The Spherical Mean Value Operators on Euclidean and Hyperbolic Spaces

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Kyung-Taek Lim

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Advisor: Fulton B. Gonzalez

Abstract

Let X be a Riemannian manifold. Fix $r > 0$ and let f be a continuous function on X. The spherical mean value operator M^r maps f to the function $M^r f$ on X whose value at each $x \in X$ is the average of f over the sphere of radius r centered at x. In this thesis, we present results pertaining to the spherical mean value operator in Euclidean space $\mathbb{R}^n$ and hyperbolic space $\mathbb{H}^n$.

There is another notion of the mean value operator when X is a homogeneous space G/K with K compact. Fix $y \in X$ and let f be a continuous function on X. The mean value operator associated to y maps f to the function on X whose value at each $x = gK \in X$ is the average of f on the translated orbit $gK \cdot y$. This mean value operator agrees with the spherical mean value operator when G/K is a Riemannian symmetric space of rank one.

In $\mathbb{R}^n$, we show that the spherical mean value operator is surjective from the space of smooth functions onto itself and from the space of distributions onto itself. We also obtain range characterizations of this operator on the spaces of compactly supported distributions and functions, respectively.

In $\mathbb{H}^3$, we obtain a range characterization of the spherical mean value operator on the space of compactly supported distributions. From this we show that this operator is surjective from the space of smooth functions onto itself.

Finally we extend some results on the spherical mean obtained by Fritz John. We derive a formula for the iterated spherical mean in $\mathbb{H}^n$. We also show that a smooth function f on $\mathbb{H}^3$ with known averages over all spheres of a fixed radius $r > 0$ is uniquely determined, if the values of f are known on certain split annuli where the sum of the thicknesses of the annuli is r. Finally we obtain an explicit solution to the inhomogeneous spherical mean value equation $M^1 f(x) = g(x)$, where g is a given function in $\mathbb{H}^3$ with a mild decay property.

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Contents

Chapter 0

Notation

Let us at the outset establish some notation.

Below is notation for some manifolds.

$\mathbb{R}^n$: n dimensional Euclidean space.

$\mathbb{H}^n$: n dimensional hyperbolic space with sectional curvature -1.

S^n : n-sphere in $\mathbb{R}^{n+1}$. If the radius is not specified, we assume it is 1.

Below is notation related to spheres and balls.

$d(x, y)$: the distance between the points x and y in the space under consideration.

$S(x, r)$: the sphere centered at x with radius r, in other words, the set $\{y \in M \mid d(x, y) = r\}$.

The space M in which the sphere lies should be determined by the context.

$B(x,r)$: the open ball centered at x with radius r, in other words, the set $\{y \in M \mid d(x,y) < r\}$. The space M in which the ball lies is determined by the context.

$\bar{B}(x,r)$: the closed ball centered at x with radius r, in other words, the set $\{y \in M \mid d(x,y) \leq r\}$. The space M in which the ball lies is determined by the context.

Ω_k : $2\,\pi^{k/2} \,/\, \Gamma(k/2)$, the surface area of the unit sphere in the k dimensional Euclidean space.[1]

c_r : the reciprocal of the surface area of the n-sphere of radius r in $\mathbb{R}^{n+1}$. The dimension $n+1$ of the Euclidean space will be clear from the context.

Below is notation for some spaces of functions or distributions.

$\mathcal{S}(\mathbb{R}^n)$: the space of rapidly decreasing functions on $\mathbb{R}^n$. Also called the Schwartz space.

$C(M)$: the space of continuous complex valued functions on a manifold M.

$C^m(M)$: the space of complex valued functions on a manifold M whose derivatives up to order m are continuous.

$\mathcal{E}(M)$: the space of complex valued smooth functions on a manifold M. (We sometimes denote this function space by $C^\infty(M)$.)

[1]The surface area of an $(k-1)$–sphere of radius r in $\mathbb{R}^k$, $k > 1$, is $r^{k-1}\,\Omega_k = 2\,\pi^{k/2}\,r^{k-1} \,/\, \Gamma(k/2)$ (See for example [19, pp.1-2]).

$\mathcal{D}(M)$: the space of complex valued compactly supported smooth functions on a manifold M.

$\mathcal{D}'(M)$: the space of distributions on a manifold M.

$\mathcal{E}'(M)$: the space of compactly supported distributions on a manifold M.

The last four vector spaces will be provided with their usual topologies as in [10, Chapter II, Section 2].

Below is some other notation.

$A\backslash B$: for sets A and B, $A\backslash B = \{x \in A \mid x \text{ is not in } B\}$.

$A - B$: for $A, B \subset \mathbb{R}^n$, $A - B = \{z \in \mathbb{R}^n \mid z = x - y \text{ for some } x \in A, y \in B\}$, where $x - y$ denotes the subtraction of vectors in the vector space $\mathbb{R}^n$.

$A + B$: for $A, B \subset \mathbb{R}^n$, $A - B = \{z \in \mathbb{R}^n \mid z = x + y \text{ for some } x \in A, y \in B\}$, where $x + y$ denotes the addition of vectors in the vector space $\mathbb{R}^n$.

D^α : for a multi-index α, D^α denotes the differential operator $(\frac{\partial}{\partial x_1})^{\alpha_1}(\frac{\partial}{\partial x_2})^{\alpha_2}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$ in $\mathbb{R}^n$.

$M(n)$: the group of rigid motions on $\mathbb{R}^n$.

o : the origin of the space under consideration. (This will be clear from the context of the discussion.)

$\mathbb{R}_+$: the set of nonnegative real numbers.

$SO_o(1,n)$: see the beginning of Section 4.1.1.

0.1 Definition of the spherical mean value operator with radius r

Fix $r > 0$ and let u be a continuous function on $\mathbb{R}^n$. The spherical mean value operator M^r maps u to the function $M^r u$ on $\mathbb{R}^n$ whose value at each $x \in \mathbb{R}^n$ is the average of u over the sphere of radius r centered at x:

$$M^r u(x) = \begin{cases} c_r \int_{y \in S(x,r)} u(y)\, dm_S(y) & \text{if } n = 2, 3, 4, \ldots \\ c_r \{u(x-r) + u(x+r)\} & \text{if } n = 1\,, \end{cases}$$

where dm_S is the standard surface area measure in $\mathbb{R}^n$. Thus the spherical mean value operator, M^r, is a map from a class of functions to a class of functions, for example, $M^r : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. With a different class of functions as the domain, we have a different map, but we use the same symbol M^r.

Let $\delta_{S(0,r)}$ be the distribution

$$\begin{aligned} \delta_{S(0,r)} : \mathcal{E}(\mathbb{R}^n) &\to \mathbb{C} \\ u(x) &\mapsto M^r u(0)\,. \end{aligned} \tag{1}$$

Then for any $u(x) \in \mathcal{E}(\mathbb{R}^n)$

$$M^r u(x) = \delta_{S(0,r)} * u(x)\,,$$

where $*$ denotes convolution. In Chapter 2 we consider the maps $M^r : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$, $M^r : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$, $M^r : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$, and $M^r : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$.

The spherical mean value operator is an example of the mean value operator on a homogeneous space G/K with K compact defined as follows.

Definition 0.1. *Let G be a Lie group and K be a compact subgroup of G. Let dk denote the normalized Haar measure on K. For any given $y \in G/K$, the mean value operator M^y assigns to each $f \in C(G/K)$ the function $M^y f$ on G/K defined as follows.*

$$M^y f(p) = \int_K f(g\,k \cdot y)\,dk \quad (p = gK \in G/K)$$

By this definition, $M^y f(p)$ represents the average of f over a certain translate of the orbit $K \cdot y$. Clearly, $M^y = M^{k\,y}$ for any $k \in K$. Thus we can restrict the superscripts of M to a transversal manifold of the set of K–orbits. For example, if G/K is a Riemannian symmetric space of non-compact type with the Iwasawa decomposition $G = KAN$, we can replace the superscript $y \in G/K$ in M^y by a unique element $a \in \overline{A^+}$, because we have the decomposition $G/K = K\overline{A^+} \cdot o$, where each $y \in G/K$ can be written as $y = k\,a \cdot o$ with $k \in K$ and a unique $a \in \overline{A^+}$. In the case G/K is of rank one, for example $\mathbb{H}^n = SO_o(1,n)/SO(n)$, we have the bijection $\overline{A^+} \to \mathbb{R}_+,\ a \mapsto d(a \cdot o, o)$. Thus we can write $M^y f(p) = M^{a \cdot o} f(p) = M^r f(p)$ with $r = d(y,o) = d(a \cdot o, o)$. Now

$$M^r f(p) = \int_K f(g\,k \cdot y)\,dk$$

is the average of f over the sphere of radius $r = d(y,o)$ centered at $p = g \cdot o$, $g =\in G$. For negative r, we have $M^r f(x) = M^{|r|} f(x)$.

Chapter 1

Introduction

The spherical mean value operators, our main interest in this thesis, are important in many ways. For example, harmonic functions are characterized by the fact that they coincide with their spherical mean values. More precisely, if $u \in C^\infty(\mathbb{R}^n)$ and L denotes the Laplace-Beltrami operator in $\mathbb{R}^n$, $Lu(x) = 0$ for all $x \in \mathbb{R}^n$ if and only if $u(x) = M^r u(x)$ for all $x \in \mathbb{R}^n$ and all $r > 0$. This characterization of harmonic functions by the mean value property can be generalized to homogeneous spaces G/K with K compact ([10, Chapter IV, Proposition 2.5 (Godement)]).

We can also view the spherical mean value operator as a generalized Radon transform that is self dual in the context of Helgason's double fibration in [10, Chapter 1, Section 3]. Thus it is natural to investigate questions pertaining to injectivity, surjectivity, range characterization, and support for this operator. These are questions which are important in integral geometry. In $\mathbb{R}^n$, hyperbolic space $\mathbb{H}^n$, and the unit sphere S^n in $\mathbb{R}^{n+1}$, there is a solution to the wave equation (or a slight variant) in terms of the spherical means of the functions given in the initial conditions ([10, Chapter II, Exercises and Further Results], [5, Theorems 2.4 and 2.5]). In particular, the spherical mean value operator appears in the discussion of Huygens' principle.

Fritz John's book *Plane Waves and Spherical Means Applied to Partial Differential Equations* ([15]) also shows the importance of the spherical mean value operator. Some of the results in this inspiring book are discussed and extended in this thesis in Chapter 4.

In Chapter 2, we prove the surjectivity of the single radius spherical mean value operator from the space of smooth functions on $\mathbb{R}^n$ to itself, and from the space of distributions on $\mathbb{R}^n$ to itself. We also obtain range characterizations of the single radius spherical mean value operator on the space of compactly supported distributions on $\mathbb{R}^n$, and on the space of compactly supported smooth functions on $\mathbb{R}^n$. The main tool is Hörmander's work on convolution operators. Then we summarize the conditions under which a surjectivity or our type of range characterization holds. In addition, we give a proof of the support theorem for the single radius spherical mean value operator for the Schwartz space, i.e. the space of rapidly decreasing functions. Although this support theorem itself is not new, we hope that the approach in our proof will be helpful in studying other integral transforms.

The spherical mean value operator on the space of smooth functions on $\mathbb{R}^n$ is not injective. This is because there exists a smooth function u with $u(x) \not\equiv 0$ and $M^r u(x) \equiv 0$. On $\mathbb{R}^1$, $u(x) = \sin(\frac{\pi}{2r}x)$ is such an example. Examples in higher dimensional Euclidean space can be found in the literature such as [28] and [1]. In Section 2.1, we provide a very brief discussion about such examples in the light of a theorem about joint eigenfunctions on $\mathbb{R}^n = M(n)/O(n)$. The injectivity of the single radius spherical mean value operator on

$L^p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $1 \le p \le 2n/(n-1)$ ($1 \le p < \infty$ when $n = 1$), was shown by Thangavelu (1994, [23, Theorem 2.2]). Later, M. Agranovsky and P. Kuchment (2011, [1, Theorem 1]) proved the support theorem for the single radius spherical mean value operator for $L^p(\mathbb{R}^n)$, $1 \le p \le 2n/(n-1)$, $n > 1$. These authors remarked in [1] that their support theorem for the single radius spherical mean value operator is implied by a result in a book by V. V. Volchkov (2003, [24, Corollary 3.3]), though it is stated in a somewhat hidden way in that book.

In Chapter 3, we characterize the range of the spherical mean value operator on the space of compactly supported distributions on $\mathbb{H}^3$, and prove that the spherical mean value operator is surjective from the space of smooth functions on $\mathbb{H}^3$ to itself. It is likely that these results can be extended to other symmetric spaces of rank 1. To do this would require proving the simplicity of zeros of certain hypergeometric functions.

It awaits the proof of the simplicity of zeros of certain hypergeometric functions.

In Chapter 4, we extend some results of Fritz John on the spherical mean ([15]) to higher dimensional Euclidean spaces or hyperbolic spaces. For example, in Section 4.1, we find a formula for the iterated spherical mean on $\mathbb{H}^n$. In Section 4.2, we consider functions with known spherical mean values over all spheres with a fixed radius $r > 0$. When $f \in C(M)$, $M = \mathbb{R}^n, \mathbb{H}^n, S^n$, we show that we can determine f if we additionally know its values on $B(o, r + \epsilon)$, $\epsilon > 0$. We explain that this follows directly from Quinto's microlocal analysis result in [21]. When $f \in C^\infty(\mathbb{R}^5)$, we present a proof that we can determine f if we additionally know its values on $\bar{B}(0, r)$. For this we use John's iterated mean value formula and work with a delay differential equation. When $f \in C^\infty(\mathbb{R}^3)$, we prove that we can determine f if we additionally know its values on a certain combination of a ball and an annulus, or a certain combination two annuli ($B(0,a) \ \cup \ B(0,2r-a)\backslash\bar{B}(0,r)$, or $B(0,r)\backslash\bar{B}(0,a) \ \cup \ B(0,2r+a)\backslash\bar{B}(0,2r)$, $0 < a < r$). For this we use John's iterated mean value formula and work with a delay equation. We are hopeful that this method works in $\mathbb{R}^n$ with general odd n. In Section 4.3, we find a solution to the inhomogeneous spherical mean value equation $M^1 f(x) = g(x)$, where $g \in C^3(\mathbb{H}^3)$ is a known function satisfying certain decay condition.

Some optional footnotes define standard ideas. Readers can ignore these footnotes.

Chapter 2

Surjectivity and range description of the spherical mean value operator on Euclidean space

In this chapter, we consider the surjectivity and the range characterization of M^r on Euclidean space. In particular, we show the four results below for any fixed $r > 0$. Here $\xi \in \mathbb{C}^n$, and $\widehat{w}(\xi)$ and $\widehat{\delta}_{S(0,r)}(\xi)$ denote the Fourier-Laplace transform of w and $\delta_{S(0,r)}$, respectively.

$$M^r : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n) \text{ is surjective.}$$

$$M^r : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n) \text{ is surjective.}$$

$$M^r\mathcal{E}'(\mathbb{R}^n) = \{w \in \mathcal{E}'(\mathbb{R}^n) \mid \widehat{w}(\xi) = 0 \text{ whenever } \widehat{\delta}_{S(0,r)}(\xi) = 0\}$$

$$M^r\mathcal{D}(\mathbb{R}^n) = \{w \in \mathcal{D}(\mathbb{R}^n) \mid \widehat{w}(\xi) = 0 \text{ whenever } \widehat{\delta}_{S(0,r)}(\xi) = 0\}$$

These results are in Sections 2.1 – 2.4, respectively. The main tool is Hörmander's results about convolution operators. For the range characterization, we also use the Weak Nullstellensatz ([6]), which is about the divisibility of holomorphic functions

In Section 2.5, we summarize which result we obtain under which condition.

