorics fails to be right-angled Artin.

Lemma 8.2.3 Let e_{i_1,j_1}, e_{i_2,j_2}, ..., $e_{i_m,j_m} \subseteq S$ be m chosen nonedges in S. Suppose that, up to relabeling, we have that $v_{i_{k+1}} = v_{j_k}$ for all $k = 1, \ldots, m-1$. Then the product of this path in Γ_{opp} is equivalent to the word $v_{i_1} v_{j_m}$ in W. Further suppose that either

1. there exists some distinct nonedge $e_{v,w} \in S$ such that $e_{v,w}$ and e_{i_1,j_m} are the missing chords to a 4-cycle in Γ, or

2. there exists a string of nonedges $e_{v,k_1}, e_{k_1,k_2}, \ldots, e_{k_l,w}$ in S (a selected cycle missing the edge $e_{v,w}$) such that $e_{v,w}$ and e_{i_1,j_m} are the missing chords to a 4-cycle in Γ,

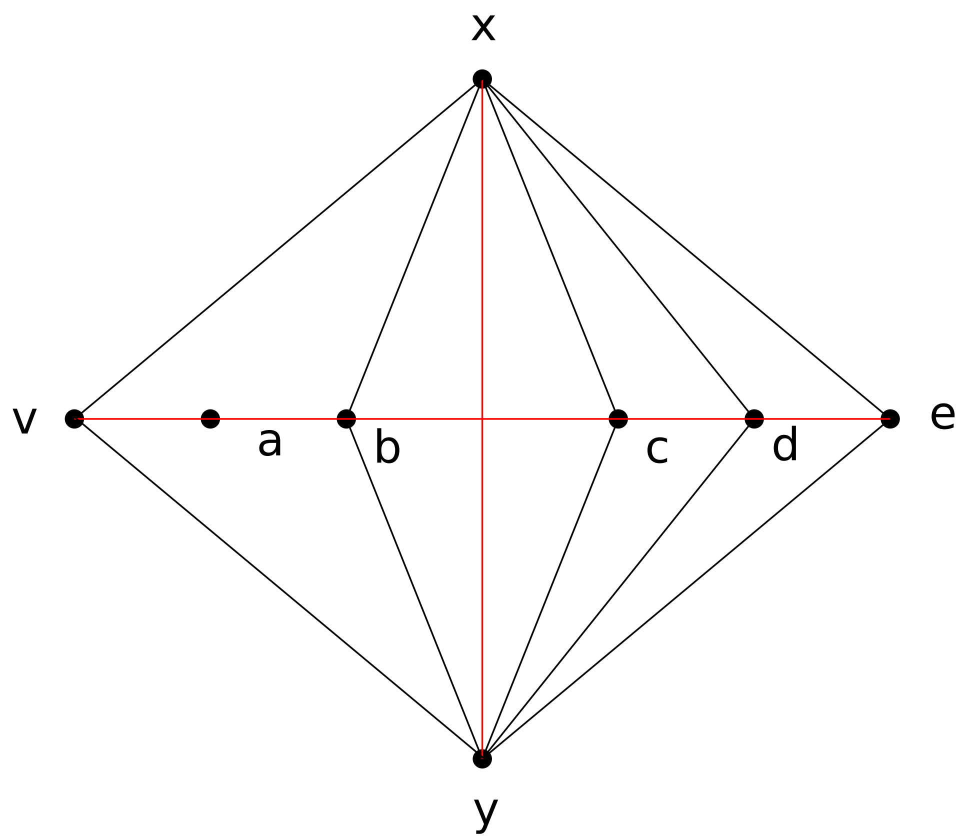
and that, for some index $1 \leq k \leq m$, the edge e_{i_k,j_k} and $e_{v,w}$ are not also the missing chords to a 4-cycle in Γ, then (A_S, S) is not a right-angled Artin system.

Proof: Let $C = \prod_{k=1}^{m} v_{i_k} v_{j_k} = v_{i_1} v_{j_m}$, and let $D = vw$, which either is in S (in case (1)) or is obtainable from S (in case (2)). By hypothesis, the word $CDC^{-1}D^{-1} = 1$ in A_S and in W. If e_{i_k,j_j} and $e_{v,w}$ were cross-chords, this relation would be a consequence of m allowable Artin relations. As this is not the case, we have produced a trivial word that is not realizable as product of generator shufflings and cancellations, and the subgroup A_S can not be right-angled Artin. □

Lemma 8.2.3 is perhaps more accessible if viewed on the graph theoretic level as a restriction of the local combinatorics of squares. Alternatively, it is another condition on hyperplanes in the Davis complex; if a product of reflections is orthogonal to another reflection wall, each reflection in the product must be, as well. It should be noted that the problematic situation highlighted here is not automatically forbidden by any aforementioned machinery; there exist low-dimension CFS right-angled Coxeter groups which are thick of order 1 and still exhibit this behavior.

