Children Merging Quantification Into Their

Qualitative Intuitions About Density

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Abstract

This study examines the role of quantification in children's understanding of basic relations implicit in the understanding of the concept of density. Even though qualitative thinking is important for learning science concepts, quantification is also essential in science education. Offering opportunities for students to make connections between quantitative and qualitative aspects of scientific concepts may allow students to construct a deeper understanding of these concepts themselves and to appreciate the values and ways of doing science, especially in the case of concepts involving intensive quantities.

The goal of the study was to evaluate whether and how experiencing the process of quantification to reflect upon the relationships among properties of objects, such as weight and volume, thus transforming these natural properties into quantities by assigning numerical values to them, helps children (1) activate and reconcile their intuitive ideas to successfully consider a second degree relationship between two first degree relationships (the relationship between weights and the relationship between volumes), and (2) to construct a linear relationship between weight and volume for a certain material.

Data come from one-hour long videotaped interviews of 20 third to fifth graders. In the intervention phase of the interview, children used a modified scale and a simplified ruler to compare and quantify the weights and sizes of specially designed cylinders and to answer questions that required merging qualitative and quantitative aspects of weight and size. Results show that, after the intervention phase of the interview, children reached a better understanding of the relationships among weight, volume, and kinds of materials and performed better in a posttest requiring an implicit understanding of density.

Keywords: Density, Weight, Volume, Proportional reasoning, Quantification, Intuition, Ratio, Covariation, Kind of material, Size.

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I worked with Paul Wagoner in both the Fulcrum Institute and the Inquiry Project, learning from him how to design and produce experiment materials. In support of this dissertation project, he contributed to the development of the materials I used in the interviews and artfully created the cylinders, the modified scale, and the simplified ruler. He went to an elementary school with me to help persuade the director of the after-school program to give me a green light to do my interviews there. He also found children from other schools for me to interview. And last but not least, as a native speaker of English, he carefully revised the final version of this dissertation.

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Chapter 1: The Proposed Study

I taught physics in middle school and high school before I came to Tufts University. Like most teachers who have experience with students of varied ages, I know that most young children are curious about the physical world and interested in physics. They tend to use their common sense to make predictions and to solve problems, and they try to make sense of their solutions after solving a problem, even though their interpretations may not be the ones currently accepted by scientists. In contrast, many students, after they move up to high school, lose the interest in physics they once had. They typically engage in what has been called "suspension of sense-making" (Hammer, 1997; Reusser, 1986; Voss, Perkins, & Segal, 1991). After solving assigned problems in school, they all-too-frequently fail to connect the problem statements and solutions with their intuitive ideas and the knowledge they have constructed in everyday life, especially when the problem solution involves numbers and equations.

One possible explanation for the above trend is that, while scientific concepts are usually dealt with in a qualitative manner in elementary and middle school, solving quantitative problems is introduced in high school and college as use of formulas, often dissociated from students' qualitative intuitions and understanding of everyday phenomena. Investigations (Halloun and Hestenes, 1985, Reif and Allen, 1992, Shaffer and McDermott, 1994, 2005, Mazur, 1996) revealed that students emerging from introductory college physics courses that focus on teaching fundamental physics principles and mathematical techniques for solving quantitative problems were unable to answer basic qualitative physics questions and exhibited many misconceptions about the motions of objects.

While qualitative thinking skills are fundamental in the development of scientific knowledge and in children's understanding of scientific concepts, quantification is necessary and essential in order to fully understand some (if not all) scientific concepts. What we need is not to avoid teaching scientific concepts quantitatively, but to help students make connections between quantitative and qualitative aspects of these concepts. In this dissertation, focusing on the relationships among volume, weight, and the materials objects are made of, basic relations in the definition and understanding of the concept of density, I argue that quantification should be part of science learning from the early years and that merging quantification into children's qualitative intuitions can help them develop a deeper understanding of science concepts.

The term "quantification," as used in this dissertation, refers to the process of assigning numerical values to physical properties of objects children can directly compare and manipulate. Rather than asking children to work with de-contextualized measurements and to carry out mathematical computations with large integers or more complicated rational numbers, they were first asked to reflect upon the relationships among the properties they deal with (weight and volume) and were given the opportunity to attribute numbers to each of the compared dimensions. The terms "intuition" and "intuitive knowledge" here mean the knowledge that children can somehow develop or acquire in everyday situations as well as while they reflect upon qualitative comparisons in more structured situations, without direct instruction based on formal definitions and procedures. I use the word “merge” to emphasize that the goal is to let children integrate quantification into their intuitions, not just to introduce another “soundtrack.”

Study Goals, Main Hypothesis, and Research Questions

The goal of this study is to evaluate whether and how the process of quantification of physical properties that are relevant in determining the density of materials helps children (1) activate and reconcile their intuitive ideas to successfully consider the second degree relationship between two first degree relationships (the relationship between weights and the relationship between volumes), and (2) to construct a linear relationship between weight and volume for a certain kind of material. The study was conducted with 20 third to fifth grade children (aged from 8 to 11 years), who may have already constructed a qualitative intuition about density (Inhelder & Piaget, 1958; Piaget & Inhelder, 1974; Smith et al., 1985) and may have already learned some basic mathematics. Each child participated in an individual interview for about one hour.

The beginning and ending sections of the interviews evaluated students’ initial and final understandings of the relationships among weight, volume, and kind of material. The middle section constituted an intervention, designed to allow children to manipulate objects (cylinders) made of different materials, to compare them using a modified weighing scale and a simplified ruler, to assigning numerical values to the weights and the volumes of the cylinders, and to answer questions that required merging qualitative and quantitative aspects of weight, size and their relationships.

I hypothesize that, even before the typical formal operations age proposed by Piaget (1974), the process of assigning numerical values to the weight and size of objects that children can directly compare and manipulate helps them coordinate these two properties to construct an explicit proportional relationship between them. I examined whether quantification of weight and volume with basic mathematics could anchor children's qualitative intuitions and extend these intuitions into considering the second degree relationship between (a) the quantitative relationship between weights of objects (W_1/W_2) and (b) the quantitative relationship between their volumes (V_1/V_2). For example, if the weight of one object is three times as much as that of another one and the size of the first object is three times as large as the second one, then those two objects could have been made of the same kind of material. I also analyzed whether their understanding could be extended into a linear relationship between weights and volumes for objects made of the certain kind of material. In the interview, they were not given any definition of density or instruction about the relationships between weight and volume, but rather were guided to make comparisons of cylinders made of the same or different kinds of materials and answer questions that could help them figure out these relationships by themselves.

I addressed the following research questions:

1. What ideas do children bring to tasks that require working with the relationships among weight, volume and the kind of material objects are made of?

2. How do children use and transform these different ideas to further their understanding of the relationships among weight, volume, and the kind of material objects are made of?

3. Can children generalize a linear relationship between weight and volume for a certain material?

4. Can children from ages 8 to 11 reason proportionally in the context of density?

5. What roles does quantification play (1) in the process of activating and reconciling children's initial ideas to construct more coherent ideas when solving problems and (2) in generalizing a linear relationship between weight and volume for a certain material?

6. What difficulties do children encounter in the process of quantification of weight and volume, in solving problems relating weight, volume, and the kind of material, and in generalizing a linear relationship between weight and volume?

7. What strategies do children use in solving problems related to the proportional relationships among weight, volume and density?

Organization of the Chapters

The dissertation is organized into 11 chapters:

Chapter 1: This Chapter, a brief description of the study and an overview of the rationale;

Chapter 2: A section on theoretical foundations and research perspectives on the development of children's knowledge in science and mathematics;

Chapter 3: A review of previous studies on students' understanding of density;

Chapter 4: The method of the study, including materials, participants, procedure and a brief overview of data analysis;

Chapter 5: Results of several Tasks related to quantification of weight and size;

Chapter 6: Results of Tasks 6 and 7 on inferring materials;

Chapter 7: Results of Task 8 on application;

Chapter 8: Results of Task 9 on generalization;

Chapter 9: Results of the first part of the pretests and posttests (Tasks 1–3 and Task 10) on why different objects have different weights;

Chapter 10: Results of the second part of the pretests and posttests (Task 4 and Task 11) on inferring materials; and

Chapter 11: Conclusion: main findings from the study, relation of these findings to those from previous studies, discussion of their theoretical implications and consideration of their relevance for science education.

Chapter 2: Theoretical Perspectives on the Development of Children's Knowledge

Any evaluation of the role of quantification in the development of scientific concepts, starting from students' qualitative intuitions, must take into account theoretical views and previous studies on cognitive development and learning. Piaget and his colleagues have given us a wealth of data on the way children at different ages think and come to develop an understanding about topics related to logic, mathematics, and science. Following them, educational researchers began to pay careful attention to what students were saying and doing on a variety of topics in science and mathematics. Some researchers even focused on children in their late high school and early college years, ages beyond those Piaget usually studied, and on factors that were not the focus of Piagetian studies, such as school instruction.

In the area of science and mathematics education it has been found that (a) students do not come to instruction as blank slates and that they have developed some intuitions about scientific phenomena in everyday life, (b) intuitive knowledge is often inconsistent with the accepted mathematical and scientific concepts presented in instruction, and (c) intuitive knowledge can be strongly held and resistant to change (Clement, 1982; Driver & Erickson, 1983; Driver, 1985; Gilbert & Watts, 1983; Osborne & Freyberg, 1985; Wiser; 1989).

Despite wide agreement that intuitive knowledge in science and mathematics learning deserves consideration, two different views on the nature of students' knowledge predominate among researchers in science and mathematics education. Some, such as McCloskey (1983), who holds a "theory theories" point of view, contend that naive knowledge is coherent and consistent, even theory-like, limited in number, and compactly characterizable. They emphasized the difference between students' and experts' knowledge characterizing novices' reasoning as concrete and that of experts as abstract (Chi, Feltovich, & Glaser, 1981; Larkin, 1983). For them, the purpose of instruction is to provoke theory change: to expose and confront students' intuitive theory with evidence and argumentation so that they can switch theories.

A different view was proposed by diSessa (1993) who argues that the theory-like conception of students' intuitions is a misleading representation of the actual state of students' ideas and that "intuitive physics is a fragmented collection of ideas, loosely connected and reinforcing, having none of the commitment or systematicity that one attributes to theories." (diSessa, 1993, p. 50). This "knowledge in pieces" point of view of students' knowledge is exemplified by diSessa's (1993) data on how elementary school children and "physics naive" adults answered four variations of a question related to circular motion, specifically on what happens when a ball moving in a circular tube leaves the tube. He found that participants in the study frequently gave multiple kinds of predictions and explanations, and that their answers changed according to the circumstances of presentation, thus showing that nonscientists' ideas are inconsistent, fragmentary, and context-bound.

For diSessa (1993), intuitive physics consists of a rather large number of fragments or phenomenological primitives (p-prims), described as "simple abstractions from common experiences that are taken as relatively primitive in the sense that they generally need no explanation" (diSessa,1993, p. 52). Some examples of candidates for p-prims, are "Force as a mover" and "Dying away" (all motion gradually dies away). He claimed that understanding intuitive physics necessarily means understanding the kind of pieces students used in their responses and he gave emphasis to the continuity between students' and experts' knowledge. Smith and diSessa's (1993) analysis of previous expert-novice studies suggested that the intuitive knowledge of physics novices contained both a sense of surface structure and a sense of deep structure but novices appear less abstract because the deeper structure they perceive is not normally tapped in the assessments of expert-novice studies. From an instructional point of view, Smith and diSessa (1993) argued "Confrontation essentially denies the validity of students' ideas. It communicates to students that their specific conceptions and their general efforts to understand are fundamentally flawed" (p. 126).

Advocating the "knowledge in pieces" point of view, Hammer, (2004a) argues: "student knowledge and reasoning is better modeled in terms of a manifold ontology of more fine-grained, context sensitive resources" (p. 12). Much of his work aims at motivating a shift from focusing exclusively on student difficulties and misconceptions toward a better comprehension of the productive aspects of student knowledge and reasoning (the raw material from which students may construct a physicist's understanding) and of the underlying dynamics of the difficulties and misconceptions students often have in that construction.

In Hammer's (2000) view, instructors who expect productive resources would be inclined to look for them in their students' reasoning and even to help students look for them themselves. For him, a resources-based account does not rule out confrontation as an instructional strategy. However, the role of confrontation will be different from that proposed by the theory theories point of view. The role confrontation plays "may be seen as helping to destabilize a stable set of resource activations, such as by activating conflicting resources, to promote further thought that may result in different activations of resources" (p. 8).

Hammer's (2000) resources-based view of knowledge also suggests that two distinct needs for the development of a scientific understanding are (1) the formation of intellectual resources and (2) the reorganization and application of these resources to align with scientific knowledge and practices. He suggests that early science education, before high school and college, should mostly address the need of formation of intellectual resources and advocated approaches to instruction along the lines of what David Hawkins (1974) called ''messing about in science.'' Hammer also claimed that "messing about, in hands-on activities or in playful, student-controlled conversations may be more productive than experiences crafted to guide students toward correct understandings of the concepts" (p. S58). He believed that "efforts to promote students' correct understanding at this early stage, and in particular their correct use of terminology, may be counter-productive, impeding children's construction and application of productive resources" (p. S58).

Research on everyday mathematics shows that children or adults with little access to school instruction and formalization can come to develop and use mathematical concepts by solving problem imbedded in everyday life and work (see, for examples, Abreu & Carraher, 1989; Acioly, 1993; Gay & Cole, 1967; Gerdes, 1986, 1988; Harris, 1987, 1988; Lave, 1977, 1988; Millroy, 1992; Nunes, Schliemann, & Carraher, 1993; Schliemann, 1985; Schliemann & Acioly, 1989; Saraswathi, 1988, 1989; Saxe & Moylan, 1982; Saxe, 1991; T. N. Carraher, Carraher, & Schliemann, 1982, 1985; T. N. Carraher, 1986; Ueno & Saito, 1994; Zaslavsky, 1973). However, as Schliemann (1995) and Carraher and Schliemann (2002) pointed out, even though everyday experiences of informal learning may provide us with a wide range of procedures and concepts that may be crucial in understanding many important concepts of elementary mathematics, naturally occurring everyday situations and experiences are not sufficiently varied and provocative to capture the spectrum of mathematical inquiry. They believe that activities in classroom situations should challenge students to go beyond their everyday experience, to experience a wider range of situations and tools for using mathematical concepts, relations, and representations that allow them to explicitly focus on mathematical concepts from different perspectives. Carraher and Schliemann (2002) also advocate documenting the variety of ways people represent and solve problems through self-invented means or through methods commonly used in special settings. By explicitly recognizing these alternative methods of conceiving and solving problems, teachers can be more likely to understand more clearly how students think and to appreciate the chasms they must sometimes cross to advance students' knowledge.

While agreeing with diSessa's "knowledge in pieces" point of view and with Hammer and Carraher and Schliemann's call for better comprehension of the productive aspects of students' knowledge and reasoning and for offering students chances to construct intellectual resources, as a former teacher, I suspect that "messing about" is not enough. In keeping with Carraher and Schliemann's views, I believe that, in school, we need to provide more structured environments for students in order to promote the formation of intellectual resources. I also doubt the idea of waiting until high school for the reorganization of the resources students already have to align with scientific knowledge. My position is based on some non-systematic observations of students in the classroom and some previous studies in science education:

First, it is very important for students to have a chance to mess about, but that should not be the end of the story. For some properties of objects with which children have already had a lot of messing about experience in their everyday lives, or after children have a chance to mess about in the classroom, instruction should offer environments or tools that are not often available in everyday life or are different from those for messing about, to deepen or anchor children's understanding of these properties and the relationships among them. For example, most students already have a lot of messing-about experience with weight and size before school, but most of them still do not have an explicit concept of density. Instruction should offer some different environment, which would allow students to do more symmetrical and quantitative comparisons of the weights and sizes of objects, in order for children to achieve a clearer concept of density.

Second, sometimes, if not most of the time, children need to reorganize some of their resources to align with scientific knowledge in order to form more new resources. For example, if students do not know that a pendulum comes to a halt at the highest point, how could they form a resource regarding what the motion of the weight would be if the weight were cut from the string at the highest point? Therefore, it will be better for students to reorganize the resources they already have to align with scientific knowledge as early as practical.

Third, I agree with Hammer that efforts to promote students' correct understanding might impede children's construction and application of productive resources. Indeed, we do need to remember that use of correct terminology does not necessarily mean correct understanding. Before introducing the terminology for a given idea it is first necessary to help students develop the idea (Arons, 1997) and to offer them opportunities to construct knowledge based on the resources they already have. However, I think we should not exclude providing good questions or good tools to invite children to reflect on their thoughts and refine them at an appropriate moment.

Fourth, I appreciate the importance of student-controlled conversations, but sometimes "experiences crafted" are necessary too. Some interventions, such as eliminating some distracting aspects of objects, processes, or events, will help students draw attention to those core aspects that may not be salient to them otherwise. As Lehrer (2009) pointed out, "when different inscriptions are brought into contact, they referentially coordinate aspects of the phenomena that were originally isolated" and "Reduction, in turn, is balanced by amplification" (p. 761). These kinds of interventions will facilitate the process of understanding the relationships among different properties of objects or different variables used to describe processes or events.

Fifth, a good intervention in science education not only guides students toward correct understandings of the concepts, but also introduces them to important aspects of doing science, as is the case with quantification. Pickering (1995) pointed out that nature passively resists the quantitative capturing of phenomena. Different aspects of measurement involve distinctive challenges that emerge when attempting to accurately and precisely answer a question. Science has developed some specialized ways of addressing these challenges. Scientists devise material arrangements to frame natural events, in a somewhat artificial way, so particular features of natural phenomena become systematically manipulable. As Ford (2005) stated, it is important for students to learn how to transform a natural phenomenon into scrupulously framed events by assigning numerical values to the features of interest.

This dissertation study attempted to help students both to reorganize their previous resources to align with scientific knowledge and to construct new resources. I examine how students' previous intuitive resources play a role in their independent construction of new knowledge and how the important scientific practice of quantification allows students to activate different resources, reconcile them, and develop new resources that can be used in a variety of new situations. The variety of methods students used creatively in solving problems is also documented.

Chapter 3: Understanding Density

This chapter focuses on theories and findings from previous studies in science and mathematics education that are related to understanding the concept of density. It includes proposals and data about the importance of quantification, children's knowledge about density, and ways of helping children learn about this concept. In science, the formal definition of density usually refers to the mass density of a material defined as its mass per unit volume. However, in previous studies on young children's ideas about density, researchers used the words "weight" and "size," rather than "mass" and "volume" in their interviews, since children at these ages use the words "weight" and "size" in everyday life but would not have learned the formal concepts of "mass" and "volume".

Quantification and the Definition of Density

As I have proposed above, it is important to examine the role of quantification in science learning. However, before discussing how to develop such studies, some questions need to be answered. They are: Why is it important to introduce quantitative concepts in science? How to develop quantitative concepts? Why are bridges between quantitative methods and qualitative intuition needed? This session aims at answering these questions.

The merits of the quantitative method.

Carnap (1966) pointed out that the concepts of science as well as those of everyday life may be conveniently divided into three main groups: classificatory, comparative, and quantitative. I use weight as an example to clarify the meaning of these three kinds of concepts. A classificatory concept, such as “heavy” or “light”, merely places an object into a class. A comparative concept, such as “heavier” or “lighter” or “equal in weight”, tells us how the weight of an object compares to the weight of another object. A quantitative concept attributes units of measure to objects. It was only after science developed the quantitative concepts of weight and mass, which can be measured, that it became possible to say that the mass or weight of an object is 5 kilograms or 49 Newtons. Carnap (1966) pointed out that, in most cases, before quantitative concepts can be introduced into a field of science, they are preceded by comparative concepts which later become the basis for quantitative ones.

Quantitative concepts are not given by nature. They arise from our practice of applying numbers to natural phenomena. As in an example given by Carnap (1966),

> … If you examine both stones, you will not come upon any numbers or find any discrete units that can be counted. The phenomenon itself contains nothing numerical--- only your private sensations of weight… It is we who assign numbers to nature (p. 109).

Why should we use numbers? What are the advantages of doing this? Carnap (1966) summarized three merits of quantification as follows:

1. With quantification there is an increase in the efficiency of our vocabulary. Before a quantitative concept is introduced, we have to use dozens of different qualitative terms or adjectives to describe a magnitude of an object property. For example, without quantification of temperature, we

have to speak of something as “very hot”, “hot”, “warm”, “lukewarm”, “cool”, “cold”, “very cold” and so on. After we correlate the magnitudes with numbers, we have only one term to memorize. The order of magnitude is immediately supplied by the order of the numbers.

2. Quantitative concepts allow us to express the relationships among concepts more briefly by formulating quantitative laws.
3. Once we have the law in numerical form, we can employ that powerful part of mathematics and, in that way, make efficient and precise predictions.

I will use the concept of weight and volume to illustrate the above three merits. We can determine the relative weights of two objects by holding each object in each hand and feeling their weights. We may be right in most cases, but we are likely to be wrong when the weights are close, or when other properties of the objects such as material, shape, volume, or temperature are different. Shaw (2001) pointed out that what you feel when putting an object in your hand is pressure rather than weight. Data by Liu (2009) shows that even adults could not correctly order cylinders with same base area by weight just by using their hands. They tended to feel that cylinders made of a denser material were heavier than they actually were. Ordering objects by weight, even without requiring determining the quantitative relationships among weights, thus requires using a scale as a measuring tool.

Even more difficult than comparing weights is to compare the volumes of objects without measuring tools. Without precise measurement, it is not uncommon to reach wrong relative judgments (bigger, smaller or equal in size), since evaluating volume requires a combination of three dimensions. We are often misled by a long side or by a big base area (Carraher & Cayton-Hodges, 2011). Therefore, developing a quantitative concept of length and a derived concept of volume is necessary for successful judgments about the volume of objects being compared.

Once weight and volume are quantified, one can succinctly express the relationships between weight and volume, and even develop other concepts, such as the concept of density. For example, we can use the equation $\frac{Weight_{Al_1}}{Volume_{Al_1}} = \frac{Weight_{Al_2}}{Volume_{Al_2}}$ to express that the ratio of weight to volume for a given aluminum object is equal to the ratio of weight to volume for another aluminum object. The relationship characterizes each kind of material. For different materials, their ratios are different, for example, for a piece of aluminum and a piece of iron, $\frac{Weight_{Al}}{Volume_{Al}} \neq \frac{Weight_{Iron}}{Volume_{Iron}}$. We can even define a new concept to represent the ratio of weight to volume, which in science is depicted by the word "density," that is, $Density_{Al} = \frac{Weight_{Al}}{Volume_{Al}}$. For different materials their densities are ordinarily different, that is, $Density_{Al} \neq Density_{Iron}$.

In addition, we can say that, for objects made of the same kind of material[1], the weight of the object is proportional to its volume. For objects made of different kinds of material, the ratio between their densities is directly

[1] At certain temperature and certain pressure, the density of one kind of material is a constant.

proportional to the ratio between their weights and inversely proportional to the ratio between their volumes.

From a comparative concept to a quantitative concept.

Given the merits of quantitative concepts, the question to be answered from an educational perspective is: How do we transform a comparative concept into a quantitative concept? For example, how do the comparative concepts of less weight and of equal weight can lead to a concept of weight that can be measured and expressed by numbers?

Carnap (1966) stated that, to give meaning to a quantitative concept, we must have rules for the process of measuring. These rules tell us how to assign a certain number to a certain object, substance, or process, so we can say that this number represents the value of the magnitude for that object/ substance/ process. There are two kinds of concepts in physics that can be measured with different methods. They are extensive quantities and intensive quantities. A large number of concepts are "extensive quantities" and are measurable with the aid of "three-rule schemas"(Carnap, 1966): the rule of equality, the rule of additivity, and the unit rule. For example, the concept of weight is an extensive quantity. To measure weight with the "three-rule schemas", we need to take the following steps:

- The rule of equality: If two bodies have the same weight (or mass), when they are placed on each side of a two-pan balance scale, the scale remains in equilibrium.

- The rule of additivity: We put object A on the scale and determine its weight (or mass). We replace it with object B and determine its weight (or mass). Then we put both objects on the scale and determine their joint weigh (or mass). This new object will have a weight (or mass) that is the arithmetical sum of the weights (or masses) of A and B.
- The unit rule: We choose an object that can be easily reproduced and define the unit of value in terms of that object. For example, the kilogram, based on an international prototype in Paris, or, using an arbitrary unit, we can choose an object A and determine the weight of other objects in terms of how many As they weigh.

Thus, with the three rules, if we call the mass of the international prototype in Paris 1kg, then the mass which is twice the mass of the prototype is 2kg, and so on. In the description of my study, I will propose a method to help children merge quantification into their qualitative intuitions guided by the "three-rule schemas". However, instead of using the standard weight measures, I will use an arbitrary unit of weight determined by a modified scale, as described later.

One must be aware, however, that, for some concepts, the additive principle is not appropriate. For example, density is a non-additive magnitude. Such concepts are called intensive quantities. How to give meaning to an intensive quantity, which is usually a derived quantity? As Carnap (1966) pointed out, when rules of measurement have been given for some quantities, like spatial length, length of time, and mass, then, on the basis of those "primitive" quantities, we can introduce other quantities by definition. These are called "defined" or "derived" quantities. The value of a derived quantity can always be determined indirectly, with the help of its definition, from the values of the primitive quantities involved in the definition. Using density as an example, in science, we define that density = mass / volume. It is important to notice that the left-hand side of the equation is defined and can only be defined by the right-hand side: the two sides of the equation are different ways of expressing the same thing. The defined quantity clearly does have its own 'identity', but it cannot have its meaning fully explained without recourse to its defining equation (Gamble, 1986). Its measurement rests on the measurement of the primitive magnitudes length and mass. We directly measure the volume and the mass of an object and then determine its density via the quotient of the mass divided by the volume[2]. Thus, to fully understand a derived quantity like density, quantifying the primitive quantities (weight and volume) is *essential*.

The need to merge quantification into children's qualitative intuition.

The great advances of the last centuries, especially in physics, would not have been possible without the use of quantitative methods. However, we should not overlook the great value that a qualitative intuition has for the discovery of new facts, the development of new theories, and the application of knowledge to new problems. In fact, there have been some criticisms of quantitative methods by some, who believe that we lose something when we describe the world with numbers.

[2] Notice, however, that it is also possible to construct an instrument that will measure an intensive quantity, for example, to measure the density of liquid directly, by means of a hydrometer.

In the middle of the last century, the philosopher Arthur Schopenhauer said that the mathematician has compelled you to admit the truth of the theorem, but you have gained no real understanding. It is as if you had been led through a maze. You suddenly walk out of the maze and say to yourself: "Yes, I am here, but I really do not know how I got here." (Carnap, 1966, p. 112).

This reminds us, in doing mathematics and in applying mathematics to solve physics problems, that we should pay attention to the intuitive understanding of what we are doing at each step along the way towards a proof or a solution to a problem. As a physics teacher, I have witnessed the feeling of being led through a maze experienced by some students who, after working on a problem by building and solving one or more complicated equations, were not able to make sense of the results and lost the meaning of the problem in the process of solving the equations (see also Liu, 2009).

Measurements with standard tools (just recording the readings from scales and rulers) and meaningless calculations will not ensure that the relationships among weight, volume, and density learned in school will connect to the qualitative intellectual resources children may have developed through their experiences in everyday life. As Schliemann and Nunes (1990) pointed out about understanding proportionality, when school-taught procedures are dissociated from the out-of-school model for proportionality, the school procedures are poorly learned and quickly forgotten. We need to find a way to interconnect the quantitative and qualitative aspects of the relationships. In this study, I attempted to help children merge quantification into their intuitive knowledge about the relationships among weight, size, and kind of material by offering them a chance to experience the process of quantification by themselves, explicitly using Carnap's (1990) "three-rule schemas" (the rule of equality, the rule of additivity, and the unit rule), by activating different intellectual ideas they have, and by reconciling their previous ideas toward constructing a more coherent understanding.

Studies on Understanding Density

Density is a core concept in science. This single concept plays a role in explaining a variety of phenomena in everyday life such as floating and sinking, convection, and pressure. Floating and sinking is also the context for many studies of children's understanding of density since Inhelder & Piaget's (1958) analysis. If an object is less dense than the liquid it is placed in, it will float. If it is denser, it will sink. If the object has exactly the same density as the liquid, then its buoyancy equals its weight and it will tend neither to sink nor float: it will be suspended in the liquid. A ship floats although it is made of steel, which is denser than water, because it encloses a volume of air and the resulting shape has an average density that is less than that of the water. If water finds its way to take over the space initially occupied by air, the ship will sink because the average density of steal and water is bigger than water's.

Many other physical processes involving density take place almost everywhere and every time, but people may be less aware of them. Convection is one such process. It occurs because heated fluids, due to their lower density, rise, and cooled fluids fall. A heated fluid will rise to the top of a column, radiate heat away and then fall to be reheated, rise again, and so on. Examples of this are atmospheric heat-driven convection, oceanic convection (also due to varying salinity), and mantle convection. It can also explain why the bottom of a lake is the coldest part of the lake in summer and the warmest part of the lake in winter.

The word “Density” is also used with other meanings. For example, population density, force density (the gradient of pressure within the bulk of a fluid), optical density, energy density, number density, charge density, and electricity density. In this dissertation, I will not discuss these concepts. When I use the word density here, it refers exclusively to mass density or weight density. However, understanding mass density will provide prototype for other density concepts.

Although density is a very useful concept, it is a challenging concept in science education. Previous studies by science educators and psychologists consider that the concept of density is a higher-order property and appears late in the child’s cognitive development (Karplus et al., 1981; Piaget & Inhelder, 1974; Smith, Carey, & Wiser, 1985). As an intensive quantity, a formal understanding of density is intimately tied to the development of proportional reasoning which was found to come into use only after about age 12 (Karplus,1981; Piaget & Inhelder, 1951, 1958; Piaget et al., 1968).

Piaget and Inhelder (1974), in a study on children’s understanding of differences in density, showed children a cork and a smaller but heavier stone and asked them to indicate which of the two was lighter and which was heavier and to explain their answers. They also asked children to explain why two stones of the same shape and size had different weights. Answers were classified into four stages of development. During Stage I, the child fails to dissociate the weight from both the volume and the quantity of matter and believes that the larger the body, the heavier it must be. During Stage II, children discover that the weight of an object does not solely depend on its volume but also on "what is inside." At Stage III, children consider weight and its coordination with the quantity of matter: the reason the smaller pebble is heavier than the larger cork is that it contains more matter (e.g., the pebble is heavier "Because it is fuller"). Finally, at Stage IV (after 12 or 13 years old, in the formal operations period), differences in density are attributed to the compression or decompression of the elements. The child appreciates not only that the weight of an object is proportional to its quantity of matter but also that its molecules can be packed together more or less tightly. Children expressed this by saying, for example, "Because the stone is tighter… It's more squashed" or "The stone is made out of squashed sand. The bits of sand are pressed together.").

Piaget and Inhelder's work is a landmark in the study of students' understanding of density and has inspired many other studies that helped clarify issues regarding factors involved the development of the density concept among children. However, by focusing on use of the schema of compression and decompression to explain differences in density, in their studies about the development of children's understanding of density, they didn't analyze their ideas regarding the proportional relationships between weight and volume of objects made of the same kind of material.

In their analysis of children's understanding of density, Smith et al. (1985) found that some 8- and 9-year-olds had reached the point where they had distinct density and weight concepts. They proposed that children at this time probably still have not developed the idea of a standard unit of volume and hence conceptualize density qualitatively as "heaviness for size," rather than as weight per unit volume. Furthermore, they thought that, lacking such standard units, children would not yet attempt to calculate densities numerically or realize that there is a unique number, which defines the density of a substance under ordinary conditions.

In the studies by Piaget and Inhelder and by Smith et al., quantification of weight and size were not available to participant children. It is therefore possible that, if quantitative relationships between weights and between sizes were easily accessible, children would be able to express ideas about the coordination between weight and size, and even to realize that there was a unique number, which defines the property of a substance. This is a main hypothesis examined in this study.

A formal understanding of an intensive quantity, such as density, is intimately tied to the development of proportional reasoning. The concept of proportion has been widely studied by Piaget and Inhelder in the field of probability (Piaget & Inhelder, 1951), the equilibrium of two-pan balance scales, the projection of shadows (Inhelder & Piaget, 1958), and the relationship between the size of fish and amount of food they eat (Piaget et al., 1968). From the results of these interview studies, Piaget's proposed the following developmental stages for children's proportional reasoning: (a) children do not understand the problem or their responses are based on only one of the variables; (b) they relate the two variables in the problem in a qualitative manner; if they attempt to quantify, they do so through additive compensation, paying attention to the difference between the quantities for one variable and applying that difference to the quantities of the other variable; (c) they know that the differences between quantities change with the size of the numbers, but they do not know that they need to consider a ratio; here they may also use the easiest ratio 1:2, but are not able to use other ratios; and (d) they fully understand the logical relationships among the four terms of a proportion.

Karplus (1981) also found that students academically classified as upper-track or upper middle-class students only come to use proportional reasoning after about age 12, and that only a small fraction of urban low-income and academically classified as lower-track students can use it even after age 14.