In addition, we provide an application of some of the lemmas we develop in the above mentioned study. This application is in Section 2.6, which proves the support theorem for the single radius spherical mean value operator for the class of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$. The result itself is not new, but we present its proof with the hope that it serves a model for study of some other integral transforms.

2.1 Surjectivity of the single radius spherical mean value operator from $\mathcal{E}(\mathbb{R}^n)$ to $\mathcal{E}(\mathbb{R}^n)$

In this section, we prove the following theorem which shows that the map $M^r : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ is surjective.

Theorem 2.1. *Let $r > 0$, and $\delta_{S(0,r)}$ be the compactly supported distribution defined by (1). The convolution equation $\delta_{S(0,r)} * u(x) = f(x)$ has a solution $u \in \mathcal{E}(\mathbb{R}^n)$ for every $f \in \mathcal{E}(\mathbb{R}^n)$.*

The method we use is Hörmander's work on convolution equations ([14]). Let X_1 and X_2 be open subsets of $\mathbb{R}^n$, and let $\mu \in \mathcal{E}'(\mathbb{R}^n)$. According to [14], the convolution equation $\mu * u = f$ has a solution $u \in \mathcal{E}(X_1)$ for every $f \in \mathcal{E}(X_2)$, if X_1, X_2, and μ satisfy certain criteria. We will summarize that criteria in Section 2.1.1. In Section 2.1.2, we prove some lemmas needed to show that those criteria are satisfied in our problem where $\mu = \delta_{S(0,r)}$ and $X_1 = X_2 = \mathbb{R}^n$, and then prove Theorem 2.1.

Before moving on, let us recall our remark in Chapter 1 that the map $M^r : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ is not injective. It is not difficult to find a smooth function u with $u(x) \not\equiv 0$ and $M^r u(x) \equiv 0$. For this we use [10, Chapter IV, Proposition 2.4]. Suppose G/K is a homogeneous space with G a Lie group and K compact. According to that proposition, if E_μ is a joint eigenspace of $\mathbb{D}(G/K)$ (the algebra of all left G invariant differential operators on G/K), $E_\mu \neq 0$, $f \in E_\mu$, and $\phi \in E_\mu$ is a spherical function, then

$$\int_K f(xkyK)\, dk = f(xK)\, \phi(yK), \quad x, y \in G\,. \tag{2.1}$$

Now let $G = M(n)$ and $K = O(n)$ so that $\mathbb{R}^n = G/K$. Let L be the Laplace-Beltrami operator in $\mathbb{R}^n$. Fix $r > 0$, and $\lambda \in \mathbb{C}$. Since $\mathbb{D}(\mathbb{R}^n)$ is generated by L, the class of functions E_λ defined in the following manner is a joint eigenspace of $\mathbb{D}(\mathbb{R}^n)$.

$$E_\lambda = \{f \in \mathcal{E}(\mathbb{R}^n) \mid Lf(x) = -\lambda^2 f(x)\}$$

Let $f_{\lambda,w}(x) = e^{i\lambda<x,w>}$, where w is a unit vector in $\mathbb{R}^n$ and $< x, w >$ denotes the usual dot product of two vectors $x, w \in \mathbb{R}^n$. It is straightforward to check that $Lf_{\lambda,w}(x) = -\lambda^2 f_{\lambda,w}(x)$. Thus $f_{\lambda,w} \in E_\lambda$. Now let $\phi_\lambda(x) = j_{(n-2)/2}(\lambda|x|)$, where $j_{(n-2)/2}$ is the function defined in Definition 2.7. Since $j_{(n-2)/2}(\lambda|x|) = (1/\Omega_n)\int_{w\in S^{n-1}} e^{-i\lambda<w,x>}\, dm_S(w)$, it is straightforward to check that $L\, j_{(n-2)/2}(\lambda|x|) = -\lambda^2\, j_{(n-2)/2}(\lambda|x|)$. So $\phi_\lambda \in E_\lambda$. Now let us check whether ϕ_λ is a spherical function by [10, Chapter IV, Section 2, Definition]. We have $\phi_\lambda(0) = 1$, and $\phi_\lambda(x) = \phi_\lambda(k \cdot x)$ for any $k \in K = O(n)$, $x \in \mathbb{R}^n$. Hence ϕ_λ is a spherical function. Thus by

(2.1), we have

$$M^r f_{\lambda,w}(x) = f_{\lambda,w}(x)\,\phi_\lambda(y), \quad x, y \in \mathbb{R}^n, |y| = r\,. \tag{2.2}$$

Now let r_0 be a zero of $j_{(n-2)/2}$, and let $\lambda = r_0/r$. Then the right hand side of (2.2) is zero for any x. Thus $f_{\lambda,w}(x) \not\equiv 0$ and $M^r f_{\lambda,w}(x) \equiv 0$.

2.1.1 Hörmander's criteria

The result by Hörmander which we use directly for our proof of the surjectivity of the map $M^r : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ is Theorem 2.5 below. However, we first give the definitions of some terms appearing in it. As in [14], if $u \in \mathcal{E}'(\mathbb{R}^n)$, $\widehat{u}$ denotes the Fourier-Laplace transform of u,

$$\widehat{u}(\xi) = u_x(e^{-i<x,\xi>}),\ \xi \in \mathbb{C}^n\,, \tag{2.3}$$

which is a holomorphic function on $\mathbb{C}^n$. Let us first explain what an "invertible" distribution is.

Theorem 2.2. *(Hörmander, [14, Part of Theorem 16.3.9 and part of Theorem 16.3.10]) Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Then the following statements are equivalent.*

(i) There is a constant $A > 0$ such that for any $\xi \in \mathbb{R}^n$ we have

$$\sup\ \{\ |\widehat{u}(\zeta)|\ |\ \zeta \in \mathbb{C}^n,\ |\zeta - \xi| < A\ln(2 + |\xi|)\ \}\ >\ (A + |\xi|)^{-A}\,. \tag{2.4}$$

(ii) If $w \in \mathcal{E}'(\mathbb{R}^n)$ and $\widehat{w}/\widehat{u}$ is an analytic function, then $\widehat{w}/\widehat{u}$ is the Fourier transform of a distribution in $\mathcal{E}'(\mathbb{R}^n)$.

(iii) For any $w \in \mathcal{E}'(\mathbb{R}^n)$, $w * u \notin \mathcal{E}(\mathbb{R}^n)$ or $w \in \mathcal{E}(\mathbb{R}^n)$.

Definition 2.3. *(Hörmander, [14, Definition 16.3.12]) Let $u \in \mathcal{E}'(\mathbb{R}^n)$ We say that u is invertible, if any of the statements (i) – (iii) in Theorem 2.2 holds for u.*

Theorem 2.5 below justifies the choice of the word "invertible" for this kind of distribution. Now let us explain what it means for a pair of open sets to be μ–convex for supports, where $\check{\mu}$ is defined by $\left[\check{\mu}(\phi) = \mu(\check{\phi}),\ \check{\phi}(x) = \phi(-x)\right]$.

Definition 2.4. *(Hörmander, [14, Definition 16.5.4]) Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$, and (X_1, X_2) be a pair of non-empty open subsets of $\mathbb{R}^n$ satisfying $X_2 - \operatorname{supp}(\mu) \subset X_1$. We say that the pair (X_1, X_2) is μ–convex for supports if the following condition holds. For every compact set $K_1 \subset X_1$, there exists a compact set $K_2 \subset X_2$ such that $\operatorname{supp} v \subset K_2$ whenever $v \in \mathcal{D}(X_2)$ and $\operatorname{supp} \check{\mu} * v \subset K_1$.*

Now we state the main theorem that we will use for proving the surjectivity of the spherical mean value operator from $\mathcal{E}(\mathbb{R}^n)$ to itself.

Theorem 2.5. *(Hörmander, [14, Part of Theorem 16.5.7]) Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$, and let $X_1, X_2 \subset \mathbb{R}^n$ be non-empty open sets. Suppose μ, X_1 and X_2 satisfy $X_2 - \operatorname{supp}(\mu) \subset X_1$. Then the following statements are equivalent.*

(i) The convolution equations $\mu * u = f$ has a solution $u \in \mathcal{E}(X_1)$ for every $f \in \mathcal{E}(X_2)$.

(ii) The distribution μ is invertible, and the pair (X_1, X_2) is μ–convex for supports.

Note that the statement (i) means that the convolution equation $\mu * u = f$ is invertible.

2.1.2 Proof of Theorem 2.1.

In this section we prove Theorem 2.1 by showing that $\delta_{S(0,r)}$ is invertible and that the pair $(\mathbb{R}^n, \mathbb{R}^n)$ is $\delta_{S(0,r)}$–convex for supports, and then using Theorem 2.5.

The lemma below is used for checking the slow decrease condition (i) in Theorem 2.2.

Lemma 2.6. *Fix $R > 0$. There exists a constant $A > 0$ such that for any $\xi \in \mathbb{R}$*

$$\sup \{ \, | \cos(R\,\zeta) | \;\; | \;\; \zeta \in \mathbb{R}, \; |\zeta - \xi| < A \ln(2 + |\xi|) \, \} \; > \; (A + |\xi|)^{-A} \, .$$

Proof. Let $A = \max\{2, 2\pi/(R\ln(2))\}$. Let $\xi \in \mathbb{R}$. We can find a $\eta \in \mathbb{R}$ satisfying

$$| \, R\,\eta - R\,\xi \, | \leq \pi \text{ and } | \cos(R\,\eta) \, | = 1 \, .$$

For such η, we have

$$| \, \eta - \; \xi \, | \leq \frac{\pi}{R} < \frac{2\pi}{R} \leq A \ln(2) \leq A \ln(2 + |\xi|) \, ,$$

and

$$| \cos(R\,\eta) \, | = 1 > 2^{-2} \geq A^{-A} \geq (A + |\xi|)^{-A} \, .$$

Thus we have

$$\sup \{ \, | \cos(R\,\zeta) | \;\; | \;\; \zeta \in \mathbb{R}, \; |\zeta - \xi| < A \ln(2 + |\xi|) \, \} \; > \; (A + |\xi|)^{-A} \, .$$

□

Definition 2.7. *Let $\nu \in \mathbb{R}$. We define the normalized Bessel function j_ν as follows.*

$$j_\nu(z) = \Gamma(\nu + 1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, \Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^{2k} \, , \quad z \in \mathbb{C} \, . \tag{2.5}$$

Note that it is clear from the series above that j_ν is an entire function. Let J_ν be the Bessel function of the first kind of order ν. Since $J_\nu(x) = \sum_{k=0}^{\infty}\{(-1)^k(x/2)^{2k+\nu}\}/\{k!\,\Gamma(k+\nu+1)\}$, we can easily see the following relationship between j_ν and J_ν.

$$j_\nu(x) = \frac{2^\nu\,\Gamma(\nu+1)\,J_\nu(x)}{x^\nu}, \quad x > 0\,. \tag{2.6}$$

The lemma below is used for checking the condition (i) in Theorem 2.2. We use the asymptotic behavior of the Bessel function of the first kind $J_\nu(x)$.

Lemma 2.8. *Fix $R > 0$, and $n \in \{2, 3, 4, \ldots\}$, and let $\nu = (n-2)/2$. There exists a constant $A > 0$ such that for any $\xi \in \mathbb{R}^n$*

$$\sup\,\{\;|\,j_\nu(\,R\,|\zeta|\,)\,|\;\;|\;\;\zeta \in \mathbb{R}^n,\;|\zeta - \xi| < A\ln(2+|\xi|)\;\}\;>\;(A+|\xi|)^{-A}\,, \tag{2.7}$$

where j_ν is the function defined in Definition 2.7.

Proof. Let $x > 0$. Let us define a function $f(x)$ on $(0,\infty)$ by the following equation with $C_\nu := 2^\nu\;\Gamma(\nu+1)\;\sqrt{\frac{2}{\pi}}$.

$$j_\nu(x) = C_\nu|x|^{-(\nu+1/2)}\{\cos(x - \frac{\nu\pi}{2} - \frac{\pi}{4}) + f(x)\} \tag{2.8}$$

From the large x behavior of $J_\nu(x)$ ([20, Section 11.6]) and (2.6), we have

$$f(x) = O(x^{-1})\,. \tag{2.9}$$

Now let $c = (2\pi)^{-(\nu+1)}\,R^{-(\nu+1)}\,C_\nu$. Because of (2.9), there exist positive numbers y_0 and M such that

$$\forall y > y_0 \quad |f(y)| \;\leq\; M\,y^{-1}\,.$$

Let

$$R_\nu := 2\pi + \max\{y_0, \frac{4}{c^2}, 2M\} \ . \tag{2.10}$$

By the definition of R_ν, we have

$$\forall y > (R_\nu - 2\pi) \quad |f(y)| \ \leq \ M \, \frac{1}{y} \ < \ M \, \frac{1}{2M} \ = \ \frac{1}{2} \ . \tag{2.11}$$

On the other hand, we have

$$\forall y > (R_\nu - 2\pi) \quad \frac{1}{c} \frac{1}{\sqrt{y}} < \frac{1}{c} \frac{1}{\sqrt{4/c^2}} = \frac{1}{2} \ . \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$\forall y > (R_\nu - 2\pi) \quad 1 + f(y) > \frac{1}{2} > \frac{1}{c} \frac{1}{\sqrt{y}} \ ,$$

and thus

$$\forall y > (R_\nu - 2\pi) \quad c \, y^{-(\nu+1/2)} \{1 + f(y)\} > y^{-(\nu+1)} \ . \tag{2.13}$$

If $x > 2\pi$ and $|y - x| \leq 2\pi$, then $0 < x - 2\pi \leq y \leq x + 2\pi$, and thus $y^{-(\nu+1)} \geq (x + 2\pi)^{-(\nu+1)}$. Moreover, if $x > 2\pi$, then $x + 2\pi < (2\pi x)$, and thus $(x + 2\pi)^{-(\nu+1)} > (2\pi x)^{-(\nu+1)}$. From these facts, we have

$$\forall x > R_\nu \ \forall y \in \mathbb{R} \quad |y - x| \leq 2\pi \ \Rightarrow \ y^{-(\nu+1)} > (2\pi x)^{-(\nu+1)} \ . \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$\forall x > R_\nu \ \forall y \in \mathbb{R} \quad |y - x| \leq 2\pi \ \Rightarrow \ c \, y^{-(\nu+1/2)} \{1 + f(y)\} > (2\pi x)^{-(\nu+1)} \ . \tag{2.15}$$

By multiplying $(2\pi)^{\nu+1} R^{\nu+1}$ in (2.15), we obtain

$$\forall x > R_\nu \ \forall y \in \mathbb{R} \quad |y - x| \leq 2\pi \ \Rightarrow \ C_\nu y^{-(\nu+1/2)} \{1 + f(y)\} > (x/R)^{-(\nu+1)} \ . \tag{2.16}$$

Now we define constants r, A_1, A_2, and A_3 depending on the constant R_ν defined in (2.10).