For the sake of convenience, we introduce the following terminology:

Definition 8.2.4 Let $W(\Gamma)$ and S be as usual. If S contains no set of selected edges satisfying the hypotheses of Lemma 8.2.3, we say that S satisfies the *chain-chord condition*, and that A_S does, as well.



Above, we present an example of a chain-chord in a graph Γ. The 6-generator visible subgroup S marked by the red edges exhibits one version of the pathology demonstrated in the previous lemma. The outside 4-cycle $\{x, v, y, e\}$ contains, among others, the 4-cycles $\{x, v, y, b\}$, $\{x, b, y, c\}$, $\{x, c, y, d\}$, and $\{x, d, y, e\}$, each involving the opposite pair of vertices x and y. The product in $W(\Gamma)$ of the 5 selected horizontal visible generators is ve, which commutes with xy in W and hence in the subgroup S. The commutator of the product of the horizontal visible generators, ve, and the vertical visible generator, xy, is trivial in the groups. Theorem 3.1.1 implies that, if S were a right-angled Artin group, this trivial word is reducible to the identity by commuting generators and their inverses to adjacency and canceling. Because no 4-cycle $\{x, v, y, a\}$ (equivalently, $\{x, a, y, b\}$) exists, this shuffling is not possible, and S is not right-angled Artin.

We are now able to state the major result of this section.

Theorem 8.2.5 Let $W = W(\Gamma)$ be a right-angled Coxeter group, and let S be some selection of non-edges of Γ. If the set S satisfies the star-cycle and chain-chord conditions, then the subgroup A_S defined as above is a right-angled Artin group.

Proof: Suppose that S satisfies both conditions, and denote $S = \{e_1, ..., e_k\} \subseteq E(\Gamma_{opp})$. For an element $e_i \in S$, let v_{i_1} and v_{i_2} be the corresponding Coxeter generators in W. Define a right-angled Artin group A with generators $E_1, ..., E_k$ with relations $[E_i, E_j] = 1 \in A$ if and only if $[v_{i_1}v_{i_2}, v_{j_1}v_{j_2}] = 1 \in W$. As A_S is embedded in $W(\Gamma)$, we can again apply the deletion and exchange conditions for right-angled Coxeter groups for cancellations in A_S, and we know that this happens exactly when e_i and e_j are missing cross-chords to a 4-cycle in Γ. Let $A_S = \langle v_{1_1}v_{1_2}, \dots v_{k_1}v_{k_2}\rangle$. Let $\phi : A \to A_S$ by $E_i \mapsto v_{i_1}v_{i_2}$.

To considerably condense notation, we write e_i for the generator $v_{i_1}v_{i_2}$ in A_S. Let $w' = e_{i_1}...e_{i_n}$ be a reduced word in A_S. Consider the element $w = E_{i_1}...E_{i_n} \in A$. We would like to conclude that $\phi(w) = w'$. The word $w \in A$ is reducible if and only if its image in the subgroup $A_S \leq W$ is reducible as a length $2n$ element of the Coxeter group W, because we defined the generators of the abstract Artin group A to commute if and only if their images under the map ϕ commute. We note that cancellation of Coxeter letters in A_S occurs in pairs. By hypothesis, the word w' is not expressible in fewer than n generators. If w can be shortened in the Artin group A, then there's a way to shuffle a generator of A and its inverse to adjacency in the word w; by definition, this shuffling is then also possible in the group A_S, which would reduce the length of w', and is a contradiction. Hence ϕ is surjective.

Let $w = E_{i_1}...E_{i_n} \in ker(\phi)$. Then $e_{i_1}...e_{i_n} = 1 \in A_S$. As above, the embedding of A_S in W implies that this relation is achievable via shuffling and cancellation of Coxeter generators as in Theorems 2.1.3 and 2.1.4. Since E_A obeys the star-cycle condition, no string of Coxeter generators in the image $\phi(w)$ can be internally canceled to reduce word length. Thus, any reduction of the length of $\phi(w)$ occurs via a shuffling to adjacency of some $e_\alpha = v_{\alpha_1}v_{\alpha_2}$ and e_α^{-1} in the target group A_S. As S obeys the chain-chord condition, those letters commute with every $e_{\alpha'}$ between their original positions. More explicitly, satisfying the chain-chord condition means that e_α isn't simply hopped over next to e_α^{-1} in $\phi(w)$ - it actually commutes with every intermediate letter. By definition of A, the same shuffling can then also be performed in A on the letters E_α and E_α^{-1} in w.