Although there have been many previous studies of proportional reasoning, few of them have focused on children's proportional reasoning in the context of density. One exception is Leoni and Mullet's study (1993), on intuitive mastery of the relationships between the concepts of mass, volume, and density in subjects of different ages, from nursery school to university in France. They found that the most frequently used methods in inferring weight, volume or density when given information about two of three properties were a unifactorial rule (involving only one factor) and additive rule (such as mass-volume). Leoni and Mullet used cards, on each of which appeared a drawing of a balance beam. For example, the left pan of the balance was shown as containing an object having a given volume and made of a given substance. A response appeared in the lower portion of the card in a balance scale with 19 graduations. The left end of the response scale was labeled Light and the right end Heavy. The levels for volume were low volume, average volume, high volume. The density levels were iron, wood, and plastic. In their study, the quantification of weight, volume and density was ordinal rather than interval. This format of data collection may have made it difficult for the children to express proportional reasoning. In my study, I examined children's proportional reasoning in the context of density with quantification based on scales of accurate, evenly spaced values where the weight of objects could be compared with each other, the lightest of them serving as unit of measurement.

Helping Children Learn about Density

Two main methods for helping students to understand density are currently found in the literature: one focuses on modeling and analogy; the other builds on mathematics.

Theorists of conceptual change have proposed that analogies from other domains can guide model building and can be a powerful source of new ideas (Clement, 2008). For example, Smith and Unger (1997) introduced a dots-per-box model to teach 7th-grade students about the distinction between mass and density, in which the quantities of total boxes, total dots, and dots per box map onto the quantities of volume, mass, and density, respectively. Their data reveal limitations of the model, with children "mapping weight onto total boxes, mapping both weight and size onto total boxes, mapping weight onto dots per box, mapping weight onto a conflated dimension of total dots and dots per box, or developing other kinds of idiosyncratic codes" (p. 158).

In traditional school curricula, density may be introduced in the form of mathematical procedures for calculating the relationship between mass and volume. Data show that a procedural mathematical approach might not be a very fruitful foundation for understanding density. For example, in earlier research, when high school students were taught procedures to plot coordinate graphs representing mass-volume relationships as lines, students failed to understand that the line signaled an invariant ratio and thus, did not understand the line as a representation of density (Rowell & Dawson, 1977a). Moreover, despite accurate calculations, many students could not understand the equivalence among various algebraic expressions of mass, volume, and density (Hewson, 1986; 1983; Leoni & Mullet, 1993; Rowell & Dawson, 1977b; Smith & Unger, 1997).

However, as pointed out by Lehrer, Schauble, Strom, and Pligge (2001), when mathematical ideas are cultivated and supported, they can serve as invaluable resources for reasoning about science. Moreover, mathematizing students' intuitive qualitative conceptions provides a firm foundation onto which their conceptions can be anchored.

Lehrer et al.'s (2001) study started with mathematics instruction on the idea of similarity in geometric figures represented in graphs of linear functions. This was followed by instruction about density, where fifth graders were encouraged to use mathematical ideas to model material kind. They separated rather than "integrated" strands of instruction for mathematics and science because, as they believed, these two disciplines have contrasting epistemic roots. Mathematics often relies on logic of certainty, illustrated canonically by proof as a form of explanation. In contrast, science relies on logic of reasoning about uncertainty, moderated through models or other means of inscribing the world. They argued that a too early integration of mathematics and science runs the risk of shortchanging one at the expense of the other. In addition, their instruction did not ignore the physical grounding afforded by material kinds. Students hefted, submerged, compared, and ordered objects by weight, volume and material kind, and then proceeded to quantification of differences and invariants like density. Classroom instruction was oriented toward using mathematics as explanations and extensions of these emerging intuitions. They found that most of the students were able to reason about density situations from graphical representations of the kind developed in their classroom.

I agree with Lehrer et al’s ideas that mathematizing students' intuitive qualitative conceptions provided a firm foundation onto which students' intuitive conceptions could be anchored. But I doubt that similarity is a good analogy for promoting understanding of density, given that, for geometric figures, one deals with the ratio between measurements of the same property, length, while, in the case of density, one has to work with the ratio between measurements of different properties, namely, weight and volume, which have different sensory foundations, different ways of measurement, and different units to be considered. Most importantly, children may have constructed a qualitative intuition about density from experiences in their everyday life, which may not be the case for intuitions about similarity. Since similarity is not less abstract than density, why use an abstract idea as analogy to understand another abstract one for which the physical intuitions are already available. Therefore, I propose to expand children's qualitative intuitions about density through activities that involve quantification of weight and volume. They will experience the process of transforming intuitive judgments of weight and size into scrupulously framed quantities by assigning numerical values to them, thus making the relationship between weight and volume accessible to them. I assume that we do not need to wait to teach the concept of density until children can understand similarities and graphs.

Chapter 4: Method

This chapter describes the participants, materials, and procedures used, and gives a brief overview of the data analysis plan presented in detail in Chapters 5–10.

Participants

The 20 participants in this study were eight third graders, eight fourth graders, and four fifth graders, all of them attending public schools in Massachusetts. Their ages ranged from 8 to 11. Eight were boys and twelve were girls.

Materials

Materials for Tasks 1–3 and for Tasks 5–10.

The stimuli used in the first part of the pretest (Tasks 1–3), the first part of the posttest (Tasks 10a–10c) and in the intervention (Tasks 5–9) were the 31 cylinders shown in Figure 1, a simplified ruler shown in Figure 2, and a modified ("color") scale (where regular intervals were labeled with color stickers rather than numbers) shown in Figure 3. Some cylinders were made of brass, some of aluminum, and some of Delrin (a kind of plastic). The ratios of the densities of brass, aluminum, and Delrin are roughly 6:2:1. The cylinders had different heights but the same base area of about 10cm^2. Some cylinders were covered with white paper so children could not see what kind of material they were made of.

Figure 1: Cylinders used in Tasks 1–3 and 5–10

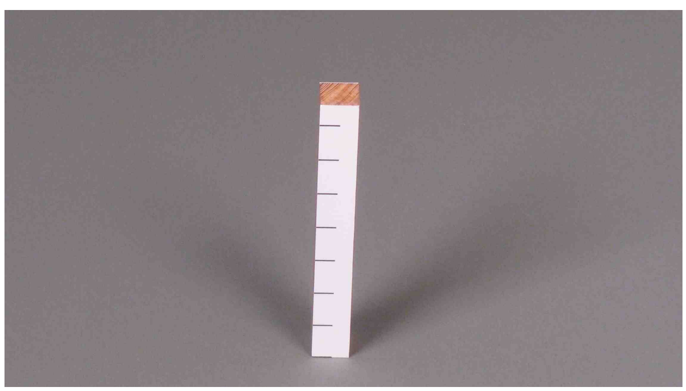


Figure 2: The simplified ruler
Each space between two marks (2 cm) is
the same as the height of the smallest cylinders
(excepting the single very short cylinder, R).

Figure 3: The modified scale.
It is labeled with colors, instead of numbers, denoting different weights. Pink, red, gray, brown, black and green represent 60g, 120g, 180g, 240g, 300g, and 360g respectively.

Despite suggestions that judgment of weight by holding objects in one's hands could be important in children's development towards understanding density (Smith et al., 1985), the path towards quantification requires recognition that weighting objects with one' hands may lead to wrong answers when the objects are made of different materials. A reliable way to quantify weight is to use a scale. However using a scale may bring other difficulties related to use of large numbers, non-integral numbers, the tolerance of the scale, and being disassociated from children's intuitions. To avoid interference of these difficulties, I used a modified scale that could help children quantify weight by considering the weight of each cylinder in relation to the weight of the lightest one. Thus, the children only needed to work with small integers. The modified scale also allowed them to experience the whole process of quantification of weight by figuring out how the scale works, using an arbitrary unit, labeling the scale with numbers, and assigning numerical values to weights of the cylinders by themselves.

The modified dial was marked with color stickers: pink, red, gray, brown, black and green (each corresponding to 60g, 120g, 180g, 240g, 300g, and 360g respectively, but the children did not know that). Thus, “Red” is twice as heavy as “Pink”, “Gray” is three times as heavy as “Pink”, and so on. I used color markers as an intermediary step between qualitative seriation and quantitative measurement. Marking the scale with colors makes it easy for children to establish that “Green” > “Gray” > “Red” > “Pink”. By allowing children to compare the different weights, I was guiding them to quantify the weights in relation to each other (e.g., one red weighs the same as two pinks, one gray weighs the same as three pinks) and then label the scale with numbers using “pink” as an arbitrary weight unit.

To minimize difficulties in measuring volume, the cylinders had bases of same area, allowing for comparing volumes by considering only one dimension. The simplified ruler, with only one level of same-space marks (each space was the same as the height of the smallest available cylinders), allowed children to compare and quantify the volume by counting how many spaces the height of a cylinder matched the marks on the ruler. Since all the cylinders had the same base, if a short cylinder was 1/3 of the height of the tall one, it had 1/3 of its volume.

The process of quantification of weight and size followed Carnap's (1966) "three-rule schemas" (the rule of equality, the rule of additivity, and the unit rule). If a child puts cylinder E on the scale, the pointer goes to pink; if she replaces it with cylinder A, the pointer will also go to pink; that means that E and A have the same weight. If one puts E and A together on the scale, the pointer goes to red: that means that red represents the sum of the weights of cylinder E and A, which is twice as heavy as the weight represented by pink.

As I mentioned above, in this study, arbitrary units, instead of standard units, were used, which allowed children to work only with integral numbers. I also simplified the evaluation of volumes by keeping base areas of the cylinders the same. These artificial modifications made it easy for children to evaluate and understand the relationships among weights and those among volumes, and to keep their understanding associated with their intuitions. However, this study did not aim at leading the children to a fully developed understanding of density, since children at these grade levels have not learned the formal concepts of mass and volume. The goal of this study is to help the children to construct the idea of density in these particular cases (cylinders of same base area and of certain heights). Hopefully this idea can be later transferred to other contexts (objects of different shapes and volumes).

Materials for Tasks 4 and 11.

The 13 cylinders shown in Figure 4, as well as a regular two-pan balance scale, were used in Task 4 (the latter part of the pretest) and Task 11 (the latter part of the posttest).

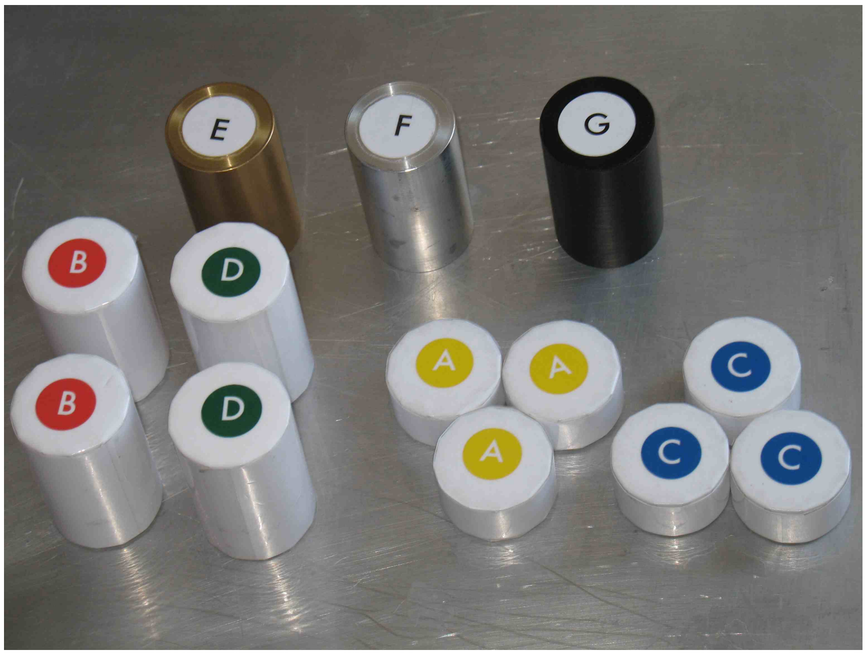


Figure 4: Cylinders used in Tasks 4 and 11.

Procedure

The Interview Approach.

Each child participated in an individual interview, which lasted for about one hour. The interview was divided into three sections. The first and last sections constituted a pretest and a posttest, each including the same questions, designed to evaluate participants' understandings before and after the interview. The middle section constituted an intervention designed to allow children to measure and manipulate the cylinders, answer questions about the relations between their weights and between their sizes, and thus develop an understanding of weight, size, and kind of material. In the intervention section of the interview, children worked on five tasks. In the interviews I used the word “size” instead of “volume”, because, as I mentioned earlier, most children at these ages have not learned the formal concept of volume yet. They may have an intuitive understanding of volume but this often translates into the word “size”.

The intervention section of the interview was based on the principles of the open-ended clinical interview approach as proposed by Piaget and his colleagues (1976) and later modified by Ginsburg (1997). It included pre-prepared open-ended questions and alternative follow-up questions. Different from Piaget’s approach, which aimed at determining children’s reasoning, in my interview I also aim at helping children develop new understandings. Follow-up questions were chosen and used depending on the interviewee's responses to the previous question. The following criteria constituted guidelines for the design and enactment of the interviews in this study: (a) Use children’s language and include some demonstrations to make clear the information about the cylinders (for example, to show that the diameters of the cross-sections of all cylinders are the same, I aligned the ends of pairs of cylinders and told the children that the ends of all these cylinders were the same); (b) Establish rapport (for example, I said: “Now I am going to ask you some questions that are a little difficult. You may or may not get the answer right away, but that is ok and you can always change your mind, as long as the new answer makes sense to you.”); (c) Motivate the children with interesting questions; (d) Abandon adult preconceptions and follow where the child’s thought leads; (e) Begin the interview with open-ended questions to let children express their initial understanding before introducing any guidance; (f) Have ready many kinds of follow-up questions to clarify and further probe children's understanding; (g) Focus on both children's answer to the problems and their reasoning and strategies towards solutions; (h) Offer various options of answers that help to indicate the different strategies of solutions; (i) Pay attention not only to children's final solution, but also to the development of their solutions; (j) Pay attention both to what children are saying and to what they are doing (I videotaped all the interviews); (k) Offer children enough time to think deeply and employ rich thought (I have tried my best, but when I interviewed some children in their school, I had to limit the interview time); (l) Make sure physical surroundings are comfortable and conductive to diligent work. (Most children were interviewed in a school library, some in a quiet office at TERC or at Tufts, and some in the kitchen of their home.)

Interview tasks and techniques were refined on the basis of four pilot interviews with graduate students (Liu, 2009) and two rounds of pilot interviews with children aged from 7 and 11, the first with two children, the second with four children. The design of the tasks also benefited from my experience in coding data related to the "inferring material" task for the Inquiry Project.

The 11 tasks in the interview are the following:

Tasks 1 to 3: Pretest on reasons objects have different weights

Task 4: Pretest on inferring materials

Task 5: Quantification

Task 6: Inferring materials with extra copies

Task 7: Inferring materials without extra copies

Task 8: Application

Task 9: Generalization

Task 10: Posttest on reasons objects have different weights

Task 11: Posttest on inferring materials

The pretest and posttest: Tasks 1–4 and Tasks 10 and 11.

The pretest and posttest used the same questions (although Task 10 used different cylinders from those used in Tasks 1–3). The questions examined children's understanding of the relationships among weight, size, and kind of material by inviting children to explain why objects have different weights (the first part), and to determine whether (and, if so, to explain why) two objects might be made of the same kind of material (the last part).

Tasks 1 to 3 and Tasks10a–10c included three questions. Table 1 shows the materials and questions used. Each question in the posttest used only two cylinders since the third cylinder proved unnecessary.

Table 1: Cylinders and Questions Used in the First Part of Pretest and Posttest

Task 1

Task 10

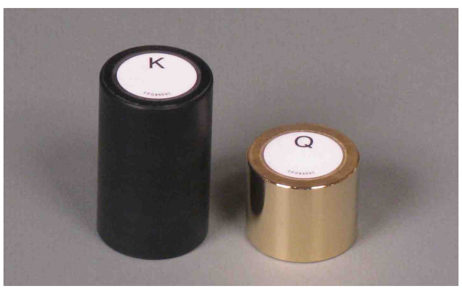
Why do you think they have different weights
even though they are the same size?

Task 2

Task 10

Why do you think they have different weights?

Task 3



Task 10

How come this one is smaller but is heavier than that one?

The 13 cylinders used in the last part of the pretest and posttest (Task 4 and Task 11) (see Figure 4), were originally designed by Carraher, Wagoner, Noble and Liu (2007) and used in interviews on children's understanding of density analyzed by Schliemann, Liu, Wagoner, and Carraher (2011). Cylinders E, F and G were made of brass, aluminum, and Delrin respectively; cylinders A, B, C, and D, wrapped with white paper, were made of brass, aluminum, aluminum, and polypropylene, respectively.

The questions to be answered by the children in this last part of the pretest and posttest, for each of the four cylinders B, D, A, and C, were: “Could this cylinder (B, D, A, and C) be made of the same kind of material as one of these cylinders (E, F and G), or would it have to be something else?” Extra copies of A, B, C and D were available. The availability of copies of the small cylinders allowed children to stack three small target cylinders, thus obtaining a composed cylinder that had the same size as the tall cylinders E, F, and G. The two-pan balance scale was available in this part of the pretest and posttest so children could compare the relative weights of two cylinders or of one tall cylinder against a cylinder composed by a combination of the smaller ones.

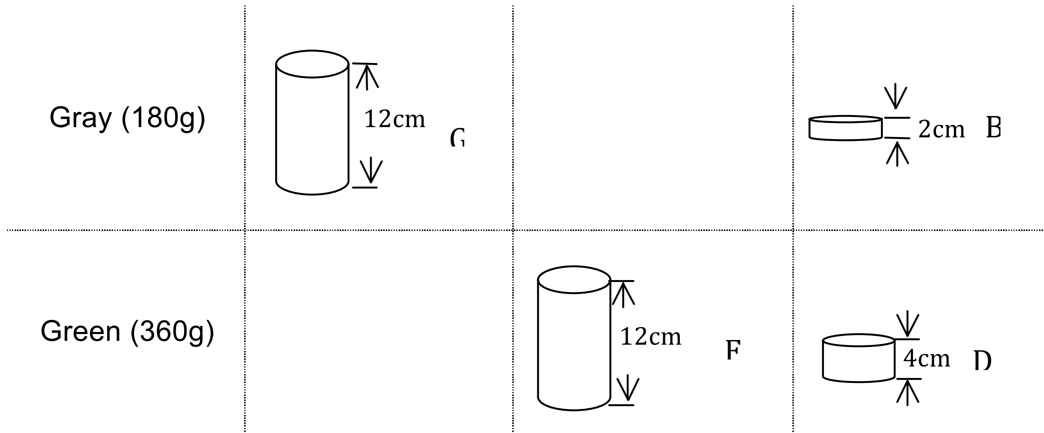
The intervention section: Tasks 5 to 9.

The five tasks in the intervention constitute four main sections: (a) quantification (Task 5), (b) inferring materials (Tasks 6 and 7), (c) application (Task 8), and (d) generalization (Task 9). The quantification task (5) aimed at offering children a chance to understand the process of quantification of weight. The inferring materials tasks (6 and 7) allowed children to activate their resources about the relationships among weight, size and kind of material and to reconcile them as they were gradually given opportunities to quantify weights and sizes. In the application task (8), I examined how well the children applied their refined knowledge to solve problems using the variations of the relationships among weight, size and kind of material, whether children at these ages use proportional reasoning in this context, and what kind of strategies they tended to use. The generalization (Task 9) gave the child an opportunity to generalize the law that, for objects made of the same material, the relationship between weight and volume is the same and that this relationship is different for different materials.

(a) Quantification (Task 5): In this task children were first asked to order seven covered cylinders (see Table 2 and cylinders A, B, C, D E, F and G in Figure 1) by weight using only their hands, hefting them or placing them on the palm of their hands. A, C, and F are made of aluminum; B and D are made of brass; E and G are made of Delrin. The ratio among the densities of brass, aluminum and Delrin is roughly 6:2:1. Each cylinder has approximately the same base area of about 10cm^2. These cylinders are wrapped in white paper to conceal the material from which they are made. F and G are the same height (12 cm); E, C, and D are all precisely 1/3 the height of F (and G); and A and B are both precisely 1/6 the height of F (and G). After the children ordered the cylinders by comparing them with their hands, I asked them to use the modified "color" scale to compare the weights of the cylinders, figure out the weight relationships among them, and label the points marked by colors on the scale with numbers.

Table 2: Seven Wrapped Cylinders Labeled A-G

Weight	Material		
	Delrin (1.5 g/cm^3)	Aluminum (3.0 g/cm^3)	Brass (9 g/cm^3)
Pink (60g)	4cm E	2cm A	
Red (120g)		4cm C	



(b) Inferring materials: This section included two tasks (Tasks 6 and 7). The first one involved determining which of three short covered cylinders could be made of the same kind of material as a tall covered cylinder, with multiple copies of the short cylinders available so they could be stacked up to produce a combined cylinder of the same height as the tall one. The second subtask involved the same question in relation to a tall cylinder to be compared to two short cylinders, but this time no extra copies of the short cylinders were provided.

The stimuli in the first task (Task 6) were four covered cylinders F, C, D and E, three extra identical copies of C and D, five extra identical copies of E (see Figure 5), and the modified scale. The cylinders were all covered with white paper. C and F were both made of aluminum, D was made of brass, and E of Delrin. The ratio among the densities of brass, aluminum and Delrin is roughly 6:2:1. F has a height of 12 cm; C, D, and E are the same height which is 1/3 the height of F. C is made of the same kind of material as F. D has the same weight as F. Children had measured the weights of these cylinders in the quantification task.

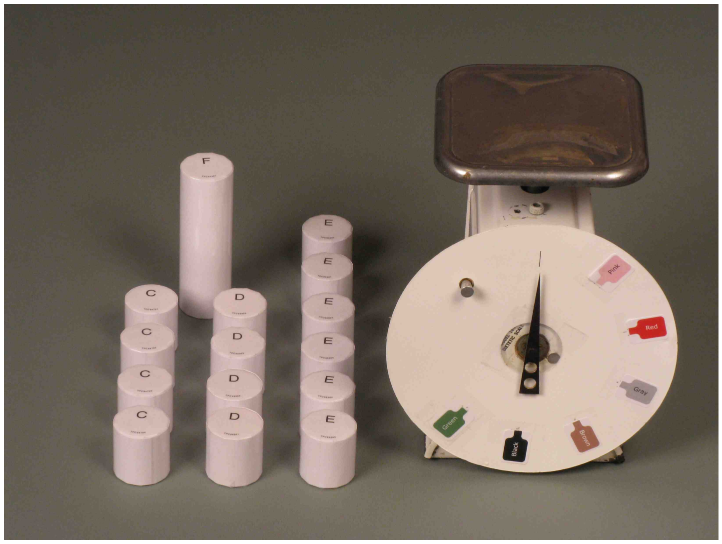


Figure 5: Cylinders and a scale used in the first subtask of inferring materials

Each child first received only a single copy of each cylinder, was told that F was made of aluminum, and was asked: “Could C, E or D be made of the same kind of material as F?” The correct answer to this question is that C is made of the same material as F, but children who think that “the same weight” is the only indicator for being made of the same kind of material would answer D.

After answering the question, children were offered the extra copies of C, D, and E and asked whether all these cylinders together could help them figure out which one could be made of the same kind of material as F. If they had already given an answer, I asked whether all these cylinders together could help show why that cylinder could be made of the same kind of material as F. Constructing a same-size cylinder by using extra copies invited children to activate and make explicit their intuitions about the covariation of weight and volume.

The stimuli in the second task of inferring material (Task 7) were cylinders F, A, and B (see Figure 6), the modified scale, and the simplified ruler. A was made of aluminum; B was made of brass. A and B had the same height, which is 1/6 the height of F. The question in this task was “Could A or B made of the same kind of material as F?” As already mentioned, here children did not have the extra copies of the small cylinders and had to estimate the relationships of weights and that of sizes with their eyes and hands, or by using the color scale and the simplified ruler. If they did not use these measuring tools, I suggested: “Could the scale or the ruler help you?”

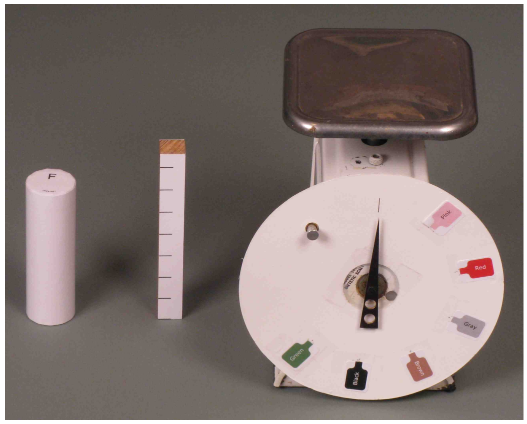
This task aimed at inviting children to think explicitly about the quantitative relationships between weights and between sizes and then to consider the second degree relationship between the two relationships. This allowed them to extend their understanding of “being made of the same kind of material because they have the same weight and same size” to a more advanced understanding of “being made of the same kind of material because one object has 1/n weight and 1/n size of the other one” (or $W_1/W_2=V_1/V_2$).

Figure 6: Cylinders and tools used in the second subtask of inferring materials

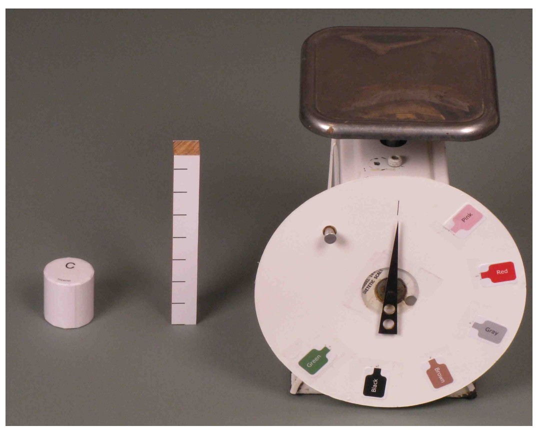
(c) Application (Task 8): In this task the children were asked to apply their new understandings to solve the six different problems shown in Table 3, with the corresponding objects or pictures presented to them. The first three were missing value problems and the last three were ratio comparison problems.

Table 3: Six Problems in the Application Task

Problem 1: We already know that F is made of aluminum and its weight is green. What would be the height of a cylinder, if it is also made of aluminum and its weight is gray?



Problem 2: If we have another aluminum cylinder that is twice as tall as the short cylinder C, how heavy would this cylinder be?



Problem 3: C is made of aluminum, but E is not. If I need an aluminum cylinder that has the same weight as E, how tall would the cylinder be?

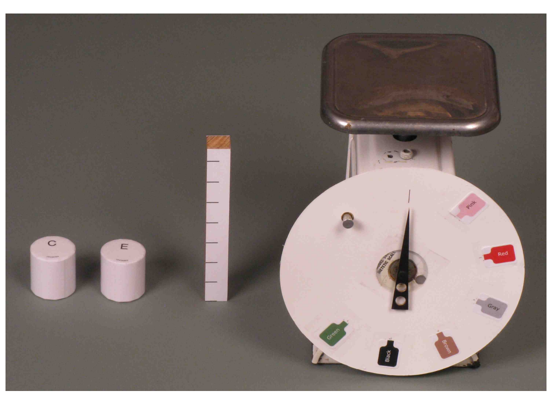
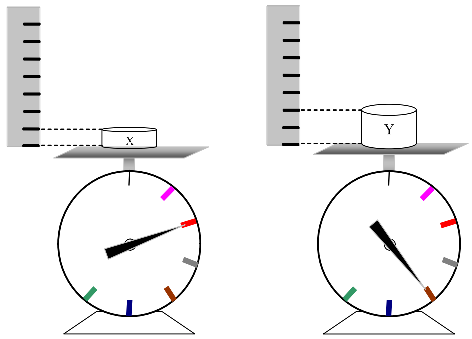
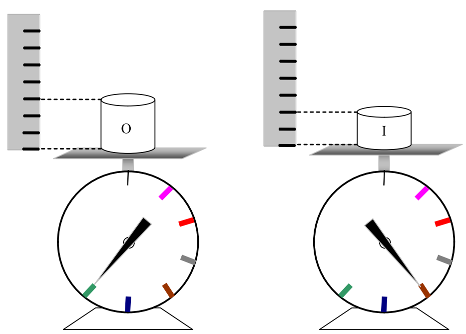


Table 3 (continued): Six Problems in the Application Task

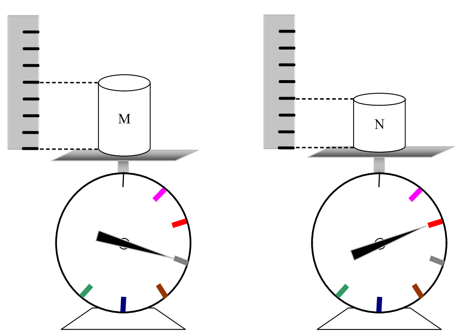
Problem 4: Could X and Y be made of the same kind of material or not?



Problem 5: Could O and I be made of the same kind of material or not?

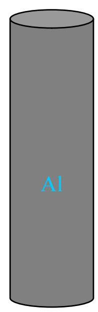
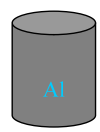
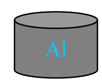


Problem 6: Could M and N be made of the same kind of material or not?



(d) Generalization (Task 9): Here children were asked to compare the weights and the sizes of three bare aluminum cylinders of different sizes, to fill out a table (Table 4) with their relative weights and sizes, and answer the question: “What do you notice about these numbers?” Then they did the same for brass cylinders. This part aimed at helping children to progress from knowing the co-variation of weight and size to knowing that weight/volume is a constant for the same material and is different across different materials.

Table 4: One of the Tables Used in the Generalization Section

Aluminum	 Weight= Size =	 Weight= Size =	 Weight= Size =

Preparing for Data Analysis

First, I transcribed more than half of the interviews (12 out of 20) and identified different answers and reasoning provided by the children, the tools and strategies used by them, the difficulties they had, and changes they experienced, uncovering some important emergent themes and categories for each theme. Then I coded all interviews, task by task, using the categories first detected. I also transcribed parts of the other eight interviews, when different categories appeared in them. These new categories were added to the initial list.

The number of children’s answers in each category was determined and presented as frequency distributions and as percentages, as shown in the following chapters. The Wilcoxon Signed-Rank Test was used to compare children's responses in the pretest to those in the posttest and determine the significance of possible differences. Throughout the analysis, examples of different responses are displayed as transcripts of relevant parts of the interviews. Children's names used here are pseudonyms.

Chapter 5: The Process of Quantification

In the interviews for this study, opportunities for children to quantify weight started from Task 5, when children were asked to seriate covered cylinders by weight using only their hands, and then to measure weight using a modified color scale. A kind of quantification of size and weight also took place in Task 6 when children realized that three short cylinders were the same height as the tall one. Opportunities to quantify size using the simplified ruler started from Task 7, when children were asked to infer the material a cylinder was made of without extra copies of smaller cylinders. Quantification of weight and size occurred as well in Task 8 (application) and in Task 9 (generalization). This chapter examines how difficult it was for the interviewed children to compare the weights of cylinders using only their hands, how well they understood the process of quantification, and the difficulties they encountered across the different tasks while measuring weight and size with the modified color scale and the simplified ruler.

Quantification of Weight

In Task 5, I presented the children with seven cylinders, each wrapped in white paper and described in detail in Table 5. Each child were then given four subtasks:

(a) Order the cylinders by weight using just their hands; (b) Measure the weights of the cylinders with the modified color scale; (c) Order the cylinders by measured weight and compare the ordering obtained from measured weight with the ordering obtained using hands only; and (d) Determine the relationships among weights represented by colors and, accordingly, label the scale with numbers.

Comparing cylinders by weight using hands.

The goals of subtask 5a were: (a) to familiarize the child with the weights of the cylinders, (b) to explore how well children can order cylinders by weight with their hands, and (c) to record this order for later comparison with the order produced with the modified color scale.

Table 5: Seven Wrapped Cylinders Labeled A-G

Weight	Material		
	Delrin (1.5 g/cm^3)	Aluminum (3.0 g/cm^3)	Brass (9 g/cm^3)
Pink (60g)	4cm E	2cm A	
Red (120g)		4cm C	
Gray (180g)	12cm G		2cm B
Green (360g)		12cm F	4cm D

The stimuli in the first subtask are the covered cylinders listed in Table 5. I placed them in front of the child and asked: "These cylinders are like the cylinders over here (pointing to the uncovered cylinders used earlier in Tasks 1 to 3). They are all solid and there is no hollow part. But we wrapped them with white paper. We labeled them with letters, A, B, C, D, E, F and G (pointing to each covered cylinder). Could you arrange them by weight, from the lightest to the heaviest?"

Figure 7: Anthony comparing weights of the seven cylinders with his hands

Figure 7 shows one of the children, Anthony, comparing the weights of the seven cylinders with his hands. The correct ordering was [E/A] C [G/B] [F/D]. That is, E and A are the same weight; C is heavier than E or A; G and B are the same weight and heavier than C; F and D are the same weight and heaviest of all.

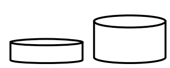
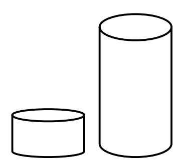
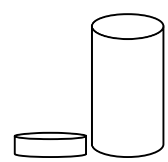
None of the children produced the correct ordering. Instead, they produced 10 different orderings. Table 6 lists the three most common results with their frequencies, with "EAGCFDB" being the most frequent order. When an ordering appeared only once or twice, I put it in the others category.

Table 6: Orderings of Weights Produced by Using Hands.

Ordering of weights	EAGCFDB	EACGFBD	EGACFBD	Others
Number of Children	6	3	3	8

Some pairs were particularly difficult for the children to rank by weight. Tables 7 and 8 show the number of children who gave wrong orderings for pairs of cylinders of the same weight and for those of different weights.