Since j_ν has isolated zeros (because zeros of a non-constant analytic function on $\mathbb{C}$ are isolated), we can choose an $r \in [0, R_\nu]$ satisfying $j_\nu(r) \neq 0$.

Let $A_1 = \max\{2\pi/(R\ln(2)), 2R_\nu/(R\ln(1.9))\}$, $A_2 = 1.1\,|j_\nu(r)|^{-1/(\nu+1)}$, and $A_3 = \nu + 1$.

The numbers 1.9 and 1.1 were chosen as numbers smaller than 2 and larger than 1, respectively, in order to obtain the strict inequalities in (2.7).

Let $\xi \in \mathbb{R}^n$. We will consider the case $|\xi| > \frac{R_\nu}{R}$ and the case $|\xi| \leq \frac{R_\nu}{R}$ separately.

I) $R|\xi| > R_\nu$

Let θ be a number satisfing

$$\cos(\theta - \frac{\nu\pi}{2} - \frac{\pi}{4}) = 1\ ,\ \theta > 0\ ,\ \text{and}\ \ |\,\theta - R\,|\xi|\,| \leq 2\pi\ . \tag{2.17}$$

Put $\eta = \frac{\theta}{R|\xi|}\,\xi$. Then we have

$$\begin{aligned} |R\eta - R\xi| &= |R|\eta| - R|\xi|| = |R\frac{\theta}{R|\xi|}|\xi| - R|\xi|| = |\,\theta - R|\xi|\,| \\ &\leq\ 2\pi\ \leq\ R\,A_1\ln(2)\ < R\,A_1\ln(2+|\xi|)\ . \end{aligned} \tag{2.18}$$

So we have

$$\begin{aligned} |\,j_\nu(\,R\,|\eta|\,)\,| &= |\,C_\nu\,|R\eta|^{-(\nu+1/2)}\{\cos(R\frac{\theta}{R|\xi|}|\xi| - \frac{\nu\pi}{2} - \frac{\pi}{4}) + f(|R\eta|)\}\,| \quad \text{(because of (2.8))} \\ &= |\,C_\nu\,|R\eta|^{-(\nu+1/2)}\{1 + f(|R\eta|)\}\,| \quad \text{(because of (2.17))} \\ &> |R\xi/R|^{-(\nu+1)} \quad \text{(because } |R|\eta| - R|\xi|| \leq 2\pi \text{ from (2.18), and (2.16))} \\ &= |\xi|^{-(\nu+1)} \\ &> (A_2 + |\xi|)^{-A_3}\ . \end{aligned} \tag{2.19}$$

Therefore we have the following from (2.18) and (2.19).

$$\forall \xi \in \mathbb{R}^n$$

$$|\xi| > R_\nu / R \Rightarrow$$

$$\sup \{ \, | \, j_\nu(\, R \, |\zeta| \,) \, | \quad | \quad \zeta \in \mathbb{R}^n, |\zeta - \xi| < A_1 \ln(2 + |\xi|) \, \} \; > \; (A_2 + |\xi|)^{-A_3} \, . \tag{2.20}$$

II) $R|\xi| \leq R_\nu$

Let us choose an $\eta \in \mathbb{R}^n$ satisfying $R|\eta| = r$. Since $R|\xi| \leq R_\nu$ and $R|\eta| \leq R_\nu$, we have $|R\xi - R\eta| \leq 2R_\nu$. Note that, by definition of A_1, we have $2R_\nu \leq RA_1 \ln(1.9)$. Thus we have

$$|R\xi - R\eta| \leq 2R_\nu \leq RA_1 \ln(1.9) < RA_1 \ln(2 + |\xi|) \, . \tag{2.21}$$

We also have the following.

$$|j_\nu(R|\eta|)| = |j_\nu(r)| = \{|j_\nu(r)|^{-1/(\nu+1)}\}^{-(\nu+1)} = (A_2/1.1)^{-A_3} > A_2^{-A_3} \geq (A_2 + |\xi|)^{-A_3} \tag{2.22}$$

So we have the following from (2.21) and (2.22).

$$\forall \xi \in \mathbb{R}^n$$

$$|\xi| \leq R_\nu / R \Rightarrow$$

$$\sup \{ \, | \, j_\nu(\, R \, |\zeta| \,) \, | \quad | \quad \zeta \in \mathbb{R}^n, |\zeta - \xi| < A_1 \ln(2 + |\xi|) \, \} \; > \; (A_2 + |\xi|)^{-A_3} \tag{2.23}$$

By combining (2.20) in the case I) and (2.23) in the case II), we obtain

$$\forall \xi \in \mathbb{R}^n \quad \sup \{ \, | \, j_\nu(\, R \, |\zeta| \,) \, | \quad | \quad \zeta \in \mathbb{R}^n, |\zeta - \xi| < A_1 \ln(2 + |\xi|) \, \} \; > \; (A_2 + |\xi|)^{-A_3} \, .$$

Now let $A = \max\{A_1, A_2, A_3\}$. Then we have

$$\forall \xi \in \mathbb{R}^n \quad \sup \{ \, | \, j_\nu(\, R \, |\zeta| \,) \, | \quad | \quad \zeta \in \mathbb{R}^n, |\zeta - \xi| < A \ln(2 + |\xi|) \, \} \; > \; (A + |\xi|)^{-A} \, .$$

$\square$

In checking the condition (i) in Theorem 2.2, it would be helpful to express $\widehat{\delta}_{S(0,r)}$ in terms of well known functions. The lemma below is that step. In the case of $\delta_{S(0,r)}$, the result of the lemma is already very well known. However, we write it in the form of a lemma. Inside its proof we briefly explain a way to obtain this kind of result. When one applies our argument in this section to a convolution operator other than $\delta_{S(0,r)}$, one may need nontrivial computation for this step.

Lemma 2.9. *Let $r > 0$, and $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ be the distribution defined by (1), and $\widehat{\delta}_{S(0,r)}(\xi)$ be its Fourier transform. Then for any $\xi \in \mathbb{R}^n$*

$$\widehat{\delta}_{S(0,r)}(\xi) = j_{(n-2)/2}(r|\xi|) \,, \tag{2.24}$$

where $j_{(n-2)/2}$ is the function defined in Definition 2.7.

Proof. Let us first consider the case $n = 1$. Since $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R})$, the Fourier transform of $\delta_{S(0,r)}$, denoted by $\widehat{\delta}_{S(0,r)}(\xi)$, is the following smooth function on $\mathbb{R}$.[1]

$$\forall \xi \in \mathbb{R} \;\; \widehat{\delta}_{S(0,r)}(\xi) = \frac{1}{2}\{e^{-i<\xi,0-r>} + e^{-i<\xi,0+r>}\} \tag{2.25}$$

Now let us compute the right hand side of (2.25).

$$\begin{aligned} \widehat{\delta}_{S(0,r)}(\xi) &= \frac{1}{2}\{e^{+ir\xi} + e^{-ir\xi}\} \\ &= \cos(r\xi) \\ &= j_{-1/2}(r|\xi|) \end{aligned}$$

Now consider the case $n > 1$. Again since $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R})$, the Fourier transform of $\delta_{S(0,r)}$, denoted by $\widehat{\delta}_{S(0,r)}(\xi)$, is the following smooth function on $\mathbb{R}^n$, where dm_S denotes the

[1]See for example [4, Theorem 8.4.1].

standard surface area measure in $\mathbb{R}^n$.

$$\forall \xi \in \mathbb{R}^n \quad \widehat{\delta}_{S(0,r)}(\xi) = c_r \int_{y \in S(0,r)} e^{-i<y,\xi>} \, dm_S(y) \tag{2.26}$$

We need to compute the right hand side of (2.26). However, it is easier to borrow the computation already done: [10, Introduction, Lemma 3.6]. Thus we obtain

$$\widehat{\delta}_{S(0,r)}(\xi) = j_{(n-2)/2}(r|\xi|) \ ,$$

where $j_{(n-2)/2}$ is the normalized Bessel function defined in Definition 2.7. ☐

Since $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$, $\delta_{S(0,r)}$ has its Fourier-Laplace transform $\widehat{\delta}_{S(0,r)}(\xi)$ which is the holomorphic extension of its Fourier transform. From (2.24), it is clear that

$$\widehat{\delta}_{S(0,r)}(\xi) = \Gamma(\frac{n}{2}) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+n/2)} \left(\frac{r}{2}\right)^{2k} (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^k \ , \quad \xi \in \mathbb{C}^n \ . \tag{2.27}$$

Lemma 2.10. *Let $r > 0$, and $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ be the distribution defined by (1). $\delta_{S(0,r)}$ is invertible in the sense of Definition 2.3.*

Proof. We will first consider separately the cases $n = 1$ and $n > 1$.

First consider the case $n = 1$. By Lemma 2.6, there exists $A > 0$ such that for any $\xi \in \mathbb{R}^n$

$$\begin{aligned} &\sup \{ \, | \, \widehat{\delta}_{S(0,r)}(\zeta) \, | \ \ | \ \ \zeta \in \mathbb{C}^n, |\zeta - \xi| < A \ln(2 + |\xi|) \, \} \\ &\geq \ \sup \{ \, | \, \widehat{\delta}_{S(0,r)}(\zeta) \, | \ \ | \ \ \zeta \in \mathbb{R}^n, |\zeta - \xi| < A \ln(2 + |\xi|) \, \} \\ &= \ \sup \{ \, | \, \cos(\, r \, \zeta \,) \, | \ \ | \ \ \zeta \in \mathbb{R}, |\zeta - \xi| < A \ln(2 + |\xi|) \, \} \\ &> \ (A + |\xi|)^{-A} \ . \end{aligned}$$

This shows that $\delta_{S(0,r)}$ satisfies the condition (i) in Theorem 2.2. Hence $\delta_{S(0,r)}$ is invertible by Definition 2.3.

Next consider the case $n > 1$. By Lemma 2.8, there exists $A > 0$ such that for any $\xi \in \mathbb{R}^n$,

$$
\begin{aligned}
&\sup \{ |\widehat{\delta}_{S(0,r)}(\zeta)| \mid \zeta \in \mathbb{C}^n, |\zeta - \xi| < A \ln(2 + |\xi|) \} \\
&\quad \geq \sup \{ | \widehat{\delta}_{S(0,r)}(\zeta) | \mid \zeta \in \mathbb{R}^n, |\zeta - \xi| < A \ln(2 + |\xi|) \} \\
&\quad = \sup \{ | j_\nu(r |\zeta|) | \mid \zeta \in \mathbb{R}^n, |\zeta - \xi| < A \ln(2 + |\xi|) \} \\
&\quad > (A + |\xi|)^{-A} .
\end{aligned}
$$

So by Definition 2.3, $\delta_{S(0,r)}$ is invertible. □

Lemma 2.11. *Let $r > 0$, and $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ be the distribution defined by (1). The pair $(\mathbb{R}^n, \mathbb{R}^n)$ is $\delta_{S(0,r)}$–convex for supports in the sense of Definition 2.4.*

Proof. We need to show that, for every compact set $K_1 \subset \mathbb{R}^n$, there exists a compact set $K_2 \subset \mathbb{R}^n$ such that $\operatorname{supp} v \subset K_2$ whenever $v \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp} \check{\delta}_{S(0,r)} * v \subset K_1$.

Now let $K_1 \subset \mathbb{R}^n$ be compact. Let

$$
K_2 = ch\ K_1 - \{y \in \mathbb{R}^n \mid |y| \leq r\} ,
$$

where ch denotes convex hull. Since K_2 is closed and bounded, it is compact in $\mathbb{R}^n$. Let $v \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \check{\delta}_{S(0,r)} * v \subset K_1$. Since $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{D}(\mathbb{R}^n) \subset \mathcal{E}'(\mathbb{R}^n)$, we use [12, Theorem 4.3.3] and obtain

$$
ch\ \operatorname{supp} \delta_{S(0,r)} * v = ch\ \operatorname{supp} \delta_{S(0,r)} + ch\ \operatorname{supp} v . \tag{2.28}
$$

So we have

$$
\begin{aligned}
ch \text{ supp}\, v &\subset ch \text{ supp}(\delta_{S(0,r)} * v) - ch \text{ supp}\, \delta_{S(0,r)} \quad \text{(because of (2.28))} \\
&= ch \text{ supp}(\delta_{S(0,r)} * v) - \{y \in \mathbb{R}^n \mid |y| \le r\} \\
&= ch \text{ supp}(\check{\delta}_{S(0,r)} * v) - \{y \in \mathbb{R}^n \mid |y| \le r\} \quad (\text{because } \check{\delta}_{S(0,r)} = \delta_{S(0,r)}) \\
&\subset ch\ K_1 - \{y \in \mathbb{R}^n \mid |y| \le r\} \\
&= K_2 \, .
\end{aligned}
$$

□

For convenience, let us repeat the statement of Theorem 2.1.

> Let $r > 0$, and $\delta_{S(0,r)}$ be the compactly supported distribution defined by (1). The convolution equation $\delta_{S(0,r)} * u(x) = f(x)$ has a solution $u \in \mathcal{E}(\mathbb{R}^n)$ for every $f \in \mathcal{E}(\mathbb{R}^n)$.

Below is the proof of this theorem.

We apply Theorem 2.5 for the proof. From Lemma 2.10 we know that $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ is invertible, and from Lemma 2.11 we know that the pair $(\mathbb{R}^n, \mathbb{R}^n)$ is $\delta_{S(0,r)}$–convex for supports. Hence the theorem follows.

2.2 Surjectivity of the single radius spherical mean value operator from $\mathcal{D}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$

In this section, we show that the map $M^r : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is surjective (Theorem 2.15 below).

We need the definition below in order to use Hörmander's Theorem 2.13 below, our main tool in this section.

Definition 2.12. *(Hörmander, [14, Definition 16.5.13]) Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$, and (X_1, X_2) be a pair of non-empty open subsets of $\mathbb{R}^n$ satisfying $X_2 - \operatorname{sing\,supp}(\mu) \subset X_1$. We say that the pair (X_1, X_2) is μ–convex for singular supports, if the following condition holds. For every compact set $K_1 \subset X_1$, there exists a compact set $K_2 \subset X_2$ such that $\operatorname{sing\,supp} v \subset K_2$ whenever $v \in \mathcal{E}'(X_2)$ and $\operatorname{sing\,supp} \check{\mu} * v \subset K_1$.*

Again, in the above definition $\check{\mu}$ is defined by $\check{\mu}(\phi) = \mu(\check{\phi})$, $\check{\phi}(x) = \phi(-x)$.

Theorem 2.13. *(Hörmander, [14, Corollary 16.5.19]) Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$, and let $X_1, X_2 \subset \mathbb{R}^n$ be non-empty open sets. Suppose μ, X_1 and X_2 satisfy $X_2 - \operatorname{supp}(\mu) \subset X_1$. Then $\mu * \mathcal{D}'(X_1) = \mathcal{D}'(X_2)$ if and only if μ is invertible and the pair (X_1, X_2) is μ–convex for supports and singular supports.*

The lemma below verifies one condition for applying Theorem 2.13 above to our problem. The other condition for applying that theorem is that $\delta_{S(0,r)}$ is invertible. This has been verified in the previous section.