As $\phi(w)$ is trivial in A_S by hypothesis, we must be able to reduce it to the identity by iterating this process with appropriate letters. By construction, we therefore have that $w = 1$ in A as well, and ϕ is also injective. □

While it is, in general, quite difficult to prove that a given group is or is not right-angled Artin, much of the weight of the previous proof is carried by existing machinery from the theory of Coxeter groups. It should be noted that it is actually rather difficult for a set S to satisfy the two defined conditions (in particular, the star-cycle condition). One can see this by taking a random graph Γ, choosing nonedges for S iteratively from the remaining allowable nonedges, and watching the available pool of nonedges quickly shrink. In general, S is a rather small subset of $E(\Gamma_{opp})$.

8.3 Reflections and fundamental domains in the Davis complex

In this section, we discuss a method for checking the index of general visible subgroups of right-angled Coxeter groups. Though it will not often be necessary, we will restrict our focus to the right-angled Artin subgroups which are the focus of this document.

Suppose that Γ is a finite simple graph. As before, let $\Sigma = \Sigma(W)$ be the Davis complex associated to the right-angled Coxeter group $W = W(\Gamma)$. We recall from chapter 1 that Σ is a CAT(0) cube complex on which W acts geometrically; let K be the finite subcomplex that is the strict fundamental domain for this action.

Unless Γ is a complete graph and W is finite, the complex Σ must be infinite. For each pair v, w of unrelated vertices of Γ, the action of the composition of reflections vw translates K in Σ. We'll refer to this extensively as a *translation direction* in Σ for the action of the full Coxeter group. If v, x, w, and z form a 4-cycle in Γ, the two translation directions vw and xz are orthogonal. If not, the usual treeing behavior is observed in Σ.

Example 8.3.1 Suppose that the entire graph Γ a a 4-cycle $\{v, x, w, z\}$. The standard construction of Σ yields an integer lattice, with the fundamental domain K being a unit square. Let $S = \{vw\}$, so A_S is simply $\langle vw \rangle$. The fundamental domain K' decomposes as a product in the expected way. In the vw translation direction, it's two squares (twice the size of the original fundamental domain). As the action of A does not translate or reflect in the orthogonal direction, we see that K' is infinite, or unbounded, in the xz direction. $\diamond$

As illustrated in the above example, for an unrelated pair v, w, the infinite cyclic group $\langle vw \rangle$ is a visible subgroup of W. Its action on Σ is by translation in the vw direction; the fundamental domain for this action is unbounded in every direction except vw. A choice of generators of a visible subgroup exactly corresponds to a choice of translation directions to explicitly bound.

It is crucial to point out here that naive choices of generators are often redundant; if generators vw and wx are chosen, the composition of these two translations (four reflections) is a translation in the vx direction. Thus, choosing vw and wx implicitly bounds the fundamental domain in the vx direction, as well. We notice that a choice of all three mentioned generators would violate the star-cycle condition, rendering the visible subgroup not right-angled Artin, so it's fortunate for our purposes that this is the case.

We can conclude that a visible subgroup A is of finite index k in W exactly when the fundamental domain K' for the action of A on Σ is a finite number k of copies of the original fundamental domain K. This is explicitly exhibited in the original Davis-Januszkiewicz construction [14]. They started with a right-angled Artin group $A = A(\Gamma)$ and embedded it in a right-angled Coxeter group as a subgroup of index 2^n, where n is the order of Γ.

8.4 Finding maximal visible RAAGs

In this section, we introduce a method for classifying visible right-angled Artin subgroups of a right-angled Coxeter group $W(\Gamma)$. While the goal is to identify "large" Artin subgroups and combinatorially detect candidates for finite-index subgroups, this initial naive approach is not sufficient. In the next section, it will be modified so that it is less elegant, less intuitive, and more correct.

Let Γ be a finite, simple, CFS graph and let $W(\Gamma)$ be the associated right-angled Coxeter group. For a subset S of $E(\Gamma_{opp})$, we define $\langle S \rangle := \langle vw \mid \{v,w\} \in S \rangle \leq W$.