Table 7: Children's Performance in Comparing Weights of Two Cylinders of Same Weight by Using Hands

Pairs of cylinders of same weight			
	A and E	D and F	B and G
Ratio of weight	1:1	1:1	1:1
Ratio of density	2:1	3:1	6:1
Number of children who thought that the denser one is heavier	18/20	19/20	20/20

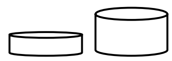
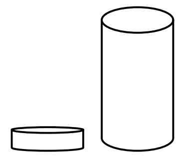
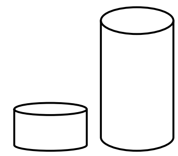
The cylinders in each pair in Table 7 had the same weight, but nearly all the children thought that the denser (or smaller) cylinder in each pair was heavier.

The size of the cylinders seemed to visually influence children's judgment of weight in a process that could be described in this way: One would first expect that small objects should be lighter than big ones; if the small object has the same weight as the larger one, one would perceive the small object heavier than in fact it is and thus heavier than the larger one.

Data from Table 8 show that, for the pairs of cylinders of different weights, when the two cylinders in a pair had different densities (pairs BF and CG), it was much more difficult for the children to find the correct answer with their hands only, in comparison to when the two cylinders in a pair had the same density (pair AC). All of the interviewed children produced a correct answer for pair AC while only a few did so for the other two pairs. Most of the children who gave a wrong judgment about the two pairs of cylinders with different densities thought the denser cylinder in each pair was the heavier one; two children judged B and F to be the same weight.

In sum, none of the children could correctly order the cylinders by weight using only their hands. They felt that the cylinders made of denser material were heavier than they were in fact. Therefore a more precise way for comparing the weights of objects is necessary if one is to solve problems involving the relationships of weights. This was later made clear in Task 6 where, when children did not use the scale or extra copies, it was difficult for them to figure out which short cylinder could be made of the same kind of material as a tall aluminum cylinder.

Table 8: Children's Performance in Comparing Weights of Two Cylinders of Different Weights with Their Hands Only

Pairs of cylinders of different weight	 A and C	 B and F	 C and G
Ratio of weight	1:2	1:2	1:1.5
Ratio of density	1:1	3:1	2:1
Number of children who gave correct answers	20	6	4
Number of children who thought that the denser one is heavier	0	12	16
Number of children who thought that the two cylinders had the same weight	0	2	0

Quantification of weight with the color scale.

After the children had attempted to order the cylinders by weight using their hands, I introduced the color scale to them. Their responses to questions about what the scale showed made it clear that all of them understood that the heavier the object was, the harder it would push the top of the scale and the farther the pointer would move from the starting point. Moreover, most of them could understand the rule of equality, that is, if the pointer goes to the same color, when you put different objects on the scale, the objects have the same weight. However, a few children pointed out that two objects might move the pointer in the scale to the same color but have different weights, because they felt that the objects would have different weights by using their hands. For some children, the perceptual judgment of weights still had an influence on their evaluation of the weights, even though they all said that they trusted the scale more than their hands. However, determining the relationships of the weights represented by the different colors and after labeling the scale with numbers, in subsequent tasks all of the children made their decisions based exclusively on the measured weights from the scale. In Task 5, children put color stickers on the top of cylinder to record their weights. In the later tasks, some children felt the cylinders first and then put them on the scale; some just put them on the scale without feeling them carefully in their hands; others referred to the color stickers to remind them of the weights of the cylinders. In addition, the children also came to understand the rule of additivity, that the weight of two objects is the sum of the weights of two objects. Labeling the scale with numbers was not easy for some children. The fact that the each interval represented the same weight was not intuitive for them. However, with extra copies of E (E "weighs pink," the unit of the scale) they all labeled the scale with numbers correctly. Finally, they demonstrated understanding of the rule of the unit by saying "One [on the scale] is the lightest," and they flexibly changed the units as needed in Task 8.

Table 9 shows examples of children's responses to the questions related to the process of quantification.

Table 9: Responses from the Children to the Questions Related to the Process of Quantification.

Questions	Responses
How Does the Scale Work?	Rachel (10 years old, 4th grade) Interviewer: You can push the top of the scale to see how it works. Rachel: Oh, I got it. If something very heavy is put on here, it will push down which turns this pin. If it goes all the way around, this one got all the way to the bottom, then it is very heavy. If you only push it a little bit like that, the pin won't go very far.
Rule of equality: When these two cylinders have the same color, what does that mean?	Jessica (10 years old, 4th grade) Jessica: It means they are basically the same weight.
	Aaron (9 years old, 4th grade) Interviewer: Are they [two cylinders labeled with gray] the same weight? Aaron: This one feels real light; this feels real heavy. Interviewer: But on the scale they all went to gray. Aaron: [Putting them on the scale one by one] This is gray; that is gray. Interviewer: So when the pointer goes to gray for each of them, what does that mean? Aaron: They weigh in the same category, but the weight is like one is heavier than the other, but barely.
	Caroline (10 years old, 5th grade) Caroline: It is hard to believe that these two [F and B] are the same weight, because my hands said this one [B] is heavier.
	Rachel (10 years old, 4th grade) Rachel: I don't think these are exactly the same. I think this one [B] is a little bit heavier. Interviewer: Yes, that one really feels much heavier than

Questions	Responses
	it is. But when you put these on the scale, the pointer all went to gray, right? Rachel: Yeah. Interviewer: What does that mean? Rachel: It means they are both gray. But the pointer went to here for one and went to here for the other [pointing to different parts of gray]. They are both gray, but the one, which was lower down, would be heavier. Interviewer: Let me see. [I put B on the scale.] It is in the middle of gray. How about the other one? Rachel: [She put G on the scale.] This one is a little bit heavier, because it's right there. Interviewer: But just a little bit. Rachel: Not much. Interviewer: We can think they are almost the same. [In fact, this is different from what she felt. She thought B was heavier, but the scale indicated that G was a little bit heavier.] Interviewer: Do you trust the scale more than your hands or your hands more than the scale? Rachel: I think I trust that scale more.
Rule of additivity: What do you think two pink objects weigh together? If you put these two pink objects on the scale, where will the pointer go?	Randy (9 years old, 3rd grade) Randy: If I put two pink objects on the scale, the pointer probably go to red. Interviewer: You can check on the scale. Randy: Yes, it went to red. Interviewer: So what does that mean? Randy: That means pinks are the same weight. They are both the same weight. And a pink plus a pink equals to a red. Interviewer: So how many times heavy is this red one [C] compared to this one pink [E]? Randy: One times two equals two. Interviewer: You put these two on the scale, the pointer

Questions	Responses
	goes to red. If we have another pink object that you could add on the scale, where will the pointer go? Randy: I don't really… one pink is equal to pink. If we add one more pink, that probably goes in the middle right here [between red and gray], I think. I don't really know. Interviewer: Actually we have some extra copies of E. You see these Es are exactly like this E. Randy: Are they the same? Interviewer: You can feel them. If they are the same, every E will weigh… Randy: Ok, I believe you. [Putting three pinks on the scale] Oh, it actually goes to gray. I think that would be different. Oh, I see what is wrong. Because one pink [cylinder] equals the whole pink [on the scale]. It will go down one full away. [Adding another E] E will go down one full length away. And another E will go down one full length. [Adding another E] one full length. [Adding another E] one full length…These [pink cylinders] will be like ones. C will be twos. Gray will be threes. Green will be fours. [see Figure 8, page 56.] Interviewer: Uh-huh Randy: Green will be five. [Counting on the scale] Ones, twos, threes, fourths, fives, oh, green will be sixes.
	Rachel (10 years old, 4th grade) Interviewer: What do you think two pink objects weigh together? Rachel: I think they weigh about red. Interviewer: You can check them on the scale. Rachel: [Putting A and E both on the scale] Interviewer: Yes, you are right. What does that mean? Rachel: It means… I think two pinks are equal to red. Two reds equal to gray. And two grays

Questions	Responses
	equal to brown. Two browns equal black and two blacks equal to green.
	Interviewer: Actually we have some extra copies of this E. You can feel with you hands to see whether they are the same. Before you put these two on the scale the pointer went to red, right?
	Rachel: Uh-huh.
	Interviewer: How about you add another pink on the scale.
	Rachel: I don't think it will go to gray. I think it will just about into between [pointing to the middle between red and gray]. [Then putting another E on the scale]
	Interviewer: It goes to gray, right?
	Rachel: [Nod]
	Interviewer: How about adding another one?
	Rachel: I think it will go to brown. [Adding another one and the pointer goes to brown].
	Interviewer: How about adding another one?
	Rachel: I think it will go to black. [Adding another one and the pointer goes to black]
	Interviewer: And another one.
	Rachel: I think it will go to green. [Adding another one and the pointer goes to green]
	Interviewer: So how many pinks are equal to green?
	Rachel: One, two, three, four, five, six.
	Interviewer: Six. How many pinks are equal to gray?
	Rachel: [taking off three Es] Three.
	Interviewer: How many pinks are equal to brown?
	Rachel: [Adding one on] four.
	Interviewer: How many pinks are equal to black?
	Rachel: Five.
	Interviewer: And you already know how many pinks are equal to red.

Questions	Responses
	Rachel: Yes. Two.
	Jessica (10 years old, 4th grade) Interviewer: What do you think these two pink objects weigh together? Jessica: It will go to red. Interviewer: You can check it on the scale. Jessica: [putting two pink [A and E] on the scale.] Interviewer: What does that mean? Jessica: It means these two together is that [pointing to the red object [C]. Interviewer: How does the weight of this red object compare to the weight of this pink object? First, which one is heavier? Jessica: C. Interviewer: How many times heavier is this one as much as that one? Jessica: It is half. Interviewer: Which is which half? Jessica: This [E] needs one more half to equal to that [C]. Interviewer: Before when you put these two pinks objects on the scale the pointer went to red. If you add another pink object, where would the pointer go? Jessica: it will go to gray. ...... Interviewer: How many pinks are equal to green? Jessica: Four and half. Interviewer: How did you figure that out? Jessica: You need another half to get to the red, plus one [pointing to gray], so [taking down some cylinders] I will show you. [one pink on the scale] You will need half to get red. Interviewer: Half of what?

Questions	Responses
	Jessica: Half of E. Interviewer: So the weight of E is pink. The weight of A is Jessica: Pink. Oh, it needs one more, so it is five. [She did not count the first one; she figured it out later.]
Label the scale with numbers	Rachel (10 years old, 4th grade) Interviewer: Now could you label this with numbers? Because you have figured out the relationship of them. We have this blank sticker. You can write a number on it and put them on the scale to label the scale. Rachel: I think the pink would be one. Red would be two. Gray would be three. Brown would be four. Black would be five. And Green would be six.
Rule of unit: When you put 6 over here, it is 6 of what?	Aaron (9 years old, 4th grade) Interviewer: When you put 6 over here, it is 6 of what? Aaron: [Pause] Interviewer: What is one? Aaron: One is the lightest.
	Isabel (9 years old, 3rd grade) Interviewer: When you put this [F] one the scale. It is six. It is six of what? Isabel: Six of A.

Figure 8: Randy adding Es on the scale to figure out the relationships of the weights represented with different colors.

From the children's responses to the question related to the rule of equality, we can see that although most of the children, as Jessica did, claimed that objects were the same weight if the pointer went to the same position (or color), for some children, the weights the children felt still had an influence on their measurement of cylinders with the color scale. The rule of equality was natural to them, but they suspected the preciseness of the scale, because, with their hands, they felt that one cylinder was heavier than the other. For example, Aaron said, "They weigh in the same category, but the weight is like one is heavier than the other, but barely." It was good to consider the preciseness of the scale, but their feeling could be the other way around. For example, Rachel thought that B was a little bit heavier than G, although the scale had suggested that G was a little bit heavier than B. That was clearly interference from their perceptual judgment (a denser object was felt to be heavier).

In the question related to the rule of additivity, most children correctly guessed that when you put two pink objects together on the scale the pointer would go to red. This shows they had an intuition of the rule of additivity in comparing quantities. However, in further determining the relationships among different colors, the fact that the same interval on the scale represented the same weight was not taken for granted by the children. But then with identical copies of the cylinder E they could easily determine the numerical relationships among the different weights. Some children, like Randy and Jessica, at the beginning, did not guess that the same interval represents the same weight. They thought that if three Es were on the scale, the pointer would be in at a mid-point between red and gray. Jessica might have been misled by the size of A. At some points she thought the weight of cylinder A was half of pink. Some other children, like Rachel, first thought that increasing one interval on the scale meant that the weight had been doubled (which is true only for the intervals on each side of pink). However, by the end of the task, all children figured out the relationships of different marks (or colors) by adding identical cylinders on the scale and they correctly labeled the scale with numbers. For example, Randy said: "Oh, it actually goes to gray. I think that would be different. Oh, I see what is wrong. Because one pink [cylinder] equals the whole pink [on the scale]. It will go down one full away. [Adding another E] E will go down one full length away."

From these responses to the question related to the rule of unit, we can see that the children understand the role of the unit in the process of measurement. It is likely that they did not mention the unit spontaneously because they did not have a handy word to name it or because they had not developed a habit of describing the value of a quantity with both a number and a unit.

Once the children figured out how to compare weights with the color scale (in Task 5) they used the scale to measure weights of cylinders correctly in subsequent tasks.

Quantification of Size

No child had any problem in using the simplified ruler to determine the heights of the cylinders they could handle. They knew that if two cylinders matched up with the same mark on the ruler, they had the same height and the children were able to correctly match the mark to the number of "shortest cylinders." Unlike the case of the color scale, they did not have any problem with the equally spaced marks on the ruler. However, when it came to the images of rulers in the pictures in Task 8, which clearly included a line at the end of the ruler (see Figure 9), 10 out of 20 children counted that first line at the bottom as one instead of zero.

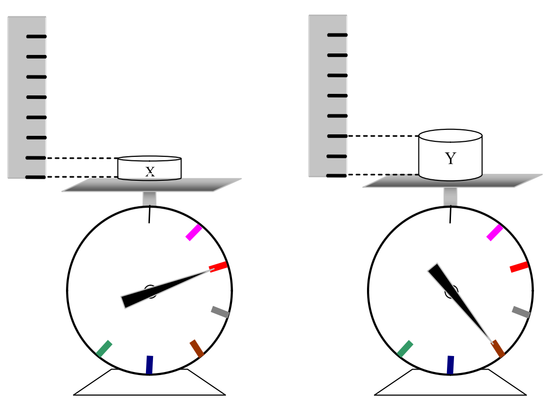


Figure 9: The picture used in Problem 4 of Task 8

The comparison of the ruler in the picture with the real simplified ruler helped them realize that the line at the end of the ruler is zero, not one. Here is an example of how that happened as children solved a ratio comparison problem in Task 8 (see Table 10).

Table 10: Example of Misreading Rulers and Correcting Errors

Action	Dialogue	
Misreading rulers	Caroline (10 years old, 5th grade)	
	Caroline:	This is red. Is that brown? (see Figure 9.)
	Interviewer:	Yes.
	Caroline:	[Thinking]
	Interviewer:	[Handing her a pen] If you need to write down something, you can.
	Caroline:	So this is two; this is three [on rulers]. So...
	Interviewer:	Could you tell me how you got this number?
	Caroline:	Well, Two. There are two lines.
	Interviewer:	Two lines. It's like ...[pointing to the real ruler].
	Caroline:	One [pointing to the line at the beginning of the ruler], two. I think. It's actually one, if you think about it.
	Interviewer:	It is just like this [A] tall, right?
	Caroline:	Oh, yeah. [Correcting the values for the heights of them] one and two. So this [height of x] is already half of this [height of Y]. And this [weight of x] is already half of this [weight of Y]. [Writing down the values of weights of them] This is half of this [in height] and this is also half of this [in weight], so this is possible that they could be the same substance.

Summary of the Results

The analysis above shows the importance of quantification by using measuring tools and the fact that the children at ages from 8 to 11 could understand the process of quantification of weight and size, transforming these properties of the objects into quantities, by assigning numerical values to them.

It was difficult for the interviewed children to evaluate relative weights using only their hands. None of them correctly ordered the cylinders by weight using only their hands and they thought that denser cylinders were heavier than they were in fact. Also, the evaluation of the relationships among the heights of cylinders by looking at them became difficult when the relationship was in the ratio of one to six. In these cases the children needed precise ways to evaluate the relationships among weights and the relationships among sizes.

The data shows that the modified scale, the simplified rule, and the cylinders, carefully designed to lead to integer measurements, played important roles in children's understanding and use of the "three-rule schemas" of measurement, when they were attempting to determine the relationships among weights and that among sizes. The identical cylinders [Es], whose weights corresponded to the weight unit on the scale, were very helpful to children in figuring out that each interval between marks on the scale represented the same weight and in correctly labeling the scale with numbers. However, it must be pointed out that some children counted the first line at the bottom of the ruler as one, instead of zero, when they dealt with the rulers in the pictures. Comparison of the rulers in the picture with the real simplified ruler helped them correct their first answer and read the rulers correctly.

These results strongly suggest that experiencing the whole process of quantification allowed the children to make sense of the numerical relationships among weights and among sizes, probably because it connected their intuitive understanding of these two properties with the numerical measured values obtained with the simplified measuring tools.

The results of this analysis addressed the first part of the sixth research question: What difficulties do children encounter in the process of quantification of weight and size? In Chapter 11, I will further discuss this question, the relation of these findings to those from previous studies, their theoretical implications, and their relevance for science education. In Chapters 6 to 10, I will also examine what role the quantification of weight and size played in the later tasks.

Chapter 6: Merging Quantification into Intuitions: Analysis of Tasks 6 and 7

Tasks 6 and 7 aimed at inviting the children to merge the quantification of weight and size into their intuitions about how to determine whether two objects were made of the same kind of material. The goal is to examine whether (and if so, how) quantification might help children further develop their understanding of the relationships among weight, size, and kind of material.

In Task 6, children first received a single copy of each of four covered cylinders C, D, E, and F, and were asked to determine which of the three short covered cylinders (C, D, and E) could be made of the same kind of material as the tall covered cylinder (F). After answering the question, children were offered extra copies of C, D and E and asked whether all these cylinders together could help them figure out which one could be made of the same kind of material as F. Because the tall cylinder is precisely three times as tall as the short ones, the children could stack short cylinders to produce a combined cylinder of the same height as the tall one. Task 7 posed the same question regarding two short cylinders, A and B, each of which is precisely 1/6 the height of the same tall cylinder (F). But this time no extra copies of the short cylinders were provided. With these two tasks, I examined what measuring tools were necessary for children to infer materials, what difficulties they encountered, and the role of the extra copies of the short cylinders.

Task 6

In Task 6 I asked children: “Could C, D, or E be made of the same kind of material as F?” Cylinders used in this task (see Figure 10) were C (aluminum), E (Delrin), D (brass), and F (aluminum).

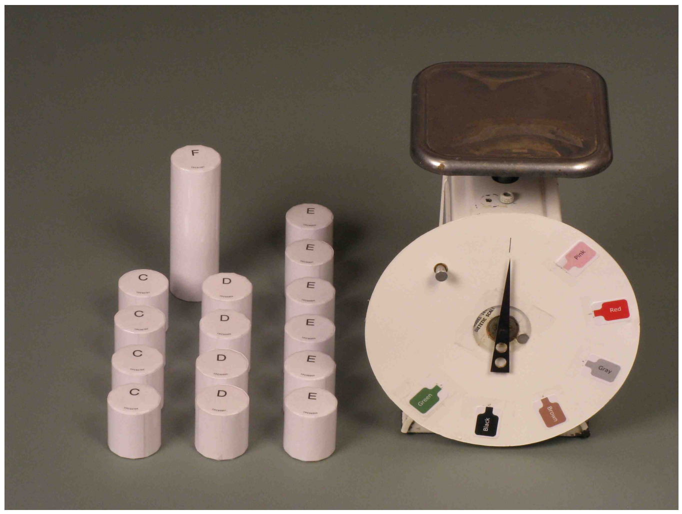


Figure 10: Cylinders and a scale used in Task 6

The goals of Task 6 were (a) to examine strategies 8 to 11 year-old children use to infer the materials objects of different sizes may be made of, (b) to invite children to notice the need to quantify weight and volume, (c) to see whether constructing the same volume by using extra copies would help the children infer the material a cylinder was made of, (d) to determine whether children understood and were able to consider the co-variation of weight and volume, (e) to observe difficulties they encountered, and (f) to analyze how the children’s understanding of “being made of the same kind of material” evolved during the interview.

Table 11 shows that the use of extra copies of the short cylinders had a strong impact on children’s ability to correctly infer which short cylinder could be made of the same material as the tall cylinder. Before being given extra copies, only one of the 20 children (5%) gave the correct answer (along with an incomplete quantitative explanation); six children (30%) gave the correct answer with an ambiguous explanation; one child (5%) gave the correct answer with a wrong explanation; and 12 children (60%) gave a wrong answer. The 12 children who gave an incorrect answer either said that D was made of the same kind of material as F or that none of the comparison cylinders were made of the same kind of material as F. No child thought that E could be made of the same kind of material as F. After extra copies of the small cylinders were available for the children to make further comparisons, 18 children (90%) gave the correct answer along with an explanation that correctly considered that cylinders of same size and same weight would be made of the same material; only two children (10%) failed to provide a correct answer.

Table 11: Answers in Task 6, without and with Extra Copies

Answers	Explanation	Without Extra Copies		With Extra Copies	
		Number of children	Percent	Number of children	Percent
Correct (C)	Same size same weight	1	5%	18	90%
	Approximate comparisons	6	30%	0	0%
	Wrong explanation	1	5%	0	0%
Incorrect (D or None)	Same weight	12	60%	2	10%

Children's explanations for their answers varied across the different moments of the interview and across children, as exemplified and discussed next.

Without extra copies.

Table 12 shows examples of children's responses before I gave them extra copies of the small cylinders.

Table 12: Examples of Children's Responses without Extra Copies in Task 6

Answers and Reasoning	Dialogue
Without extra copies Correct answer Same size same weight (Incomplete quantitative explanation)	Lily (9 years old, 4th grade) Interviewer: Could any of these cylinders be made of the same kind of material as F? Lily: Maybe C. Interviewer: How do you know that? Lily: This [D], since they were the same weight, they couldn't be made of the same material. This [D] is much smaller than that [F]. And this one [E] seems a little bit too light to be made of the same thing. Interviewer: Why do you think C could be made of the same kind of material as F? Lily: I kind of try to feel like that there are three of them and they all stack up together and they are the same size as this [F]. (I have not given her the extra copies yet; she just imagined three Cs stacking up together)
Without extra copies Correct answer Approximate comparisons	Rachel (10 years old, 4th grade) Rachel: As the scale told us, they [D and F] were almost the same weight. Oh, this one [D] is smaller. I think they are made of different material. I think this [D] might made of brass, because this is smaller, but they are the same

	weight. Interviewer: So you said D is not made of the same material as F. Rachel: I think they are made of different kinds of materials. Interviewer: The reason is that you think D is smaller. Rachel: Yeah, it is smaller but weighs the same. I think it is made of brass, which is heavier than the aluminum. Interviewer: So how about these two [C and E]? Could any of these be made of the same kind of material as F? Rachel: I think C is made out of aluminum as well. Because it is kind of heavy, but it's small. And I am guessing that this [D] is brass and this [C] is aluminum. The aluminum is lighter than brass. This one [C] is smaller and lighter. So I think it is made out of aluminum. Interviewer: How about that one [E]? Could E be made of the same kind of material as F or not? Rachel: I don't think so, because it is a lot of lighter. I am guessing this one [C] is made of the same material. And this [E] is a lot of lighter than this one. I think it is made out of plastic.
Without extra copies Correct answer Wrong explanation (same weight) or considering "density"	Aaron (9 years old, 4th grade) Aaron: [Hold D and F, then C and F] This one [C]. Interviewer: How did you figure that out? Aaron: Because they weigh just about the same. (In fact, C does not weigh the same as F. C is one third of the weight of F.)
Without extra copies	Isabel (9 years old, 3rd grade) Isabel: D could be made of the same material as F.

Wrong answer (D)	Interviewer: Why do you think that? Isabel: They [F and D] are kind of weigh the same amount. They are made of the same thing. Except there are more storage in D.
Without extra copies Wrong answer (None of them)	Diana (9 years old, 4th grade) Interviewer: Could any of these cylinders be made of the same kind of material as F? Diana: [Thinking for a while] I don't think they are. Interviewer: You don't think any of them could be made of the same kind of material as F? Why not? Diana: Because none of them feel the same.

When no extra copies were available, only one child (Lily) attempted to give a quantitative explanation by imagining three Cs stacked up together to compose a cylinder the same size as F. However, her explanation was not complete: she did not articulate that three Cs would have the same weight as F. Less than one third of the children (Rachel, for example) only gave an approximate qualitative explanation, such as "This one [C] is smaller and lighter." More than two thirds of the children gave a wrong explanation by only considering the weights of the cylinders. Some of them thought D was the same weight as F; some thought C was the same weight as F; and the others thought that none of them was the same weight as F. This shows the limitations of comparing weights with only hands when one needs more precise evaluation of weight for inferring the material objects are made of. It was also possible that some children, such as Aaron, who said that two cylinders (C and F) that clearly had different weights (F was three times as heavy as C) were the same weight, might have used the word "weight" in a way that was closer to our meaning of density. However, since I didn't question him any further, it is not possible to determine whether this was the case.

With extra copies.

After the children explained their thinking based on single examples of C, D, and E, I offered the children extra copies of each of those cylinders (three each of C and D as well as five of E). The color scale was still available on the table and occasionally I suggested jointly considering the relationships between weights and that between sizes. Except for one child, all provided the correct answer and gave a quantitative proportional reasoning explanation for their answers. All of them referred to the color scale: three children (15%) remembered the weights of the cylinders and did not need to measure them again; 13 (65%) spontaneously measured the weights of cylinders C and F again, or the weight of three Cs together on the color scale; and four children (20%) used the color scale after suggestion (see Table 13).

Table 13: Use of Scale to Measure Weights in Task 6

	Without measuring on the scale again	Measuring on the scale again spontaneously	Measuring on the scale after suggestion
Number of children	3	13	4
Percent	15%	65%	20%

After using the scale and extra copies, most children (75%) used the "same weight and same size" strategy to justify that C could be made of the same kind of material as F. Five children (25%) figured out the relationship between the weights of F and C and the relationship between their sizes but did not spontaneously conclude that C could be made of the same kind of material as F (see Table 14). In these cases, I guided them to consider the two relationships together. Then, except for one of them, they realized that C and F could be made of the same kind of material.

Table 14: Guidance on Jointly Considering Weight and Size in Task 6

	No need of guidance	Needing guidance
Number of children	15	5
Percent	75%	25%

Two examples in Table 15 show how the interview progressed after I offered the children extra copies of short cylinders. One is from a child who needed only to use the extra copies and the color scale. The other is from a child who needed not only the extra copies and the color scale, but also guidance on jointly considering the relationship of weights and that of sizes.

Table 15: Examples of Children's Responses with Extra Copies in the Task 6

Tools and guidance	**Dialogue**
With extra copies Color scale	Without extra copies, Jake (9 years old, 3rd grade) thought that D could be made of the same kind of material as F, because "it labeled green like F". Then I handed him the extra copies of C, D and E. Interviewer: Could all these cylinders together help you figure out whether C or E or D be made of

the same kind of material as F?

Jake: D. [grabbing several Ds] No, D is too heavy to be made of the same material as F. So it's excluded. My two guesses are pretty much E and C, but... [stacking up three 3 Cs]. Three of these [3Cs] [putting 3 Cs on the color scale] are equal to this [green]. When you put them like this [stacking them up], so they are the same amount. This [F] and this [3 Cs] both equal to green.

Interviewer: You mean this [3 Cs] and this [F] are made of the same material?

Jake: Yes, they are just smaller, so they are different weights.

Interviewer: Why did you stack up three together, not 4 or 5?

Jake: Because that is how tall these are.

Interviewer: Just explain to me a little bit more why do you think 3 Cs are made of the same material as F.

Jake: This [F] equals this three [3 Cs] together. This is just a smaller piece of it. And compare to the weight of this 3 Es [putting 3 Es on the scale], these equal to gray. So it can't be this [F]. And three these [putting 3 Ds on the scale]…

Interviewer: That is too heavy for this scale. (The weight of three Ds is out of the measuring range of this color scale.)

Jake: Way too heavy for it. This [F] is lighter [than 3Ds] and this [3 Cs] is green [putting 3 Cs on the scale again] which is what this [F] was labeled. It has to be these [3 Cs].

Interviewer: You said these three Cs could be made of the same material as F. How about one C? Could one C be made of the same material as F?

Jake: Yes.

Interviewer: How do you know that?

Jake: Because these together weighs the same. This is just one of these. One of these

		chopped into 3 pieces equal this.
	Interviewer:	How does the weight of C compare to the weight of F?
	Jake:	Weight is different?
	Interviewer:	How many times heavy is this one as that one? You can use the scale to tell me.
	Jake:	Two times heavier. [I guess he knew that "two times heavier" had the same meaning as "weight three times as much as."]
	Interviewer:	How does the size of C compare to the size of F?
	Jake:	It's smaller. C is smaller than F.
	Interviewer:	How much smaller?
	Jake:	It's like this is one third of this.
With extra copies Color scale Guidance on jointly considering weight and size	Without extra copies, Aaron (9 years old, 4th grade) thought that C could be made of the same kind of material as F, because "they weigh just about the same." Thus, he gave the correct answer for a wrong reason, because, in fact, C does not have the same weight as F. Then I handed him extra copies of C, E and D.	
	Interviewer:	So could all these cylinders together help you show me why C could be made of the same material as F?
	Aaron:	[Holding F and C and looking at them]
	Interviewer:	How does the weight of C compare to the weight of F?
	Aaron:	This [C] is a tiny bit lighter.
	Interviewer:	You have put numbers to label the scale. So...
	Aaron:	Yeah. Two and six, but they feel like the same. This [C] may actually be the black (the black was the color sticker on the scale which was numbered as 5 by Aaron in the previous task).
	Interviewer:	Tell me how does the size of C compare to the size of F?
	Aaron:	[Stacking up three Cs together next to F] we need 3 [Cs] to get to this [F].

Interviewer: So is C made of the same kind of material as F or not.

Aaron: No.

Interviewer: Why not?

Aaron: [Pause]

Interviewer: You said three Cs have the same height as F. So how many Cs will have the same weight as F [pointing to the scale].

Aaron: [Put Cs on the scale] One, not two, three. So three Cs have the same weight as F.

Interviewer: You said three Cs have the same weight as F and three Cs have the same height as F.

Aaron: Yes.

Interviewer: So could C be made of the same kind of material as F or not?

Aaron: No.

Interviewer: Why not?

Aaron: Because it would weigh the same. Wait, they might not weigh the same. They would [be] pretty close in weight. [Holding F and C] Like, this feels heavier than this… a lot of heavier than this.

Interviewer: So you said C is not made of the same kind of material as F, right? Could 3 Cs be made of the same kind of material as F?

Aaron: Yes.

Interviewer: Why do you think that?

Aaron: This [C] could be one third of this [F] weight, these two could be two third of this, this is three third of this.

Interviewer: Do you mean the weight or the height?

Aaron: The weight.

Interviewer: The weight. How about the height?

Aaron: This would be one third of the height, two thirds and three thirds.

Interviewer: Did you say three Cs could be made of the same kind of material as F?

Aaron:	Yes.
Interviewer:	How about one C? Could one C also be made of the same kind of material as F?
Aaron:	No.
Interviewer:	No. Why not?
Aaron:	[Pause]
Interviewer:	What does the same material means to you? How to decide if something is made of the same kind of material or not?
Aaron:	Like its weight, these three [Cs] go into this [F]. Actually it could, because if these [Cs] were unwrapped, it were like this [3Cs stacking up together], it will actually weigh the same as this.
Interviewer:	How about one C?
Aaron:	It probably is the same, because if these two are there [on the top of one C], it will be the same [as F]. This [C] is just lighter than this [F].

Once Jake grabbed D cylinders together in one of his hands, he realized that D could not be made of the same kind of material as F because three Ds in a pile that is the same size as F, are much heavier than F. He promptly revised his previous strategy (same weight means same material) and began to use a “same size and same weight” strategy to infer the cylinders’ material.

I asked all the children the questions “How about one C? Could one C be made of the same material as F?” and asked about the relationships between F and C in weight and in size. With these requests, I wanted them to extend their “same size and same weight” strategy to considering proportional relationships between weight and size. That is, if C were one third of F in both weight and size, C could be made of the same kind of material as F. Thus, even when there were no extra copies, as in Task 7, they would still correctly determine the cylinders' materials.

Aaron, on the other hand, could give the correct answer only after a prompt to jointly consider the relationship of weights and that of sizes. As shown in Table 12, Aaron, most of the time, used "same weight" as an indicator of being made of the same kind of material as F. Without the extra copies, he thought C could be made of the same kind of material as F, because they felt the same weight. By looking at the scale, he found out that the weight of C is two units and the weight of F is six. He then expressed that C could not be made of the same kind of material as F. At this point, although he also had found out that three Cs had the same height as F, he did not make a connection between the relationship in weight and that in size.

I repeated his comments "You said three Cs have the same weight as F and three Cs have the same height as F." But he still did not think one C could be made of the same kind of material as F, because F and C did not have the same weight. Then I let him judge three Cs, instead of just one. I asked him whether three Cs could be made of the same kind of material as F. He said yes, because three Cs have the same weight as F. Obviously he still used "same weight" as an indicator of being the same material.

Then I drew his attention to the heights and asked again whether one C could be made of the same kind of material as F. He still said no. I guess he still did not think the sizes of the cylinders had anything to do with inferring material. Then I asked him how to decide whether two things could be made of the same kind of material. He mentioned the weight and, at this point, realized that one C could be made of the same kind of material as F, stating that: "It probably is the same, because if these two are there [on the top of one C], it will be the same [as F]. This [C] is just lighter than this [F]." He may have noticed that one C was part of three Cs and since three Cs was the same material as F, C must be too.