Lemma 2.14. *Let $r > 0$, and $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ be the distribution defined by (1). The pair $(\mathbb{R}^n, \mathbb{R}^n)$ is $\delta_{S(0,r)}$–convex for singular supports in the sense of Definition 2.12.*

Proof. We need to show that, for every compact set $K_1 \subset \mathbb{R}^n$, there exists a compact set $K_2 \subset \mathbb{R}^n$ such that $\operatorname{sing\,supp} v \subset K_2$ whenever $v \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{sing\,supp} \check{\delta}_{S(0,r)} * v \subset K_1$.

Now let $K_1 \subset \mathbb{R}^n$ be compact. Let

$$K_2 = ch\ K_1 - \{y \in \mathbb{R}^n \mid |y| \leq r\}\ ,$$

where ch denotes convex hull. Since K_2 is closed and bounded, it is compact in $\mathbb{R}^n$. Let $v \in \mathcal{E}'(\mathbb{R}^n)$ with $\operatorname{sing\,supp} \check{\delta}_{S(0,r)} * v \subset K_1$. Since $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ is invertible by Lemma 2.10 and $v \in \mathcal{E}'(\mathbb{R}^n)$, we have, by [14, Corollary 16.3.15, (16.3.19)], $ch\ \operatorname{sing\,supp} v \subset ch\ \operatorname{sing\,supp}(\delta_{S(0,r)} * v) -\ ch\ \operatorname{sing\,supp} \delta_{S(0,r)}$. So we have

$$\begin{aligned}
ch\ \operatorname{sing\,supp} v &\subset ch\ \operatorname{sing\,supp}(\delta_{S(0,r)} * v) -\ ch\ \operatorname{sing\,supp} \delta_{S(0,r)} \\
&= ch\ \operatorname{sing\,supp}(\delta_{S(0,r)} * v) - \{y \in \mathbb{R}^n \mid |y| \leq r\} \\
&= ch\ \operatorname{sing\,supp}(\check{\delta}_{S(0,r)} * v) - \{y \in \mathbb{R}^n \mid |y| \leq r\} \quad (\text{because } \check{\delta}_{S(0,r)} = \delta_{S(0,r)}) \\
&\subset ch\ K_1 - \{y \in \mathbb{R}^n \mid |y| \leq r\} \\
&= K_2\ .
\end{aligned}$$

□

Theorem 2.15. *Let $r > 0$, and $\delta_{S(0,r)}$ be the compactly supported distribution defined by (1). Then $\delta_{S(0,r)} * \mathcal{D}'(\mathbb{R}^n) = \mathcal{D}'(\mathbb{R}^n)$.*

Proof. $\delta_{S(0,r)}$ is invertible by Lemma 2.10. The pair $(\mathbb{R}^n, \mathbb{R}^n)$ is $\delta_{S(0,r)}$–convex for supports by

Lemma 2.11. Moreover, the pair $(\mathbb{R}^n, \mathbb{R}^n)$ is $\delta_{S(0,r)}$–convex for singular supports by Lemma 2.14. So by Theorem 2.13, $\delta_{S(0,r)} * \mathcal{D}'(\mathbb{R}^n) = \mathcal{D}'(\mathbb{R}^n)$ □

2.3 Range of the single radius spherical mean value operator on $\mathcal{E}'(\mathbb{R}^n)$

In this section, we characterize the range of M^r on $\mathcal{E}'(\mathbb{R}^n)$ (Theorem 2.17 below). We will introduce our key Proposition, which is about the divisibility by $\widehat{\delta}_{S(0,r)}$, and then prove Theorem 2.17.

Proposition 2.16. *Let $r > 0$, $\widehat{\delta}_{S(0,r)}(\xi)$ be the function as in (2.27), and g be a holomorphic function on $\mathbb{C}^n$. If $g(\xi) = 0$ whenever $\widehat{\delta}_{S(0,r)}(\xi) = 0$, then the function $g(\xi)/\widehat{\delta}_{S(0,r)}(\xi)$ is holomorphic on $\mathbb{C}^n$.*

The proof of this Proposition is in Section 2.3.1. In the case one wants to prove this Proposition with $\delta_{S(0,r)}$ replaced by ones own $\mu \in \mathcal{E}'(\mathbb{R}^n)$, the only thing one needs to re-work with μ in Section 2.3.1 is Lemma 2.22.

Theorem 2.17. *Let $r > 0$, $\delta_{S(0,r)}$ be the compactly supported distribution defined by (1). Then we have*

$$M^r\mathcal{E}'(\mathbb{R}^n) = \{w \in \mathcal{E}'(\mathbb{R}^n) \mid \widehat{w}(\xi) = 0 \textit{ whenever } \widehat{\delta}_{S(0,r)}(\xi) = 0\}\ , \tag{2.29}$$

where $\widehat{w}(\xi)$ and $\widehat{\delta}_{S(0,r)}(\xi)$ denote the Fourier-Laplace transform of w and $\delta_{S(0,r)}$, respectively.

Proof. Let A be the set in the right hand side of (2.29). We will first show that $M^r\mathcal{D}(\mathbb{R}^n) \subset A$, and then show that $M^r\mathcal{D}(\mathbb{R}^n) \supset A$.

Let $\mu \in M^r\mathcal{E}'(\mathbb{R}^n)$. Then $\mu = \delta_{S(0,r)} * u$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$. So $\widehat{\mu}(\xi) = \widehat{\delta}_{S(0,r)}(\xi)\,\widehat{u}(\xi)$. Thus we have $\mu \in A$.

Now let $v \in A$. Then by Proposition 2.16, $\widehat{v}(\xi)/\,\widehat{\delta}_{S(0,r)}(\xi)$ is holomorphic on $\mathbb{C}^n$. Moreover $\delta_{S(0,r)}$ is invertible by Lemma 2.10. Therefore by Theorem 2.2, $\widehat{v}(\xi)/\,\widehat{\delta}_{S(0,r)}(\xi)$ is the Fourier transform of a distribution in $\mathcal{E}'(\mathbb{R}^n)$. Hence for some $u \in \mathcal{E}'(\mathbb{R}^n)$, we have $\widehat{v}(\xi) = \widehat{\delta}_{S(0,r)}(\xi)\,\widehat{u}(\xi)$. Since the Fourier transform is injective on $\mathcal{E}'(\mathbb{R}^n)$, we obtain $v = \delta_{S(0,r)} * u$, and thus have $v \in M^r\mathcal{E}'(\mathbb{R}^n)$. □

2.3.1 Proof of Proposition 2.16

Consider the set of all germs of holomorphic functions at $0 \in \mathbb{C}^n$. With the the usual addition and multiplication of germs of complex valued functions, this set forms a ring. Moreover, this ring is an integral domain. Throughout this section, let

$$\mathcal{O}_n$$

denote this integral domain. Let us recall the definiton of a unit and the definition of an irreducible element in the integral domain $\mathcal{O}_n$.[2]

- An element f of $\mathcal{O}_n$ is a unit if there exists a $g \in \mathcal{O}_n$ such that $fg = 1$.
- Suppose $f \in \mathcal{O}_n$ is nonzero and is not a unit. Then f is called irreducible in $\mathcal{O}_n$ if whenever $f = gh$ with $g, h \in \mathcal{O}_n$, at least one of g and h must be a unit in $\mathcal{O}_n$. Otherwise f is said to be reducible.

[2]Here the terms "unit" and "irreducible element" are used just as in typical abstract algebra textbook such as [3], etc.

For example, $f(z) = (z_1^2 + z_2^2 + \ldots + z_n^2)^2$ is a reducible element in $\mathcal{O}_n$, and $f(z) = z_1$ is an irreducible element in $\mathcal{O}_n$. We will use the following basic fact in this integral domain. For any $f \in \mathcal{O}_n$,

$$f \text{ is a unit if and only if } f(0) \neq 0\,. \tag{2.30}$$

The following theorem plays an important role in our proof.

Theorem 2.18. *[Weak Nullstellensatz. A corollary in [6, Chapter 0, Section 1]] Let $f, h \in \mathcal{O}_n$. If f is irreducible and $h(z) = 0$ whenever $f(z) = 0$, then f divides h in $\mathcal{O}_n$.*

Now let us prove some lemmas for our problem. After proving four lemmas, we prove Proposition 2.16.

Lemma 2.19. *Let $f \in \mathcal{O}_n$. Suppose f is not a unit. Then the following statement holds. If there exists a $j \in \{1, 2, 3, \ldots, n\}$ such that $\frac{\partial f}{\partial \xi_j}(0) \neq 0$, then f is irreducible.*

Proof. We will prove this lemma by proving the contrapositive: if f is not irreducible, then $\frac{\partial f}{\partial \xi_j}(0) = 0$ for any $j \in \{1, 2, 3, \ldots, n\}$.

Suppose $f \in \mathcal{O}_n$ is not irreducible. Then, by definition of an irreducible element of an integral domain and the condition in the Theorem that f is not a unit, either f is a zero or $f(\xi) = g(\xi)h(\xi)$ on a neighborhood of 0 with some $g, h \in \mathcal{O}_n$ where neither g nor h is a unit.

In case f is a zero, we have $\frac{\partial f}{\partial \xi_j}(0) = 0$ for any j.

In the other case, we have, for any j,

$$\frac{\partial f}{\partial \xi_j}(0) \;=\; g(0)\,\frac{\partial h}{\partial \xi_j}(0) \;+\; \frac{\partial g}{\partial \xi_j}(0)\,h(0)\;.$$

Since neither g nor h is a unit, $g(0) = h(0) = 0$ by (2.30). So, for any j, we have

$$\frac{\partial f}{\partial \xi_j}(0) \ = 0 \ .$$

$\square$

Lemma 2.20. *Let h be a holomorphic function in $\mathbb{C}^n$. Suppose $\eta \in \mathbb{C}^n$ is a zero of h, and that there exists a $j \in \{1, 2, 3, \ldots, n\}$ such that $\left.\frac{\partial h}{\partial \xi_j}\right|_{\xi=\eta} \neq 0$. Let f be a function defined by $f(\xi) = h(\xi + \eta)$. Then f is an irreducible element in $\mathcal{O}_n$*

Proof. Since η is a zero of h, $f(0) = h(\eta) = 0$. Thus by (2.30), f is not a unit in $\mathcal{O}_n$. Therefore we can apply Lemma 2.19, and conclude that f is irreducible in $\mathcal{O}_n$. $\square$

Lemma 2.21. *Let u be a holomorphic function such that for any $\eta \in \mathbb{C}^n$ with $u(\eta) = 0$, there exists $j \in \{1, 2, 3, \ldots, n\}$ such that $\left.\frac{\partial u}{\partial \xi_j}\right|_{\xi=\eta} \neq 0$. Let g be a holomorphic function on $\mathbb{C}^n$ such that $g(\xi) = 0$ whenever $u(\xi) = 0$. Then the function $g(\xi)/u(\xi)$ is holomorphic on $\mathbb{C}^n$.*

Proof. Let

$$V_u := \{\xi \in \mathbb{C}^n \mid u(\xi) = 0\} \ .$$

We will show that $g(\xi)/u(\xi)$ is holomorphic on a neighborhood of any point in $\mathbb{C}^n \backslash V_u$, and then show that $g(\xi)/u(\xi)$ is holomorphic on a neighborhood of any point in V_u.

Let $\eta \in \mathbb{C}^n \backslash V_u$. Since V_u is closed, $\mathbb{C}^n \backslash V_u$ is open. Hence there is a neighborhood N of η with $N \subset \mathbb{C}^n \backslash V_u$. Since $N \cap V_u = \emptyset$ and both g and u are holomorphic on $\mathbb{C}^n$, $g(\xi)/u(\xi)$ is holomorphic on N.

Let $\eta \in V_u$, $f(\xi) = u(\xi + \eta)$, $h(\xi) = g(\xi + \eta)$. Then $f, h \in \mathcal{O}_n$. By Lemma 2.20, f is irreducible. By the condition of this lemma, $h(\xi) = 0$ whenever $f(\xi) = 0$. So by Theorem

2.18, f divides h in $\mathcal{O}_n$. This means that there exists $\nu \in \mathcal{O}_n$ satisfying $f(\xi)\,\nu(\xi) = h(\xi)$ in a neighborhood of $0 \in \mathbb{C}^n$. Thus $h(\xi)/f(\xi) = (f(\xi)\,\nu(\xi))/f(\xi) = \nu(\xi)$ in a neighborhood of $0 \in \mathbb{C}^n$. Hence $h(\xi)/f(\xi)$ is holomorphic on a neighborhood of $0 \in \mathbb{C}^n$. Therefore $g(\xi)/u(\xi)$ is holomorphic on a neighborhood of η. □

Lemma 2.22. *Let $r > 0$, and $\widehat{\delta}_{S(0,r)}(\xi)$ be the function in (2.27). If $\eta \in \mathbb{C}^n$ satisfies $\widehat{\delta}_{S(0,r)}(\eta) = 0$, then there exists $j \in \{1, 2, 3, \ldots, n\}$ such that*

$$\left.\frac{\partial\,\widehat{\delta}_{S(0,r)}}{\partial \xi_j}\right|_{\xi=\eta} \neq 0\,.$$

Proof. Note that $\widehat{\delta}_{S(0,r)}(0) = 1$ from (2.27). So $\eta \neq 0$.

We will consider the cases $n = 1$ and $n > 1$ separately.

First consider the case $n = 1$. Note that $\widehat{\delta}_{S(0,r)}(\xi) = \cos(r\xi)$. So $\widehat{\delta}'_{S(0,r)}(\eta) = -r\sin(r\eta)$. Since $\widehat{\delta}_{S(0,r)}(\eta) = 0$, $\cos(r\eta) = 0$. If we assume $\sin(r\eta) = 0$, we have $(e^{ir\eta} + e^{-ir\eta})/2 = 0$ and $(e^{ir\eta} - e^{-iR\eta})/(2i) = 0$. This set of two equations implies $e^{ir\eta} = 0$, which is a contradiction. Thus $\sin(r\eta) \neq 0$, and thus $\widehat{\delta}'_{S(0,r)}(\eta) \neq 0$.

Next consider the case $n > 1$. Note that

$$\begin{aligned}
\widehat{\delta}_{S(0,r)}(\xi) &= \Gamma(\frac{n}{2})\,\{\,\frac{1}{((n-2)/2)!} \\
&+ \frac{(-1)^1\,(r/2)^2}{1!\,(1+(n-2)/2)!}\,(\xi_1^2 + \ldots + \xi_n^2)^1 \\
&+ \frac{(-1)^2\,(r/2)^4}{2!\,(2+(n-2)/2)!}\,(\xi_1^2 + \ldots + \xi_n^2)^2 \\
&\vdots \quad\ \}\,. \qquad (2.31)
\end{aligned}$$

Since $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \neq 0$, there is an $m \in \{1, 2, 3, \ldots, n\}$ satisfying $\eta_m \neq 0$. With this

m, we have

$$\left.\frac{\partial\widehat{\delta}_{S(0,r)}}{\partial\xi_m}\right|_{\xi=\eta} = 2\eta_m\Gamma(\frac{n}{2})\,\{\frac{(-1)^1\,(r/2)^2}{1!\,(1+(n-2)/2)!} + \frac{(-1)^2\,(r/2)^4\,2}{2!\,(2+(n-2)/2)!}\,(\eta_1^2+\eta_2^2+\ldots+\eta_n^2)^1 + \frac{(-1)^3(r/2)^6\,3}{3!\,(3+(n-2)/2)!}\,(\eta_1^2+\eta_2^2+\ldots+\eta_n^2)^2 \;\vdots\; \}\,. \tag{2.32}$$

Let $z \in \mathbb{C}$ satisfy $z^2 = \eta_1^2 + \eta_2^2 + \ldots + \eta_n^2$. Since $\eta \neq 0$, $z \neq 0$. From (2.31), and (2.5), we have $\widehat{\delta}_{S(0,r)}(\eta) = j_{(n-2)/2}(rz)$. Since $\widehat{\delta}_{S(0,r)}(\eta) = 0$, rz is a zero of $j_{(n-2)/2}$. Now from the relationship between $j_{(n-2)/2}$ and $J_{(n-2)/2}$, the Bessel function of the first kind of order $(n-2)/2$, rz is also a zero of $J_{(n-2)/2}$. Since zeros of $J_{(n-2)/2}$ are simple zeros ([25, Section 15.2]), $J'_{(n-2)/2}(rz) \neq 0$, and thus $j'_{(n-2)/2}(rz) \neq 0$. From (2.32) and (2.5) we have

$$\left.\frac{\partial\widehat{\delta}_{S(0,r)}}{\partial\xi_m}\right|_{\xi=\eta} = \frac{\eta_m r}{z}\; j'_{(n-2)/2}(rz)\,.$$

Therefore

$$\left.\frac{\partial\widehat{\delta}_{S(0,r)}}{\partial\xi_m}\right|_{\xi=\eta} \neq 0\,.$$

□

Proposition 2.16 follows from Lemma 2.22 and Lemma 2.21.