Definition 8.4.1 Define a *graded collection* $\mathfrak{C} = \mathfrak{C}(\Gamma)$ of subgraphs of Γ as follows:

- $\mathfrak{C}_0$ is the empty graph
- $\mathfrak{C}_1$ is the set of induced 4-cycles of Γ
- For $i > 1$, define the intermediate set of induced subgraphs $\mathfrak{C}_i' = \{\Delta \cup v \mid \Delta \in \mathfrak{C}_{i-1}$ and $v \in V(\Gamma \backslash \Delta)\}$. Then the set $\mathfrak{C}_i \subseteq \mathfrak{C}_i'$ is obtained by discarding all $\Delta \in \mathfrak{C}_i'$ where Δ is not CFS.

The first nontrivial level of this collection then consists of the induced 4-cycles of Γ. To move up a level, we append an external vertex to the graphs in the previous step, discarding if the new induced subgraph of Γ is not CFS.

We note a few relevant properties of $\mathfrak{C}$:

Proposition 8.4.2 Suppose that Γ has n vertices.

1. $\mathfrak{C}_{n-3}$ is nonempty.
2. $\mathfrak{C}_i$ is empty for $i \geq n-3$.
3. $\mathfrak{C}$ has a natural poset structure given by subgraph inclusion, and some chain in this poset terminates in $\mathfrak{C}_{n-3}$.

Proof: The first claim follows from the fact that Γ is CFS; $\mathfrak{C}_{n-3}$ consists of the graph Γ itself. The other claims are immediate from the construction (and Zorn's lemma).$\square$

The collection $\mathfrak{C}$ can be constructed algorithmically for any choice of CFS starting graph Γ, and is essentially an exhaustion of the CFS structure. Each element is an induced subgraph of Γ, and hence corresponds to a special subgroup of $W(\Gamma)$ which is also a CFS right-angled Coxeter group. At the first nontrivial level of the collection, the graphs are induced 4-cycles, which are obviously virtually RAAG via the obvious $\mathbb{Z} \oplus \mathbb{Z}$ subgroup marked (the missing chords). One might hope to use induction to move up the graded collection to the full graph Γ, marking viable new nonedges at each step and preserving the desired property of containing a finite index right-angled Artin subgroup. This fails for three reasons, forcing a modification of the initial machinery.

8.5 Obstructions

The most easily remedied failing of the previous construction is that not all right-angled Coxeter groups associated to CFS graphs are virtually Artin. Recall from the previous chapters that the Coxeter group on any graph containing an induced n-cycle for $n \geq 5$ contains an embedded hyperbolic surface subgroup, and hence is quasi-isometrically distinguishable from a right-angled Artin group. One can immediately fix this problem by restricting our set of inputs to the algorithm to the appropriate subclass of graphs, but it's important to put a pin in this for now.

The second problem lies in the inductive step of the proposed proof. As presented, $\mathfrak{C}$ is a well-defined partially-ordered set of subgraphs of Γ, and the first nontrivial set of these, as previously noted, correspond to virtually Artin Coxeter groups. Moving from a graph Δ in $\mathfrak{C}_{i-1}$ to a graph $\Delta \cup v$ in $\mathfrak{C}_i$ is not immediately well-defined, as this involves making a selection of some nonedges between the newly adjoined vertex v and vertices in the existing subgraph Δ. It is possible that graph symmetries will produce multiple isomorphic choices of marked Artin subgroup, and even possible that algebraically distinct finite-index Artin subgroups might exist.

Lastly, the critical problem with this plan is that it doesn't actually work. It turns out that the class of virtually Artin right-angled Coxeter groups is considerably smaller than one might expect. This is a very surprising result of its own merit, and should be stated precisely and proven before discussing its implications. The proof will exhibit a pathological graph with as few as 6 vertices that illuminates a very common class of forbidden minor in the defining graph.

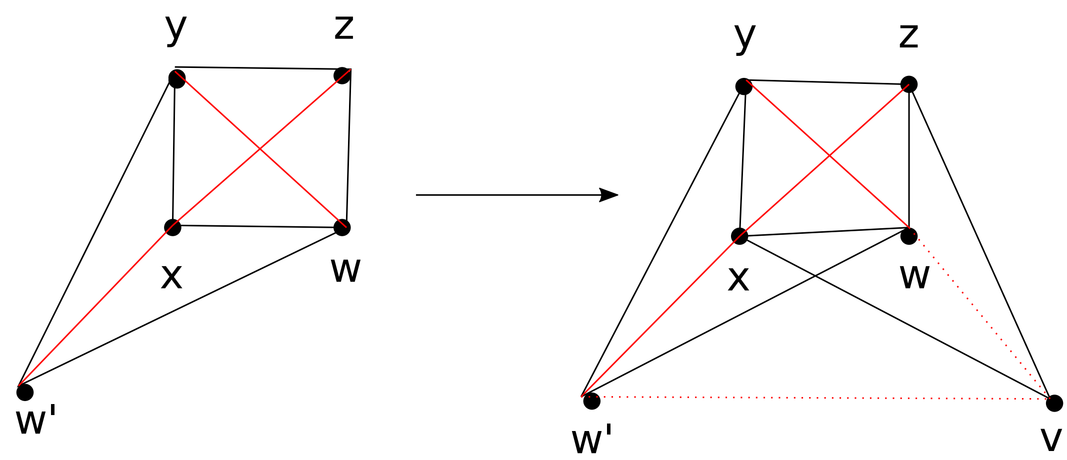
Theorem 8.5.1 There exists a finite, simple, CFS graph Γ with no long cycles such that the right-angled Coxeter group W associated to Γ contains no finite index visible right-angled Artin subgroup.