In this task, Aaron activated two ideas: first he thought that "same weight" indicated "same material"; then he activated another idea, that is, if one thing is part of another thing, they should be made of the same kind of material. But it is not just the case that the second idea won. Although he did not express it clearly here, he reconciled these two ideas by considering the relationship between sizes. Later, in Task 7 with cylinders A, B, and F, he had a clearer coordination between weight and size. Then, in Task 8, he often used "same height and same weight," instead of just "same weight," as a criterion for being made of the same kind of material through most of the problems.

Summary of Task 6 results.

From children's answers and explanations during Task 6, one can conclude that the extra copies played an important role in their performance in inferring materials and in refining their understanding of how the co-variation of weight and volume can be used to determine if two objects are made of the same kind of material. Using the color scale was also necessary for successfully inferring materials in Task 6. Even after having weighed the cylinders in Task 5, some children still felt that C was the same weight as F, or that none of the cylinders C, D, and E was the same weight as F, and gave a wrong judgment or a wrong justification. It was only after they once more weighed the cylinders using the scale, or compared their weights using the extra copies, that they reached correct answers. A few children did encounter some difficulty in jointly considering both weight and size but, with slight guidance and by spontaneously activating and reconciling different ideas, these children were able to explicitly consider both weight and size.

Task 7

The question for Task 7 was “Could A or B made of the same kind of material as F?” Cylinders in this task were A, B, and F (see Figure 11). A and F are made of aluminum and B is made of brass. Therefore the correct answer was that A was made of the same material as F.

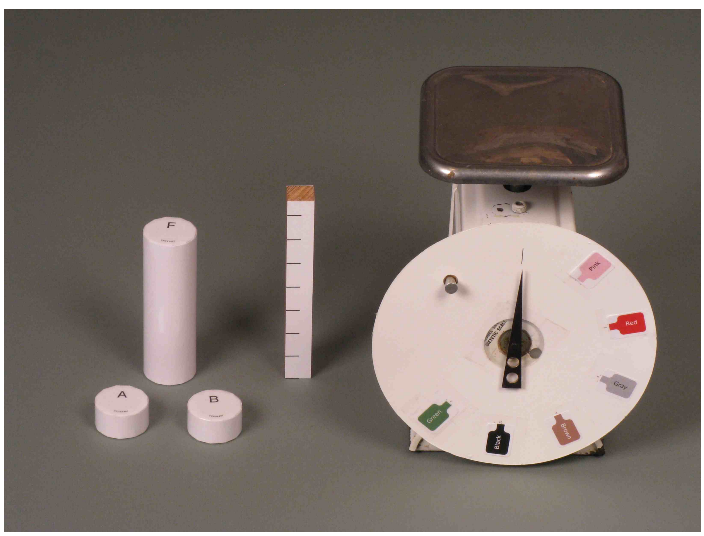


Figure 11: Cylinders and tools used in Task 7

The goals of this task were: (a) to determine whether children could infer the material of the target cylinder without the help of extra copies; (b) to give children a chance to find out or to explicitly express that, when inferring material, one must consider not only the weights of the objects, but also their sizes; (c) to let children realize the need of a more precise measurement of weight and height than using only their hands and eyes; (d) to determine whether, when extra copies were not available, making the quantification of weight and size available with the color scale and the simplified ruler would help children infer the materials cylinders of different sizes and weights were made of; and (e) to determine whether the children could further their understanding of "being made of the same kind of material because they have same weight and same size" to a more advanced understanding of "being made of the same kind of material because one object has 1/n weight and 1/n size of the other one (or $W_1/W_2=V_1/V_2$)."

In Task 7, without extra copies of A or B, children had to estimate the relationships of weights and that of sizes with their eyes and hands, until hey resorted to the measuring tools (the color scale and the simplified ruler). Most of them did not get precise estimations or felt that it was hard to do so without the measuring tools.

Size.

Only 5 out of 20 children (25%) guessed correctly that F was six times as tall as A or B by only looking at the cylinders or by placing the shorter cylinder against the longer one; two children (10%) found the relationship between the size of F and that of the shorter cylinders by comparing the size of A or B with C[3] and guessing that A or B is half of C, since they knew that three Cs are the same

[3] The cylinders C, E, and D they used in Task 6 were still on the table. I put them aside, but the children still could see them.

height as F; and the remainder (13 children; 65%) came to a correct judgment of the size relations only with the use of the ruler (see Table 16).

Table 16: Comparing Sizes in Task 7

	Estimating by looking or measuring iteratively	Using C as a mediator	Using the simplified ruler
Number of children	5	2	13
Percent	25%	10%	65%

Examples in Table 17 show the different ways they used to compare sizes and the role of the simplified ruler in successfully determining the cylinder material. It also exemplifies how some children thought that F was four or five times as tall as the smaller cylinders, until they used the simplified ruler.

Table 17: Examples of Different Ways of Comparing Sizes

Tools and actions	**Dialogue**
Measuring iteratively	Caroline (10 years old, 5th grade) Caroline: [Stacking up A and B next to F] This will be into six I believe. [Lifting A and B together along cylinder F to measure the height of F iteratively]
Using C as a mediator	Rachel (10 years old, 4th grade) Interviewer: How about A? Could A be made of the same kind of material as F or not? Rachel: I think that it could. My guess is that it's [A] made of the same material because for the Cs it took 3 Cs... and it was the same height and same weight [pointing to F].

		And for this [A], it's half smaller I say. So it will take six of these [A] to make it the same height [pointing to F]. I am guessing the same weight.
Using the simplified ruler	Lily (9 years old, 4th grade)	
	Interviewer:	I see. So now just focus on A. Is there some way you can show me why do you think A could be made of the same kind of material as F?
	Lily:	[Putting A and F next to each other to compare them]
	Interviewer:	You can use the scale.
	Lily:	Ok. [Putting A on the scale] Yeah, it's pink.
	Interviewer:	So if A is made of the same kind of material as F, what can you say about the size of A?
	Lily:	It's like a fifth of F.
	Interviewer:	Why do you think it's a fifth of F?
	Lily:	[Comparing the size of A with F in a iterative way] Maybe a fourth.
	Interviewer:	We have this kind of thing you can use. [Handing her the simplified ruler]
	Lily:	[Measuring them] Actually it looks more like sixth.
	Interviewer:	Sixth. The size of this [F] is six times as much as this one [A]. How do the weights of A and B [I meant F and she knew what I meant.] compare to each other.
	Lily:	This one is definitely a lot of lighter. Maybe six of these would be equal to this.
	Interviewer:	How do you know that?
	Lily:	If they are made out of the same material, then six of these, if you stacked them up, should be about the same size. If they are made of the same material, their weights should equal to each other.

Weight.

In Task 7, only 2 out of 20 children (10%) did not need to use the scale again, because they remembered the relationships among the weights as represented by the colors on the cylinders. Ten children (50%) went back to the color scale spontaneously to measure the cylinders again or figured out the relationships of the weights by looking at the labels on the scale. Eight children (40%) could not find the relationships among the weights of A, B and F, until I suggested them to use the scale (see Table 18). Table 19 shows an example of using the scale after a suggestion.

Table 18: Using the Color Scale to Compare the Weights in Task 7

	Without measuring on the scale again	Measuring on the scale again spontaneously	Measuring on the scale after a suggestion
Number of children	2	10	8
Percent	10%	50%	40%

Table 19: An Example of Using the Scale after a Suggestion in Task 7

Tools	**Dialogue**
Using the scale to find the relationships between weights after a suggestion	Rachel (10 years old, 4th grade) Rachel: If there are more As, I could stack them up to see if it takes six to equal one F and to see if it takes the same amount on the scale to equal green. Interviewer: But we do not have any other As. So could you figure that out with this scale? Is there some ways you can tell if F weighs six times as much as A?

> Rachel: If I weigh it [A] once, it's pink. If I take it off and weigh it again and it's pink, and we know that two pink equal one red. So we can just see how many it would go to green.

Jointly considering the relationships in weight and in size.

Even though there were no extra copies available in Task 7, most children (75%) could figure out the relationships among weights of A, B and F and the relationships among their sizes, and jointly consider the two relationships to determine that A could be made of the same kind of material as F. Five children (25%), even though they had determined that F was six times as heavy as A and six times as tall as A, still did not realize that A could be made of the same kind of material as F (see Table 20). Three of them were also among the five children who needed guidance on jointly considering weight and size in Task 6. In these cases, after I guided them to jointly consider the relationship between the weights and the relationship between the sizes, four of them reached the correct answer. The remaining child came to the correct answer only when we returned to this task after doing Task 9; on that occasion he didn't need guidance on jointly considering weight and size.

Table 20: Jointly Considering the Relationships in Weight and in Size in Task 7

	Spontaneously jointly considering the relationships in weight and in size	Needing guidance on jointly considering the relationships in weight and in size
Number of children	15	5
Percent	75%	25%

In Table 21, there are two examples of responses from the children who spontaneously considered jointly the relationships between weights and between sizes and one example from a child who needed guidance on jointly considering these relationships.

Table 21: Examples of Children's Responses When Inferring Material in Task 7 with or without Guidance on Jointly Considering the Relationship between Weights and that between Sizes

Tools and guidance	**Dialogue**
Spontaneously jointly considering the relationship between weights and that between sizes	Henry (8 years old, 3rd grade) Henry: I don't think B is the same material. I think it's heavier. It looks like one sixth as tall, but weigh half its weight. Interviewer: How about A? Henry: I think A is the same material. Because it's one sixth as tall and it's one sixth of the weight.
Spontaneously jointly considering the relationship between weights and that between sizes Using both the scale and the simplified ruler	Jim (9 years old, 4th grade) Jim: [stacks up A with B to compare with F] So that is two, equaling C. So times… pink [put A on the scale]. One [start from red], two [gray], three [brown], four [black], five [green] and six [pointing to some place next to green]. And B is gray, so one [gray] then [pointing to green]. I don't think so. Interviewer: You don't think so what? Jim: Gray is heavier than F. A is pink, but it will go over if you add too many. (That is because he miscounted the markers on the scale by starting from red.) Interviewer: Do you mean neither of these two are made of the same material as F.

	Jim:	Yes.
	Interviewer:	Actually, we have this kind of thing. It's kind of like a ruler. Do you think it will help you figure out the sizes of them?
	Jim:	[counted on the ruler, then counted on the scale] Wait. Pink [A] probably the same weight [he meant the same material] as green [F].
	Interviewer:	Why?
	Jim:	[Pointing to the scale] One A [pointing to pink], this is 6 [pointing to green]. One six times. Because I noticed it goes 6 times on this [the ruler]. So, that is 6 of 1 [pointing to the scale]. So I think they will be the same.
	Interviewer:	I just want to know how did you figure out that A is made of the same material as F.
	Jim:	You can put A up 6 times on this [the ruler] to get to F. And 6 pinks will be equaling green.

Needing guidance on jointly considering the relationship between weights and that between sizes

At first, Isabel (9 years old, 3rd grade) thought that B could be made of the same material as F, because "they are kind of the same weight" and "A is lot of lighter than F and B." After a suggestion of using the scale, she found that B did not have the same weight as F. Then I drew her attention to the relationships between A and F in weight and that in size at the same time.

Using both the scale and the simplified ruler	Interviewer:	How does the size of A compare to the size of F?
	Isabel:	It will take six of As to make one of F.
	Interviewer:	How does the weight of A compare to F?
	Isabel:	A is a lot of lighter and F is heavier. F and B, they feel they're the same, but they are not really the same weight.
	Interviewer:	I just want to focus on A. We have the scale over here, could you tell me how many times heavy is this one [F] as that one [A].
	Isabel:	How many what it [A] takes to fill this [F]?

Interviewer:	I mean the weight, not the size.
Isabel:	[Feeling them in hands then putting A on the color scale] It will take six of these to make one of these [F].
Interviewer:	So you have found out that the weight of this F compared to A is six times as heavy as that one [A], right? And the size of F is six times as big as that one [A]. So could A be made of the same kind of material as F?
Isabel:	[Nod]
Interviewer:	Aha.
Isabel:	No.
Interviewer:	Why not?
Isabel:	Yes, it could.
Interviewer:	Why they could?
Isabel:	Because [inaudible] it will take six of A to make F... [inaudible] If you try to make the same height, you have to have six As to make F.

Henry remembered the weights of the cylinders, so he did not need to go back to the scale to check them. He also carried out a surprisingly accurate evaluation of the relationships among the sizes of F, A, and B by just looking at them. Then he spontaneously coordinated the weights and the sizes of the three cylinders and figured out that A could be made of the same kind of material as F.

However, Jim needed to use both the scale and the simplified ruler, and then spontaneously jointly considered both weight and size. Jim figured out the height of A by comparing A and B together with C. I guess, at that moment, he figured out that F was six times as tall as A, although he did not articulate it. He tried to find the weight of six As by counting six marks on the scale. But he miscounted the labels on the scale, because he counted red instead of pink as one.

So he mistakenly found that six As would go beyond green, which was the weight of F. Thus, he did not think that A could be made of the same kind of material as F. Then since he did not articulate that the relationship between the size of A and that of F, at that moment, I thought he had not figured that out, so I handed him the simplified ruler. After counting the marks on the ruler, he imediately went back to count the marks on the scale again. This time he got the counting right and then gave the correct answer. Children's responses like this show the importance of accurate measurements when attepting to infer the cylinders' material.

Isabel had a really hard time in dealing with the two inferring material tasks (Tasks 6 and 7). In Task 6 she did not think that C could be made of the same kind of material as F even with extra copies, the scale, and guidance on considering jointly the relationship between weights and that between sizes. We then moved on to Task 7. At the beginning, Isabel thought that same weight was the only indicator of being made of the same kind of material and answered that B could be made of the same kind of material, because "they are kind of the same weight." That was the reason that even after she found that it would take six Bs or As to make F in size, using the simplified ruler, and that it would take six As to make one F in weight, using the color scale, she still did not think that A could be made of the same kind of material as F. Only after I guided her to focus on both the relationships between the weights and between the sizes of A and F, did she realize that A could be made of the same kind of material as F, by considering the importance of having six As to make a pile the same size as F. Due to limitations of the quality of the video and her low voice at some moments, I could not catch her complete reasoning. After Task 7, we went back to Task 6 with the cylinders C, D and, E and she very easily concluded that C could be made of the same kind of material as F by stacking three Cs together to make them the same height as F and putting three Cs on the scale to see whether they had the same weight as F. We can therefore conclude that at the end of the interview she had developed an understanding that one can determine that two cylinders are made of the same kind of material if they have the same weight and the same size.

Children's improvement in Task 7 as compared to Task 6.

Table 22 compares children's performance in Tasks 6 and 7. In Task 6, without extra copies, only 8 out of 20 children (40%) gave the correct answer and only one of them gave a quantitative (though incomplete) explanation. In Task 7, with the scale and the ruler, all the children except one (95%) arrived at the correct answer and an explanation that proportionally took into account weight and size. Therefore, 18 children (90%) made progress in these two inferring material tasks with the help of the scale, the ruler, and the scaffolding of the extra copies. With one exception, the children in this study understood that, for the objects made of the same kind of material, the weight covaried with the size, or, in other words, that if two objects were made of the same kind of material, the relationship between them in weight should be the same as the relationship between them in size. The only exception, Anthony, a third grader, thought in both Tasks 6 and 7 that "same weight," regardless of size, was the indicator of being made of the same kind of material. However, after Task 9, we went back to

Tasks 6 and 7 and he then figured out that C and A could be made of the same kind of material as F, considering both weight and size. From then on, he correctly coordinated weight and size in the later tasks (Tasks 8 to 11).

Table 22: Improvement from Task 6 to Task 7

Answers	Explanation	Without extra copies in Task 6		Without extra copies in Task 7	
		Number of children	Percent	Number of children	Percent
Correct	Same size and same weight (or 1/n size and 1//n weight) means same material	1	5%	19	95%
	Approximate comparisons	6	32%	0	0%
	Wrong explanation	1	5%	0	0%
Wrong		12	58%	1	5%

Analysis of Task 7 shows that the lack of extra copies invited the children to recognize the need to measure the weights and sizes of cylinders A, B, and F in order to make productive comparisons to F. Accurate measurements of weights and sizes with the color scale and the simplified ruler played important roles for most children in successfully inferring the material of the small cylinders, one sixth of the size of F. All the children, except one, could further their understanding of cylinders "being made of the same kind of material because they have same weight and same size" to a more advanced understanding of "being made of the same kind of material because one object has 1/n weight and 1/n size of the other one." It is interesting to notice that many children still justified their answers by imagining six As would make an F in weight and in size. It is clear that they kept using the "same size and same weight" strategy to support and/or make sense of the "1/n size and 1/n weight" strategy.

Children's Performance in Task 6 and 7 by Grade Levels

Now let's look at children's performance in Tasks 6 and 7 by grade levels. Table 23 shows that, in Task 6, there is no difference across the grades in the percentage of children who needed guidance on jointly considering weight and size. In Task 7, however, a larger percentage of third graders (38%) needed guidance on jointly considering weight and size, in comparison to fourth graders (13%) and fifth graders (25%).

Table 23: Percentage of Children Who Needed Guidance on Jointly Considering the Relationships in Weight and in Size by Grade Levels

Grade	Total children in each grade	Task 6		Task 7	
		Number of children	Percent	Number of children	Percent
Third	8	2	25%	3	38%
Fourth	8	2	25%	1	13%
Fifth	4	1	25%	1	25%

Table 24 shows the improvement in the performance of children in different grades in Task 7 compared to Task 6. In Task 6, without extra copies, a larger percentage of third graders (75%) gave wrong answers in comparison to fourth graders (50%) and fifth graders (50%). In Task 7, all fourth and fifth graders gave correct answers with good explanations and 87% of third graders gave correct answers. The third graders' increase in performance (from 25% of correct answers in Task 6 to 87% correct in Task 7) was therefore larger than the increase in performance among the fourth and fifth graders (from 50% to 100%).

Table 24: Improvement in Task 7 Compared to Task 6 by Grade Levels

Answers	Explanation	Without Extra copies in Task 6			Without Extra copies in Task 7		
		Third Grade	Fourth Grade	Fifth Grade	Third Grade	Fourth Grade	Fifth Grade
Correct	Same size same weight means same material	0	1 (13%)	0	7 (87%)	8 (100%)	4 (100%)
	Approximate comparisons	2 (25%)	2 (25%)	2 (50%)	0	0	0
	Wrong explanation	0	1 (13%)	0	0	0	0
Wrong		6 (75%)	4 (50%)	2 (50%)	1 (13%)	0	0

Summary of Results

The scaffolding of the extra copies, the modified scale and the simplified ruler play important roles in children merging quantification into their intuitions. Without extra copies at the beginning of Task 6, only 40% of the children gave the correct answer and, among them, only one child attempted to give a quantitative explanation by imagining three Cs stacked up together to form the same size as F. After I offered them extra copies, with only one exception, the children provided the correct answer and gave a quantitative proportional reasoning explanation. In Task 7, even without extra copies, with the modified scale and the simplified ruler, all the children except one arrived at the correct answer and an explanation that proportionally took into account weight and size. Even though some third graders had a harder time working with these two tasks, third graders learned more from the tasks.

This chapter's analysis answers the first two research questions: What ideas do children bring to tasks that require working with the relationships among weight, size and the kind of material objects are made of? How do children use and transform these different ideas to further their understanding of the relationships among weight, size, and the kind of material objects are made of? It also addresses the first part of the fifth research question: What roles does quantification play in the process of activating and reconciling children's initial ideas to construct more coherent ideas when solving problems?. In Chapter 11, I will discuss the answers to these research questions, the relation of these findings to those from previous studies, their theoretical implications, and their relevance for science education.

Chapter 7: Applying Newly Constructed Knowledge in New Situations in Task 8

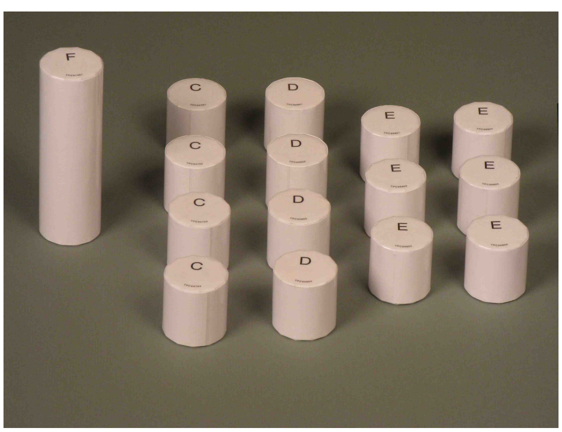
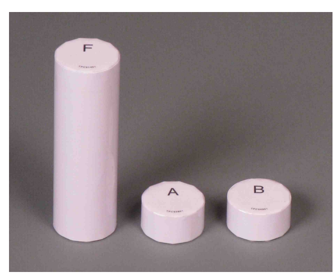
In Task 8 children were asked to solve three missing-value problems and three ratio-comparison problems. These application problems aimed at consolidating children's new understandings about the relationships among weight, size, and kind of material and at determining how well they would use the new understandings to solve more abstract problems in new situations. The analysis also aimed at exploring the kinds of strategies the children tended to use in particular situations.

Missing value problems and ratio comparison problems are two fundamental types of problems involving proportionality analyzed by researchers in mathematics education (see Schliemann & Carraher, 1992). Missing value problems require one to determine the value of a particular variable given three values: the corresponding value of the other variable as well as an additional pair of values relating the two variables. Ratio comparison problems provide information on the values of two variables (e.g., price and weight or volume and weight), in two different situations, and require determining which situation is relatively costlier, denser, etc.

The first three problems in Task 8 were missing value problems in which children were given one or two cylinders and asked to imagine a target cylinder (made of the same material) of a certain size (or weight) and to infer its weight (or its size). The last three problems were ratio comparison problems. For each of these problems, I presented the children with a picture showing a pair of hypothetical cylinders and their readings on the ruler and on the scale (see Figures 15, 16, and 18) and asked whether the pair of cylinders could be made of the same material or not.

Task 8 was given immediately after the inferring material tasks (Tasks 6 and 7, where the children had found that A and C were made of the same kind of material as F) and all the cylinders (A, B, C, D, E, and F; see Table 25) used in these two previous tasks were still on the table. In addition, the missing value problems in Task 8 took the weights and sizes of C, E, and F as given information. Thus, in Task 8, some children spontaneously considered some of the cylinders from the two previous tasks, in addition to the cylinders C, E, and F, which were in front of them and referred to in the questions.

Table 25: Cylinders Used in Tasks 6 and 7
(shown here because children referred to them as they worked on Task 8)

Cylinders used in Task 6	Cylinders used in Task 7
	

General Analysis

In Task 8 all 20 children correctly answered Problems 1, 2 and 4. For Problem 3, 80% of the children gave the correct answer, and 55% of children gave the correct answer to Problem 5. Most of the children who gave correct answers to Problem 5 provided clear explanations for their answers. Ten children who did not give the correct answer or did not give a good explanation to Problem 5 were asked to answer the problem again, after they had worked on Task 9. At that time, four of them provided the correct answer. In Problem 6, half of the children gave the correct answer and, of those, fewer than half gave a good and complete explanation (see Table 26).

Table 26: Frequency of Correct and Wrong Answers and Explanations in Task 8

		Number of children					
Answers	Explanation	P1	P2	P3	P4	P5	P6
Correct	Correct	20	20	16	20	10	4
	Wrong	0	0	0	0	1	6
Wrong	Wrong	0	0	4	0	9	10

Problems 1 to 3: Missing Value Problems

On the basis of previous studies one may categorize strategies for solving proportionality problems as: (a) constant difference or purely additive strategies, (b) scalar strategies by addition or building up, (c) scalar strategies by multiplication, and (d) functional strategies (see Table 27).

Table 27: Strategies for Solving Proportionality Problems

<table>
<tr><td>Strategies leading to errors</td><td colspan="2">Constant difference/purely additive strategy</td></tr>
<tr><td rowspan="3">Strategies leading to correct answers</td><td rowspan="2">Scalar strategy</td><td>Scalar strategy by addition
(or building-up strategy)</td></tr>
<tr><td>Scalar strategy by multiplication</td></tr>
<tr><td colspan="2">Functional strategy</td></tr>
</table>

Constant difference or purely additive strategies consist of establishing an additive relationship within a ratio and extending it, also additively, to the second ratio (Hart, 1981; Inhelder & Piaget, 1958; Karplus & Karplus, 1972, Tourniaire & Pulos, 1985). This strategy leads to errors since it doesn't consider proportionality.

The scalar strategy, described by Vergnaud (1983) as opposed to the functional strategy, carries out parallel transformations on both variables, thereby maintaining their values in the same proportional relationship. That is, when there is an increase in one variable, there is a proportional increase in the other variable. Parallel transformations are carried out either by multiplication or by addition. When parallel transformations are carried out by addition, some researchers, like Hart (1981), called it a building-up strategy. In the analyses of this study, I will call it a scalar strategy by addition in order to indicate the feature of parallel transformations on both variables. And when parallel transformations are carried out by multiplication, I will call that a scalar strategy by multiplication.

The functional approach consists of finding the ratio between the two variables and using this function operator to determine the missing term of another pair of values of the two variables. The function operator represents the coefficient of the linear function of variables of different nature.

Problem 1.

As already described in the Methods section, the materials used in Problem 1 are those in Figure 12 and the question posed to the children was:

"We already know that F is made of aluminum and its weight is green. What would be the height of a cylinder, if it is also made of aluminum and its weight is gray?"

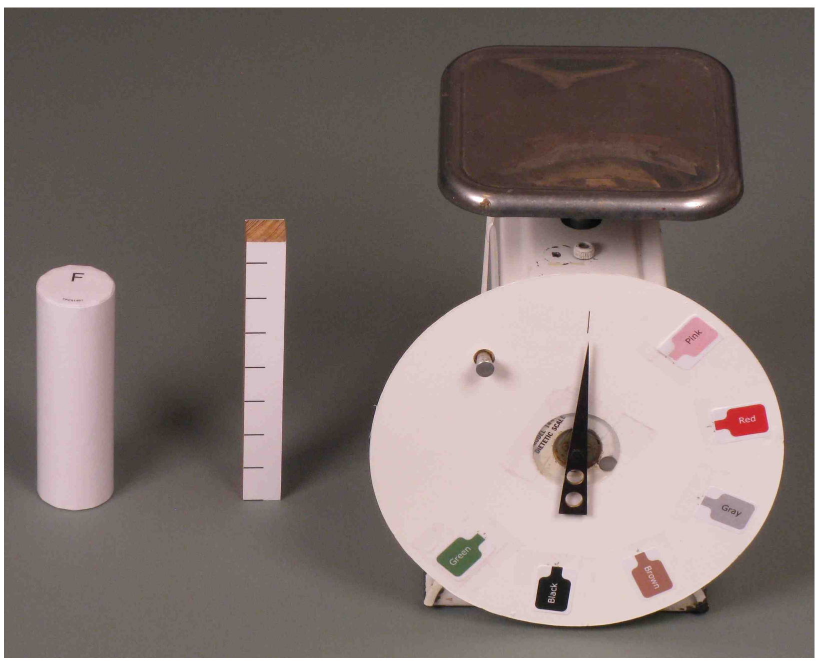


Figure 12: The material used in Problem 1 of Task 8.
(The weight of F is green).

All 20 interviewed children either gave the correct answer right away, as did Henry (see excerpt in Table 29), or did so after a wrong answer and a second presentation of the problem, as did Jennifer (see excerpt in Table 29).

Three kinds of strategies were found for solving this problem: matching numbers, scalar strategy by addition, and scalar strategy by multiplication (see Tables 28 and 29). The matching numbers strategy was different from all strategies found in the previous studies described above. The children who used this strategy did not articulate any mathematical operation. They just enumerated the weights and the sizes of other cylinders made of the same kind of material as the target cylinder and inferred the size of the target cylinder based on its weight so that the number for the weight was the same as the number for the size (see the excerpt in Table 29). This may not be a new strategy, but it is hard to tell what hidden operations the children were using, because the numbers for weight and size were the same (that is, the functional relationship between weight and size is 1).

As shown in Table 28, more than half of the children applied a scalar strategy by multiplication with the simplest relationship (double or half) to find and explain the answer to the problem, as exemplified by the transcription of Henry's response in Table 29. Three children used a scalar strategy by addition of real cylinders, stacking up A and C [A and C were both made of aluminum] to represent the target cylinder, in order to find the height of the target cylinder (see Jennifer's response in Table 29). Four children matched the number of the weight with the number of the size, as Caroline did (see the excerpt in Table 29).

Table 28: Strategies Used by Children in Problem 1 of Task 8

Strategies	Number of children	Percent
Scalar strategy by multiplication (double or half)	13	65%
Scalar strategy by addition	3	15%
Matching numbers	4	20%

Table 29: Examples of Strategies Used in Problem 1 of Task 8

Approaches	Excerpts of Responses
Scalar strategy by multiplication (Half)	Henry (8 years old, 3rd grade) Henry: It would be this tall [Pointing to the third mark on the simplified ruler]. Interviewer: How did you figure that out? Henry: Because it's half of this tall and half the weight, which make them the same material.
Scalar strategy by addition	Jennifer (10 years old, 5th grade) Jennifer: Should be as tall as that B is. Interviewer: Why do you think that? Jennifer: Because this is gray. Interviewer: I mean another cylinder that is also made of aluminum, the same kind of material as F. If the weight of this aluminum [cylinder] is gray, how tall should that cylinder be? Jennifer: If that was made out of aluminum? Interviewer: Yes. And has the weight of gray. Jennifer: [Put A and C together on the scale; the pointer goes to gray; take them off and stack them up; put them on the scale again] This tall. [Then use ruler to measure] It's like three of these.

Matching numbers

Caroline (10 years old, 5th grade)

Caroline: Well, this one [A, short cylinder, 1/6 of the height of F] is only this one [pointing to pink on the scale]. One of these [the space between two markers next to each other on the ruler; one division] and this is one [pointing to pink on the scale]. And red [pointing to the cylinder C] is two [on the scale] and it is number two [pointing to the ruler] and this [F] is number six and it's six. So my guess is that gray is going to be number three [on the ruler].

Problem 2.

Figure 13 shows the material used in Problem 2. The question was: "If we have another aluminum cylinder that is twice as tall as the short cylinder C, how heavy would this cylinder be?"

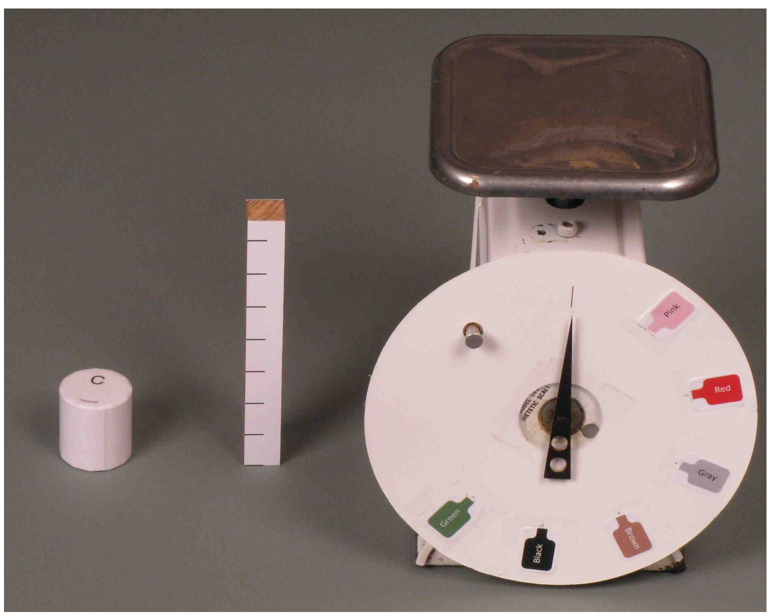


Figure 13: The material used in Problem 2 of Task 8.
(The weight of C is red.)

All children correctly answered this question, easily and even more quickly than was the case for Problem 1. Most of them (75%) used a scalar strategy by addition and the others used a scalar strategy by multiplication with the simplest relationship (double or half) (see Table 30). Table 31 shows examples of these two strategies the children used in this problem.

Table 30: Strategies used by children in Problem 2 of Task 8

Strategies	Number of children	Percent
Scalar strategy by addition	15	75%
Scalar strategy by multiplication (double or half)	5	25%

Table 31: Examples of Strategies Used by Children in Problem 2 of Task 8

Approaches	**Excerpts of Responses**
Scalar strategy by addition	Caroline (10 years old, 5th grade) Caroline: It would be brown. Interviewer: How did you figure that out? Caroline: I believe I know. I know that red is double pink. Interviewer: Yes. Caroline: So this is pink; this is red [on the scale]. It's one, two. That is one [from zero to red] and one, two [from red to brown]. That will be brown. So this is one red and this is another red and that is the third red [from brown to green].
Scalar strategy by multiplication (double)	Lily (9 years old, 4th grade) Lily: It should be 4. Interviewer: You mean [pointing to the scale] Lily: Twice of two is four.

Problem 3.

Figure 14 shows the material used in Problem 3. The question was:

"Look at C and E. C is made of aluminum. If I need an aluminum cylinder that has the same weight as E, how tall would that cylinder be?"