2.4 Range of the single radius spherical mean value operator on $\mathcal{D}(\mathbb{R}^n)$

In this section, we find a characterization of the range of M^r on $\mathcal{D}(\mathbb{R}^n)$. Our main tool is Theorem 2.17, and a result by Hörmander (the condition (iii) in Theorem 2.2).

Theorem 2.23. *Let $r > 0$, $\delta_{S(0,r)}$ be the compactly supported distribution defined by (1). Then we have*

$$M^r\mathcal{D}(\mathbb{R}^n) = \{w \in \mathcal{D}(\mathbb{R}^n) \mid \widehat{w}(\xi) = 0 \text{ whenever } \widehat{\delta}_{S(0,r)}(\xi) = 0\} \ , \tag{2.33}$$

where $\widehat{w}(\xi)$ and $\widehat{\delta}_{S(0,r)}(\xi)$ denote the Fourier-Laplace transform of w and $\delta_{S(0,r)}$, respectively.

Proof. Let A be the set in the right hand side of (2.33). We will first show that $M^r\mathcal{D}(\mathbb{R}^n) \subset A$, and then show that $M^r\mathcal{D}(\mathbb{R}^n) \supset A$.

Let $\mu \in M^r\mathcal{D}(\mathbb{R}^n)$. Then $\mu = \delta_{S(0,r)} * u$ for some $u \in \mathcal{D}(\mathbb{R}^n)$. So $\widehat{\mu}(\xi) = \widehat{\delta}_{S(0,r)}(\xi)\,\widehat{u}(\xi)$. Thus we have $\mu \in A$.

Now let $v \in A$. By Theorem 2.17, $v \in M^r\mathcal{E}'(\mathbb{R}^n)$. So $v = \delta_{S(0,r)} * u$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$. Since $\delta_{S(0,r)}$ is invertible by Lemma 2.10, we have, by Theorem 2.2, $\delta_{S(0,r)} * u \notin \mathcal{E}(\mathbb{R}^n)$ or $u \in \mathcal{E}(\mathbb{R}^n)$. Since $v = \delta_{S(0,r)} * u \in A \subset \mathcal{D}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$, $u \in \mathcal{E}(\mathbb{R}^n)$. Recall that we also have $u \in \mathcal{E}'(\mathbb{R}^n)$. Thus we have $u \in \mathcal{D}(\mathbb{R}^n)$. Thus $v \in M^r\mathcal{D}(\mathbb{R}^n)$. □

2.5 A summary for study of other convolution operators

In this section we summarize under which set of conditions we obtain which surjectivity or range characterization result.

Consider a compactly supported distribution μ on $\mathbb{R}^n$. Let us first list some conditions that μ might satisfy.

1. μ is invertible.
2. The pair $(\mathbb{R}^n, \mathbb{R}^n)$ is μ–convex for supports.
3. The pair $(\mathbb{R}^n, \mathbb{R}^n)$ is μ–convex for singular supports.
4. If $\eta \in \mathbb{C}^n$ satisfies $\widehat{\mu}(\eta) = 0$, then there exists $j \in \{1, 2, 3, \ldots, n\}$ such that
$$\left.\frac{\partial \widehat{\mu}}{\partial \xi_j}(\xi)\right|_{\xi=\eta} \neq 0 \, .$$

Now we write which result we obtain under which set of conditions.

- If $\mu \in \mathcal{E}'(\mathbb{R}^n)$ satisfies 1 and 2, then $\mu * \mathcal{E}(\mathbb{R}^n) = \mathcal{E}(\mathbb{R}^n)$.
- If $\mu \in \mathcal{E}'(\mathbb{R}^n)$ satisfies 1, 2, and 3, then $\mu * \mathcal{D}'(\mathbb{R}^n) = \mathcal{D}'(\mathbb{R}^n)$.
- If $\mu \in \mathcal{E}'(\mathbb{R}^n)$ satisfies 1 and 4, then
$$\mu * \mathcal{E}'(\mathbb{R}^n) = \{w \in \mathcal{E}'(\mathbb{R}^n) \mid \widehat{w}(\xi) = 0 \text{ whenever } \widehat{\mu}(\xi) = 0\} \, ,$$
and
$$\mu * \mathcal{D}(\mathbb{R}^n) = \{w \in \mathcal{D}(\mathbb{R}^n) \mid \widehat{w}(\xi) = 0 \text{ whenever } \widehat{\mu}(\xi) = 0\} \, .$$

2.6 Application of some of our lemmas: proof of the support theorem for the single radius spherical mean value operator for $\mathcal{S}(\mathbb{R}^n)$

As an application of our lemmas in the previous sections to other study, we present a quick proof of the support theorem (Theorem 2.29 below) for the single radius spherical mean value operator for $\mathcal{S}(\mathbb{R}^n)$. This support theorem is not new, since the support theorem for the single radius spherical mean value operator for $L^p(\mathbb{R}^n)$ with $p \leq 2n/(n-1)$ was already proved by V. V. Volchkov ([24, Corollary 3.3]) and by M. Agranovsky and P. Kuchment (in [1, Theorem 1]), and $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. Our approach, however, is different, and may serve as an additional model for studying support theorems for other integral transforms. In our proof, we need to show little about the shape or size of $\operatorname{supp} f$, except that $\operatorname{supp} f$ is compact. Then we use a support theorem for Pompeiu transforms integrating on geodesic spheres of fixed radius, which is a result from microlocal analysis, to show that $\operatorname{supp} f$ is contained in the ball $\bar{B}(0, R)$.

Let us clarify some notation. $S(y, r)$ denotes the $(n-1)$–sphere of radius r centered at y in $\mathbb{R}^n$, not in $\mathbb{C}^n$. Also $\bar{B}(y, r)$ is the closed ball in $\mathbb{R}^n$. For, $k \in \{1, 2, 3, \ldots\}$ and $A \subset \mathbb{C}^k$, $A \cap \mathbb{R}^k$ denotes the set $\{\xi \in A \mid \operatorname{Im} \xi = 0\}$.

Roughly speaking, the outline of our proof is as follows. We start with $f \in S(\mathbb{R}^n)$ satisfying $\operatorname{supp} M^r f \subset \bar{B}(0, R + r)$. Let $\delta_{S(0,R)}$ be the distribution corresponding to M^r given in (1) so that we have $M^r f = \delta_{S(0,r)} * f$. For convenience, we let $w(x) := \delta_{S(0,r)} * f(x)$.

For any $u \in \mathcal{E}'(\mathbb{R}^n)$, let $\widehat{u}$ denote the Fourier-Laplace transform of u given by (2.3). We will show that $\widehat{w}(\xi)$, $\xi \in \mathbb{C}^n$, vanishes whenever $\widehat{\delta}_{S(0,r)}(\xi)$ vanishes. Then we use our proposition about the divisibility by $\widehat{\delta}_{S(0,r)}$ (Proposition 2.16) to show that $\widehat{w}/\widehat{\delta}_{S(0,r)}$ is holomorphic. This holomorphicity and the fact that $\delta_{S(0,R)}$ is invertible (Lemma 2.10) will enable us to use a theorem by Hörmander (Theorem 2.2) to show that f has compact support. Then we use a result of Quinto (Lemma 2.27) to show that $\operatorname{supp} f \subset \bar{B}(0,R)$.

Now let us present some additional lemmas and then our support theorem. The lemma below is a well known result, whose proof we present in Section A.1 for the sake of completion of our argument.

Lemma 2.24. *Assume $n \geq 2$, and fix $\lambda > 0$. Let $M := \{\xi \in \mathbb{C}^n \mid \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 = \lambda^2\}$. The Lie group $SO(n,\mathbb{C})$ acts transitively on M.*

Lemma 2.25. *Assume $n \geq 2$, and fix $\lambda > 0$. Let $M := \{\xi \in \mathbb{C}^n \mid \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 = \lambda^2\}$. Suppose that $g : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and g vanishes on $M \cap \mathbb{R}^n$. Then g vanishes on M.*

Proof. By Lemma 2.24, M is diffeomorphic, as a complex manifold, to $SO(n,\mathbb{C})/SO(n-1,\mathbb{C})$. Since $SO(n,\mathbb{C})$ is connected, M is connected. So, if g vanishes on a non-empty open subset of M, then by its holomorphicity, g vanishes on M.

It therefore suffices to show that there exists a non-empty open subset of M on which g vanishes. Note that there is a connected neighborhood $U \subset M$ of the point $p = (0,0,\ldots,0,\lambda)$ on which $\phi : (\xi_1,\xi_2,\ldots,\xi_n) \mapsto (\xi_1,\xi_2,\ldots,\xi_{n-1})$ is a coordinate map. Let $V = \phi(U)$. Then $V \cap \mathbb{R}^{n-1} \neq \emptyset$, $\phi^{-1}(V \cap \mathbb{R}^{n-1}) \subset U \cap \mathbb{R}^n$, and V is a connected open set in $\mathbb{C}^{n-1}$. Moreover, since g is holomorphic, $g \circ \phi^{-1}$ is holomorphic on $V \subset \mathbb{C}^{n-1}$. Since g vanishes on $U \cap \mathbb{R}^n$, the holomorphic function $g \circ \phi^{-1}$ vanishes on $V \cap \mathbb{R}^{n-1}$. So $g \circ \phi^{-1}$ vanishes on V. Since $\phi^{-1}(V) = U$, g vanishes on U. □

Lemma 2.26. *Fix $R > 0$ and $r > 0$. If $f \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp} M^r f \subset \bar{B}(0, R+r)$, then $f \in \mathcal{D}(\mathbb{R}^n)$.*

Proof. Let $\delta_{S(0,r)}$ be the distribution defined by (1), and let $w(x) := \delta_{S(0,r)} * f(x) = M^r f(x)$. Since $\delta_{S(0,r)} \in \mathcal{E}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, we have

$$\widehat{w}(\xi) = \widehat{\delta}_{S(0,r)}(\xi)\ \widehat{f}(\xi)\ , \tag{2.34}$$

for $\xi \in \mathbb{R}^n$, where $\widehat{\delta}_{S(0,r)}$, $\widehat{f}$, and $\widehat{w}$ are the Fourier transforms of $\delta_{S(0,R)}$, f, and w, respectively.[3]

Since $w(x) = M^r f(x)$ and $\operatorname{supp} M^r f \subset \bar{B}(0, R+r)$, $w \in \mathcal{D}(\mathbb{R}^n)$. Also $\delta_{S(0,R)} \in \mathcal{E}'(\mathbb{R}^n)$. Thus $\widehat{w}(\xi)$ and $\widehat{\delta}_{S(0,r)}(\xi)$ extend to holomorphic functions on $\mathbb{C}^n$. According to (2.34),

$$\text{if } \xi \in \mathbb{R}^n \text{ and } \widehat{\delta}_{S(0,r)}(\xi) = 0, \text{ then } \widehat{w}(\xi) = 0\ . \tag{2.35}$$

Now we will to show that

$$\text{if } \xi \in \mathbb{C}^n \text{ and } \widehat{\delta}_{S(0,r)}(\xi) = 0, \text{ then } \widehat{w}(\xi) = 0\ . \tag{2.36}$$

Let us first consider the case $n = 1$. Since $\widehat{\delta}_{S(0,r)}(\xi) = \cos(r\xi)$ (see (2.27)) and $\{\xi \in \mathbb{R} \mid \cos(\xi) = 0\} = \{\xi \in \mathbb{C} \mid \cos(\xi) = 0\}$, (2.35) implies (2.36). Now let us consider the case $n \in \{2, 3, 4, \ldots\}$. Note that $\widehat{\delta}_{S(0,r)}(\xi) = j_{(n-1)/2}(r|\xi|) = 2^{(n-2)/2}\Gamma(n)\ (r|\xi|)^{-(n-2)/2}\ J_{(n-2)/2}(r|\xi|)$ and $\widehat{\delta}_{S(0,r)}(0) \neq 0$ (see Definition 2.7, Lemma 2.9, and (2.27)). Here $J_{(n-2)/2}$ is the Bessel

[3]See for example [12, Theorem 7.1.15], [4, Theorem 8.4.2], etc. At this stage of the proof, we do not know whether f has compact support. Thus we do not claim (2.34) for $\xi \in \mathbb{C}^n$.

function of the first kind of order $(n-2)/2$. Since $J_\nu(z)$, $\mathrm{Re}(\nu) > -1$, has only purely real zeros ([25, Section 15.25]) and the zeros of a non-constant holomorphic function in $\mathbb{C}$ is isolated and thus at most countable , we have the following description of the zero set of $\widehat{\delta}_{S(0,R)}(\xi)$.

$$\{\xi \in \mathbb{C}^n \mid \widehat{\delta}_{S(0,r)}(\xi) = 0\} = \bigcup_{p\in\mathbb{N},\, \lambda_p \neq 0} \{\xi \in \mathbb{C}^n \mid \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 = \lambda_p^2/r^2\} \,,$$

where $\{\lambda_p\}_{p\in\mathbb{N}} \subset \mathbb{R}$ is the zero set of $J_{(n-2)/2}(z)$. For a $p \in \mathbb{N}$ with $\lambda_p \neq 0$, let $\lambda := |\lambda_p|/r$. Then (2.35) indicates that $\widehat{w}$ vanishes on $\{\xi \in \mathbb{C}^n \mid \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 = \lambda^2\} \cap \mathbb{R}^n$. Hence by Lemma 2.25, $\widehat{w}$ vanishes on $\{\xi \in \mathbb{C}^n \mid \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 = \lambda^2\}$. This estabilishes (2.36).

Now let us analyze the ratio $\widehat{w}/\widehat{\delta}_{S(0,r)}$. By (2.36) and Proposition 2.16, $\widehat{w}/\widehat{\delta}_{S(0,r)}$ is holomorphic in $\mathbb{C}^n$. Therefore by Lemma 2.10 and Theorem 2.2, $\widehat{w}/\widehat{\delta}_{S(0,r)}$ is the Fourier transform of a distribution in $\mathcal{E}'(\mathbb{R}^n)$.