Proof: Let Γ' be the left graph below. The marked red edges represent choices of generators for a maximal visible right-angled Artin subgroup $A' = \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})$ of the Coxeter group W associated to Γ'. This choice of marking is unique up to graph symmetry: for example, the generator zw' could be chosen in place of either zx or xw'. In the chosen marking, the only nonedge unaccounted for corresponds to the absent generator zw'; this nonadjacency represents a pair of parallel reflection walls in the Davis complex, and hence an (a priori) translation direction unbounded in the fundamental domain for the action of A' on $\Sigma(W')$. This translation direction can be obtained by composing the chosen generators zx and xw'; their product acts on Σ by translation in the zw' direction, and hence the fundamental domain is bounded in that, and all, directions. Hence A' is finite index in W'.

Consider the graph Γ on the right, obtained by appending a new vertex v to Γ' along with some new edges vx and vz. Let W be the right-angled Coxeter group associated to Γ. Working from the maximal visible Artin subgroup inherited from W', we observe three unbounded translation directions when considering the action of A' on W - namely, vw, vw', and vy - and mark two candidates. By the star-cycle condition, we may add one of either vw or wy to our generating set, and we notice that we obtain the other of the two for free by multiplication with wy. By the same condition, the proposed generator vw' cannot be chosen along with either of the other two, as $((vw')^{-1}(vw))^2$ and $((vy)^{-1}(vw))^2$ are trivial in W . Hence, no combination of chosen generators produces a translation in all three newly added translation directions, and no maximal visible Artin subgroup is finite index in W .

Similar arguments can be made for alternate choices of A'.

As both graphs Γ and Γ' are CFS and long cycle-free, this proves the claim. □



Notice that the pathology exhibited here is not an obscure one; when appending a new vertex and some edges to an existing graph, one must actually be quite careful. If the external vertex v is connected to an existing vertex w, no new generators vw' may be chosen for *any* w' in the link of w. Consequently, it is extraordinarily easy to construct CFS, long cycle-free graphs whose associated right-angled Coxeter groups are not virtually visible Artin.

Chapter 9

A subclass of thick RACGs

In this chapter, we define a new class of graphs that avoids the pathology exhibited in the previous chapter and show that the right-angled Coxeter groups associated to graphs in this class admit finite-index visible right-angled Artin subgroups.

9.1 Link-satisfied graphs and the set $\mathfrak{A}$

We begin by introducing new terminology for the local graph combinatorics explored in the previous section.

Definition 9.1.1 Let Γ be a finite simple graph and let $v \in V(\Gamma)$. Let $\mathfrak{K}$ be the set of cliques of Γ which are not contained in a clique containing v. If, for all $C \in \mathfrak{K}$, $\{v,w\} \in E(\Gamma_{opp})$ for at most one vertex $w \in C$, then we say that v is *link-satisfied* in Γ. If, v is link-satisfied for all $v \in V(\Gamma)$, we say that Γ is a *link-satisfied graph.*

This condition is pure graph theory, and could be presented independently of any other topic in this paper. We first note a few subtleties about this property. If v is contained in a k-clique C, it is adjacent to all $(k-1)$ vertices of the subclique $C \backslash v$. This is explicitly allowed by our definition, as this C is not an element of $\mathfrak{K}$. If $\{w,w'\} \in E(\Gamma)$ is contained in some $C \in \mathfrak{K}$, then at least one of $\{v,w\}$ and $\{v,w'\}$ is in $E(\Gamma)$.

A crucial property of this condition is that it is inherited by induced subgraphs:

Lemma 9.1.2 Suppose that Γ is a finite, simple, link-satisfied graph and $\Delta \subseteq \Gamma$ is an induced subgraph. Then Δ is a link-satisfied graph.