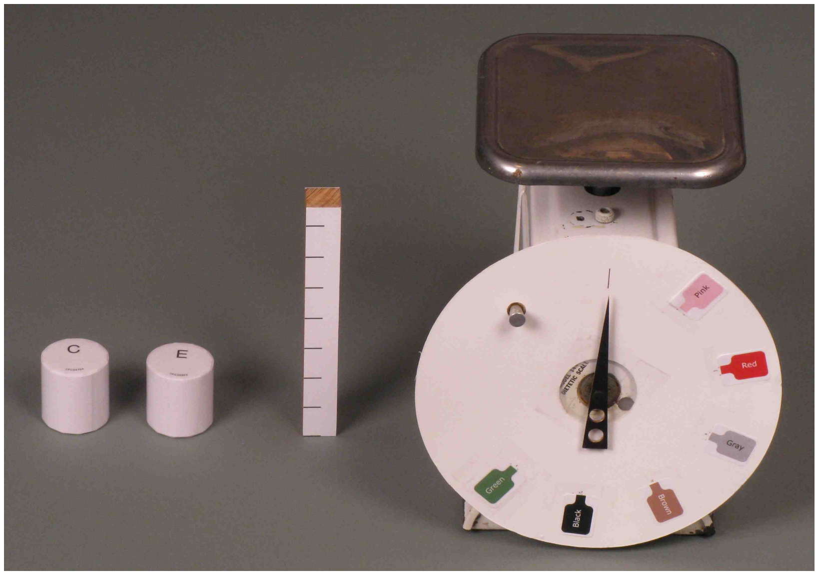


Figure 14: The material used in Problem 3 of Task 8.
(The weight of C is red and the weight of E is pink.)

Problem 3 was solved correctly by 80% the children. In contrast to the first two problems, two children said that they did not know the answer and two gave a wrong answer, as in the case of Isabel (see Table 33). These two children who gave wrong answers used a scalar strategy by multiplication, but it seemed like they thought the same weight meant same material. Table 33 shows an example of wrong answers.

Table 32: Children's Performance in Problem 3 of Task 8

Answer	Number of children	Percent
Correct	16	80%
Wrong	2	10%
Don't know	2	10%

Table 33: An Example of a Wrong Answer to Problem 3

Approaches	Excerpts of Responses	
Scalar strategy by multiplication (double)	Isabel (9 years old, 3rd grade)	
	Interviewer:	Look at C and E. C is made of aluminum. If I need an aluminum cylinder that has the same weight as E, how tall would the cylinder be?
Same weight means same material	Isabel:	It will be red.
	Interviewer:	It will be red? I am talking about the height, not the weight.
	Isabel:	It will be four spaces tall?
	Interviewer:	Why do you think that?
	Isabel:	Because two will be four?
	Interviewer:	You mean if that is an aluminum cylinder and has the same weight as this one [E], it would be how tall?
	Isabel:	It will be four.
	Interviewer:	Why do you think that?
	Isabel:	Because I double it [E], it could be made of aluminum.

It is not clear whether Isabel did not understand my question or whether she went back to thinking that same weight means being made of the same kind of material, as she had shown at the beginning in previous inferring material tasks (Tasks 6 and 7).

Among the children who gave the correct answer, eight used a scalar strategy by multiplication; three used a scalar strategy by addition; and five found that the target cylinder should be the cylinder A (see Table 34). However, there were no clearcut divisions among these strategies, as showed in Table 35. Some of them were a combination of a scalar strategy by multiplication and a scalar strategy by addition, like Henry. I coded the children's responses by considering the main strategy they used. Also, one child used a scalar strategy by multiplication and found out that the target cylinder should be the cylinder A. I coded the children's responses as "Identifying the target cylinder as cylinder A," as long as they found out that the target cylinder should be cylinder A and used the cylinder to explain their answers. Table 35 shows examples of these three strategies the children used in this problem.

Table 34: Strategies Used by Children Who Gave Correct Answers in Problem 3 of Task 8

Strategies	Number of children	Percent
Scalar strategy by multiplication (double or half)	8	50%
Scalar strategy by addition	3	19%
Identifying the target cylinder as cylinder A	5	31%

Table 35: Examples of Strategies Used by Children in Problem 3 of Task 8

Approaches	**Excerpts of Responses**
Scalar strategy by multiplication (Half)	Caroline (10 years old, 5th grade) Caroline: So it is the same... Interviewer: Material as C. Caroline: It's aluminum [the imaginary cylinder]. It is the same weight as this [E]? Interviewer: Yes. Caroline: This [E] is pink. It should be half of this [C]. Which is only one line [pointing to the ruler].

Approaches	Excerpts of Responses	
	Interviewer:	How did you figure that out?
	Caroline:	Since this is pink, and I know pink is half of red. So the cylinder itself should be the half of red [in size].
Scalar strategy by addition combined with a scalar strategy by multiplication	Henry (8 years old, 3rd grade)	
	Interviewer:	Look at C and E. C is made of aluminum. If I need an aluminum cylinder that has the same weight as E, how tall would the cylinder be?
	Henry:	It would be one of these [the space between two marks on the ruler] tall.
	Interviewer:	How do you know that?
	Henry:	Because this [the weight of C] is twice as much as this [E]. If you took away half of material, it would be half of much.
Scalar strategy by multiplication using A as a referent	Rachel (10 years old, 4th grade)	
	Rachel:	It should be half of the height as this [C]. I am guessing that would be A.
	Interviewer:	How do you know that?
	Rachel:	Because they [lifting A and E] are both pinks, so they are both about the same weight, except this one [A] is smaller. It's half the size of this one [E]. And one plus one [pointing the pink and red on the scale] is equal to two. It takes two pinks... takes two of these pinks [A] to equal one red [pointing to the C].
	Interviewer:	So the height of that cylinder would be...
	Rachel:	Would be two, I think.
	Interviewer:	I mean the cylinder that made of aluminum and has the same weight as E. How tall would that cylinder be?
	Rachel:	If it's the same weight as E?
	Interviewer:	But it is made of aluminum.
	Rachel:	I think it would be one of this [A].
	Interviewer:	You mean like this tall.
	Rachel:	Yes.

Approaches	Excerpts of Responses
	Interviewer: It is one [on the ruler], right? Rachel: Yes.

Henry's approach in this problem was a combination of scalar strategy by addition and scalar strategy by multiplication, because children knew that what they needed to cut is half of C. The approach of identifying the target cylinder as cylinder A was a scalar strategy with the help of the cylinder A as an example (or referent) in front of them.

In fact, Problem 3 would be similar to Problem 1 and Problem 2, if it were phrased as: "If we have another cylinder that is also made of aluminum and its weight is pink, how tall should that cylinder be?" In fact, the children who gave the correct answer did interpret the information "an aluminum cylinder that has the same weight as E" as equivalent to "an aluminum cylinder whose weight is pink" (see Caroline' response above in Table 35). However, for some children, the description of "the weight of the target cylinder is the same as the weight of the cylinder E" confused them, because the cylinder E was made of a different material from either cylinder C or the imaginary cylinder. Thus, they either did not know how to answer or gave a wrong answer.

Since these children do not have a clear concept of density at this point, none of them solved this problem using an inversely proportional approach (the sizes of two objects of different materials should be inversely proportional to their densities in order to have the same weight). For this problem the reasoning could be like this "E and C had the same size and E was half as heavy as C, so the material of which C was made of (aluminum) was twice as dense as the material of which E was made of. Thus, in order to have an aluminum cylinder that had the same weight as E, the size of it would be half of the size of E." However, as shown in Table 36, 88% of children displayed this kind of reasoning at the end of Task 9, when I asked them a similar question: "If an aluminum cylinder has the same weight as a brass cylinder, how do their sizes compare to each other?" (see detailed analysis of Task 9.) This change illustrates the impact of Task 9 on children's reasoning.

Table 36: Improvement of Children's Performances in the Inversely Proportional Problems

	The number of children who gave a correct answer with an inversely proportional strategy	
The sizes should be inversely proportional to the densities in order to have the same weight	Problem 3 of Task 8	Similar Problem in Task 9
	0	15/17 (88%)

Problems 4 to 6: Ratio Comparison Problems

Even though problems 4 to 6 were all ratio comparison problems, children's performances across these three problems were dramatically different. As already shown in Table 26, all children gave the correct answer to problem 4 but less than half did so for problems 5 and 6.

Problem 4.

Figure 15 shows the picture used in Problem 4. The question was: "Could X and Y be made of the same kind of material or not?"

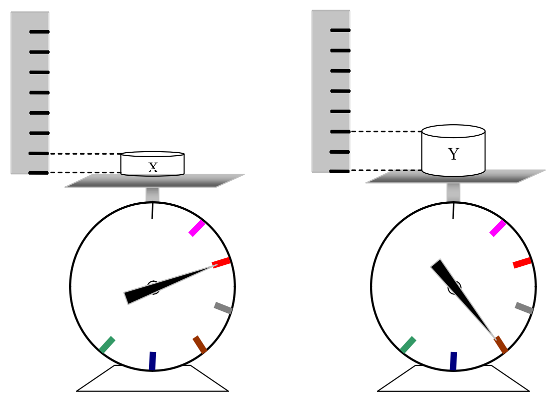


Figure 15: The picture used in Problem 4 of Task 8

In Problem 4, (see Table 37), about two thirds of the children used the scalar strategy by multiplication approach by pointing out that both the size and the weight of X are half of those of Y or that both the size and the weight of Y are twice (or double) those of X. Less than one third of the children used the scalar strategy by addition approach, adding another X to the original X to make a cylinder that was the same size as Y and showing that the size and the weight of two Xs were equal to these of Y. Two of these children used one C to represent X and two Cs to represent Y. One child used a functional approach and one child gave the correct answer, but did not give a complete explanation. Table 38 shows the excerpts of these responses.

Table 37: Strategies Used by Children in Problem 4 of Task 8

Strategies	Number of children	Percent
Scalar strategy by multiplication (double or half)	12	60%
Scalar strategy by addition	6	30%
Functional approach	1	5%
Incomplete justification	1	5%

Table 38: Examples of Strategies Used by Children in Problem 4 of Task 8

Approaches	**Excerpts of Responses**
Scalar strategy by multiplication (double or half)	Aaron (9 years old, 4th grade) Interviewer: Could X and Y be made of the same kind of material or not? Aaron: Yes. Interviewer: How did you figure that out? Aaron: Because this [pointing to the height of Y] is two and half of two is one [pointing to the height of X]. This goes to two [the weight of X], if you double that [X], it goes to there [pointing to the weight of Y].
Scalar strategy by addition (Adding another X and thinking of X as C)	Jennifer (10 years old, 5th grade) Interviewer: Could X and Y be made of the same kind of material or not? Jennifer: Do you have X and Y? Interviewer: No, I don't have. You just imagine that there are two cylinders. Jennifer: Yes, they could. Interviewer: How did you figure that out? Jennifer: Because this is two, wait, this is four [pointing to the real scale]. So it is like this [put one C on the scale and add another one]. This and another C… this is Y. This C by itself is X.

Approaches	Excerpts of Responses
	They are made out of the same material because it's like this is two… this is just adding two [marks]. [pointing to the scale in the picture] This is just add one more up. [showing me on the real scale] Two and adding two.
Scalar strategy by addition (Adding another X)	Jake (9 years old, 3rd grade) Jake: Because I said that red plus red is equal to brown. Y is brown and X is red.
Functional approach	Anthony (3rd grade) Interviewer: Could you give me a clue to how did you figure that out? Anthony: So that is one (pointing to the ruler in the picture) like one inch equals to two (pointing to the scale in the picture). Two times two equals to four (pointing to brown on the real scale)

It is important to notice that Jennifer adjusted the unit size when solving this problem. In Task 7, the unit size was the size of A, so the size of C was two units. In this problem 5 of Task 8, she assumed that the unit size was the size of C, so the size of C was now one. Thus she could use C to represent X and 2Cs to represent Y, showing flexibility in using different units of quantities.

Anthony used a functional approach, as he tried to point out the relationship between size and weight, that is, 1 to 2. He was the only child to use this approach in this problem and also the only one who did Task 9 before Task 8. This suggests that Task 9 had an effect on children's responses to Task 8, which is consistent with the later analysis of the changes in children's performance in Problem 5 after Task 9.

Problem 5.

Figure 16 shows the picture used in Problem 5. The question was:

"Could O and I be made of the same kind of material or not?"

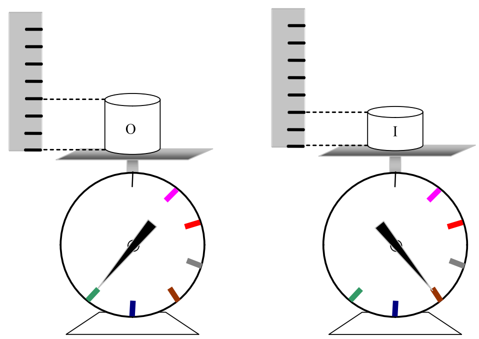


Figure 16: The picture used in Problem 5 of Task 8

In this problem, before Task 9, less than half of the children gave correct answers with correct explanations, one child gave the correct answer with a wrong explanation and half of the children gave a wrong answer (see Table 39).

Table 39: Strategies Used in Problem 5 of Task 8

Strategies			Number of children
Functional approach			3
Scalar strategy by multiplication	Correct Answer	(Two thirds)	1
		Scalar strategy by multiplication combined with splitting	1
	Wrong answer	(Double or half)	1

Table 39 (continued): Strategies Used in Problem 5 of Task 8

Strategies			Number of children
Scalar strategy by addition	Correct Answer	Scalar strategy by addition combined with scalar strategy by multiplication	1
		Scalar approach by addition and considering X [from Problem 4] as C, Y [from Problem 4] or I as C + C and O as C+C+C	3
	Wrong answer	Add another whole I	1
Constant difference strategy (Wrong answer)			8
Correct answer but no justification			1

Responses from children who gave wrong answers.

Children who gave a wrong answer used correct strategies inappropriately or used the constant difference approach (see Table 39). Table 40 shows examples of the responses from the children who gave wrong answers.

Table 40: Examples of Responses from Children Who Gave Wrong Answers in Problem 5 of Task 8

Approaches	**Excerpts of Wrong Responses**
Scalar strategy by multiplication (double or half)	Rachel (10 years old, 4th grade) Interviewer: Could O and I be made of the same kind of material or not? Rachel: I don't think they are made of the same kind of material, because I is brown. For the height, it's two. For this one [O], the height is three. So if you took two of these [I], it would equal

Approaches	Excerpts of Wrong Responses	
		four. It would equal the same amount. Also, if this one [I] is brown, four plus four is eight. And this one [O] is green, which is six.
	Interviewer:	How tall is this one [O]?
	Rachel:	Three.
	Interviewer:	I remember you said that if you added another I, right?
	Rachel:	Uh-huh.
	Interviewer:	Why did you take another I?
	Rachel:	If you took two Is, it could be four [on the ruler], because two plus two is four. But this one [O] is only three [on the ruler]. So it is not the half of the height, but the half of the weight.
	Interviewer:	It's the half of the weight?
	Rachel:	No. No, it's not the half of the weight. It's less than half of the weight.
	Interviewer:	So, are they made of the same kind of material or different kind of material?
	Rachel:	Different.
	Interviewer:	Okay.
Scalar strategy by addition	Jake (9 years old, 3rd grade)	
	Interviewer:	Could O and I be made of the same kind of material or not?
	Jake:	Because brown plus brown does not equal to green.
	Interviewer:	Tell me the height of I.
	Jake:	It's two inches [for I] and three inches from O. And brown plus brown does not equal to green.
	Interviewer:	Why did you use brown plus brown?
	Jake:	Because brown is just on the scale. You can... brown is pretty much equal to 4.
	Interviewer:	Right.
	Jake:	A green is equal to... It's really one is pink, two is red, three is gray, four is brown, five is

Approaches	Excerpts of Wrong Responses	
		black, six is green.
	Interviewer:	I am just wondering... because you said the height of this one [I] is two, the height of this one [O] is three.
	Jake:	And the brown plus brown is not equal to green, so it won't work. They are not the same material. Two different materials.
Constant difference approach	Isabel (9 years old, 3rd grade)	
	Interviewer:	Could O and I be made of the same kind of material or not?
	Isabel:	No.
	Interviewer:	Why not?
	Isabel:	Because O is just one space taller than I. If you move I one space [on the scale] it went on blue, not green.

Rachel still looked for a double or half relationship with a scalar approach, but in this case the weight and the size of I is 2/3, not 1/2, of O. In fact, if she had been able to use a functional approach, the double or half relationship would still work, because the ratio of the weight to the size was 2:1. However a combination of a double or half relationship with a scalar approach was not conducive to a solution to this problem.

Jake was also unsuccessful in trying to solve the problem by using a scalar approach. Although he clearly knew that the height of I was two and the height of O was three, he chose to add brown plus brown and found that the result was not equal to green (the reading of O on the scale). The fact that in previous problems cylinders were in a relation of 2 to 1 may have led the children to attempt to equalize the cylinders only by doubling or halving one of them. In fact, he could have gotten the correct answer if he had used a scalar approach by addition properly, adding half of I to the original I, to build an O.

Eight children who got a wrong answer used a constant difference strategy, as Isabel did. It is important to notice that, in Problem 4, all children gave the correct answer and most of them used a scalar strategy by multiplication or by addition. However, in Problem 5, when the two correct approaches were hard to use, eight children used the incorrect constant difference approach.

Responses from children who gave correct answers.

For the 10 children who answered correctly, three used a functional approach; one used a scalar approach by multiplication with the ratio 2:3; one used a scalar approach with the help of splitting; four used a scalar approach by addition (one child added half of I to I to make an O; three children considered I as 2 of cylinder X and O as 3 of cylinder X, or F); and one did not know how to explain his correct answer (see Table 39). Table 41 shows examples of these responses.

Table 41: Examples of Responses from Children Who Gave Correct Answers to Problem 5 of Task 8

Approaches	Excerpts of Responses
Functional approach	Jim (9 years old, 4th grade) Interviewer: Could O and I be made of the same kind of material or not? Jim: No, same size, different weights. Interviewer: Not same size. Jim: Oh, no. [Pause a little bit] No. Interviewer: Why not?

Approaches	Excerpts of Responses
	Jim: This one is up by one [on the ruler], but you can't reach that point [green on the scale]. You can't reach that point [green] by adding one. That is one two [gray and brown] and one two [pink and red]. Wait, they could be the same. Interviewer: They could? Jim: Because it is up by one [on the ruler]. If it is equal to one, I mean two [on the scale]. Each line [on the ruler] equals two [on the scale]. One two [from zero to red], one two [from red to brown]. That is what that is. And one two [pointing to the scale for O] you get there [brown], this is one more [counting from red to brown on the scale for O].
Scalar approach by multiplication (Two thirds)	Henry (8 years old, 3rd grade) Interviewer: Could O and I be made of the same kind of material or not? Henry: I don't think so. Interviewer: Why not? Henry: Because this is one shorter and two of these. Interviewer: Two of these? Henry: It's only two tall [I]. This is three tall [O]. Wait, wait. I think they are the same material actually. Interviewer: Okay. Why do you think they are the same material? Henry: Because it's two thirds of this tall and two thirds of the weight. Interviewer: Two thirds of this tall? Henry: Two thirds of the height.
Scalar strategy by multiplication combined with splitting	Caroline (10 years old, 5th grade) Caroline: I don't think they are made of the same material. Interviewer: Why not? Caroline: Since this is one third of this [in height]. It is not half.

Approaches	Excerpts of Responses
	Interviewer: What do you mean by one third. Caroline: Oh, two thirds. Sorry. If you count this [O] be a whole object, you will count this [I] as two thirds, because one will be one line. Since it is two lines, it is two. It's two of this, since it is only one more line. Actually they could be the same object. Because if this is two [she labeled the brown on the scale for I as 2] (see Figure 17, page 115), yeah, if this is three [she labeled the green on the scale for I as 3], this [brown] could be two, this could be one [she labeled the red on the scale for I as 1]. So it is possible. This is one, two, three [she labeled 1, 2, and 3 on the scale for O too]. Interviewer: Could you explain more about why do you think O and I might be made of the same material? Caroline: So this [I] is two thirds of this [O] [on ruler]. If you could split this [the scale for I in the picture]... This [the reading on the scale for O] is six. Six is a factor or multiple. It's both. No it's multiple. It is a multiple of three. So it can be split up into three parts too. So I split it up into three parts. So there are two parts in each whole, in each section. This one [I] since there are two [on the ruler], it weighs number two (see Figure 17, page 114). Even though this [the height of O] is only one more than this is [the height of I], it equals to three [the number she labeled on the scale for O]. Does that make sense?
Scalar approach by addition combined with scalar approach by multiplication	Aaron (9 years old, 4th grade) Interviewer: So could O and I be made of the same material? Aaron: Yes. Interviewer: Tell me how did you figure that out? You can draw anything, if that will help you show me. Aaron: If you have this regular [I] and then this [I] cut in half and stack this there [on the top of the original I], it would be the same height and it will also weigh the same.

Approaches	**Excerpts of Responses**	
	Interviewer:	How do you know this?
	Aaron:	Because this [half of I] will weigh two. Two plus two is four. If you add that [the half one on the top of I], it will go to green. Because this [I] weighs four; this [the half] weighs two.
	Interviewer:	You mean the half of this [I] weighs two.
Scalar approach by addition and considering X [from Problem 4] as C, Y [from Problem 4] or I as C + C and O as C+C+C	Jennifer (10 years old, 5th grade)	
	Interviewer:	Could O and I be made of the same kind of material or not?
	Jennifer:	Yes, they could.
	Interviewer:	Could you show me how did you figure that out?
	Jennifer:	[put D on the scale] This is green. These two [put 2Cs on the scale]. Yes, brown. The same thing as Problem 4.
	Interviewer:	[handing her the picture for Problem 4]
	Jennifer:	This one… this is like these two [put 2 Cs on the scale] for this one [problem 5] you adding two more. [the pointer go above green, then she take one off] one more, so it makes it made of the same material as all these [X, Y, O and I]
	Interviewer:	Why did you add one more?
	Jennifer:	Because this [C] by itself is red and two [on the scale], adding one more makes it four, adding another one makes it six.

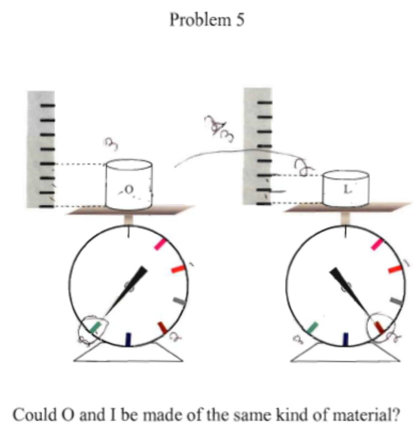


Figure 17: Caroline's work on Problem 5 of Task 8

Functional approach.

You can see that although Jim finally used a functional approach getting the correct answer, he first used a constant difference approach until he found that "Each line [on the ruler] equals two [spaces on the scale]." When he found the functional relation, he immediately abandoned the constant difference approach and used the functional approach successfully. It is also possible that the constant difference approach may be a misuse of a scalar approach by addition; Jim might have assumed that one in size corresponded to one in weight. Once he found that the assumption was not correct, he focused on the correct relationship between weight and size.

Scalar approach by multiplication (two thirds).

Henry also first used a constant difference approach. Then he found out that the relationship between O and I in size (2/3) was the same as the relationship between their weights (2/3). Then he concluded that O and I could be made of the same kind of material. The ratio of 2/3 was not salient for other children.

Scalar approach by multiplication combined with splitting.

Apparently Caroline used a scalar approach because she found that the relationship between I and O was two thirds and she tried to decide whether the relationship was also true for weights. By splitting, she found that it was. This tells us that splitting could be a good method to help children understand a relationship when it is represented by a non-simplified fraction.

Scalar approach by addition combined with scalar approach by multiplication.

Two main factors in Aaron's solution were that he determined that a half of I would correspond to two units of weight and that you need to add one half of I to the original I to make one O, a flexibly joint use of a multiplicative and an additive relation.

Scalar approach by addition and considering a specific cylinder (C).

Sometimes Jennifer only considered the weight, which was why she thought the cylinder D could be equal to O at one point. However, as she was reasoning about this problem, she was able to coordinate weight and size. As she had done in Problem 4, Jennifer adjusted the unit size and assumed that it was the size of C. Thus, she could use two Cs to represent I and three Cs to represent O.

Children's performance in Problem 5 after Task 9.

Task 9 was designed to draw children's attention to the relationship between weight and size (the functional relation).

In Task 8 there were ten children had difficulty with Problem 5; either they did not give the correct answer or they did not give a correct explanation. After those children completed Task 9, I asked them to reconsider Problem 5.

After Task 9, four of the ten children were able to answer Problem 5 correctly, three of them using a functional approach and one not explaining how he found it. Task 9 seems to have guided children to focus on the relationship between the weight and the size of each cylinder. For example, before Task 9, Jake misused a scalar strategy by addition and gave a wrong answer to problem 5 of Task 8 (see above in Table 40). After Task 9, he gave the correct answer using a functional approach (see Table 42).

Table 42: Example of a Functional Approach in Problem 5 of Task 8, after Task 9

Approach	Excerpts of Responses
Functional approach	Jake (9 years old, 3rd grade) Jake: Inches of two and inches of three. They could be made of the same material. One inch is about two pounds. So they could be the same material. Interviewer: How do you know that? Jake: Because if you said one inch is two pounds. This [I] would be 4 pounds; that [O] was 6. So this [O] is one inch more.

Jake first determined the relationship between size and weight and then explained that, by multiplying the size of O (3 units) and the size of I (2 units) by the functional operator, one would arrive at the weights displayed by the scale, that is, 6 pounds for cylinder O and 4 pounds for cylinder I.

Summary of Problem 5 results.

The results show that Problem 5 was difficult for children in the age range included in this study, most likely because the ratio between the heights of the two cylinders and the ratio between their weights were not unit ratios. Children who failed to solve the problem when it was first given seemed to try to closely extend the procedure they had used in Problem 4 or to resort to a wrong additive approach. Those who successfully addressed the problem flexibly used different approaches and were able to use their newly constructed knowledge to solve ratio comparison problems, even when the ratio between the weights and between the sizes were not unit ratios. After help from the focus on the functional relation provided by Task 9, more, but not all, of the children could solve the problem.

In addition, it seems that considering a small quantity as a unit and considering larger quantities as multiples of the smaller one is very natural for children at these ages. Difficulties arose when they (like Rachel and Jake) could not find a unit of which other quantities were multiples. The difficulties were overcome when they could create a unit (for example, Aaron imagined cutting I in half and considered half of I as a unit) or found a cylinder that was outside of the problem as a unit, as Jennifer did, using cylinder C as a unit and considering I as 2 Cs and O as 3 Cs.

Problem 6.

Figure 18 shows the picture used in Problem 6. The question was:

"Could M and N be made of the same kind of material or not?"

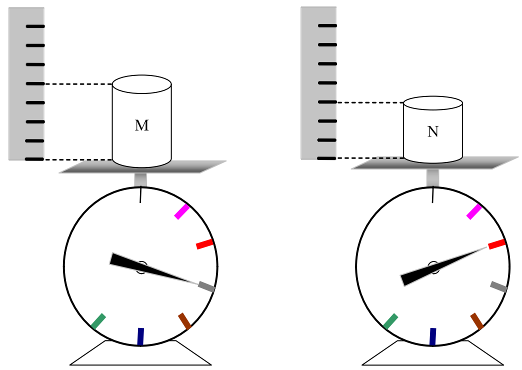


Figure 18: The picture used in Problem 6 of Task 8

Responses from children who gave wrong answers.

Among the ten children who gave wrong answers, seven used a constant difference approach; two used a scalar approach by addition incorrectly; and one used a functional approach incorrectly (see Table 43). Table 44 shows examples of different strategies.

Table 43: Strategies Used by Children Who Got a Wrong Answer in Problem 6 of Task 8

Strategies	Number of children
Constant difference approach	7
Scalar approach by addition incorrectly	2
Functional approach incorrectly	1

Table 44: Examples of Responses from Children Who Gave Wrong Answers in Problem 6 of Task 8

Approaches	**Excerpts of Responses**
Constant	Lily (9 years old, 4th grade)

Approaches	Excerpts of Responses	
difference approach	Interviewer:	Could M and N be made of the same kind of material?
	Lily:	No.
	Interviewer:	Why not?
	Lily:	Because, wait. Actually I think they could, because here it's gray; here it's red. And this is only one more smaller than this [pointing to the ruler], so it could go one back [on the scale].

Approaches	Excerpts of Responses
Scalar approach by addition (incorrectly)	Aaron (9 years old, 4th grade) Aaron: [counting the height of them] Yes. Interviewer: How did you... Aaron: If you were to add that [drawing a shorter cylinder on the top of N] up there, it will be the same height [pointing to the height of M] and this [the shorter one he drew] will weigh one, this [N] will be two, this [M] weighs three. Two plus one equal to three, so it will be there [gray]. Since this is gray [the weight of M], so it will be the same height and same weight, if this is stack up there. Interviewer: Sorry, how tall is this [N] one? Aaron: Three. Interviewer: How tall is this one [M]? Aaron: Four. Interviewer: So that is three; that is four, so what did you say? Aaron: So if this [N] is 3 in height. And this [the shorter one he added] is one in height and this [M] is four in height. One plus three is equal to four. Would be up here, which is the same height as this. Interviewer: How about the weight? Aaron: This [the one he added] weighs one; this [N] weighs two; this [M] weighs three. Interviewer: Why? Aaron: One plus two. It stacks up here. It would weigh three. Interviewer: Where did you get two? Aaron: That is how much it [N] weighs. And this [M] weighs three. If you cut this [N] in half, wait, no, not in half. A little bit more... a little bit taller [drawing the one he added a little bit taller than before]. It will weigh one. If you add one to this, it will go to gray and this is what this [M] is.

Approaches	Excerpts of Responses
Functional approach (incorrectly)	Earl (9 years old, 3rd grade) Earl: Yes, it could, because it could be pink. That could be pink [pointing to the scale for M]. Have one more, it could go one more to gray. This [N] could be pink, so it goes one back to red. Interviewer: What do you mean it would be pink? Which one… Earl: So you count by one to pinks. So as count by ones, it goes ahead one more [pointing to the scale for M]. That's [N] backwards, because that [M] is one more higher. It would be equal to one, because that is pink. That would be pink, but it is one smaller, so it would be equal to one less. Interviewer: I just do not understand…when you said it would be pink, what do you mean? Earl: It has to be pink, because it counts by ones. Look [counting on the ruler for N] one, two, three, [counting on the scale for N] One, oh [he found that the reading was two on the scale of N], it doesn't have to be pink. But still it is something.

Constant difference approach.

Lily used a constant difference approach. She claimed that M and N could be made of the same kind of material, because the difference between M and N in weight was the same as the difference between them in size.

Scalar approach by addition (incorrect).

At the beginning, Aaron thought that the height of one half of N was one, although he mentioned that the height of N was three. He also thought that one in height corresponds to one in weight, so if he added one in height, there would be one more in weight. Even though, at the end, he realized that the half of N is "a little bit taller than [one]," he still thought that half of N will weigh one and did not realize that half of N plus N was not equal to M in height. The reason he thought that one in height corresponds to one in weight might be because he realized the relationship of the differences in weight and in size between M and N was 1:1. But he did not notice that this was not consistent with the relationship of the weight and the size for either M or N.

Functional approach (incorrect).

I consider Earl's strategy as functional approach, because, in Problem 5, he correctly used a functional approach and said: "One more in size is equal to two more pinks." Here, in Problem 6, he thought that one in size is equal to pink, thus, since M is one more higher, the weight of N would go backwards by one. At the end, he realized the height of N was three, but the weight of N was not three. He knew his assumption (one in size is equal to pink) was not correct, but he did not attempt to correct it, a task that would prove difficult since he would have had to deal with non-unit fractions.

Responses from the children who gave correct answers.

Although 10 children gave the correct answer to Problem 6, this problem was even harder for the children. Only three children provided a good explanation, and they all used a functional approach. Of the others who gave correct answers, one child used a scalar approach with an incomplete explanation, five children gave the correct answer with a wrong explanation, and one child could not explain his answer (see Table 45). Table 46 shows some examples of the responses from children who gave correct answers.