Since $\widehat{f}(\xi) = \widehat{w}(\xi)/\widehat{\delta}_{S(0,r)}(\xi)$, f has compact support. □

To prove Lemma 2.28 below, we use [21, Theorem 1.1] by Quinto. Here we state it almost verbatim from [21] without stating all the definitions of symbols and terms used in it.

Lemma 2.27. *(Quinto, [21, Theorem 1.1]) Let M be a real analytic Riemannian manifold, and let $\mathcal{A} \subset M$ be open and connected. Let I_y denote the injectivity radius of the exponential map at $y \in M$. Let $r > 0$ and assume for each y with $\bar{B}(\mathcal{A}, y) \leq r$, $I_y > r$. Let P_μ be a Pompeiu transform on geodesic spheres in M of radius r with nowhere zero real analytic weight μ. Assume $f \in \mathcal{D}'(M)$ with $P_\mu f(y) = 0$ for all $y \in \mathcal{A}$ and assume, for some $y_0 \in \mathcal{A}$, the ball $\bar{B}(y_0, r)$ is disjoint from* supp f. *Then for all $y \in \mathcal{A}$, $\bar{B}(y, r)$ is disjoint from* supp f.

Lemma 2.28. *(A special case of [21, Theorem 1.1]) Fix $R > 0$ and $r > 0$. If $f \in \mathcal{D}(\mathbb{R}^n)$ and* $\operatorname{supp} M^r f \subset \bar{B}(0, R+r)$, *then* $\operatorname{supp} f \subset \bar{B}(0, R)$.

Proof. To examine the applicability of Lemma 2.27, we can let $M = \mathbb{R}^n$, $\mathcal{A} = \mathbb{R}^n \setminus \bar{B}(0, R+r)$. Then $I_y = \infty$ for any $y \in M$. So $I_y > r$. We can also let the weight μ be an appropriate nonzero constant function, and let the Pompeiu transform P_μ be M^r. By the condition of Lemma 2.28, we have $f \in \mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$, $M^r f(y) = 0$ for all $y \in \mathcal{A}$. Since f has compact support, there is a $y_0 \in \mathcal{A} = \mathbb{R}^n \setminus \bar{B}(0, R+r)$ satisfying $\bar{B}(y_0, r) \cap \operatorname{supp} f = \emptyset$. Thus by Lemma 2.27, we can conclude that $\bar{B}(y, r) \cap \operatorname{supp} f = \emptyset$ for all $y \in \mathcal{A}$. This implies $\operatorname{supp} f \subset \bar{B}(0, R)$. □

Theorem 2.29. *Assume $n \geq 1$. Fix $R > 0$ and $r > 0$. If $f \in \mathcal{S}(\mathbb{R}^n)$ and* $\operatorname{supp} M^r f \subset \bar{B}(0, R+r)$, *then* $\operatorname{supp} f \subset \bar{B}(0, R)$.

Proof. By Lemma 2.26, we have $f \in \mathcal{D}(\mathbb{R}^n)$. Hence by Lemma 2.28, we have $\operatorname{supp} f \subset \bar{B}(R; 0)$. □

Chapter 3

Surjectivity and range description of the spherical mean value operator on $\mathbb{H}^3$

In this chapter, we characterize the range of the spherical mean value operator on the class of compactly supported distributions on $\mathbb{H}^3$, and show that the spherical mean value operator is surjective from the space of smooth functions on $\mathbb{H}^3$ to itself. We will explain why we have results only on $\mathbb{H}^3$. As a historical remark, we would like to mention an interesting surjectivity result in symmetric space by Helgason. He showed in 1973 that if D is a nonzero G–invariant differential operator on a symmetric space $X = G/K$ of noncompact type, then D is surjective from smooth functions on X to itself ([8, Theorem 8.2]).

We use some of Hörmander's results on convolution operators in Euclidean space and some of Helgason's works on symmetric space. Our results are in Theorems 3.1 below. In our range characterization in Theorems 3.1, we use the Fourier transform on symmetric space introduced in [7, 1965, Helgason]. So we give its brief definition here. We continue to use the notation defined here throughout Section 3.1 below, where we prove our theorem. We mostly follow the notation in [9].

Let $X = G/K$ be a symmetric space of noncompact type; that is, G is a connected noncompact semisimple Lie group with finite center and K is a maximal compact subgroup. Let us further assume that G/K has the Iwasawa decomposition $G = KAN$. Let M be the centralizer of A, and let $B = K/M$. Let $\mathfrak{a}$ be the Lie algebra of A. Let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$, and $\mathfrak{a}^*_{\mathbb{C}}$ be the complexification of $\mathfrak{a}^*$. For an $H \in \mathfrak{a}$, let $\rho(H) = \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}\,H|_{\mathfrak{n}}$, where $\mathfrak{n}$ is the Lie algebra of N. For $(gK, kM) \in X \times B$, let $A(gK, kM)$ denote the element H in $\mathfrak{a}$ satisfying $gK = kn\exp(H)K$ with $n \in N$. Let dx be a G–invariant measure on X. Now we let $\widetilde{f}$ denote the Fourier transform of a function f on X:

$$\widetilde{f}(\lambda, b) = \int_X e^{(-i\lambda+\rho)(A(x,b))}\, f(x)\, dx\,, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ b \in B\,, \tag{3.1}$$

assuming the integral converges absolutely. Let db denote the the K–invariant measure on B with total measure 1. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let ϕ_λ denote the spherical function $\phi_\lambda(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}\, db$.

Theorem 3.1. *Let $r > 0$. Let $\phi_\lambda(x)$ be a spherical function in $\mathbb{H}^3 = SO_o(1,3)/SO(3)$. With fixed $a \in \mathbb{H}^3$ with $d(a,o) = r$, let $V_r = \{\lambda \in \mathbb{C} \mid \phi_\lambda(a) = 0\}$. For $w \in \mathcal{E}'(\mathbb{H}^3)$, let $\widetilde{w}(\lambda, b)$ denote the Fourier transform of w. Then*

$$M^r\mathcal{E}'(\mathbb{H}^3) = \{w \in \mathcal{E}'(\mathbb{H}^3) \mid \widetilde{w}(\lambda, b) = 0 \text{ for all } b \in S^2 \text{ whenever } \lambda \in V_r\}\,,$$

and the map $M^r : \mathcal{E}(\mathbb{H}^3) \to \mathcal{E}(\mathbb{H}^3)$ is surjective.

In Section 3.1 below, we prove the theorem above by utilizing an expression of the spherical function in terms of hypergeometric function. This is in Lemma 3.7, which is for a Riemannian symmetric space $X = G/K$ of noncompact type of rank one with the Iwasawa decomposition $G = KAN$. With that expression we verify the simple zero condition (i) and the slow decrease condition (ii) in Lemma 3.6. Thus we expect that Theorem 3.1 can be extended to other Riemannian symmetric space of noncompact type of rank one with Iwasawa decomposition. Though we did not show here, we succeeded in verifying the slow decrease condition in certain symmetric spaces of rank one other than $\mathbb{H}^3$. We are still working on verifying the simple zero condition in certain symmetric spaces of rank one other than $\mathbb{H}^3$.

3.1 Proof of Theorem 3.1

In this section we prove Theorem 3.1. Earlier in this chapter, we set up some notation. Here we continue to set up notation from there under the same $X = G/K$. Thereafter we prove some lemmas and then prove the theorem.

Let $\mathcal{K}(\mathfrak{a}^*_{\mathbb{C}} \times B)$ be the space of all functions $\psi \in \mathcal{E}(\mathfrak{a}^*_{\mathbb{C}} \times B)$ such that there exists a triple of nonnegative constants (C, N, R) satisfying

$$|\psi(\lambda, b)| \leq C\,(1 + |\lambda|)^N\, e^{R\,|\,\mathrm{Im}\,\lambda|} \quad \text{for all } (\lambda, b) \in \mathfrak{a}^*_{\mathbb{C}} \times B\,.$$

Let M' be the normalizer of A, and let W denote the Weyl group M'/M. Let $\mathcal{K}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$

be the space of all functions ψ in $\mathcal{K}(\mathfrak{a}_{\mathbb{C}}^* \times B)$ satisfying

$$\int_B e^{(is\lambda+\rho)(A(x,b))}\psi(s\lambda, b)\, db = \int_B e^{(i\lambda+\rho)(A(x,b))}\psi(\lambda, b)\, db \quad \text{for any } s \in W, \lambda \in \mathfrak{a}_{\mathbb{C}}^*, x \in X. \tag{3.2}$$

Let $\mathcal{K}_W(\mathfrak{a}_{\mathbb{C}}^*)$ be the space of all functions in $\mathcal{K}(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$ constant in $b \in B$. We will use the fact that the Fourier transform is bijective from $\mathcal{E}'(X)$ to $\mathcal{K}(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$ ([9, Chapter III, Corollary 5.9]).

Fix an orthonormal basis $A_1, \ldots, A_l$ of $\mathfrak{a}$ with respect to the Killing form. Then the map $\lambda \mapsto (\lambda(A_1), \ldots, \lambda(A_l))$ is a linear bijection from $\mathfrak{a}_{\mathbb{C}}^*$ to $\mathbb{C}^l$. Thus we can identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^l$.

For a fixed $y \in X$ with $y \neq o$, we let δ_{S_y} be the distribution in $\mathcal{E}'(X)$ defined as follows, where M^y is as in Definition 0.1.

$$\begin{aligned} \delta_{S_y} : \mathcal{E}(X) &\to \mathbb{C} \\ u(x) &\mapsto M^y u(o) \end{aligned} \tag{3.3}$$

The lemma below is a lemma about divisibility of holomorphic functions uniformly with an additional parameter.

Lemma 3.2. *Fix any index set B. Let $d\lambda$ denote the Lebesgue measure in $\mathbb{C}^n$. Suppose functions $W : \mathbb{C}^n \times B \to \mathbb{C}$ and $U : \mathbb{C}^n \to \mathbb{C}$ satisfy the conditions (i) – (v) below.*

(i) For each fixed $b \in B$, the functions $W(\zeta, b)$ and $U(\zeta)$ are holomorphic in ζ.

(ii) There exists a triple of constants $(C_W, N_W, R_W) \in \mathbb{R}_+^3$ such that for any $(\zeta, b) \in \mathbb{C}^n \times B$

$$|W(\zeta, b)| \leq C_W (1 + |\zeta|)^{N_W} e^{R_W |\operatorname{Im} \zeta|} .$$

(iii) There exists a triple of constants $(C_U, N_U, R_U) \in \mathbb{R}^3_+$ *such that for any* $\zeta \in \mathbb{C}^n$

$$|U(\zeta)| \le C_U (1+|\zeta|)^{N_U} e^{R_U |\operatorname{Im}\zeta|} .$$

(iv) There exists a constant $A > 0$ *such that for any* $\xi \in \mathbb{R}^n$

$$\sup \{ \, |U(\zeta)| \mid \zeta \in \mathbb{C}^n, \ |\zeta - \xi| < A \ln(2+|\xi|) \, \} \ > \ (A+|\xi|)^{-A} .$$

(v) For each fixed $b \in B$, $\frac{W(\zeta,b)}{U(\zeta)}$ *is holomorphic in* ζ.

Then there exists a triple of constants $(C, N, R) \in \mathbb{R}^3_+$ *such that for any* $(\zeta, b) \in \mathbb{C}^n \times B$

$$\left| \frac{W(\zeta,b)}{U(\zeta)} \right| \le C \, (1+|\zeta|)^N e^{R \, |\operatorname{Im}\zeta|} .$$

Proof. By the conditions (i) and (iii) in this lemma and the Paley-Wiener-Schwartz theorem, U is the Fourier-Laplace transform of a compactly supported distribution in $\mathbb{R}^n$. Hence by [14, Theorem 16.3.10] and the condition (iv) in this lemma, U satisfies the hypothesis (B.8) in Lemma B.3. Hence the conclusion of Lemma B.3 implies this lemma. □

Lemma 3.3. *Let* $X = G/K$ *be a symmetric space of noncompact type of rank* l *with the Iwasawa decomposition* $G = KAN$. *Let* $B = K/M$. *Let* $\phi_\lambda(x)$ *be a spherical function in* X. *Fix* $y \in X$. *View* $\phi_\lambda(y)$ *as a function in* $\lambda \in \mathbb{C}^l$ *and let* $V_y = \{\lambda \in \mathbb{C}^l \mid \phi_\lambda(y) = 0\}$. *For* $w \in \mathcal{E}'(X)$, *let* $\widetilde{w}(\lambda, b)$ *denote the Fourier transform of* w. *Suppose* $\phi_\lambda(y)$ *satisfies conditions (i) and (ii) below.*

(i) If $\eta \in \mathbb{C}^l$ *satisfies* $\phi_\eta(y) = 0$, *then there exists* $j \in \{1, 2, 3, \dots, l\}$ *such that*

$$\left. \frac{\partial \phi_\xi(y)}{\partial \xi_j} \right|_{\xi=\eta} \neq 0 .$$

(ii) There exists a constant $A > 0$ such that for any $\xi \in \mathbb{R}^l$

$$\sup \{ \, |\phi_\zeta(y)| \mid \zeta \in \mathbb{C}^l, \; |\zeta - \xi| < A \ln(2 + |\xi|) \, \} \; > \; (A + |\xi|)^{-A} \, .$$

Then

$$M^y \mathcal{E}'(X) = \{ w \in \mathcal{E}'(X) \mid \widetilde{w}(\lambda, b) = 0 \textit{ for all } b \in B \textit{ whenever } \lambda \in V_y \} \, . \qquad (3.4)$$

Proof. Let A be the set on the right hand side of (3.4). We will show that $M^y\mathcal{E}'(X) \subset A$, and then show that $M^y\mathcal{E}'(X) \supset A$.

Let $\mu \in M^y\mathcal{E}'(X)$. Then there is a $u \in \mathcal{E}'(X)$ with $\mu = \delta_{S_y} * u$, where δ_{S_y} is as defined in (3.3). Since $\widetilde{\mu}(\lambda, b) = \phi_\lambda(y)\, \widetilde{u}(\lambda, b)$ for any $\lambda \in \mathbb{C}^m$ and $b \in B$, we have $\mu \in A$.

This time let $\mu \in A$.

We will first check whether the conditions in Lemma 3.2 are satisfied with $W(\zeta, b)$ replaced by $\widetilde{\mu}(\zeta, b)$, $U(\zeta)$ by $\phi_\zeta(y)$, and n by l. Since $\widetilde{\mu}(\zeta, b) \in \mathcal{K}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ and $\phi_\zeta(y) = \widetilde{\delta_{S_y}}(\zeta) \in \mathcal{K}_W(\mathfrak{a}^*_{\mathbb{C}})$, the conditions (i) – (iii) in Lemma 3.2 are satisfied. The condition (iv) in Lemma 3.2 is satisfied by the condition (ii) in Lemma 3.3. The condition (i) in Lemma 3.3 and our assumption that $\mu \in A$ enable us to apply Lemma 2.21 and conclude that the Paley-Wiener condition (v) in Lemma 3.2 is satisfied.