Proof: Let $v \in \Delta$. If v is not link-satisfied, there exists a clique $C \in \mathfrak{K}_{\mathfrak{d}}$ and vertices w, w' in C such that $\{v,w\},\{v,w'\} \notin E(\Delta_{opp})$. Since Δ is an induced subgraph, we know that the inclusion of C in Γ is also not contained in any clique containing

v. This means that at least one of the two mentioned edges must be in $E(\Gamma)$, and thus was in $E(\Delta)$, as well. This is a contradiction; therefore, all vertices v are link-satisfied in Δ and Δ is itself a link-satisfied graph. □

Unfortunately, the property of being CFS is not inherited by induced subgraphs. To see this, take any CFS graph Γ with some vertex v contained in only one 4-cycle. If w is any other vertex in that 4-cycle, the graph $\Gamma \backslash w$ is not CFS - although it may still be connected - as no 4-cycle in any component of the 4-cycle graph Γ^4 contains the vertex v. This is actually not an issue for our construction of $\mathfrak{C}$, as we explicitly discard at each step any induced subgraph which is not CFS. Thus, we obtain the following corollary:

Corollary 9.1.3 If Γ is a finite, simple, CFS, long cycle-free, link-satisfied graph, then all induced subgraphs $\Delta \in \mathfrak{C}$ are, as well.

Proof: It is clear that Δ is finite, simple, and long cycle-free; by explicit construction of $\mathfrak{C}$, Δ is also CFS. By Lemma 9.1.2, Δ is also link-satisfied. □

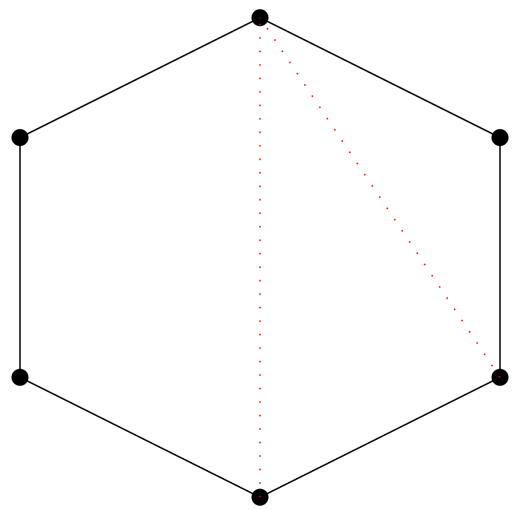
The problem from the end of the previous chapter is was that a newly introduced external vertex v might be nonadjacent to both some $w \in \Gamma$ and some $w' \in link(w)$. In that case, the star-cycle condition forbids the selection of both vw and vw' as visible Artin generators, and results in an unbounded translation direction in Σ. The graph Γ being link-satisfied is exactly the condition that resolves this issue, by eliminating this commonly occurring forbidden minor from consideration.

We note one other property of link-satisfied graphs which is incidental to our purposes, but which may be of broader interest. The property of being link-satisfied actually implies that a graph is long cycle-free; the former is a stronger condition which implies the latter. We briefly prove this result here, and will return to consider its consequences in the next chapter.

Theorem 9.1.4 Let Γ be a link-satisfied graph. Then Γ is long cycle-free.

Proof: Suppose by way of contradiction that Γ contains an n-cycle C for some $n \geq 5$. Distinguish some vertex $v \in C$, and number the remaining vertices in C sequentially $w_1, ..., w_{n-1}$, so that the first and last are adjacent to v.

We note that as C is induced, the edge $\{w_2, w_3\}$ is contained in a clique not containing v (and perhaps is a maximal clique itself). Thus, v must be adjacent to at least one of w_2 and w_3; in either case, C was not an induced n-cycle in Γ, and is in fact at most an $(n-1)$-cycle. Repeated applications of this argument show that any proposed long cycle in Γ must be reduced to at most a 4-cycle. □



We will include the two properties, though they are redundant, in the statement of the final theorem of the chapter, to make the result more clear and the proof more explicit. One could omit mention of long cycles in this entire discussion and simply rely on the stronger condition, but we've chosen to include it for several reasons. It makes the results intersect more clearly with related results in the field, and the link-satisfied condition was discovered relatively late in the project. Many key insights about these groups and graphs came from observing the consequence of long cycles in the Morse boundary, and it would be somewhat disingenuous to sweep mention of them under the rug.

If we append our new condition to the growing list, we can state the main theorem of this chapter:

Theorem 9.1.5 Let $\mathfrak{A}$ be the set of finite, simple, CFS, long cycle-free, link-satisfied graphs. If $\Gamma \in \mathfrak{A}$, then $W(\Gamma)$ contains a visible right-angled Artin subgroup of finite index.