Table 45: Strategies Used by Children who Gave Correct Answers in Problem 6 of Task 8

Strategies	Number of children
Functional approach	3
Incomplete scalar approach	1
Wrong reason	5
No justification	1

Table 46: Examples of the Responses from the Children Who Gave Correct Answers in Problem 6 of Task 8

Approaches	Excerpts of Responses
Functional approach (correct explanation): Inferring from the relationship of the weight and the size of N	Henry (8 years old, 3rd grade) Interviewer: Could M and N be made of the same kind of material or not? Henry: I don't think so. Interviewer: Why not. Henry: Because M is one taller in height and one taller in weight.... If you add it to this [making N the same height as M), it will be between in there (between red and gray) Interviewer: Sorry, I could not hear you. [Some children were talking loudly nearby.] Henry: I just realized something. This would be half way closer to gray, because one height equals two thirds of one of these [pointing to the units in the scale for N]. If this is one tall. Interviewer: Why do you think one height equals... Henry: It's three tall [on the ruler] and it has two [on

Approaches	Excerpts of Responses
	the scale]. And two thirds three times would be... Two times will be one and one third. Another two third would be two.
Functional approach (correct explanation): Inferring from the relationship of the difference in weight and that in size between M and N	Jim (9 years old, 4th grade) Jim: Yes. Because...No. Interviewer: Why not? Jim: If they would be, at one [the ruler for N]. It would be one, two, three [pointing to pink, red and gray on the scale for N]. It will over that weighing of N [red]. That will go over by one too on this one [the scale for M]. Interviewer: I didn't get what you said. Jim: M and N would be different materials, because if one weight per one, it will go one, two, three [scale on N]. That will be more than what is there.
Scalar approach combined with splitting (incomplete)	Caroline (10 years old, 5th grade) Caroline: One, two, three, four, this [the height of M] is four. This [the height of N] is one, two, three. So it makes it here three fourth [writing ¾ between two cylinders] (see Figure 19, page 126). [Looking at the reading on the scale for N] These can't be the same substance. Interviewer: Why not? Caroline: If you try to start from the beginning [pointing the scale for M]. It is hard to split up the scale. Actually, [on the scale for M] this is zero, one, two, three. It looks like three, but if you split it up, it would be... It looks odd, but one, two, three. [Splitting the three marks into four sections on the scale for M] I did not do it evenly, but you could split in between them, so that will be equal. But if I try with three [on the scale for N], it would be one, two, could I do that? Yeah, but if this has to equal three, this is equal to two, so I have to split it up in between. But if I am using that kind of pattern, I can't

Approaches	Excerpts of Responses
	possible to get it here [gray] with one more. I have to get it here between two more [between red and gray]. Is this...
	Interviewer: I just want to know more... I see you said that you split this into how many?
	Caroline: [On the scale for M in the picture] So if you cross this out, you will make this up to like a third or something. So this would be one right here [circling the sections] (see Figure 19, page 126). Then you have a second one here. You have a third one here. And you have a fourth one [inaudible] to here. I don't have exact ones. If I try here (the scale for N), I could do it here, because here are only two lines here, I have to split it up. So I did it one here, one here and one here [circling the sections on the scale for N; see Figure 19, page 126], but there isn't any possible way using this rule to get to here [the third mark which is the same reading on the scale for M]. There is only one more. I have to be able to get there with two more.
	Interviewer: What do you mean two more?
	Caroline: Since I have to split it in half [between red and gray]… two more numbers. If this is five [pointing to the height of M], it could be possible.
	Interviewer: I see. If the height of M is five...
	Caroline: If the height of M is five, it is possible, if these two are the same substance, but it is not possible for these two to be the same substance here.
Scalar approach by addition (wrong explanation)	Rachel (10 years old, 4th grade)
	Rachel: N is three [on the ruler]. Three plus three is six. This one [M] is only four [on the ruler]. So just one more. And this one [M] is gray and this one [N] is red. And two plus two is four. It would be brown, not gray. So I think they are made out of a different kind of material.

Approaches	Excerpts of Responses
Falling behind (wrong or unclear explanation)	Jennifer (10 years old, 5th grade) Interviewer: Okay, continue with your question. Are they made of the same kind of material? Jennifer: Different. Interviewer: Why? Jennifer: Because this N is like this M. No matter how big it is, it is always going to fall behind. [Put E and A on the scale] Just these two are equal to this N. And this is [B] up to M.

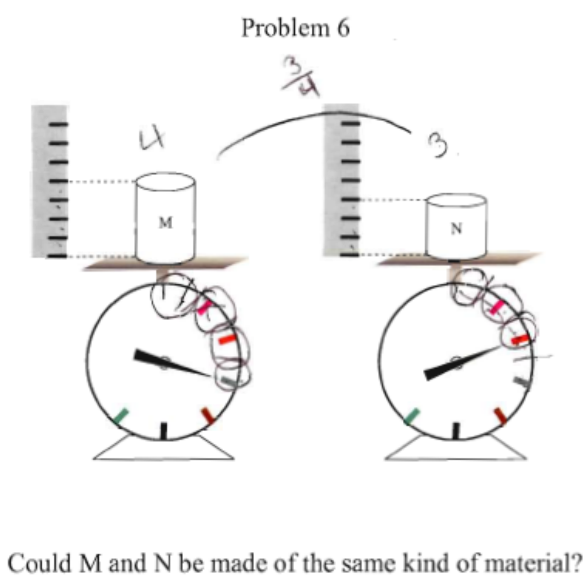


Figure 19: Caroline's work on Problem 6 of Task 8

Functional approach (correct explanation): inferring from the relationship of the weight and the size of N.

I coded Henry's strategy as a functional approach, because he considered the relationship between the weight and the size first and got the weight by multiplying the relationship or the functional operator by the size. He explained that one unit of height corresponded to two thirds in weight and that by adding one unit to the height of N he would be just short of four units in weight. He showed he could deal with fractions, trying to explain why the weight of M would be between red and gray if M and N were made of the same kind of material. The other two children who also used the functional approach only estimated that the weight of M would be between red and gray, if they were made of the same material, but did not precisely explain why the weight would be between red and gray.

Functional approach (correct explanation): Inferring from the relationship of the difference in weight and that in size between M and N.

I coded Jim's approach as a functional approach, because he found the relationship between the difference in weight and that in size and compared that relationship with the relationship between the weight and the size for M or for N. My interpretation is that, when Jim said that "if one weight per one," he meant that since the difference in weight between M and N was one and the difference in size between M and N was one too, if they are made of the same kind of material, one size would corresponded to one weight. But this was not true when considering the relationship between the weight and the size for either M or N, so M and N could not be made of the same kind of material. It is also possible that, for some reasons, he just thought that in order to be made of the same kind of material, one weight should correspond to one size.

Scalar approach combined with splitting (incomplete).

Caroline split the three marks on the scale for M into four and the two marks on the scale for N into three. When she said: “Yeah, but if this has to equal three, this is equal to two, so I split it up in between [marks],” it seemed that she wanted to split the two marks into three equal parts. But when she said: “I am using that kind of pattern. I can't possible to get it here with one more. I have to get it here between two more,” My interpretation is that the pattern might be the sections on each scale are the same and each notch on the ruler would correspond to each section on the scales. If that was her pattern, she should expect the sections on each scale be the same, which conflicted with splitting two marks into three parts on the scale for N. But she did not notice this conflict.

She did notice that since M is only one notch taller than N, the increase in weight should be one section which is smaller than the space between red and gray. However, what she said, “I have to be able to get there with two more,” was not correct, because both the section on the scale for M (3/4) and the section on the scale for N (2/3) were bigger than one half. That is why I think her reasoning was not complete, precise, or clear. We can see that Caroline’s strategy was still clever, but awkward.

It is clear that neither she nor any of the other children noticed the fact that the fraction 2/3 is not the same as 3/4. None of the children mentioned that M and N were not made of the same kind of material for the reason that 2/3 is not the same as 3/4. We can see that when the ratios were not the same, the problem became more challenging for them. It is also possible that they had no idea of 2/3 or 3/4 as ratios, that the children were only thinking about two sets of numbers, (2,3) and (3,4). Some children might have not learned fractions or ratios in school. They might only understand one half, one third, or one sixth from everyday life.

Scalar approach by addition (wrong explanation).

In this problem, as in the previous problem (Problem 5), Rachel began by thinking about whether one cylinder was half the size of the other. However, in Problem 5, suddenly she thought that O and I could be made of the same kind of material and the strategy she used was a functional approach: “The height is half of the weight for both of them.” With the functional approach, she could still use “half.” If she used a scalar approach, she would need to deal with the ratio of 2 to 3. I guess that this was one reason she could more easily solve Problem 5 with a function approach. However, in Problem 6, either a scalar approach or a function approach needed to work with a non-unit ratio. Thus, although she had a feeling that M and N could not be made of the same kind of material, she gave a wrong reason.

Falling behind (wrong explanation).

Jennifer, as in the previous problems, still wanted to use the real cylinders to represent the cylinders in the picture. But in this case it did not work, because neither M nor N could have been made of the same kind of material as any of the real cylinders in front of her. Jennifer used E and A to represent N, but did not realize that E and A were not made of the same kind of material. She used B to represent M, but did not notice that B and M did not have the same size. It seems that she focused only on the weight of them at that time. In addition, it is interesting that she used the words “fall behind.” Did she mean that the weight fell behind the size for either M or N? But that was not a plausible reason for M not to be made of the same kind of material as N. However, it might be a plausible reason for her. She might think that the ratio of weight to size should not be smaller than one, or in her words, “No matter how big it is, it is always going to fall behind.” Perhaps she was thinking that if she could make N grow taller until it was as tall as M, the weight of N would not increase fast enough to match the weight of M. That is a true description of the relationship between N and M, but there is not enough evidence to know whether that is what she meant. I should have asked more to understand her ideas better.

We can see that Problem 6 was a difficult problem for most of the children, (a) because M and N were not made of the same kind of material, that is, the ratio between the size of the two cylinders (4/3) was different from the ratio between their weights (3/2) and; (b) because these two are non-unit ratios.

Summary of Children’s Approaches in Solving Different Problems

Most children used different approaches to solve different problems. Table 47 relates the nature of the problems and the number of children who used particular approaches.

Table 47: The Number of Children Who Used Each Approach in Different Problems

		Missing value problems			Ratio comparison problems		
		P1	P2	P3	P4	P5	P6
Correct answer	Functional approach				1	3	3
	Scalar approach by multiplication	13	5	8	12	2	
	Scalar strategy by addition	3	15	3	6	1	
	Identifying the target cylinder as a real cylinder			5		3	
	Matching numbers	4					
	Wrong reason or incomplete explanation				1	1	7
Wrong answers	Constant difference approach					8	7
	Functional approach						1
	Scalar approach by multiplication			2		1	
	Scalar approach by addition					1	2
Don't know				2			

The first three were missing value problems. No child used a functional approach for any of these. When the description of the problem specified the value of the quantities (Problem 1), all of the children gave correct answers and most children used a scalar approach by multiplication. When I specified the relationships of the sizes between the target cylinder and the real cylinder in front of them (Problem 2), all of the children gave correct answers and most children used a scalar strategy by addition. I used the word “twice” in the description of the problem. It seems that to them the word “twice” meant “adding another identical one” more frequently than “times two.” When the description of the problem specified that the weight of the target cylinder was the same as a real cylinder, 80% of the children gave correct answers. Among the children who gave the correct answer, half of them used a scalar approach by multiplication and one third of them found a real cylinder to represent the target cylinder. The description “the weight of the target cylinder is the same as the weight of cylinder E” compounded the difficulty.

The three ratio comparison problems all specified values of weights and sizes illustrated with the drawings of scales and rulers. I helped children clarify the readings of the rulers if they miscounted them. Thus, all children got the correct values of the weights and the sizes of the cylinders in the pictures. When the relationships were simple, such as 1:2 with a scalar approach and 2:1 with a functional approach (Problem 4), all of the children gave correct answers and most children used a scalar approach by multiplication. When the relationships were 2:1 with a functional approach and 3:2 with a scalar approach (Problem 5), only half of the children gave the correct answer. We can see that the ratio of 3:2 could be a barrier in using a scalar approaches. Some children switched to using a functional approach and succeeded with the simple ratio 2:1, and some switched to a wrong approach (constant difference approach) and failed. The ratios in Problem 6 were 3:2 and 4:3 (not equal), but the differences between the sizes and that between the weights were the same. That increased the difficulty of deciding whether the two cylinders were the same material. Although 10 children got the correct answer, only three of them gave a correct reasoning. One reason this problem was so difficult is that the correspondence between the difference in sizes and that in weights was so salient. A second possible reason is that the meanings of the ratios of 3:2 and 4:3 were not familiar to the children. None of the children could figure out that M and N could not be made of the same kind of material based on the fact that 3:2 does not equal 4:3. It was possible that children in these grades were not used to reasoning with ratios, unless they were unit ratios such as 1:2, 3:1 or 1:6. To the children, the four values of weights and sizes of two objects were just two pairs of numbers. These numbers were not combined into two ratios (or fractions) in their minds.

Relationship between Children's Performance in Task 8 and Grade Levels

Table 48 shows no relationship between children's performance and grade levels. All third, fourth, and fifth graders gave correct answers to Problems 1, 2 and 4. Problem 3 was slightly more difficult, with wrong answers from one third grader and three fourth graders. In Problem 5, the proportion of correct answers, given either before or after Task 9, was in fact higher for third and fourth graders. And, for Problem 6, third graders did better than the children in the other grades.

Table 48: Number of Correct Answers to Each Problem in Task 8 by Grade Levels

Grade	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
Third (N=8)	8	8	7	8	6*	5**
Fourth N=8	8	8	5	8	6*	3***
Fifth N=4	4	4	4	4	2	2****

*Two correct answers were given after task 9

** Three with wrong explanations

*** Two with wrong explanations

****One with wrong explanation

Summary of Results

In sum, the children were able to apply the knowledge they constructed in Task 6 and Task 7 to solve the new problems. Children's performance in solving problems related to the relationships between weight and size, to children's understanding of "being made of the same kind of material," to their abilities to deal with ratios (or fractions), and to the way the problems were described. Children chose to use the strategies that were easy to apply. The ease of applying a strategy may have depended on children's familiarity with the strategy, their understanding of the concepts the problem involved, and the complexity of the ratios they needed to work with. It is also interesting to note that children were very flexible in using the units of quantities and in combining the use of different strategies.

The analysis of this Chapter answers the fourth research question (Can children from ages 8 to 11 reason proportionally in the context of density?) the second part of the sixth research question (What difficulties do children encounter in solving problems relating weight, size, and the kind of material?), and the seventh question (What strategies do children use in solving problems related to the proportional relationships among weight, volume and density?). In Chapter 11, I will discuss the answers of these research questions, the relation of these findings to those from previous studies, their theoretical implications, and their relevance for science education.

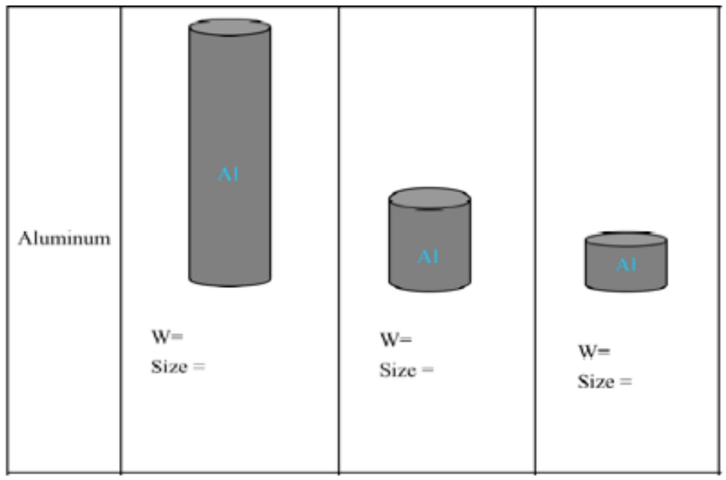
Chapter 8: Toward a Formal Expression of the Relationship between Weight and Size: Analysis of Task 9

With Task 9, I examine whether the quantification of weight and volume, together with some systematic comparisons, allowed the children to develop a formal understanding of the quantitative relationship between weight and volume, the foundation of the concept of density. That relationship could be stated in this way: (a) For objects made of the same kind of material, the ratio of weight to volume is a constant, and (b) The constants are different for different materials.

The goals of Task 9 are (a) to give the child an opportunity to generalize a law that, for a single material, the ratio of weight to size for each object is a constant and that the constant depends on the material; (b) to observe children's difficulties in this process; (c) to see whether the child can determine that, when the sizes are the same, the weights of objects of different materials are proportional to the ratios of weight to size; and (d) to see whether the child can determine that, when the weights are the same, the sizes of objects made of different materials are inversely proportional to the ratios of weight to size.

As already described in Chapter 4 (Method), at the start of Task 9, I presented the child with three bare aluminum cylinders and a copy of Table 49. Then I asked the child to compare the sizes and weights of the three cylinders and to fill out the table with the values for the size and weight of each cylinder. Once the table was completed, I asked: "What do you notice about these numbers?"

Table 49: The Table Used in Task 9 for Comparing Weights and Sizes of Three Aluminum Cylinders



After children had described the relationship between the weight and size of these three actual aluminum cylinders, I presented Table 50, which repeated the values for the three actual aluminum cylinders and gave partial descriptions of three hypothetical aluminum cylinders. I asked them to determine the weight or the size of each of those cylinders, given its size or weight.

Table 50: Weights and Sizes of Objects Made of Aluminum

Aluminum

Object 1	Object 2	Object 3	Object 4	Object 5	Object 6
Weight = 6	Weight = 2	Weight = 1	Weight =	Weight = 24	Weight = 90
Size = 6	Size = 2	Size = 1	Size = 4	Size =	Size =

A similar process followed regarding three bare brass cylinders given to each child, along with Table 51 and then followed by Table 52.

Table 51: The Table Used in Task 9 for Comparing Weights and Sizes of Three Brass Cylinders

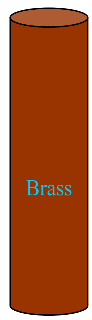
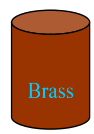
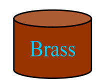
Brass	 Weight= Size =	 Weight= Size =	 Weight= Size =

Table 52: Weights and Sizes of Objects Made of Brass

Brass					
Object 1	Object 2	Object 3	Object 4	Object 5	Object 6
Weight =18	Weight = 6	Weight = 3	Weight =	Weight = 24	Weight = 90
Size = 6	Size = 2	Size = 1	Size = 4	Size =	Size =

Finally, I placed together Tables 50 and 52 and said: "Please compare the aluminum cylinders and the brass cylinders. How are they different?" Then I asked the following two questions to determine how well they understood the meaning of the quantified relationships between weight and size for the aluminum and brass cylinders: "If an aluminum cylinder has the same size as a brass cylinder, how do their weights compare to each other?" "If an aluminum cylinder has the same weight as a brass cylinder, how do their heights compare to each other?"

Determining the Relationship between Size and Weight for Aluminum

After the children filled out Table 49 with the sizes and the weights of the three aluminum cylinders, all of them, except for one third grader (Randy, who said that none of the cylinders has a height more than six), expressed something equivalent to "The weight is the same number as the size." or "For the aluminum, the size, how many tall, is equal to the amount of the weight." As we will see in the transcript presented below, one child, Rachel, even used a square in shape as a metaphor to explain the relationship between the weight and the size of aluminum cylinders. Table 53 shows two examples of children's responses to this question.

Table 53: Two Examples of Children's Responses to the Question "What do You Notice about These Numbers? (Aluminum)"

Reasoning	Dialogue
The weight is the same number as the size	Lily (9 years old, 4th grade) Interviewer: What do you notice about these numbers? Lily: Well, for each one, the weight is the same number as the size. It weighs 6; the size is 6. And this weighs 2; the size is 2. Interviewer: Do you think the size and the weight are exactly the same thing for aluminum or are there some other ways to express that? Lily: Well, the size and the weight aren't the same thing. Size is how much something measured, if it's taller or wider. Weight is how heavy something is. Interviewer: So when you say the weight and the size are the same, what do you mean? Lily: Well, on this [scale], it goes to the same

Reasoning	Dialogue
	number. It doesn't mean they are the same.
The weight is the same number as the size Using a metaphor	Rachel (10 years old, 4th grade) Rachel: The weight and the height are both the same number. Interviewer: The same number. What do you mean the same number? Do you mean the weight and the size are exactly the same thing? Rachel: I labeled pink as one. This one [the small aluminum] is one part of the ruler. So it's the same both ways. So it's basically like square weight and height, but not in shape. Interviewer: Say it again. Rachel: It's the same for the weight and the height, like the square is for the dimensions. So its height is one and its weight is one [putting the small aluminum on the scale]. So it's the same thing, except it's a cylinder.

Although the children said that the size and the weight are the same, most of them pointed out that the size and the weight are not the same thing and they just have the same numbers. However, none of the 19 children expressed the relationship as a ratio of the weight to the size.

Inferring the Weight or the Size of Aluminum Cylinders

The children were then asked to complete Table 50 with the weight or the size of the last three hypothetical aluminum cylinders. Except for three children, all others (85%) gave correct answers for all three cylinders. Table 54 shows examples of children's responses in inferring the weight or the size of the hypothetical aluminum cylinders.

Table 54: Examples of Children's Responses in Inferring the Weight and the Size of Aluminum Cylinders

Answers and reasoning	Dialogue
Correct answers to all three questions The weight matches the size	Jim (9 years old, 4th grade) Jim: The aluminum is supposed to goes up. The size of it matches the weight; the weight matches the size. So 4, 4, 24, 24, 90, 90. Interviewer: So you said match. What do you mean? Are they exactly the same thing or… Jim: They are same number. Interviewer: Same number. Okay.
Correct answer to the question with small number (4) Wrong answer to the question with big numbers (24) Using real cylinders	Randy (9 years old, 3rd grade) Randy: OK, the size is four. Can I use those [the real covered cylinders]? Interviewer: Sure. Randy: [He grabbed D (brass) and put it against the ruler, then put the medium aluminum cylinder on the top of D (now these two cylinders reach four marks on the ruler). Next he put D and the medium aluminum cylinder together on the scale. The pointer went beyond green (6).] I don't know what weight it would be. Are these the same? Interviewer: D is not made of aluminum. C is made of aluminum. Randy: OK, [grabbed C and compared it with the medium aluminum cylinder in his hands] they are the same weight then. Interviewer: Yeah. Randy: [Put C and the medium aluminum cylinder together on the scale.] OK, it would equal to brown, which equals to four [writing the number in the table]. [Looking at the fifth cylinder in the table] and the size for the weight is 24 [then

Answers and reasoning	**Dialogue**	
		looking at the scale].
	Interviewer:	We do not have that heavy thing, you can just imagine…
	Randy:	So the weight is 22 [the weight actually is 24], [calculating in mind] it would… the height … OK … so 24.. this is a very hard one to do.
	Interviewer:	It is OK if you can't do it.
	Randy:	I don't think I can do it.
Correct answer to the question with small number (4)	Emily (9 years old, 3rd grade)	
	Emily:	I think it might be 4.
Wrong answers to the questions with big numbers (24, 90)	Interviewer:	How about the fifth object?
	Emily:	Might be like 20, for instance, I don't expect it like this big [rising one hand to show very high].
The size did not increase as much as the weight	Interviewer:	Okay, you can write it down. How about the last one [the weight of it is 90]?
	Emily:	It will definitely not be that [inaudible]. It might be like 70 sort of.
	Interviewer:	Could you explain more about why do you think it couldn't be 90?
	Emily:	It couldn't be 90, because it may weight 90, but this [B, which is a covered cylinder made of brass] weights a lot but is not much [in size]. Because this equals gray, but it is totally in height, it is one.
Correct answer to the question with small number (4)	Aaron (9 years old, 4th grade)	
	Aaron:	I think it's going to be 4.
Wrong answers to the questions with big numbers (24, 90)	Interviewer:	How did you figure that out?
	Aaron:	I don't really know.
	Interviewer:	Okay. You don't have to explain it to me. Are you sure it is four?
Something inside of me just said it was	Aaron:	Yes.
	Interviewer:	You can write down the number.

Answers and reasoning	Dialogue	
	Aaron:	[Writing down 4]
	Interviewer:	How about this one? If the weight is 24, how tall should that cylinder be?
	Aaron:	12.
	Interviewer:	How did you figure that out?
	Aaron:	I think that will be half of 24. That will be 12.
	Interviewer:	Why is it half of 24?
	Aaron:	Something inside of me just said it was.
	Interviewer:	How about this one? If the weight is 90.
	Aaron:	25. I have no idea how I know it.

Most children just wrote down the same number to match the numbers for the known weight (or size). Randy could infer the weight of the fourth cylinder (Size=4) with real cylinders, but could not do it when the numbers became bigger (24 and 90) perhaps because he could not clearly express the relationship between weights and sizes of the three aluminum cylinders at the start. This suggests that being able to describe the relationship between weight and size has an effect on the ability to infer the weight or the size of an aluminum cylinder.

Emily's and Aaron's responses are very interesting. They did not think that the sizes of the cylinders which have the weights of 24 and 90 would have the same numbers as the numbers for their weights, although, before, they had noticed that for the three real aluminum cylinders the weight and the size had the same number. It seems that, for them, when the weight became bigger, the sizes would not increase as much as the weights.

Figuring out the Weight of the Largest Brass Cylinder

After I handed the children the three brass cylinders and Table 51, they found that the largest brass cylinder was too heavy to be measured with the color scale and they tried to find another way to figure out its weight. All the children, except one (Randy), succeeded in doing that. (I had prepared another color scale with a larger measuring range, but it was never used, because the children were so resourceful.) Table 55 shows the number of children who used particular strategies to determine the weight of the large cylinder. Table 56 shows examples of children's responses for each approach.

Table 55: The Number of Children Who Used Each Strategy in Finding out the Weight of the Large Brass Cylinder

Strategies	Number of children
Functional approach	4
Scalar approach by multiplication	6
Combination of a scalar approach by addition and a scalar approach by multiplication	4
A scalar approach by addition with the help of cylinders D	3
A scalar approach by addition by counting by 3	2
Could not determine the weight.	1

Table 56: Examples of Children's Responses by Using Different Approaches in Figuring out the Weight of the Largest Brass Cylinder

Approaches	**Dialogue**
Functional approach	Jake (9 years old, 3rd grade) Jake: [Measures the height of the big one with the

Approaches	Dialogue
	simplified ruler]. Two inches equal 6 pounds, so one inch equals three pounds. [He is doing some mental calculation]. So this is equal to 18. Interviewer: Could you tell me how did you figure that out? Jake: It's because one inches is equal to 3 pounds for brass, so that is 6 times 3 will equal 18.
Scalar approach by multiplication	Rachel (10 years old, 4th grade) Rachel: Because for the size, one [the height of the small brass] plus one is two [the height of the medium brass], except two [the height of the medium one] plus two is four, not six [the height of the large brass]. Two times three is six. So it's a third in size. Maybe the weight [of the medium one] would be a third of this [large], which would be eighteen.
A scalar approach by addition followed with a scalar approach by multiplication	Aaron (9 years old, 4th grade) Aaron: This [small brass] weighs three. [Stacks up the small brass and the medium brass together next to the big one and puts ruler against them, writing down the height of the larger one] Probably weight [lift the larger one in one hand and the stack of the small and medium ones in the other hand] I think it weights 12. Interviewer: How did you figure that out? Aaron: This [the small and medium brass] weights 9, this [big] will be 18? Interviewer: Tell me how did you figure that out? Aaron: If 9 is the weight of half of this [large brass] in size, you must double 9 to get 18.
A scalar approach by addition (3 Ds)	Jennifer (10 years old, 5th grade) Jennifer: Is weight 18? The weight is 18 for that 6 [the size of the large brass]. Interviewer: Why is that 18? Jennifer: Yes. D is 6, right? [D was also made of brass, just wrapped with write paper] Interviewer: Yes.

Approaches		Dialogue
	Jennifer:	[Put D on the scale and put medium on the scale] 6 plus 6 equals 12.
	Interviewer:	But that is too heavy for the scale.
	Jennifer:	But if the scale has more, like shop scales, you could add like another D.
	Interviewer:	We have another D.
	Jennifer:	Yeah, another D and you add another D. Then it will make the same weight as this [F]. Because they all the brass.
	Interviewer:	How did you figure out it is 18?
	Jennifer:	Because when I measure these two first… If you added this [the small brass] for three times, it's going to the height number. I knew something is wrong with this one [large brass]. If you added one more time [I guess she meant to add 3 small brass one more time], it will make 18.
A scalar approach by addition (3+3…)	Nancy (10 years old, 4th grade)	
	Nancy:	Oh. Could I do with this one [the small brass].
	Interviewer:	Sure.
	Nancy:	[Put it on the scale] That one is three. [Put medium one on the scale] That is green. So each one is three. [Stacking up the small one on the medium one] six and nine. Use this guy [using the ruler to measure the big piece of brass]. So each one is three. Three, six, nine, twelve, fifteen, eighteen.

As in previous tasks, the children flexibly applied different strategies or combinations of strategies to determine the weight of the larger brass cylinder. Moreover, from then on, the children began to use functional approaches more often. In fact, I did not expect the children to be able to determine the weight of the large brass cylinder before finding the relationship between weight and size for brass, which was the next question.

Determining the Relationship between Size and Weight for Brass

When asked what they noticed about the values of the weights and the sizes for the three brass cylinders (see Table 57), 70% of the children expressed, using slightly different words, that the size times 3 was equal to the weight, 10% said: "The size is counting by ones; the weight is counting by threes"; one child (5%) said: "They all have a weight different from the size and the size has different numbers"; one only focused on the relationship among the sizes of three brass cylinders; and one child said: "I don't know." One child was not asked this question. Table 58 shows examples of their different responses.

Table 57: The Number of Children Who Gave Different Answers to the Question "What Do You Notice about These Numbers?" (Brass)

Responses	The number of children
The size times 3 is equal to the weight	14
The size is counting by ones; the weight is counting by threes.	2
They all have a weight different from the size and the size has different numbers.	1
Only focused on the relationship among the sizes of three brass cylinders	1
I don't know	1
Not asked	1

Table 58: Examples of Children's Responses to the Question "What Do You Notice about These Numbers?" (Brass)

Patterns	**Dialogue**
The size times 3 is equal to the	Lily (9 years old, 4th grade)

Patterns	**Dialogue**
weight	Lily: Everything times 3. The size times 3 is equal to the weight. 6 times 3 is 18; 2 times 3 is 6; 1 times 3 is 3.
	Henry (8 years old, 3rd grade) Henry: One tall is always three weight.
	Jake (9 years old, 3rd grade) Jake: One inch is equal to 3 pounds for brass
	Nancy (10 years old, 4th grade) Nancy: Oh, double. No. It is tripled. Interviewer: What do you mean? Nancy: You just triple the size and you get the weight.
The size is counting by ones, the weight is counting by threes.	Adele (9 years old, fourth grade) Adele: The size is counting by ones, the weight is counting by threes. So they are not the same at all.
They all have a weight different from the size and the size has different numbers.	Isabel (9 years old, 3rd grade) Isabel: They all have a weight different from the size and size has different number.
Only focusing on the relationship among the sizes of three brass cylinders	Rachel (10 years old, 4th grade) Rachel: It's half for the first two, but a third for the second. Interviewer: What is the same for any cylinders made of brass? Is there something the same for all of them? Rachel: Well, the weight and the height are kind of in a pattern. It's like... so this one [the height of the small one] is half of this one [the medium one], this one [the height of the medium one] is a third of this one [the height of the large one]. And this one [the weight of the small one] is half of this one [the medium one], this one [the weight of the medium one] is a third

Patterns	Dialogue
	of this one [the weight of the large one]. So it's kind of half, third. So I am guessing it would go on to be half and third.

Here Rachel noticed only that the increase in size was the same as the increase in weight. But later, in the next question, she figured out that the size was a third of the weight. Three children could not find the relationship between the weight and the size for brass cylinders. Thus, finding the relationship between the weight and the size for brass cylinders was more difficult than for aluminum cylinders.

Inferring the Weight or the Size of Brass Cylinders

Table 59 shows the number of correct answers and types of strategies used to infer the weight or the size of brass cylinders. For the small numbers (4 and 24), a few children used a scalar approach by addition successfully, but for the largest number (90), none of them used a scalar approach by addition successfully. Children who used functional approaches could always give the correct answer. Moreover, in this task more children used functional approaches than was the case before.

We can also see that whether the children noticed the relationship between the weight and the size of the brass cylinder in the above question had an effect on their performance in this set of questions. Three children who did determine the proportional relationship between the weight and the size of the brass cylinder in the previous question performed worse than other children. One child did not give a correct answer for all of these questions, one child gave two wrong answers and one gave one wrong answer. For the questions they correctly answered, they used a scalar approach by addition. However the scalar approach by addition became cumbersome for larger numbers and one child failed in the third problem and another failed in both the second and the third problem. Table 60 shows examples of responses from the children who used different approaches.

Table 59: The Number of the Children Who Used Different Approaches in Inferring the Size or the Weight for Brass Cylinders

Answers	Strategies	Problems		
		Size of 4	Weight of 24	Weight of 90
Correct answer	Functional approach with a linear function	1	1	1
	Functional approach	9	9	9
	Scalar approach by addition	2	3	0
	No explanation	4	2	1
Wrong answer		2	3	4
Don't know (It is hard)		1	1	4
Not asked		1	1	1

Table 60: Examples of Responses from the Children Who Used Different Approaches in Inferring the Size or the Weight for Brass Cylinders

Approaches	**Dialogue**
Functional approach	Rachel (10 years old, 4th grade) Rachel: The size was a third of the weight. Interviewer: What do you mean? Explain a little bit more about that. Is this just for the brass, or it is also for the aluminum? Rachel: No, the aluminum is the same. Interviewer: Okay. Rachel: Then for the brass, it takes three ones equal three, and three twos equal six, and three six

Approaches	Dialogue	
		equal to eighteen, and three four equal twelve.
	Interviewer:	Could you guess this one? The weight is 24. What would be the size of this one?
	Rachel:	If it were a third, a third of 24 would be 8. I think the size would be 8.
	Interviewer:	How about that one?
	Rachel:	If the weight were 90, a third of 90 would be... I think the size would be 60? No. I think it would be 30, because 30 plus 30 and plus 30 is 90.
Functional approach Writing a linear function	Caroline (10 years old, 5th grade)	
	Caroline:	The weight is four times three. You have to know the multiplication. Twelve. That divided by... eight. Ninety ... thirty times three, yeah.
	Interviewer:	Could you tell me how you got these numbers?
	Caroline:	I multiplied, oh. [Writing down an expression and labeling the input and output] The rule is n times three.
	Interviewer:	I can't see it. [Caroline turned the paper around.] Oh, that is output and input. That is really cool. Could you tell me what is n?
	Caroline:	N is any number that gets put into the input.
	Interviewer:	So n times three is equal to...
	Caroline:	n times three is whatever input multiplied by three and that will be output.
	Interviewer:	So output is weight, right?
	Caroline:	Yes, output is weight and input is size [for 4]. But down here [weights 24 and 90] it's kind of switched around.
	Interviewer:	What do you mean?
	Caroline:	Well, down here you only give these numbers. This will be the input; this would be the output. So down here it will be... this would be input, this will be the output. But the rule down here, is n divided by three. And n is divided by three, even though down here it is n times by three. This is input here, this is output here. So it is kind of switched around. You also have to do

Approaches	**Dialogue**
	the opposite rule (see Figure 20, page 151)
Scalar approach by addition Correct answers to the problems with small numbers (4, 24) Wrong answer to the problem with big number (90)	Adele (9 years old, fourth grade) Adele: 12. Oh, my God. 8, the weight is 24, the size would be… hold on… [counting in her head] 8… 8 counted by 3s. And 90… oh my god… 72… I am just writing down… 98. So it is not going to even of it. I don't think so. Interviewer: If you don't know, that is fine. Adele: I think that would be twenty… I am just confused. Oh, I see what I did wrong. This would be two 8s. This would be 16. 16 plus 8?... I think… I am still trying to figure it out… 24 plus another 8… 32. Maybe the size is 32. I have no idea.