Thus by Lemma 3.2, there exists a triple of constants $(C, N, R) \in \mathbb{R}^3_+$ such that for any $(\zeta, b) \in \mathbb{C}^l \times B$

$$\left| \frac{\widetilde{\mu}(\zeta, b)}{\phi_\zeta(y)} \right| \le C \, (1 + |\zeta|)^N e^{R \, |\operatorname{Im} \zeta|} \, .$$

Since $\widetilde{\mu}(\zeta, b) \in \mathcal{K}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$, $\widetilde{\mu}(\zeta, b)$ satisfies (3.2). Moreover $\phi_\zeta(y) = \phi_{s\zeta}(y)$ for $s \in W$ ([10, Theorem 4.3]). Thus the holomorphic function $\widetilde{\mu}(\zeta, b)/\phi_\zeta(y)$ satisfies (3.2). Therefore $\widetilde{\mu}(\zeta, b)/\phi_\zeta(y) \in \mathcal{K}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$.

Since the Fourier transform is bijective from $\mathcal{E}'(X)$ to $\mathcal{K}(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$, there exists $v \in \mathcal{E}'(X)$ with $\widetilde{v}(\zeta, b) = \widetilde{\mu}(\zeta, b)/\phi_\zeta(y)$. Thus $\widetilde{\mu}(\zeta, b) = \widetilde{v}(\zeta, b)\, \phi_\zeta(y) = \widetilde{\delta_{S_y} * v}(\zeta, b)$. Again because the Fourier transform is bijective from $\mathcal{E}'(X)$ to $\mathcal{K}(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$, we have $\mu = \delta_{S_y} * v$. Hence $\mu \in M^y\mathcal{E}'(X)$. □

Lemma 3.4. *(Helgason, [10, Chapter 1, Section 3, Theorem 3.7]) Let G be a locally compact group, K and H be closed subgroups. Let us follow the assumptions and notation in the double fibration of $X = G/K$ and $\Xi = G/H$ in [10, Chapter 1, Section 3]. Let $N = \{s \in \mathcal{E}'(X) \mid \widehat{s} = 0\}$, and $N^\perp = \{f \in \mathcal{E}(X) \mid s(f) = 0$ for $s \in N\}$. If K is compact and the range $\mathcal{E}'(X)^\wedge \subset \mathcal{E}'(\Xi)$ is closed, then $\mathcal{E}(\Xi)^\vee = N^\perp$.*

Lemma 3.5. *Let $X = G/K$ be a symmetric space of noncompact type with the Iwasawa decomposition $G = KAN$. Fix $y \in X$. If $M^y\mathcal{E}'(X)$ is closed in $\mathcal{E}'(X)$, then $M^y : \mathcal{E}(X) \to \mathcal{E}(X)$ is surjective.*

Proof. We will apply Lemma 3.4 bearing in mind that for a fixed $y \in X$, the mean value operator M^y is self dual in the context of Helgason's double fibration.

Let δ_{S_y} be as defined in (3.3). By Lemma 3.4, $M^y\mathcal{E}(X) = N^\perp$. On the other hand, we can use the Fourier transform and show that the map $M^y : \mathcal{E}'(X) \to \mathcal{E}'(X)$ is injective. So the kernel of this map is zero, that is, $N = \{s \in \mathcal{E}'(X) \mid \delta_{S_y} * s = 0\} = \{0\}$. Therefore $N^\perp = \{f \in \mathcal{E}(X) \mid s(f) = 0 \text{ for } s \in N\} = \mathcal{E}(X)$. So $M^y\mathcal{E}(X) = \mathcal{E}(X)$ □

Lemma 3.6. *Let $X = G/K$ be a symmetric space of noncompact type of rank l with the Iwasawa decomposition $G = KAN$. Let $B = K/M$. Let $\phi_\lambda(x)$ be a spherical function in X. Fix $y \in X$. View $\phi_\lambda(y)$ as a function in $\lambda \in \mathbb{C}^l$ and let $V_y = \{\lambda \in \mathbb{C}^l \mid \phi_\lambda(y) = 0\}$. For*

$w \in \mathcal{E}'(X)$, let $\widetilde{w}(\lambda, b)$ denote the Fourier transform of w. Suppose $\phi_\lambda(y)$ satisfies conditions (i) and (ii) below.

(i) If $\eta \in \mathbb{C}^l$ satisfies $\phi_\eta(y) = 0$, then there exists $j \in \{1, 2, 3, \dots, l\}$ such that

$$\left.\frac{\partial \phi_\xi(y)}{\partial \xi_j}\right|_{\xi=\eta} \neq 0\,.$$

(ii) There exists a constant $A > 0$ such that for any $\xi \in \mathbb{R}^l$

$$\sup\,\{\ |\phi_\zeta(y)|\mid \zeta \in \mathbb{C}^l,\ |\zeta - \xi| < A \ln(2 + |\xi|)\ \}\ >\ (A + |\xi|)^{-A}\,.$$

Then

$$M^y\mathcal{E}'(X) = \{w \in \mathcal{E}'(X) \mid \widetilde{w}(\lambda, b) = 0 \textit{ for all } b \in B \textit{ whenever } \lambda \in V_y\}\,,$$

and the map $M^y : \mathcal{E}(X) \to \mathcal{E}(X)$ is surjective.

Proof. The first conclusion of this lemma follows from Lemma 3.3.

Since the set $\{w \in \mathcal{E}'(X) \mid \widetilde{w}(\lambda, b) = 0$ for all $b \in B$ whenever $\lambda \in V_y\}$ is closed in $\mathcal{E}'(X)$, the second conclusion of this lemma follows from Lemma 3.5. ☐

Lemma 3.7. *([18, Lemma 3.2]) Let $X = G/K$ be a Riemannian symmetric space of non-compact type of rank one with the Iwasawa decomposition $G = KAN$. Let $\phi_\lambda(x)$ denote a spherical function in X. Let $\Sigma^+ = \{\alpha, 2\alpha\}$, and let $p \geq 1$ and $q \geq 0$ be the multiplicity of α and 2α, respectively. Let $\sigma = p/2 + q$. Let $s \in \mathbb{C}$ satisfy $\lambda = -is\rho$. Fix $y \in X$ and let $t = 2\,d(y, o)$ and view the $\phi_\lambda(y)$ as a function in $\lambda \in \mathbb{C}$. Then*

$$\phi_\lambda(y) = (\cosh t)^{(s-1)\sigma}\ {}_2F_1\left(\frac{1-s}{2}\sigma,\ \frac{1-s}{2}\sigma + \frac{1-q}{2};\ \frac{p+q+1}{2};\ \tanh^2 t\right)\,,$$

where ${}_2F_1$ denotes the usual Gauss hypergeometric function.

Lemma 3.8. *Let $\phi_\lambda(x)$ be a spherical function in $\mathbb{H}^3$. Fix $y \in \mathbb{H}^3$. If $\eta \in \mathbb{C}$ satisfies $\phi_\eta(y) = 0$, then*

$$\left.\frac{\partial \phi_\xi(y)}{\partial \xi}\right|_{\xi=\eta} \neq 0\,.$$

Proof. In the case $X = \mathbb{H}^3$, $p = 2$ and $q = 0$. Thus from Lemma 3.7 and the identity ([20, Section 11.3, Table 3])

$${}_2F_1\left(\alpha, \alpha + \frac{1}{2}; \frac{3}{2}; x^2\right) = \frac{(1+x)^{1-2\alpha} - (1-x)^{1-2\alpha}}{2x(1-2\alpha)}\,,$$

we have

$$\phi_\lambda(y) = \frac{\sinh(ts)}{s\sinh(t)}\,,$$

where $t = 2d(y, o)$ and $s \in \mathbb{C}$ satisfies $\lambda = -is\rho$. Let $f(s) = \frac{\sinh(ts)}{s\sinh(t)}$. It suffices to show that f has no zero with multiplicity greater than one. Let s_0 be a zero of f. Since $f(0) = \frac{t}{\sinh t} \neq 0$, $s_0 \neq 0$. From the formula of f we have $\sinh(ts_0) = 0$. Note that $f'(s_0) = t\frac{\cosh(ts_0)}{\sinh(t)} - \frac{1}{s_0}f(s_0)$. If we assume $f'(s_0) = 0$, then we have $\sinh(ts_0) = \cosh(ts_0) = 0$ which leads to a contradiction. Therefore $f'(s_0) \neq 0$. ∎

Lemma 3.9. *Let $\phi_\lambda(x)$ be a spherical function in $\mathbb{H}^3$. Fix $y \in \mathbb{H}^n$. There exists a constant $A > 0$ such that for any $\xi \in \mathbb{R}$*

$$\sup\left\{\, |\phi_\zeta(y)| \mid \zeta \in \mathbb{C},\ |\zeta - \xi| < A\ln(2 + |\xi|)\,\right\} \ >\ (A + |\xi|)^{-A}\,.$$

Proof. Let us use the same expression for ϕ_λ as in the proof of Lemma 3.8:

$$\phi_\lambda(y) = \frac{\sinh(ts)}{s\sinh(t)}\,,$$

where $t = 2\,d(y, o)$ and $s \in \mathbb{C}$ satisfies $\lambda = -is\rho$. Let $f(\zeta) = \frac{\sinh(it\zeta)}{i\zeta\sinh t} = \frac{\sin(t\zeta)}{\zeta\sinh t}$. Let $A = \max\{\frac{\pi}{t}, \frac{2\pi}{t\ln 2}, \sinh t, 2\}$.

Let $\xi \in \mathbb{R}$. We can find $\eta \in \mathbb{R}$ satisfying $|\, t\,\eta - t\,\xi \,| \le \pi$ and $|\, \sin(t\,\eta) \,| = 1$. For this such η, we have

$$|\, \eta - \xi \,| \le \frac{\pi}{t} < A \ln 2 \le A \ln(2 + |\xi|)\,,$$

and

$$\left| \frac{\sin(t\eta)}{\eta \sinh t} \right| \ge \frac{1}{|\eta \sinh t|} \ge \frac{1}{(|\xi| + \pi/t) \sinh t} \ge \frac{1}{(A + |\xi|)A} \ge \frac{1}{(A + |\xi|)^2} > (A + |\xi|)^{-A}\,.$$

Thus we have

$$\begin{aligned}
&\sup \{\, |f(\zeta)| \mid \zeta \in \mathbb{C},\ |\zeta - \xi| < A \ln(2 + |\xi|) \,\} \\
&\ge \sup \{\, |f(\zeta)| \mid \zeta \in \mathbb{R},\ |\zeta - \xi| < A \ln(2 + |\xi|) \,\} \\
&> \ (A + |\xi|)^{-A}\,.
\end{aligned}$$

□

By Lemmas 3.8 and 3.9, we obtain Theorem 3.1 as a special case of Lemma 3.6.

Chapter 4

Extension of John's results about the spherical mean

In this chapter, we extend some of John's results ([15]) about the spherical mean.

In Section 4.1, we show the hyperbolic space version (Theorem 4.2) of John's iterated spherical mean value theorem in $\mathbb{R}^n$. We use this result in showing the support theorem for single radius spherical mean value operator in $\mathbb{H}^3$ in Section D.2. We also use John's iterated spherical mean value theorem in $\mathbb{R}^n$ in Section 4.2.2 and 4.2.3.

In Section 4.2, we consider whether we can uniquely determine a function with known average values over all spheres with a fixed radius $r > 0$, if we additionally know the values of the functions on a ball. We first explain that if $f \in C(M)$, $M = \mathbb{R}^n, \mathbb{H}^n, S^n$, f is determined by its values on $B(o, r + \epsilon)$, $\epsilon > 0$ (Corollary 4.4). John already proved this for $M = \mathbb{R}^3$. We explain that Corollary 4.4 follows immediately from Quinto's microlocal analysis result in [21]. Then we give a proof that if $f \in C^\infty(\mathbb{R}^5)$, $f(x)$ is uniquely determined by its values in

$\bar{B}(0,r)$ (Theorem 4.5). John already proved this for $f \in C^\infty(\mathbb{R}^3)$. In our proof, we work with a delay differential equation which follows from John's iterated spherical mean value theorem in $\mathbb{R}^n$. We also prove a variation of John's latter claim in $\mathbb{R}^3$. A function is determined by its values in (instead of $\bar{B}(0,r)$) two split regions in $\mathbb{R}^3$: a certain ball and a certain annulus, or certain two annuli, where the sum of thicknesses of the annuli is r (Theorem 4.6). In our proof, we use a delay equation which follows from John's iterated spherical mean value theorem.

In Section 4.3, we consider the inhomogeneous spherical mean value equation $M^1 f(x) = g(x)$, where f is an unknown function and g is a known function. John finds an explicit solution to such an equation under a certain decay condition on $g \in C^3(\mathbb{R}^3)$. We find an explicit solution to such an equation under a similar decay condition on $g \in C^3(\mathbb{H}^3)$ (Theorem 4.24).

4.1 Iterated Mean Value Theorem in $\mathbb{H}^n$

For a function $f \in C(\mathbb{R}^n)$, John defines the iterated mean of a function as the spherical mean of the spherical mean of the function. More specifically, he defines, $I(x,r)$ as $M^r f(x)$ in [15]. With this notation he defines the iterated mean value of f at x with radii λ and μ as $M(x,\lambda,\mu) = M^\lambda(M^\mu f)(x) = \frac{1}{\Omega_n}\int_{\zeta \in S^{n-1}} I(x+\lambda\zeta,\mu)\,dw(\zeta)$, where dw denotes the standard surface area measure in the Euclidean space. It turns out that $M(x,\lambda,\mu) = M(x,\mu,\lambda)$.[1]

[1]This is clear in $\mathbb{R}^n$, since addition is commutative in $\mathbb{R}^n$. In a symmetric space G/K with compact K, the relation $M^x M^y f(z) = M^y M^x f(z)$ holds for $f \in C(G/K)$, since it holds for any analytic function ([9, Chapter II, Section 2, equation (30)]) and f can be uniformly approximated by analytic functions on a

To illustrate some visual intuition behind this definition, let us plot the paths of inner integrals in the double integral $M(0,\lambda,\mu) = \frac{1}{\Omega_n}\int_{\zeta\in S^{n-1}} I(0+\lambda\zeta,\mu)\,dw(\zeta)$ in $\mathbb{R}^2$ in Figure 4.1. On the left picture, $\lambda = 2$ and $\mu = 1$, and the paths are circles of radius $\mu = 1$. On the right picture, $\lambda = 1$ and $\mu = 2$, and the paths are circles of radius $\mu = 2$. In any case, we can

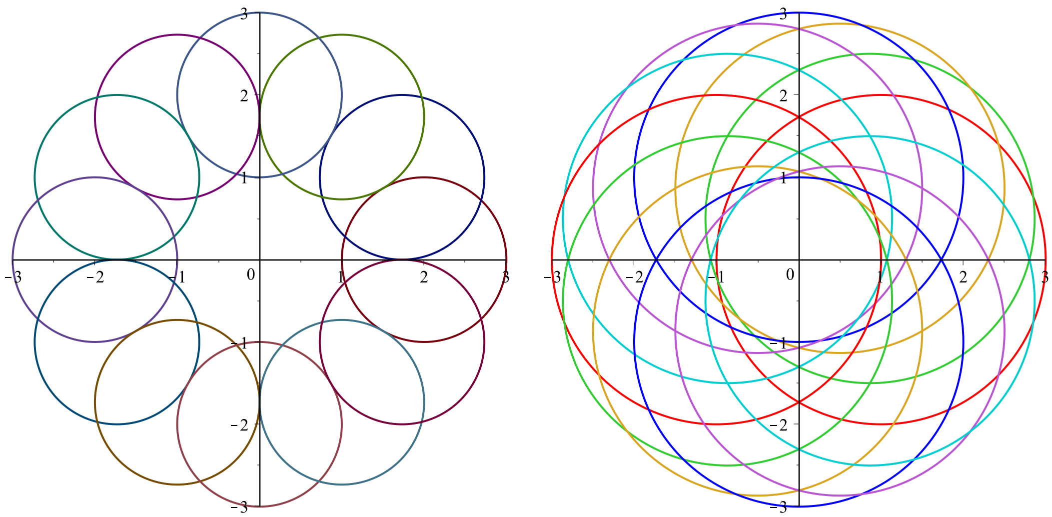

Figure 4.1: Paths of inner integrals in $M(0,\lambda,\mu) = \frac{1}{\Omega_n}\int_{\zeta\in S^{n-1}} I(0 + \lambda\zeta,\mu)\,dw(\zeta)$. On the left picture, $\lambda = 2$ and $\mu = 1$. On the right picture, $\lambda = 1$ and $\mu = 2$.

see from Figure 4.1 that the paths of inner integrals sweep the annulus with inner radius $|\lambda-\mu|$ and outer radius $\lambda-\mu$. So there may be a way to re-write $M(0,\lambda,\mu)$ as a double integral where the paths of inner integrals are concentric circles as in Figure 4.2. John's iterated mean value theorem (stated below as Theorem 4.1) gives the relation between the two ways of integration illustrated in Figure 4.1 and Figure 4.2, respectively. In the latter way of integration, the outer integral is one dimensional integral, which is a factor that makes

compact set.