Proof: Suppose $\Gamma \in \mathfrak{A}$ and $W = W(\Gamma)$ is the right-angled Coxeter group associated to Γ. We construct the graded collection $\mathfrak{C}$ of subgraphs of Γ as before. By the above corollary, all $\Delta \in \mathfrak{C}$ inherit the requisite properties of $\mathfrak{A}$. We know that $\Gamma \in \mathfrak{C}_{n-3}$, where n is the number of vertices of Γ, and that some chain $\{1\} = \Delta_0 \subseteq ... \subseteq \Delta_{n-3} = \Gamma$ is contained in $\mathfrak{C}$. We'll show by induction on i that the right-angled Coxeter group associated to each Δ_i, for $1 \leq i \leq n-3$, contains a finite-index visible right-angled Artin subgroup.

For $i = 0$ and 1, the result is clear; the respective Artin subgroups are the trivial group and $\mathbb{Z} \oplus \mathbb{Z}$, respectively. Suppose that for some $i \in \{0, ..., n\}$, Δ_{i-1} defines a right-angled Coxeter group W' that contains a finite-index visible Artin subgroup A' with visible generating set given by a set S' of nonedges of Δ_{i-1}. By Theorem 8.2.5, the set S' satisfies both the star-cycle and chain-chord conditions. Consider the graph Δ_i. By construction, $V(\Delta_i \backslash \Delta_{i-1})$ is a single vertex v.

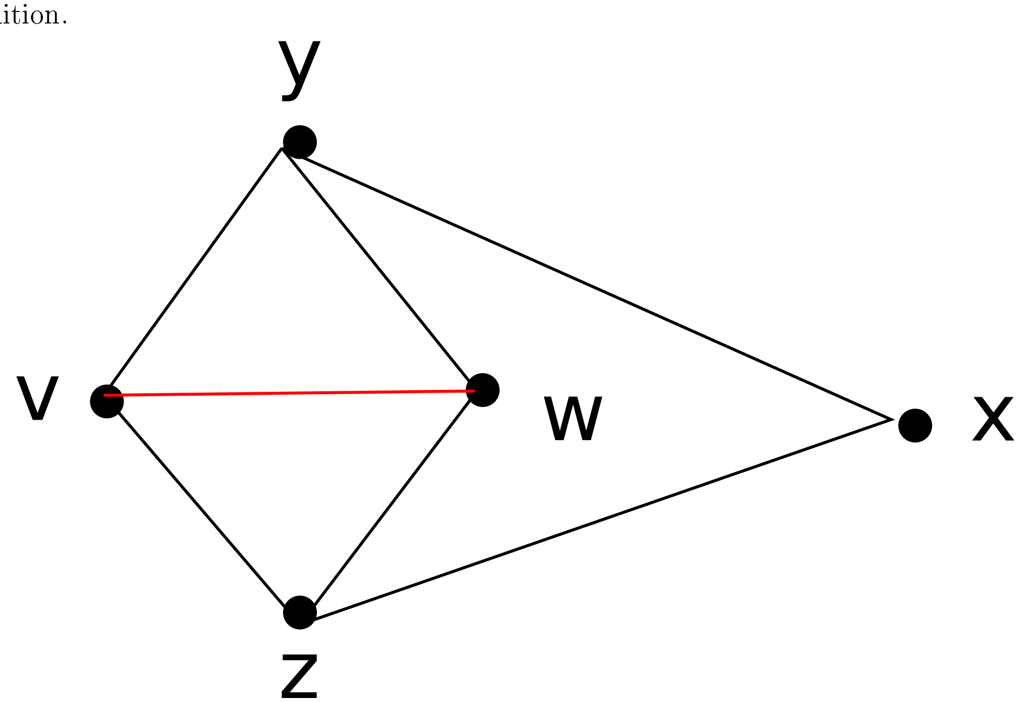
Consider the set E of nonadjacencies between v and $V(\Delta_{i-1}) \subset \Delta_i$. The set E represents translation directions in $\Sigma(W(\Delta_i))$ that are unbounded by the action of A' on that larger complex. We must construct a right-angled Artin group $A \subset W(\Delta_i)$, with $A' \subseteq A$, whose action on $\Sigma(W(\Delta_i))$ bounds each of these translation directions; this will show that A is the desired finite-index subgroup.

We first expand S' to form the generating set S of a maximal visible subgroup A of $W(\Delta_i)$. Because S' satisfies the star-cycle condition, any obstruction to that condition arising from appending new nonedges from E to S' would necessarily involve the newly added vertex v. However, we know by Theorem 9.1.2 that Δ_i is link-satisfied, so no two nonedges vw and vw' exist with $w' \in link(w)$, as then the edge $\{w, w'\}$ would be a clique not contained in a clique containing v, and at least one of vw and vw' is an *edge*, not a nonedge.