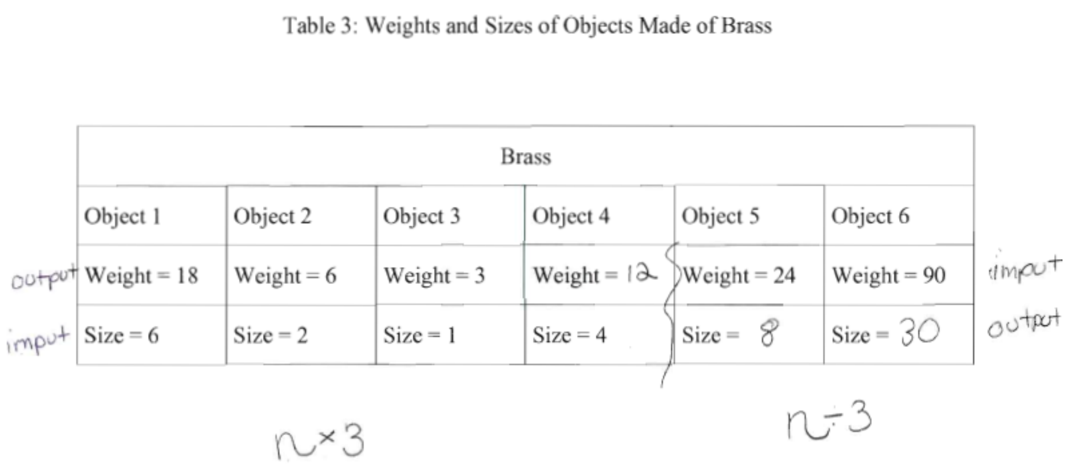


Table 3: Weights and Sizes of Objects Made of Brass

Brass							
	Object 1	Object 2	Object 3	Object 4	Object 5	Object 6	
output	Weight = 18	Weight = 6	Weight = 3	Weight = 12	Weight = 24	Weight = 90	impout
impout	Size = 6	Size = 2	Size = 1	Size = 4	Size = 8	Size = 30	output

Figure 20: Caroline's written work on inferring the weight and the size of the brass objects

The above examples strongly suggest that children's ability to perform arithmetical operations played a role in this set of questions. It is obvious that calculating a third of 90 was harder than calculating a third of 24 or 3 times 4. It is interesting that one child used a linear function representation to express the relationships between weight and size. But because she could not provide a complete expression like "size × 3 = weight," she had to use two rules. That is, when inferring weight, she used n×3; when inferring size she used n÷3. She always used n to represent input, no matter whether it was weight or size.

Direct and Inverse Proportional Reasoning

When asked the questions "If an aluminum cylinder has the same size as a brass cylinder, how do their weights compare to each other?" and "If an aluminum cylinder has the same weight as a brass cylinder, how do their heights compare to each other?" all of the children expressed that, when they have the same size, the brass cylinder would be heavier; when they have the same weight, the aluminum cylinder would be larger. For the same size question, most of the children stated that the brass cylinder would be three times as heavy as the aluminum one, if they had the same size. Among them, ten children referred to real cylinders; four gave the correct answer without referring to any cylinders. Only two children gave the difference in weight between the brass cylinder and the aluminum cylinder as an answer.

For the same weight problem, most children figured out that the aluminum cylinder would be three times as tall as the brass one if they had the same weight. Among them, seven children referred to real cylinders and eight gave the correct answer without referring to any cylinders. Only two children said that the aluminum cylinder would be taller, without specifying how much taller (see Table 61). Table 62 shows examples of children's responses to the two questions.

Table 61: The Number of Children by type of Response for each Question

Responses		Same size	Same weight
Three times	Without real cylinders	4	8
	With real cylinders	10	7
Difference		2	0
Taller		0	2
Not asked		4	3

Table 62: Examples of Children Responses to the Questions "If Same Size (or Weight), How do Their Weights (or Sizes) Compare to Each Other?"

Responses	**Dialogue**
Three times Without real cylinders	Jim (9 years old, 4th grade) Interviewer: If an aluminum cylinder has the same size as a brass cylinder, how do their weights compare to each other? Jim: Brass would be heavier. Interviewer: How many... Jim: By three. Interviewer: If an aluminum cylinder has the same weight as a brass cylinder, how do their sizes compare to each other? Jim: The brass would be smaller. Interviewer: How... Jim: By three.
Three times With real cylinders	Rachel (10 years old, 4th grade) Interviewer: So if aluminum cylinder has the same size as a brass cylinder, how does their weights compare to each other? Rachel: For the aluminum the weight is six for the largest one. And here the weight is eighteen, which is a third. Six is a third of eighteen. It's kind of like the aluminum is third of the

Responses	Dialogue	
		brass, like the size of the brass is a third of the weight.
	Interviewer:	Could an aluminum cylinder have the same weight as a brass cylinder?
	Rachel:	It could. So for the brass, this one [medium] is six [in weight] and this one [the large aluminum] is six [in weight]. So they are both the same size [she meant same weight].
	Interviewer:	Okay. If they have the same weight, how do the sizes of them compare to each other?
	Rachel:	Well, this one [the medium brass] is two [in size] and this one [the large aluminum] is six [in size]. So the size of this one [the medium brass] is a third of this one [large aluminum].
	Interviewer:	Why is the size of the brass one a third of the size of the aluminum one, if they have the same weight?
	Rachel:	Well, the brass is much heavier material than the aluminum. Aluminum is a lot of lighter, so it has to be larger to be the same as the brass.
Difference	Isabel (9 years old, 3rd grade)	
	Interviewer:	If an aluminum cylinder has the same size as a brass cylinder, how do their weights compare to each other?
	Isabel:	[She was silent]
	Interviewer:	Which one would be heavier?
	Isabel:	The brass cylinder.
	Interviewer:	How many times heavier?
	Isabel:	4 or 2, or 12 [these were the differences between the weights of small, medium and big aluminum and brass cylinders]
Starting with difference, then considering the ratio of two weights with referents	Caroline (10 years old, 5th grade)	
	Interviewer:	If an aluminum cylinder and a brass cylinder have the same size, how do their weights compare to each other?
	Caroline:	OK, if they have the same size, so I find two

Responses	Dialogue	
		and two [in size] [the medium brass and the medium aluminum], and two and six [in weight]. This is four more than the weight here. It could be possible. I will try different ones out here. These two [small ones] are the same size, are both one, but this is plus two [for small ones], and this is plus four [for medium ones]. So the size doubles the difference. So two add two is four and that is the difference of these two [medium]. And one adds one is two and that is the difference between one and three [small ones]. I am going to try this [large]. This [the large aluminum] would be eighteen I think. No. This [the large brass] is eighteen; this is six. Six and six is twelve. Twelve is the difference.
	Interviewer:	Could you find something that is the same for any size of aluminum?
	Caroline:	Like a rule.
	Interviewer:	Yeah. I mean no matter how tall it is, as long as they have the same size, there is a rule.
	Caroline:	So, it would be n times... no ... it will be n plus... no.
	Interviewer:	First, tell me if an aluminum cylinder and a brass cylinder have the same size, which one will be heavier?
	Caroline:	Definitely the brass. It will be triple. Oh yeah. So, the aluminum cylinder, no the brass cylinder is triple the weight of aluminum cylinder.
Taller	Jennifer (10 years old, 5th grade)	
	Interviewer:	Which one would have bigger size, if they have the same weight?
	Jennifer:	[Pointing to aluminum]
	Interviewer:	Why?
	Jennifer:	Because this is lighter. [Puts big aluminum cylinder on the scale] Because this is six and

Responses	Dialogue	
		this [big brass] is 18.
	Interviewer:	If there is another brass cylinder that has the same weight as this one, how tall should that cylinder be?
	Jennifer:	[Puts medium brass cylinder on the scale] this tall.
	Interviewer:	How do their sizes compare to each other?
	Jennifer:	This one [brass] is smaller, but has a lot weight in it, so it does have [inaudible] that much. This one [big Al] is a lot of taller. If it were this small [put medium Al on the scale] it wouldn't weigh anything.

In general, children's performance shows that throughout the interview they gained a good understanding of the relationships between weight and size, although they did not know the word density nor used the expression "ratio of the weight and the size." They came to acknowledge that the relationship between the weight and the size does not change for the same kind of material, that these relationships are different for different materials, and that for the less "dense" material, in order to have the same weight as the more "dense" material, the size must be bigger and inversely proportional to their "densities." However, because I did not introduce the word "density" to the children, some still used the word "heavy" to express the meaning of "density" saying, for example: "Well, the brass is much heavier material than the aluminum. Aluminum is a lot of lighter, so it has to be larger to be the same as the brass."

Children's Performance in Task 9 by Grade Levels

Table 63 shows children's performance in Task 9 by grade levels. Their performance in the items related to the relationships between weight and size for aluminum cylinders did not show much difference across grades. For brass cylinders, there were no differences across grades when small numbers for weight or size were used. However, for questions involving large numbers (24 and 90), the fifth graders tended to performed slightly better than the fourth graders and the fourth graders better than the third graders. All children correctly answered all questions comparing the weights (or sizes) of aluminum and brass cylinders of the same size (or same weight).

Table 63: The Number of the Children Who Gave Correct Answers in Different Questions in Task 9 by Grade Levels

Grade	The Number of the Children Who Gave Correct Answers										
	Aluminum				Brass					Same Size	Same Weight
	What noticed ?	4	24	90	Weight of the large piece	What noticed ?	4	24	90		
Third (8)	7	8	6	6	7	4/7*	6	5	3	6/6*	6**/6*
Fourth (8)	8	8	7	7	8	7	6/7*	6/7*	5/7*	6/6*	7/7*
Fifth (4)	4	4	4	4	4	4	4	4	3	4	4**

* Numerator is the number of children got the correct answer; denominator is the number of children who were asked this question.

** One child just mentioned that aluminum would be taller without specifying how much taller.

Summary of Results

The data show that quantification of weight and size, together with some systematic comparisons, helps children gain a formal understanding of relationships between weight and size and to use the relationships they found for aluminum and brass to infer the weight or the size of aluminum and brass cylinders. The difficulties for younger children arose when divisions with larger numbers were involved. No ratios or units were used in children's expressions of the relationships between weight and size.

The analysis in this chapter addresses the third research question (Can children generalize a linear relationship between weight and volume for a certain material?), the second part of the fifth research question (What roles does quantification play in generalizing a linear relationship between weight and volume for a certain material?), and the third part of the sixth research question (What difficulties do children encounter in generalizing a linear relationship between weight and volume?). In Chapter 11, I will discuss what the above results suggest in terms of answers to these research questions, the relation of these findings to those from previous studies, their theoretical implications, and their relevance for science education.

Chapter 9: Evaluating the Impact of the Intervention: Results of the First Part of the Pretest (Tasks 1–3) and Posttest (Task10) on Why Different Objects Have Different Weights.

As described in the Method section of this dissertation (Chapter 4), the first and final sections of the interview constitute a pretest and posttest to evaluate the impact of the intervention, that is, the middle section of the interview aimed at helping children merge quantification into their intuitions and understand the relationships of weight and size of cylinders. This chapter presents the results the first part of the pretest (Tasks 1–3) and posttest (Task 10). Chapter 10 covers the second part of the pretest (Task 4) and posttest (Task 11), where children were asked to infer cylinders' materials by considering their sizes and weights.

Tasks 1–3 examined whether children knew that the weight of an object is determined by its size and by the kind of material it is made of. I presented children with sets of three cylinders and posed questions about them. Task 10 repeated the same questions while providing different sets of cylinders (pairs, this time) to evaluate how access to quantification affected children's responses.

Pretest Results

Table 64 shows the materials and questions used in Tasks 1 to 3.

Children's responses to Tasks 1–3 show that all participant children knew that the kind of material a cylinder is made of (Tasks 1 and 3) and its size (Task 2) have an effect on its weight. In their answers to Tasks 1 and 3, most children used the word "material" (see Tables 65 and 66). Other expressions they used were "made of different substance," "made of different kinds of metals," "made of different things," "this is made of metal; this is made of plastic," and "this is more compact." One child even used the word "density".

Table 64: Materials and Questions Used in Tasks 1–3

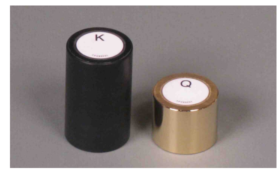
Task 1	Task 2	Task 3
		
Why do you think they have different weights even though they are the same size?	Why do you think they have different weights? Are they all made of the same kind of material or different materials? How do you know?	How come this one is smaller but is heavier than that one?

Table 65: Number of Children Who Used Particular Reasons to Explain Why K, L, and P Have Different Weights in Task 1

	Different materials	Different kinds of metals	Different things	Denser
The number of children	14	3	2	1

Table 66: Number of Children Who Used Difference in Kind of Material to Explain Why K and Q Have Different Weights in Task 3.

	Different materials	Metal vs. plastic	More compact	Different substance	Not asked
The number of children	16	1	1	1	1

However, only five children mentioned the co-variation between weight and size as a criterion to determine whether P, Q and R (in Task 2) were made of the same kind of material (see Table 67). Here, one child (Caroline) did it quantitatively, and the others, such as Gerard, just did it approximately. Some children used two or more criteria. The appearance of the cylinders was more often used, with more than half of the children considering their colors and three considering their textures. Four children thought the P, Q and R were made of the same kind of material because they were kind of heavier than usual. One child (Stella) said that P, Q and R had the same density (see excerpt in Table 68).

Table 67: Number of Children Who Used Different Criteria to Explain Why P, Q and R Were Made of the Same Kind of Material in Task 2

	Color	Texture	Smell	Heavier than usual	Weight and size co-vary	Density
The number of children	11	3	1	4	5	2

Table 68: Examples of Responses from Children Who Considered the Weight and/or Size in Task 2

Criteria	Dialogue
Weight and size co-vary	Gerard (9 years old, 3rd grade) Gerard: I think they are made of the same kind of material, because this one [R] is the smallest and it weights less. And this Q, which weighs a little bit more. P weighs the most.
Weight and size co-vary Quantitatively	Caroline (10 years old, 5th grade) Caroline: They are the same shape like round in the middle and they are not the same height. This [medium] like is half, so half of the weight of this [large]. And this is quarter of this [comparing heights]. This is a quarter of weight of this. Interviewer: Are they made of the same kind of material or different kind of material? Caroline: Same kind of material but different portion. Interviewer: How can you tell they are made of the same kind of material? Caroline: Well, they are the same color. I am guessing that if you take two of these [the medium one] and put them together, they should weigh the same amount [holding the large one in the other hand]. If you take four of these [the small one], they should weigh the same amount [holding small and large ones in each hand]. Take two of these [small] they [two small ones and the medium one] should weigh the same amount.
Same density	Stella (5th grade) Stella: When you see it [Q], you would think you could flick it across the table, but you can't. Stella: When you pick it [P] up, it's very heavy, so it is a very dense material. Interviewer: Why do you think they are made of the same kind of meteial? Stella: Same weight... same density... really

Criteria	Dialogue
	compact.

Posttest Results

Although the cylinders used in Task 10 (see Table 70) were different from those in Tasks 1–3, the questions were similar. Task 10 consisted of the same three questions from Tasks 1–3 presented with different set of cylinders (the three pairs shown in Table 70.) Table 69 shows that, before the intervention, almost all the children used qualitative expressions, such as “same,” “different,” “heavier,” or “smaller”, while, after the intervention, more than half of the children used quantitative expressions (see examples in Table 70).

Table 69: Improvement in Use of Quantitative Reasoning from Pretest to Posttest

	Qualitatively	Quantitatively
Pretest	19	1
Posttest	8	12

Table 70: Examples of Responses from Children Who Compared Cylinders Quantitatively

Cylinders	**Dialogue**
	Lily (9 years old, 4th grade) Interviewer: Why do you think they have different weights? Lily: Although they are made of the same material, because this one is bigger, it weighs more, because this is three times as the size of this.
	Jim (9 years old, 4th grade) Interviewer: Why do you think they have different weights, even though they are the same size? Jim: This one is heavier. It's a heavier object and heavier material. Interviewer: Are they the same size. Jim: They are the same size, but this one is heavier material, so it's heavier. . . I think it is going to be by three. [Put the aluminum one on the scale] Yes, it's by three.
	Nancy (10 years old, 4th grade) Interviewer: How come this one is smaller but heavier than that one? Nancy: Because it's a heavier material and if they are going to the same [weight], this one has to triple the Z. Interviewer: but it's not triple... Nancy: But it is just double, so they are not the same.

Summary of Results

Prior to the intervention part of the interview, the children knew that both the kind of material an object is made of and its size have effects on its weight, but most of them could express the relationships among weight, size, and kind of material only qualitatively, with only one child displaying the co-variation between weight and size quantitatively. This suggests that the children had intuitions about the relationships among weight, size, and kind of material and that, at these ages, they could benefit from opportunities to merge quantification into their intuitions and thus refine their understandings and their ability to reason productively. After the intervention, most of the children were able to coordinate weight, size, and kind of material quantitatively. These questions on why different objects have difference weights did not yet demand use of quantitative proportional reasoning, and some of the children still using qualitative comparisons may have thought that it was enough to mention qualitative relationships. The results from the more formal final stage of the pretest (Task 4) and posttest (Task 11) on inferring materials, described in the next chapter, show more dramatic changes in children's reasoning and explanations.

Chapter 10: Evaluating the Impact of the Intervention: Results of the First Part of the Pretest (Task 4) and Posttest (Task 11) on Inferring Material

As mentioned in Chapter 4 (Method) 13 cylinders were used in Tasks 4 and 11, the second stage of the pretest and posttest (see Figure 21). The questions to be answered by the children here, for each of the four cylinders B, D, A, and C, were: "Could this cylinder (B, D, A, or C) be made of the same kind of material as one of these cylinders (E, F and G), or of something else?" Some extra copies of A, B, C and D were available. A two-pan balance scale was available so children could compare the relative weights of two cylinders or of one tall cylinder against a combination of the smaller ones. Data in this analysis come from interviews with only 18 third to fifth grade children, because there were two children that I could not interview with this task as a posttest.

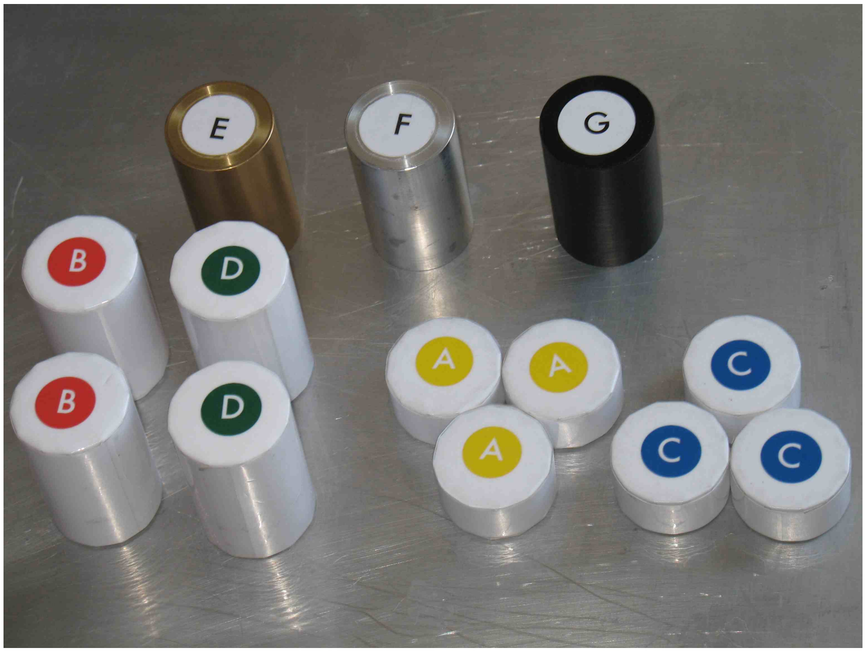


Figure 21: Cylinders used in the second part of pretest and posttest

The results show that the number of correct answers was significantly higher in the posttest, with an average of 3.9 correct answers per child, in comparison to the pretest where the average was 2.8 (Wilcoxon W = -66, z = -2.91, p = .0018). However, as we will see next, results varied greatly across the four subtasks in the pretest.

Subtask 1: What is B made of?

Results for the question *What is B made of?* are shown in Table 71. Cylinder B is the same size as cylinders E, F and G. Regardless of whether children used their hands or the balance scale, or whether they considered only weight or both size and weight, all the students, except for one in the pretest, in both occasions, correctly answered that B was made of the same kind of material as F (B and F are both made of aluminum). In the pretest, 61% of the students used the scale and nearly all of them did so in the posttest. More children expressed that they considered both the weight and the size in the posttest than in the pretest, but less than a third of them did so.

Table 71: Children's Pretest and Posttest Performance in Subtask 1

What is B made of?

		Pretest		Posttest	
		Frequency	Percent	Frequency	Percent
Answer	Aluminum (F) (Correct)	17	94%	18	100%
	Something else	1	6%	0	0
Tools	Scale	11	61%	17	94%
	Hand	7	39%	1	6%
Reason	Size and Weight	3	17%	5	28%
	Weight	14	78%	11	61%
	Balance	0	0	1	6%
	No explanation	1	6%	1	6%

Since B was the same size as the cylinders to be compared with, it was easy to estimate that B was lighter than E, heavier than G, and the same weight as F.

Subtask 2: What is D made of?

For the second question, "*What is D made of*?" (see Table 72), although D was the same size as E, F and G, only 61% of the children gave a correct answer in the pretest. In the posttest the percentage of correct answers increased to 89%. In the pretest, all the children giving a correct answer used the balance scale and 86% of those giving a wrong answer used only their hands. In the posttest, 94% of children who answered correctly used the scale and the two children who gave a wrong answer used their hands only. More children used the scale in the posttest than in the pretest. In the pretest only two children mentioned both weight and size; in the posttest only one child did so.

Table 72: Children's Performance in Subtask 2

What is D made of?

		Pretest		Posttest	
		Frequency	Percent	Frequency	Percent
Answer	Something else (Correct)	11	61%	16	89%
	Plastic (G)	7	39%	2	11%
Tools	Scale	12	67%	15	83%
	Hand	6	33%	3	17%
Reason	Size and Weight	2	11%	1	6%
	Weight	13	72%	13	72%
	Others	2	11%	0	0
	No explanation	1	6%	4	22%

D has the same size and was lighter than any of these three cylinders (E, F and G), but its density (and weight) was very close to G (the density of D was 2/3 of that of G), making it difficult for the children to tell, if they did not use the balance scale, that G was heavier than D.

The above two subtasks shows that, even in the pretest, most of the children could give a correct answer since all they had to consider was the relative weight of the objects. In subtasks 3 and 4, correct answers would require clearly determining the relationship between the heights and that between the weights of the two compared cylinders. Therefore, two subtasks described next should allow better evaluating children's strategies and possible progress.

Subtask 3: What is A made of?

Results for the third question (*What is A made of?*) are shown in Table 73. In this case, the target short cylinder A, made of brass, was heavier than cylinder G, had the same weight as F, and was lighter than E. The correct answer for this task is that cylinder A is made of the same kind of material as cylinder E. Table 73 shows that, even in the pretest, children's performance was rather high, with 78% of them giving the correct answer, and all of them answering correctly in the posttest. In comparison to the pretest, in the posttest about twice as many children stacked up 3 As, used the balance scale, and mentioned both the size and the weight, an indication of the positive effect of the intervention.

Table 73: Children's Performance in Subtask 3

What is A made of?

		Pretest		Posttest	
		Frequency	Percent	Frequency	Percent
Answer	Brass (E)	14	78%	18	100%

	(Correct)				
	Aluminum (F)	4	22%	0	0
Stacking	Yes	8	44%	16	89%
	No	10	56%	2	11%
Tools	Scale	10	56%	17	94%
	Hand	8	44%	1	6%
Reason	Size and Weight	6	33%	14	78%
	Weight	10	56%	3	17%
	Weight*	2	11%	0	0
	No explanation	0	0	1	6%

* May indicate an intuitive understanding of density

Table 74 shows examples of how children reached the correct answer in the pretest and their justifications.

Table 74: Examples of Correct Pretest Answers to the Question
"What is A Made of?"

Actions	Dialogue
Stacks up 3As Uses the scale Considers size and weight	Jim (9 years old, 4th grade) Jim: [Stacks up 3As and puts them on the balance scale] This is sort of heavy, so [puts E on the other pan of the scale]. I think it may be the same weight as E. I think it is E. It's brass. Interviewer: Could you explain to me why do you think it is made of E? Jim: It's the same weight. When stack these all together, they are the same size as E. E just looks like made of the same material.
Does not stack	Henry (8 years old, 3rd grade)

Actions	Dialogue	
Uses the scale Considers size and weight	Henry:	[Putting G and A on the balance scale] Definitely not the plastic. Maybe aluminum [putting F and A on]… I don't think it's the same material as aluminum, because it's smaller. So it's probably going to be brass [putting E and A]. I think it's brass.
	Interviewer:	Why do you think it's brass?
	Henry:	Because with this [F] [putting F and A on], even though it's even, they are different sizes. So they can't be the same material. This [A] must be heavier. And these [E and A], I think it's about the same, if you cut this [E] to the size of this [A].
Does not stack Uses hands Considers weight or maybe density	Stella (5th grade)	
	Interviewer:	Could A be made of one of these materials?
	Stella:	Maybe brass... It has the same oddly heavy feeing to it.

All of the children who stacked up 3 As reached the correct answer. However, the percentages of correct answers among children who did not stack and who used their hands only (instead of the scale) were also high (64% and 88%, respectively). In addition, half of the children who stacked up the cylinders clearly considered both weight and size. Children who gave a correct answer, even though they did not stack up the cylinders and used only their hands, in 67% of the cases stated that they just felt that A and E had the same weight ("because they are both very heavy") and 33% expressed something which was hard to classify according to whether they meant weight or an intuitive concept of density as, for example, this response: "it has the same oddly heavy feeling to it". Among those who gave a wrong answer (aluminum [F]) 75% used the scale and 75% considered only the weight when inferring the material of A.

Examples of responses from children who gave the wrong answer (aluminum [F]) are shown in Table 75.

Table 75: Examples of Wrong Answers to the Pretest Question "What is A Made of?"

Actions	Dialogue	
Does not stack Uses the scale Considers Only weight	Aaron (9 years old, 4th grade)	
	Aaron:	[Holds A and F, A and E, then puts A and E on the scale, then A and F on the scale] I think it is made out of that [F].
	Interviewer:	How do you know that?
	Aaron:	Because they feels about the same in weight.
Does not stack Uses the scale Considers size and weight	Isabel (9 years old, 3rd grade)	
	Isabel:	I think A is close to E [putting E and A on the scale], because ...or F [putting F on] I think A is closer to F. It [A] looks small but hold a lot of paper or something.
	Interviewer:	Did you say A is closer to F or E?
	Isabel:	F.
	Interviewer:	To F. Do you mean the weight or the material?
	Isabel:	The material.

Aaron used the balance scale, but he only considered the weight and did not stack the cylinders. He determined that A was made of the same kind of material as F, because they were the same weight. Even though Isabel realized that A is smaller, she did not feel it was necessary to coordinate the weight with the size, or did not have a clear idea about how to do it.

In sum, in the pretest, as long as both size and weight were considered together, the children could find the correct answer even if they did not stack the cylinders or use the scale.

All children answered the posttest question correctly; 69% stacked three short cylinders, and 78% considered weight and size. In these cases (see examples in Table 76) they expressed use of proportional reasoning and awareness of the linear relationship between weight and size for the objects made of the same kind of material.

Table 76: Examples of Posttest Answers to the Question "What is A Made of?"

Actions	Dialogue	
Does not stack Proportional reasoning Uses the scale Considers size and weight	Stella (5th grade)	
	Stella:	My suspicion is that it is made of brass… A is one tall… Brass is 3 times heavier than aluminum, so if A has the same weight as F, A is brass. [Puts A and F on the balance scale] So A is brass.
Does not stack Proportional reasoning Uses the scale Uses the ruler Considers size and weight Then stacks up 3As	Henry (8 years old, 3rd grade)	
	Henry:	[Puts G and A on the balance scale] it's not G. [Puts F and A on] I think it weighs the same as E. I think it's brass.
	Interviewer:	Why do you think that?
	Henry:	Because A is smaller than F and it's the same weight.
	Interviewer:	Is A made of F?
	Henry:	No.
	Interviewer:	Why do you think it's made of E.
	Henry:	This is level like this [F and A on the scale now], but it [A]'s smaller. And this [E] is

Actions	Dialogue
	heavier than this one [F]. Since, can I check this [pointing to the ruler]? This [A] is one tall; this [E] is three tall. Since this [E] weighs more than this [A], so it's either this material [E] brass or the material we don't have here, not one of these. (After figuring out the material of C [the next subtask], Henry then went back to E.) Henry: [Putting E and 3As on the balance scale] Yes, it's the same material.
Stacks up 3As Proportional reasoning Uses the scale Considers size and weight	Rachel (10 years old, 4th grade) Rachel: Three of these [putting 3As in one pan of the scale], I put them here and one E here. It is about the same. So I change my mind. I think they are the same material. Interviewer: Just explain it to me why do you think they are made of the same kind of material? Rachel: Because for brass, if it is a third of the height and a third of the weight, then they will be the same, because they kind of follow the pattern for the brass.

In the pretest, Stella only said that A could be made of the same kind of material as E, because “it has the same oddly heavy feeling to it.” In the posttest, she could find the correct answer using proportional reasoning. She did not use the word density but her statement that “Brass is 3 times heavier than aluminum” may indicate an understanding of density. She possibly meant that if the brass cylinder were the same size as the aluminum cylinder, the brass cylinder would be 3 times as heavy as the aluminum. She also said that A is one tall and if A has the same weight as F, A is brass. She did not explicitly mention, but may have been aware, that F was three times as tall as A.

Henry, in the pretest, explained: “I think it's about the same, if you cut this [E] to the size of this [A].” In the posttest he did not do the stacking at the beginning, but he compared the weights of A and E. Using proportional reasoning, he guessed that A was probably made of brass. Later he stacked 3As to justify his decision more confidently.

And Rachel, who in the pretest only said that A could be made of the same material as E, “because it [A] is very heavy and also very small,” in the posttest stacked up 3As and also displayed proportional reasoning to justify her answer.

In the pretest, most of the children reached a correct answer without considering the relationships between weight and size. But since they were comparing a small cylinder made of brass, a much denser material than the other cylinders’ material, they seem to have intuitively identified a property akin to density, but which could not be explored further than saying that it felt like the brass cylinder. Children’s explanations in the posttest changed dramatically, showing that they understood the co-variation of weight and size and were able to coordinate the weight and the size quantitatively, using proportional reasoning.

The last subtask, analyzed next, will better allow for analysis of children’s understandings before and after the intervention and of the role of quantification in their responses.

Subtask 4: What is C made of?

Although this task is similar to the previous one, with a target cylinder one-third the size of the comparison cylinders, children’s pretest performance in this case (see Table 77) was much lower, with only 50% of the children answering correctly. In the posttest, all the children responded correctly, stacked small cylinders to compare the composed result to the tall cylinders, and used the scale. Moreover, 69% of them explicitly considered weight and size, against only 39% in the pretest.

Table 77: Children's Performance in Subtask 4:
"What is C made of?"

		Pretest		Posttest	
		Frequency	Percent	Frequency	Percent
Answer	Aluminum (F) (Correct)	9	50%	18	100%
	Delrin (G)	2	11%	0	0
	Something else	7	39%	0	0
Stacking	Yes	7	39%	18	100%
	No	11	61%	0	0
Tools	Scale	13	72%	18	100%
	Hand	5	28%	0	0
Reason	Size and Weight	7	39%	12	67%
	Weight	9	50%	3	17%
	No explanation	2	11%	3	17%

In the pretest, all of the children who stacked up 3 Cs got the correct answer, using the balance scale or using their hands. Only 25% of the children who did not stack up 3 Cs found the correct answer. In addition, 71% of the children who stacked up the cylinders clearly considered both the weight and the size. 71% of the children who gave a wrong answer only considered the weight when inferring the material of C. Two children (11%) considered both the size and the weight but thought that C could be made of something else or G (Delrin).

Table 78 shows examples of how children reached the correct answer in the pretest and their justifications.

Table 78: Examples of Correct Answers to the Pretest Question "What is C Made of?"

Actions	Dialogue
Stacks up 3Cs Uses the scale Considers size and weight	Jim (9 years old, 4th grade) Jim: [Stack up 3Cs and put them on one pan of the balance scale] C is lighter. It might be made out of aluminum. [Put F on the other pan of the scale] I think it is aluminum. Same size, same weight.
Stacks up 3Cs Uses hands Considers size and weight	Lily (9 years old, 4th grade) Lily: [Stacking up 3 Cs together and comparing them with F with her hands.] I think C may be F. Interviewer: Why? Lily: Because they feel about the same like weight, but these [3Cs and B] are also about the same weights. I think they [3Cs and B] both would be F. Interviewer: Why did you stack up 3 Cs together? Lily: Because they are about the same size of the cylinder [F].
Does not stack Uses hands Considers only weight	Diana (9 years old, 4th grade) Diana: I think it is this one [aluminum]. Interviewer: How did you figure that out? Diana: Because it is kind of light, but is not as light as this one [Delrin].

We can see that, as was the case for cylinder A, as long as the child stacked up 3 Cs and compared them with F, they could get the correct answer, regardless of whether or not they used the scale.

In fact the weight of C is a little bit closer to G (Delrin) than to F (aluminum), but C is smaller and denser than G. Some children, who felt that C was closer to F in weight, may have implicitly considered the factor of size without mentioning it, like what Diana did. This was consistent with the findings in my pilot study (Liu, 2009).

Examples of responses from the children who gave a wrong answer (G, Delrin, or something else) are shown in Table 79.

Table 79: Examples of Wrong Answers to the Pretest Question "What is C Made of?"

Actions	Dialogue	
(Brass) Does not stack Uses the scale Considers size and weight	Henry (8 years old, 3rd grade)	
	Henry:	[Puts G and C on the balance scale] It may be this material, because it's smaller. [Puts F and C on the scale] Probably not this material. [Puts E and C on the scale] And not this material. So I think it's plastic.
	Interviewer:	Do you mean this plastic [G]?
	Henry:	This plastic.
	Interviewer:	Why do you think C is made of this plastic?
	Henry:	[Puts G and C on the scale] Because even though this [G] weighs more, that's because there are more mass.

Actions	Dialogue
(Something else) Does not stack Uses the scale Considers size and weight	Isabel (9 years old, 3rd grade) Isabel: I think C is really close to G. [Putting G and C on the scale] G is heavier. Even though C feels a lot heavier. And they are not the same height. If you put like three of these in G, it will probably feel a little heavier. C is little bit heavier than G, so I think C is closer to G kind of material. Interviewer: But it is not the same kind of material as G or the same kind of material as G? Isabel: It is not the same.
(Something else) Does not stack Uses hands Considers only weight	Aaron (9 years old, 4th grade) Aaron: [Holds C and G, C and F] I think it is made of something else. Interviewer: How do you know that? Aaron: Because this [C] is heavier than this [G]; this [F] is heavier than this [C], so I think this [C] is between this and this. Interviewer: Between G and F? Aaron: Yeah.
(Something else) Does not stack Uses the scale Considers only weight	Jessica (10 years old, 4th grade) Jessica: I will start with the lightest. [Putting C and G on the scale]. G is the lightest and it is still [inaudible] probably made of something else.

We can see that, even though Henry and Isabel coordinated the weight with the size, they could not find the correct answer, because the balance scale can only show whether or not two objects are equal in weight. Quantification of weight was not available, unless the child stacked up 3Cs.

Aaron and Jessica only considered weight. Although Aaron used his hands and Jessica used the scale, both of them thought that C could be made of something else, because both the scale and the hands showed that C was not the same weight as G or F. They did not compare C with E, perhaps because they already knew that E was much heavier than C.

The difference in children's pretest performance for this subtask, in comparison with the previous subtask, may be due to the fact that the compared cylinders did not give a clear feeling of "heaviness," or some intuition of density, as was the case for the brass cylinders. Here, in the case of cylinder C, the correct answer required the more precise coordination of size and weight by stacking cylinders.

Table 80 shows two examples of children's performance in the posttest.

Table 80: Examples of Answers to the Posttest Question
"What is C Made of?"

Actions	Dialogue	
Stacks up 3Cs Proportional reasoning Uses the scale Considers size and weight	Henry (8 years old, 3rd grade)	
	Henry:	[Putting G and C on the balance scale] I think it's the same material as G. Because G weights more, but C is smaller. Again C is one tall and G is three tall. It's [G] probably about 3 times the weight.
	Interviewer:	Could you figure out if it is 3 times as heavy?
	Henry:	[Puts 3 Cs and G on the scale] No.
	Interviewer:	How about the others? Could it be made of other materials?
	Henry:	[Put F and 3Cs on] I think it's the amount of F.

Actions	Dialogue	
	Interviewer:	Why do you think that?
	Henry:	Because it's one third as tall and three of those equal F in weight and height.
Stacks up 3Cs Proportional reasoning Uses the scale Considers size and weight	Aaron (9 years old, 4th grade)	
	Aaron:	[Stacks up three Cs and puts on the scale to compare with F] I think C is made of aluminum.
	Interviewer:	How did you figure that out?
	Aaron:	They weigh about the same.
	Interviewer:	I am still wondering why you use 3 Cs not just one C to compare with F.
	Aaron:	Because you kind of need three to get the same height.

In the pretest, Henry thought that C was made of G (Delrin), even though he tried to coordinate the weight and the size qualitatively. In the posttest, he not only answered correctly, but also showed clear proportional reasoning to justify his answer.

In the pretest Aaron thought that C was made of something else and considered only the weight of the cylinders. In the posttest, he gave the correct answer and also used proportional reasoning to justify his answer.

Summary of Results

Data from pretests and posttests on inferring material show that, after children's participation in the interview activities, all of the third, fourth, and fifth graders who had started by considering that the cylinders were made of the same material if they were of same weight or who had worked only with an ambiguous qualitative consideration of both weight and size, switched to consider a clear quantitative covariation between the weight and the size of the cylinders.

Chapter 11: Discussion

This study was designed (1) to examine third to fifth graders' initial ideas about the relationships among the volume (or size), the weight, and the materials objects are made of, (2) to evaluate how the process of quantification of physical properties that are relevant in determining the density of materials helps children activate and reconcile their intuitive ideas to construct new understandings of the relationships among weight, size, and the kind of material objects are made of, and (3) to construct a formal expression of the relationship between weight and size.

This chapter reviews the results of this study with respect to its goals and to each of its original research questions, compares these results to those of previous studies on children's understanding of density and of proportionality, and discusses the relevance of this study's findings to theoretical views on learning and development and to education.

On Research Questions and Contribution to Research

Research Question 1.

Concerning Question 1, on the intuitive ideas children bring to tasks that require determining the relationships among weight, volume, and the kind of material objects are made of, results show that different children activated different ideas and most children activated multiple ideas.

The most frequently activated ideas in the inferring material tasks (Tasks 6 and 7) referred to: (a) same weight means same material; (b) part-whole relationships (“if the whole object is made of aluminum, then part of the object is made of aluminum too”); and (c) the non-quantitative co-variation between weight and size (“If you were to break down F to the size of C, or D, or E, C would be the same weight as [that piece of] F.”; “This one [C] is lighter, only because it's smaller.”). This is in keeping with previous results by Schliemann, Liu, Wagoner, and Carraher (2011) showing that, when asked to determine the materials the objects were made of, most children only considered the weights of the objects, some children could coordinate the weight and the size qualitatively, and only few children could coordinate the two properties quantitatively.

The new results by the current study reveal that: (a) The activities developed throughout the interview allowed each child to activate various ideas, as quantification was gradually introduced; (b) With extra copies and quantification of weight and size, the children could spontaneously and independently refine their previous ideas to construct more advanced knowledge (same weight and same size means same material, and if both the weight and the size of an object is 1/n of another object, they could be made of the same kind of material); and (c) The new elements of knowledge, that is, same-size-and-same-weight strategy and “1/n size and 1/n weight” strategy, gradually came to play central roles as core knowledge in Tasks 8 to 11.

Research Question 2.

Concerning the second question, on how do children use and transform their different ideas to further their understanding of the relationships among weight, volume, and the kind of material objects are made of, the analysis of

Tasks 6 and 7 revealed that the simplified measuring tools and extra copies played important roles in facilitating children's expression of the second degree relationship, activating and refining children's previous ideas about the meaning of "being made of the same kind of material."

The extra copies in Task 6 allowed the children to feel the relationships among weights of different objects with their hands and see the relationships among their sizes with their eyes. These perceived and understandable properties of the weights and sizes of the objects being compared may have contributed to make it easier for children to consider and to express the second-degree relationship between the first-degree relationship between weights and that between sizes.

Another important role the extra copies played in Task 6 was that children's action of stacking three cylinders together activated different ideas. One was the co-variation of weight and size ("If you were to break down F to the size of C, or D, or E, C would be the same weight as [that piece of] F"; "This one [C] is lighter, only because it's smaller."). The other idea referred to part-whole relationships. When children explained why three Cs could be made of the same kind of material as F, and also why one C could be made of the same kind of material as F, they said: "if the whole object is made of aluminum, then part of the object is made of aluminum too;" "One of these chopped into 3 pieces"; and "This is just a smaller piece of it." Apparently activation of these ideas was a necessary step for children to reconcile and refine their previous ideas, making them more coherent. Some children, after grabbing three Ds together, immediately realized that their previous strategy "same weight means same material" was not appropriate when comparing two cylinders of different sizes. Stacking three cylinders together allowed children to activate and reconcile the different, so far isolated, ideas they already had toward quantitatively coordinating weight and size.

Use of the color scale and of the simplified ruler, with arbitrary units instead of standard units, made the relationship between weights and that between sizes available and understandable. With these simplified tools, children had a chance to practice the process of quantification by themselves, to understand the meaning of units, and to make sense of the numbers they assigned to different weights and sizes. This allowed them to easily determine the relationship between weights and that between sizes, and then jointly consider the second-degree relation between the two relations, even without the help of extra copies. The absence of extra copies in Task 7 (in which the children could not stack up small cylinders to build a cylinder that had the same size as the tall cylinder) also invited children to extend their understanding of being made of the same kind of material from "same-size-and-same-weight" to "1/n-size-and-1/n-weight". The analysis of the interview dialogues supports the hypothesis that quantification was essential both in activating, reconciling, and refining different ideas and also in jointly considering and proportionally coordinating two quantities.

Another possible role the measuring tools played was in helping children to differentiate pre-existing conceptions into more clearly defined, separated, but closely related conceptions. It was possible that some children, like Aaron at the beginning of Task 6, might have used the word “weight” in a way that was closer to our meaning of density, when he said that C and F could be made of the same kind of material because they had the same weight. In fact C was only one third of the weight of F and the difference in weight could be easily felt. It is possible that Aaron had felt that C and F were related in some way through the combination of vision and feeling with hands, and used the word weight to mean a quality similar to “density.” After measuring C and F on the color scale, he found that the weight of C was 2 and the weight of F was 6. Measuring the weights on the scale might have played a role in differentiating the feeling of weight from that of “density.” After these two tasks, he began to use the expressions like “weight per size” to describe the idea of “density.”

No previous study has focused on offering an environment in which children can activate and reconcile different ideas to further their understanding of the relationships among weight, volume, and the kind of material objects are made of. This study has provided us insights into the effect of this effort. It strongly suggests that merging quantification into children’s intuitions can further their understanding of the relationships among weight, volume, and the kind of material objects are made of and shows us how that may happen.

Research Question 3.

Concerning Question 3, on whether children can acquire a general expression of proportional relationships between weight and volume, the data support a positive answer. Most children expressed the relationships by stating something equivalent to “for all aluminum cylinders, the weight and the size have the same numbers", or, "for all brass cylinders, the weight is three times as much as the size." This is in keeping with previous results by Lehrer et al.'s (2001) showing that children in fifth grade used similar expressions when explaining graphs relating weight and size.

Also, children in this study showed a more general understanding of the relationships between weight and size by expressing, in Task 9, their awareness that (a) For objects made of the same kind of material, the relationship between weight and volume is a multiplicative constant; (b) That constant differs for different materials; (c) If two same-size objects are made of different materials, the relationship between their weights is the same as the relationship between the constants for the two materials; and (d) If two objects made of different materials have the same weight, the relationship between their sizes is the inverse of the relationship between the constants for the two materials. These were new ideas for the children because they focused on the relationships between weight and size and because the relationships were quantitative, not just qualitative. However, one must recall that these ideas were not totally new, since the children could make use of them before through their qualitative intuitions, such as when they would express that "stone is a heavier kind of material than wood" and "in order to have a piece of wood that has the same weight as a piece of stone, the piece of wood will be much bigger than that piece of stone." However, because these previous ideas were basically qualitative, they did not allow the children to precisely infer the weight and the size of objects.

As expected, quantification of weight and size seems to have played an important role in helping children develop a general expression of the relationship between weight and volume. As I mentioned in Chapter 3, understanding physics concepts requires understanding relationships among multiple concepts. Some concepts themselves are defined in terms of relationships between other concepts, such as velocity, density, pressure, etc. Density is defined as the relationship between weight (or mass) and volume. For young children to develop a thorough understanding of density they need to measure the weight and the volume of the objects to be compared and to reflect upon the quantitative relationships among the measures. In Task 9, the quantification with arbitrary units made the quantitative relationships between weight and size more salient than when standard measurements of weights or sizes are used, because every measurement was a comparison with the lightest cylinder or with the smallest cylinder. This allowed children to easily make sense of the numerical relationships between weight and size.

Recording and representing the weights and sizes of cylinders in the data tables used in Task 9 also facilitated the process of identifying a property of a substance and of expressing the relationship between weight and size, by making it easier for children to determine the relationship between weight and size for a certain kind of material. As I mentioned in Chapter 2, Lehrer (2009) pointed out that inscription signifies a particular aspect of the phenomena, rather than a copy of it. There is a movement away from the target phenomena and toward reduction. Reduction is also accompanied by amplification because, when different inscriptions are brought into contact, they referentially coordinate aspects of the phenomena that were originally isolated. With the data tables used in Task 9, children's attention was drawn only to the weight and the size of objects. This was a process of both reduction and amplification, that allowed children to better relate weight and size and to coordinate these originally isolated properties of objects to invent a new concept, namely the relationship between weight and size (that is, density) for a certain material.

Research Question 4.

Concerning Question 4, on whether children from ages 8 to 11 could reason proportionally in the context of density, analysis of Task 8 shows that, when they could quantify weight and size, participants in the study could indeed use proportional reasoning to solve problems related to density.

As mentioned in Chapter 3, Smith et al. (1985) found that some of the 8- and 9-year-olds in their studies had developed distinct density and weight concepts. They proposed that children at this time probably still have not developed the idea of a standard unit of volume and hence conceptualize density qualitatively as heaviness for size rather than as weight per unit volume. Furthermore, they thought that, lacking such standard units, children would not yet attempt to calculate densities numerically or realize that there is a unique number, which defines the density of a substance under ordinary conditions.

I believed that, given relevant experiences with measurement, children at these ages could develop the idea of a standard unit of volume. In fact, in this study, we can see that children from ages 8 to 11 can come to understand units of weight and size. In the interview, they used arbitrary units instead of standard units, but standard units do not seem to require more advanced (or harder) reasoning processes than is the case for arbitrary units from the knowledge development point of view. It is true that evaluation of volumes by using cylinders of the same base area simplified the task. Therefore it remains to be investigated in future studies if, with use of arbitrary units, instead of standard units, children will be able to compare the volumes of objects that differ in more than one dimension.

We can see that, in Task 6 and Task 7 of this study, when quantification of weight and size was available to the children, 12 out of the 13 children aged from 8 to 9 (children of the same ages as those in Smith et al.'s study [1985]) could quantitatively coordinate weight with size using the weight and the size of the lightest and smallest object as units of weight and of size, such as, "This [F] is six of them [A] tall. If it [F] is six times as much as the weight, it would be the same material." Also, in Task 9, although the children at these ages did not spontaneously use the ratio of weight to size to express the relationship between weight and size, they were able to find out that there was a unique number (one for aluminum cylinders; three for brass cylinders), which defines a property of a substance with arbitrary units.

Let us now compare children's proportional reasoning in this study with the results of previous studies on proportional reasoning. From the results of the studies on various topics related to proportional reasoning, Piaget found that children could only reach the formal operational stage, in which they consider second-degree operations, by ages 12 or 13. The concept of proportions was also found difficult and considered to be a late acquisition by other researchers (Cowan & Sutcliffe, 1991; Dixon & Moore, 1996; Karplus1981, Spinillo & Bryant, 1991; Squire & Bryant, 2003; Stavy & Tirosh, 2000; Strauss & Stavy, 1982).

In this study, when quantification was not available at the beginning of Task 6, some of children's responses were similar to those reported by previous studies of proportional reasoning (Piaget & Inhelder, 1951, Leoni & Mullet, 1993) for children of the same age. Without quantification, in this study, about two thirds of the children only considered one factor and used same-weight-means-same-material strategies.

In contrast, when the quantification of weight and size was available and understandable, 19 out of the 20 children aged from 8 to 11 were able to use proportional reasoning when inferring materials in Tasks 6 and 7 stating, for example, "I think A is the same material. Because it's one sixth as tall and it's one sixth of the weight," thus considering second degree operations (relations between relations in weight and in size) and understanding the quantitative co-variation between weight and size.

In Task 8, children aged from 8 to 11 could consider the relationships among weight, size, and kind of material proportionally. In missing value problems they inferred the weight and size of objects made of the same kind of material correctly and in ratio comparison problems they determined whether two objects were made of the same kind of material by considering both weight and size proportionally. They only had difficulty when working with non-unit fractions.

Understanding intensive quantities requires awareness of both direct and inverse relations between the two extensive quantities and the intensive quantity they measure. Previous studies show children's difficulties with inverse relations (e.g., Gilmore & Bryant, 2005; Howe, Nunes, & Bryant, 2010; Inhelder & Piaget, 1958; Squire & Bryant, 2003; Stavy & Tirosh, 2000). This study's results show that, with understandable quantification, children from 8 to 11 years old could deal with inverse relationships, for example, determining in Task 9 that if an aluminum cylinder and a brass cylinder had the same weight, the aluminum cylinder should be three times as tall as the brass cylinder.

Research Question 5.

The question on what role quantification plays (1) in the process of activating and reconciling children's initial ideas to construct more coherent ideas when solving problems and (2) in the process of developing a general expression of the proportional relations between weight and volume has been addressed above in the discussion of other questions. To the best of my knowledge, no previous study had specifically focused on the roles of quantification in the formation of intellectual ideas and the reorganization and application of these ideas to align with scientific knowledge and practices. This study contributes to clarify these issues. The comparison between pretest and posttest answers further shows the effect of children merging quantification into their intuitions. For example, children's performance in the fourth subtask of the pre-and post-tests on inferring material changed dramatically. In the pretest, only half of the children gave the correct answer and only 39% of the children considered both weight and size. In contrast, in the posttest, all the children gave the correct answer and 67% of the children clearly showed that they were considering the proportional relationships between weight and size.

Research Question 6.

Concerning Question 6, on what difficulties children show in the process of quantification, in solving problems, and in developing a general expression of the proportional relations between weight and volume, this study's findings can be summarized as follows.

Difficulties in the process of quantification.

The results of Task 5 show that it was difficult for children to evaluate relative weights using only their hands. Such results are consistent with what I had found before with adults (Liu, 2009). None of the adults in that study correctly ordered weight by using only their hands and they also thought that denser cylinders were heavier than they in fact were. One different finding was that, unlike adults, most of the children in this study did not consider the possibility that some of cylinders may have the same weight.

From children's responses to the questions related to the process of quantification, we have seen some difficulties experienced by some children. First, although most of them could understand the rule of equality, a few children claimed that two objects might move the pointer in the scale to the same color but have different weights, because they felt that the objects' weights were different on their hands. Second, although all of the children understood the rule of additivity, labeling the scale with numbers was not easy for some of them. The fact that each interval represented the same weight was not immediately acknowledged. However, after experimenting with the extra copies on the scale, they realized that each interval represented the same weight and correctly labeled the scale with numbers.

Although no child had any problem in using the simplified ruler they could manipulate, it is interesting to note that, when it came to the rulers in the pictures in Task 8, which clearly included a line at the end of the ruler, 10 out of 20 children counted the first line at the bottom as one, instead of zero. As often happens in school tasks, they did not focus on the meaning of the marks on the picture until I asked them to compare rulers in the picture with the real ruler. Experiencing the process of quantification may have helped the children to understand the rules of measurement, leading to a connection between the picture model and the actual ruler.

I did not find in the research literature any previous study on children's understanding of Carnap's (1966) "three-rule schemas." The above findings of this study inform us about how well children understand them and what difficulties they may have.

Difficulties in solving missing value and ratio comparison problems.

We have seen that, once the children understood the missing value problems, most did not have difficulties solving them. Some children, however, did experience difficulties in solving the ratio comparison problems. First, for some children the ratio of 3:2 in Problem 5 was a barrier to using a scalar approach. This led some children to switch to a functional approach and to succeed with the simple ratio 2:1; others switched to a wrong approach (constant difference approaches) and failed the task. Second, the ratios in Problem 6 were 3:2 and 4:3, and they were not equal, but the difference between sizes and that between weights were the same. That increased the difficulty of deciding whether the two cylinders were made of the same material. One reason this problem was so difficult for the children may have been the fact that the correspondence between the difference in sizes and that in weights was so salient, leading them to think that those two cylinders were the same in some way. A second possible reason is that the meanings of the ratios of 3:2 and 4:3 were not familiar to the children. None of the children could figure out that cylinders M and N could not be made of the same kind of material based the fact that 3:2 was not equal to 4:3. It is possible that children in these grades were not used to reasoning with ratios other than simple unit ratios such as 1:2, 1:3 or 1:6. To them, the four values of the weights and sizes of the two objects were just two disconnected pairs of numbers.

Difficulties in developing a general expression of the proportional relations between weight and volume.

Although the children could develop a general expression of the proportional relationships between weight and size, no child expressed it as a ratio or referred to units of measurement, although they clearly expressed that weight and size were not the same thing. This may be due to the fact that they did not give names to the arbitrary units they were working with. That made it difficult for them to express the relationship between weight and size with both numbers and units.

Research Question 7.

Concerning Question 7, on what strategies children use in solving problems related to the proportional relationships among weight, volume, and density, some of the findings of this study conform to previous studies and some complement previous studies.

Analysis of Task 8 informs us about the kind of strategies children tend to use, under different conditions, when solving missing value problems and ratio comparison problems in the context of density, and the factors that influence children's performance in solving these problems.

Previous studies on proportional reasoning in other contexts show that building-up strategies (similar to scalar strategies by addition) are frequently observed during childhood and adolescence (Hart, 1981; Ricco, 1982; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990; Tourniaire, 1984). While these strategies do lead to successful solutions in simple problems, they become cumbersome when the problem contains non-unit ratios. As was the case in my study, only a few students are successful in applying building-up strategies to non-unit ratio problems (Hart, 1981). Note that this dissertation uses the term "unit ratio" to describe ratios in which one number is a whole number and the other number is 1, such as 1:3, 6:1, etc. The term "integer ratio," widely used in the literature, has the same meaning.

Constant difference strategies have been widely documented (Hart, 1981; Inhelder & Piaget, 1958; Karplus & Karplus, 1972) from childhood through adulthood. It is often used as a fall-back strategy for dealing with non-unit ratios, i.e., a child may use a building-up strategy on unit ratios and then a constant difference strategy on non-unit ratios (Karplus et al., 1983; Tourniaire, 1984; Hart, 1981).

This study's results replicate the above findings. For example, consider the ratio comparison problems in Task 8. In Problem 4 scalar strategies by addition or building-up strategies lead to successful solutions; however, in Problems 5 and 6, scalar strategies by addition became difficult to be used because the problems involved non-unit ratios (2/3 and 3/4). Concerning the constant difference approach, in Problem 4 none of the children used this approach when working with unit ratios. In contrast, in Problem 5 and Problem 6, 40% and 35% of the children focused on constant differences when non-unit ratios had to be dealt with.

This study also contributes some new findings concerning students' strategies for solving proportionality problems. A large number of the children in this study successfully used scalar approaches by multiplication, especially when the comparisons involved unit ratios. The data also suggest a new way of looking at children's use of the constant difference approaches, namely that it might result from using a scalar approach by addition and mistakenly assuming that one unit in weight should correspond to one unit in size.

Another finding in this study that complements previous studies concerns how several aspects of the problems influence the choice of strategies. Karplus et al. (1983a, 1983b) found that the semantics (context) of the problem impacts the choices of strategies. For instance, a Lemonade Puzzle problem (Karplus et al., 1983a) suggests a functional relation, while a fuel consumption problem (Vergnaud, 1980) suggests a scalar comparison. The results from Task 8 in this study show that even in the same context, the choices of strategies are influenced by the complexity of the ratios the children needed to work with, the way the problems were described, children's understanding of the concept, and their ability to deal with ratios or fractions. Since the ratios would be different depending on the strategy used (for example, in Problem 5, the ratio is 2:1 for a functional approach and 2:3 for a scalar approach), the children were more likely to choose an approach that allowed using a simpler ratio. When the problem specified the relationships such as "Twice as much as", the children preferred to use a scalar strategy by addition; when the problem gave the values of different quantities, more children chose a scalar strategy by multiplication; when the problem offered the information that the weight of the target virtual cylinder was the same as the weight of a real cylinder, more children found a real cylinder to represent the target cylinder. After Task 9, when children apparently had developed a better understanding of the relationships between weight and size, more children used functional approaches. Also, the children who could deal with non-unit ratios (2/3) chose the scalar approach and solved Problem 5 successfully.

Two more aspects of children's mathematical reasoning shown by this study that had not been previously reported are the use of (a) combinations of the basic strategies (scalar strategy by addition, scalar strategy by multiplication, functional strategy, and constant difference strategy); and (b) strategies described in prior research literature that here included slight modifications or adjustments to suit the problem, such as using a scalar approach by addition combined with a scalar approach by multiplication, a scalar approach by multiplication combined with splitting, a functional approach based on inferring from the relationship of the weight and the size of N, or a functional approach inferring from the relationship of the difference in weight and that in size between cylinders M and N.

There is one aspect in the results of Task 8 of this study that does not conform to the results of previous studies. Hart (1981) and Karplus et al. (1983) found that one of the errors in solving proportionality problems is that a child might attempt to solve a problem by simply comparing the numerators of the two ratios. In this study no such error occurred in Task 8. None of the children considered only one variable. One possible reason was that these tasks were different from the ratio comparison problems in previous studies. I did not ask the children which one was denser, but rather whether they were made of the same material or not. Another important reason was that the children already knew from Tasks 6 and 7 that they needed to consider both the weight and the size of the cylinders.

Limitation

As I mentioned before, no child expressed the relationship between weight and size with ratios. A few children even had difficulty expressing it in words. A better way to address this limitation may be to ask students to discuss what kind of relationship between the weight and the size holds for all objects made of a certain kind of material. In instructional environments, relevant questions could be: Is the sum of the weight and the size the same? Is the difference between the weight and the size the same? Is the product of the weight and the size the same? Is the ratio (or quotient) of the weight to the size the same? After discussion, if students find out that the ratio (or quotient) of the weight and the size is the same, they could be asked: "What does the ratio (or quotient) mean?" This question will be challenging but productive and will help them make sense of the ratio (or quotient) with the knowledge they have from everyday experiences. To answer such a question students possibly draw on some other ratios (or rates) in everyday life, such as price or the number of candies per person.

Although in this study the children could not use the form of weight/size to express the relationship between weight and size, we could not say for sure that they did not have an understanding of the ratio of weight to size. Some children used expressions such as "one weight per one" at some points. As Schliemann et al. (2000) pointed out, students begin to understand linear functions and constant rates long before they make any sense of an expression such as $y = mx + b$.

Another aspect that could have been explored in the interview was to guide the children to focus on the general notational representation, as Schliemann et al. (2000) did in their study on linear functions and constant rates of change. Adding a task consisting of filling out tables for specific pairs of weight and size for cylinders of a given material and guiding children to use variables to represent the general relation, could have helped the children to generalize the relations among sets of measures and to describe relations among variables. As some researchers (Karplus et al., 1983; Suarez, 1977; Vergnaud, 1983) pointed out, proportionality cannot be divorced from the understanding of linear functions. Theories that deal only with arithmetic relations among particular sets of numbers and not with the relation between the variables themselves cannot satisfactorily account for proportional reasoning.

Another limitation of this study is that it did not pursue a full understanding of density. As mentioned by Roger Tobin (personal communication), a full understanding of density will require a full understanding of volume. In this study, I simplified the evaluation of volume by using cylinders that have the same base area. Further studies are needed, using objects of different shapes and tools that allow children to fully explore the quantification of volume and to develop a full understanding of density.

Theoretical Relevance of the Study

Reconciling different resources to build a more coherent resource.

Consistent with diSessa (1988) and Hammer (2005)'s point of view about children's knowledge, children's responses to the tasks in this study could be explained by considering children's intuitive physics or previous understandings as knowledge in pieces or resources that play productive roles in the acquisition of expertise. For example, children's responses to the questions in Tasks 6 and 7 (inferring material) showed that the original resources or ideas they activated in these two tasks were related to: (a) same weight means same material; (b) part-whole relationships; and (c) the co-variation between weight and size.

I do not view children's intuitions (such as, "same weight means same material") as misconceptions that are unitary, stable, and need to be replaced. Instead, this piece of knowledge carried with it some contexts of successful use. For example, when comparing two objects of same size, being the same weight could be a good indicator of being the same material. Smith and diSessa (1993) believed that persistent misconceptions could be seen as novices' efforts to extend their existing useful conceptions to contexts in which those conceptions turn out to be inadequate. In this view, the children who used a same-weight strategy to infer material just extended this piece of knowledge to contexts in which it is not productive.

I consider this piece of knowledge to be a resource that needs to be reconciled with other pieces of knowledge (such as part-whole relationships and the co-variation between weight and size). By doing this, the children could combine and refine these resources to construct more advanced knowledge (such as "Same weight and same size means same material" and "If one object is 1/n of another object both in weight and in size, then those objects could be made of the same kind of material."). The old piece of knowledge ("same weight means same material") was not replaced, but became irrelevant when the sizes of the objects compared were not the same.

Smith and diSessa (1993) pointed out that, "An adequate theory of learning must both provide richer descriptions of knowledge and explain the gradual transformation of that knowledge into more advanced states"(p. 147). The findings in this study informed us about children's refinement of a prior conception of "being made of the same kind of material" (1) by the analyses of children's initial knowledge, (2) by showing in appropriate detail how different pieces of knowledge interacted to produce the real-time reasoning and problem solving, and (3) by examining how expertise was acquired from the resources initially provided by more naive states.

Strategies used are context-sensitive.

One of the features of the point of view of "knowledge in pieces" is that children's knowledge is context-bound (diSessa, 1993). Children's knowledge elements are activated in specific contexts and the kind of knowledge activated is sensitive to context. Children's responses in Task 8 of this study showed significant context sensitivity. The application of diverse strategies was widespread throughout the 20 children; only three children used the same strategy on all six problems, namely, the scalar approach by addition.

Although Smith and diSessa (1993) emphasized the fragments of knowledge, they also value continuity in knowledge content. They pointed out, based on Smith's study (1990) on order and equivalence relations among fractions, that prior novice knowledge (divided quantity) remained productive for masters by supporting new knowledge (conversion to common denominator) that was more efficient and reliable. Master's knowledge of divided quantity, though often not directly apparent, remained at the basis of their knowledge system, underlying and justifying their more powerful numerical strategies.

In Task 8 of this study, although children used different strategies (scalar approach by addition, scalar approach by multiplication, functional approach and combinations of them), these have a common root. The root is the awareness that "same size and same weight means same material," which children often used or came up with in Tasks 6 and 7. When children used a scalar approach by addition, they were trying to make the two cylinders the same size and to see whether they would then have the same weight. This was also true when the children used functional approaches. For example, Anthony used a functional approach to figure out the weight per size for X and then he doubled the size of X to match the size of Y to see whether it would have the same weight as Y. When the children used scalar approaches by multiplication the root was not directly apparent. For example, Henry said, "Because it's two thirds of this tall and two thirds of the weight." Although he did not construct a same size cylinder and then compare their weights, his approach might extend from "Same size and same weight means same material" to "Part (2/3) of the cylinder in size would have part (2/3) of the whole weight."

Why did children use different strategies in different problems, even though all these problems related to the relationships among weight, size and kind of material? As Smith and diSessa (1993) illustrated, a rubber band can be mapped to different scientific terms, depending on the problem situation. Similarly, the applicability of different strategies in Task 8 of this study might depend directly on children's intuitive knowledge-knowledge prior to any formal scientific training.

Implications for Education and Further Research

Many generations of students through the years have found density, and indeed many other scientific concepts, difficult to learn. This study shows that meaningful quantification provides an opportunity for students to develop arguments by themselves that could, in turn, contribute to their understanding of an intensive quantity (density). During the interviews, children aged from 8 to 11 activated different ideas and reconciled them to build more coherent knowledge, discovered linear relationships between the extensive quantities (weight and volume) by themselves, and were able to analyze unfamiliar situations and apply their newly found principles to these new situations. These results lead to the recommendation that quantification should be part of science learning from the early years and that studies are needed to learn how to promote children's understanding and practice with quantification and its use to further their understanding of different concepts and the relationships among those concepts.

In a near future, I hope to evaluate the role of the activities examined in this dissertation in elementary school classrooms. The goal is to examine under classroom conditions whether these stimuli and questions will help to establish an inquiry-based arena for students to learn how to do science, including creating variables by assigning quantities to vague, everyday terms, assigning a quantitative descriptor to a variable, making sense of data, exploring thoroughly the relationships among physical concepts, and achieving robust understandings.

Also, I hope to explore similar issues on the relationships among distance, time and speed, and the relationships among force, area, and pressure. Like density, speed and pressure are also defined quantities. The defined quantity "clearly does have its own 'identity', but it cannot have its meaning fully explained without recourse to its defining equation" (Gamble, 1986, p. 356). In order to make sense of a defined quantity, quantification as a tool is not only helpful, but also indeed indispensable. The challenges in learning density, speed, and pressure may share some similar characteristics. Therefore students may need to draw on similar ideas. A proper use of quantification may play an important role in understanding all these defined quantities.

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