We can conclude that if the expansion of S' to a maximal set S violates the star-cycle condition, the obstruction is a true cycle of nonedges involving the vertex

v, and not a star-cycle. That is, there exist nonedges vw and vw' with $w' \notin link(w)$ in our new set S which close up some path P of nonedges from S' into a cycle of nonedges. In this case, we can freely discard either one of the two new nonedges, as, without loss of generality, the translation direction vw can be obtained by taking the consecutive product of every other generator in the cycle. Therefore any nonedges in E which are prohibited by the star-cycle condition from the expansion of S' to a maximal visible generating set S do not cause the subgroup A_S to fail to be finite index in W_{Δ_i}.

Suppose that $vw \in E$ is a nonedge which is prohibited by the chain-chord condition from a maximal visible generating set S containing S'. Then there exists some chain $vw, ww_1, w_1w_2, ..., w_kx$ such that the product of those elements, vx, is a cross-chord with some nonedge yz in S'; further, some nonedge in the chain is *not* a cross-chord with yz. As S' satisfies the chain-chord condition, the failure must involve the new vertex v; there must be no 4-cycle $\{v, y, w, z\}$ in Δ_i. But, by hypothesis, $\{v, y, x, z\}$ and $\{w, y, x, z\}$ must be 4-cycles in Δ_i, else there is no way for the condition to fail. This guarantees the existence of the 4-cycle $\{v, y, w, z\}$, which contradicts the assumption that the inclusion of vw in S violated the chain-chord condition.



Therefore, for all nonedges $\{v,w\} \in E$, vw either can be appended to S', or doesn't need to be in order to bound that translation direction. Consider $A = A_S$. By Theorem 8.2.5, A is right-angled Artin. The action of A' on $\Sigma(W(\Delta_{i-1}))$ with fundamental domain K' naturally extends to a action of A on $\Sigma(\Gamma(\Delta_i))$ with fundamental domain K such that K' is a subcomplex of K. As we have shown that every 'new' translation direction in $\Sigma(W(\Delta_i))$ is bounded, we have that K finite over K', and the subgroup A is finite index. □

For the converse direction, we note that if Γ contains no separating cliques (equivalently, $W(\Gamma)$ is 1-ended), failure to be CFS implies by Theorem 5.1.2 that W has super-quadratic divergence, and can thus not be virtually a right-angled Artin group.

Chapter 10

Conclusion

Some legwork integrating several recent results in the field shows that the set of 1-ended right-angled Coxeter groups which admit finite-index visible right-angled Artin subgroups is rather small. We have shown via original research that the set $\mathfrak{A}$ of such Coxeter groups is actually considerably smaller than was previously known. The restriction to 'visible' subgroups seems mild at this time, as it would be very difficult to embed a non-visible subgroup such that it is finite index in W. In this vein, we note open questions that present possible directions for future research:

Open question: Which 1-ended right-angled Coxeter groups contain a finite-index right-angled Artin subgroup that is *not* visible - that is, is not generated by unrelated generator pairs? □

Open question: If a 1-ended right-angled Coxeter group contains a finite-index right-angled Artin subgroup, does it necessarily contain a finite-index visible right-angled Artin subgroup? □

These questions seem reasonable to approach. A far more difficult problem is expanding the original question about Artin subgroups to the case where W is infinite-ended. As such a group has infinite divergence function, the machinery of thick and CFS groups we've adapted for this paper are immediately rendered useless. One could attempt an approach similar to that of the final theorem of this paper, but it's not at all clear what class of groups we would expect to be virtually Artin.

One final point of interest relates not to the problems discussed in this paper, but to an invariant that has attracted much attention in the past few years. It is now well-known that the hyperbolic surface subgroup of $W(\Gamma)$ arising from a long cycle in Γ affects the Morse boundary of the Coxeter group; in particular, it implies that the Morse boundary is not totally disconnected. For this paper, it is sufficient to note that that immediately shows that W is not virtually Artin.

As we have shown that link-satisfied is a stronger condition for graphs than being long-cycle free, and also immediately rules out the existence of a finite-index right-angled Artin subgroup, it may be worth considering whether failing this condition also disrupts the Morse boundary. That is:

Open question: If a finite simple graph Γ is not link-satisfied, can the Morse boundary of the Coxeter group $W(\Gamma)$ be not totally disconnected without the presence of an induced long cycle in Γ? $\square$

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