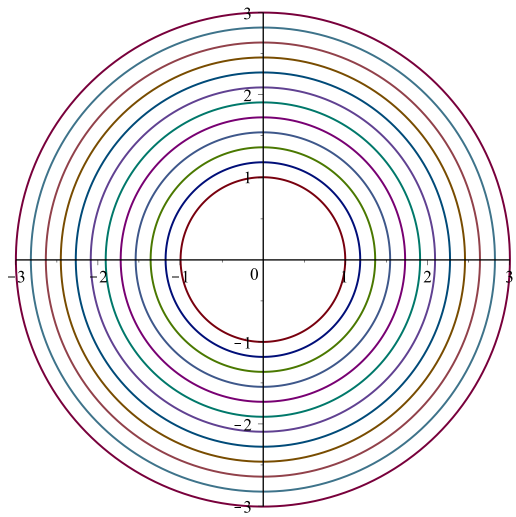

Figure 4.2: Concentric circles centered at the origin lying within the annulus centered at the origin with inner radius $|2-1|$ and outer radius $2+1$. $M(0,\lambda,\mu)$ may be written as a double integral where the paths of inner integrals are such concentric circles.

John's iterated mean value theorem useful.

Theorem 4.1. *(John, [15, Chapter IV, Equation (4.9c)]) Let dw be the standard surface area measure in $\mathbb{R}^n$. Let $f \in C(\mathbb{R}^n)$, $I(x,r) = M^r f(x)$ and $M(x,\lambda,\mu) = \frac{1}{\Omega_n}\int_{\zeta\in S^{n-1}} I(x + \lambda\zeta,\mu)\, dw(\zeta)$. For $\mu,\lambda > 0$ and $r \in [\mu-\lambda,\mu+\lambda]$, let*

$$k(n;\mu,\lambda,r) = [(r+\mu-\lambda)(r+\mu+\lambda)(\lambda+r-\mu)(\lambda-r+\mu)]^{(n-3)/2}. \tag{4.1}$$

For any $x \in \mathbb{R}^n$ and any $\lambda,\mu > 0$,

$$M(x,\lambda,\mu) = \frac{2\Omega_{n-1}}{\Omega_n(2\lambda\mu)^{n-2}}\int_{\mu-\lambda}^{\mu+\lambda} k(\mu,\lambda,r)\, r\, I(x,r)\, dr\ .$$

Let us make some technical remarks about Theorem 4.1.[2] For $\mu,\lambda > 0$ and $r \in$

[2] These technical remarks are in [15, Chapter IV].

$[\mu-\lambda, \mu+\lambda]$, no matter whether $\mu-\lambda$ is positive or negative, $(r+\mu-\lambda)(r+\mu+\lambda)(\lambda+r-\mu)(\lambda-r+\mu) \geq 0$, and thus $k(n;\mu,\lambda,r)$ is real. Furthermore $k(n;\mu,\lambda,r)$ is even in $r \in [-|\mu-\lambda|, |\mu-\lambda|]$. So $\int_{-|\mu-\lambda|}^{|\mu-\lambda|} k(\mu,\lambda,r)\, r\, I(x,r)\, dr = 0$. Thus no matter whether $\mu-\lambda$ is positive or negative, $\int_{\mu-\lambda}^{\mu+\lambda} k(\mu,\lambda,r)\, r\, I(x,r)\, dr = \int_{|\mu-\lambda|}^{\mu+\lambda} k(\mu,\lambda,r)\, r\, I(x,r)\, dr$. In addition, it turns out that we can interchange the order of λ and μ: $M(x,\lambda,\mu) = M(x,\mu,\lambda)$.

Theorem 4.2 below is the hyperbolic space version of John's iterated mean value theorem. The proof of the theorem is in Section 4.1.1.

Theorem 4.2. *Let $G = SO_o(1,n)$ and $K = SO(n)$ so that $G/K = \mathbb{H}^n$. Let $f \in C(\mathbb{H}^n)$, $I(x,r) = M^r f(x)$ and $M(x,\lambda,\mu) = \int_K I(gk\cdot y, \mu) dk$, where y is a point in $\mathbb{H}^n$ with $d(o,y) = \lambda$ and g is an element in G with $g \cdot o = x$. For $\mu, \lambda > 0$ and $r \in [|\mu-\lambda|, \mu+\lambda]$, let*

$$h(n;\lambda,\mu,r) = \left[1 - \cosh^2\lambda - \cosh^2\mu - \cosh^2 r + 2\cosh\lambda\, \cosh\mu\, \cosh r\right]^{(n-3)/2} \tag{4.2}$$

or equivalently

$$h(n;\lambda,\mu,r) = \left[4\, \sinh\frac{\lambda+\mu+r}{2}\, \sinh\frac{\lambda+\mu-r}{2}\, \sinh\frac{\lambda-\mu+r}{2}\, \sinh\frac{-\lambda+\mu+r}{2}\right]^{(n-3)/2} . \tag{4.3}$$

For any $x \in \mathbb{H}^n$ and any $\mu, \lambda > 0$,

$$M(x,\lambda,\mu) = \frac{\Omega_{n-1}}{\Omega_n\, (\sinh\lambda\, \sinh\mu)^{n-2}} \int_{|\mu-\lambda|}^{\mu+\lambda} h(n;\mu,\lambda,r)\, \sinh r\, I(x,r)\, dr\, . \tag{4.4}$$

We obtained this theorem by adapting Helgason's group theoretic version of the proof of John's iterated mean value theorem in Euclidean space ([11, Chaper VI, Section 3]). Before long, we heard that F. Rouvière independently obtained a more general result: the iterated mean value theorem in Riemannian symmetric space of noncompact type and of rank one ([22]). Our result agrees with his result specialized to $\mathbb{H}^n$.

4.1.1 Proof of Theorem 4.2

In this section we prove Theorem 4.2.

Let us start with some notation and basic facts about the hyperboloid model which we will use for the proof.[3]

Let I_k be the k by k identity block matrix, and let

$$B = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} .$$

For $x, y \in \mathbb{R}^{1+n}$, the Lorentzian inner product of x and y is defined as $<< x, y >>= x^T B y = -x_0 y_0 + x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$, where xBy^T is understood by the usual matrix multiplications. The points in $\mathbb{H}^n$ are the points on the upper sheet (with the $+x_0$ chosen to point upward) of the hyperboloid

$$S^+ = \left\{ (x_0, \ldots, x_n)^T \in \mathbb{R}^{1+n} \mid x^T B x = -1, x_0 > 0 \right\} .$$

The Riemannian metric is $ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + \ldots + dx_n^2$, and the geodesic connecting two points x and y in S^+ is the intersection of S^+ and the plane passing through x, y, and the origin 0 of $\mathbb{R}^{1+n}$. The group $G = SO_o(1, n)$ is the subset of $GL_{n+1}(\mathbb{R})$ with any $g \in G$ satisfying $<< x, y >>=<< gx, gy >>$ for any $x, y \in \mathbb{R}^{1+n}$ and $S^+ g = S^+$ for any $g \in G$. It immediately follows that G acts on $S^+ \subset \mathbb{R}^{1+n}$ transitively by matrix multiplication. The origin o in $\mathbb{H}^n$ is the bottom point $(1, 0, \ldots)^T$ on the upper sheet of the hyperboloid. The

[3] Readers unfamiliar to the hyperboloid model may see for example [26], [2], etc.

isotropy subgroup of $SO_o(1,n)$ fixing o is

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \in SO_o(1,n) \;\middle|\; \sigma \in SO(n) \right\} .$$

Now we have $\mathbb{H}^n = G/K = SO_o(1,n)/SO(n)$.

For our proof, let us define a matrix.

Definition 4.3. *For $s \in \mathbb{R}$, h_s is the following matrix.*

$$h_s = \begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & I_{(n-1)\times(n-1)} \end{pmatrix}$$

Let us describe some of the properties of h_s. The matrix h_s is an element of G. The inverse of h_s is h_{-s}. The element $h_s \in G$ moves the origin o to a point on the intersection of S^+ and x_0x_1–plane whose distance to the origin is $|s|$, that is, $d(o, h_s \cdot o) = |s|$. In terms of Lorentzian inner product, we have $<< o, h_s \cdot o >>= -\cosh s$, since $h_s \cdot o = (\cosh s, \sinh s, 0, \ldots)^T$.

Let $x \in \mathbb{H}^n$. With h_s as in Definition 4.3, $a_s = h_s \cdot o$, and $g \in G$ with $g \cdot o = x$, we can write the spherical mean value of $f \in C(\mathbb{H}^n)$ over the sphere centered at x with radius s as follows.

$$M^s f(x) = \int_K f(g\,k \cdot a_s)\,dk = \int_K f(g\,k\,h_s \cdot o)\,dk \tag{4.5}$$

Let $\mu, \lambda > 0$. We will first find out an expression for $M(o, \lambda, \mu)$ for $f \in C(\mathbb{H}^n)$.

Let $a_\lambda = h_\lambda \cdot o$. From (4.5), we have

$$\begin{aligned} M(o,\lambda,\mu) &= \int_K I(k \cdot a_\lambda, \mu)\,dk \\ &= \int_K \int_K f(k h_\lambda k' \cdot a_\mu)\,dk'\,dk \,. \end{aligned} \tag{4.6}$$

By interchanging the order of double integration and then using (4.5) again, we have

$$\begin{aligned} M(o,\lambda,\mu) &= \int_K \int_K f(kh_\lambda k' \cdot a_\mu)\, dk\, dk' \\ &= \int_K \int_K f(k \cdot (h_\lambda k' \cdot a_\mu))\, dk\, dk' \\ &= \int_K M^{d(o,h_\lambda k' \cdot a_\mu)} f(o)\, dk' \,. \end{aligned} \tag{4.7}$$

Since $h_{-\lambda} \in G$,

$$\begin{aligned} d(o, h_\lambda k' \cdot a_\mu) &= d(h_{-\lambda} \cdot o, h_{-\lambda} h_\lambda k' \cdot a_\mu) \\ &= d(h_{-\lambda} \cdot o, k' h_\mu \cdot o) \end{aligned} \tag{4.8}$$

We will find this distance using the law of cosines in hyperbolic space. Consider the triangle connecting the points o, $h_{-\lambda} \cdot o$, and $k' h_\mu \cdot o$ (See Figure 4.3.). Let $\gamma(k')$ be the angle between the geodesic connecting o and a_μ and the geodesic connecting o and $k' \cdot a_\mu$. Let $\alpha(k')$ be the angle between the geodesic connecting o and $k' \cdot a_\mu$ and the geodesic connecting o and $a_{-\lambda}$. Then $\alpha(k') = \pi - \gamma(k')$, and thus $\cos(\alpha(k')) = -\cos(\gamma(k'))$. Now by the law of cosines in hyperbolic space, we have

$$\begin{aligned} \cosh(d(h_{-\lambda} \cdot o, k' h_\mu \cdot o)) &= \cosh\lambda\, \cosh\mu - \sinh\lambda\, \sinh\mu\, \cos(\alpha(k')) \\ &= \cosh\lambda\, \cosh\mu + \sinh\lambda\, \sinh\mu\, \cos(\gamma(k'))\,. \end{aligned} \tag{4.9}$$

For convenience, let

$$F(r) = M^r f(o)\,. \tag{4.10}$$

Then by (4.7), (4.8), (4.9) and (4.10), we have

$$M(o,\lambda,\mu) = \int_K F(\, \cosh^{-1}(\, \cosh\lambda\, \cosh\mu + \sinh\lambda\, \sinh\mu\, \cos(\gamma(k'))\,)\,)\, dk' \,. \tag{4.11}$$

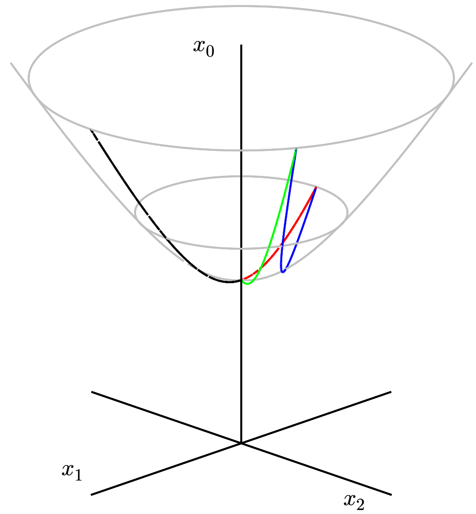

Figure 4.3: Triangle bounded by the green, red, and blue geodesics. The green curve connects o and $k'h_\mu \cdot o$. The red curve connects o and $h_{-\lambda} \cdot o$. The blue curve connects $k'h_\mu \cdot o$ and $h_{-\lambda} \cdot o$. The black curve, which is not part of the triangle, connects o and $h_\mu \cdot o$. The figure shows the case in $\mathbb{H}^2$ with $\lambda = \sinh(1)$, $\mu = \sinh(2)$, and k' being a rotation by $\frac{\pi}{3}$ about the x_0–axis.

Imagine an infinitesimal sphere centered at o in $\mathbb{R}^n$ spanned by x_1–axis, x_2–axis, …, and x_n–axis. Let us align the north pole of the sphere to $\overrightarrow{o\ a_\mu}$. Then $\gamma(k')$ is the vector from o to the north pole and $\overrightarrow{o\ k' \cdot a_\mu}$. Thus by the usual way we change an integral over $SO(n)$ to an integral over a sphere, we can write (4.11) as follows.

$$M(o, \lambda, \mu) = \frac{1}{\Omega_n} \int_0^\pi F(\ \cosh^{-1}(\ \cosh\lambda\ \cosh\mu + \sinh\lambda\ \sinh\mu\ \cos(\theta)\ )\ )\ (\sin\theta)^{n-2}\ \Omega_{n-1}\, d\theta$$

Now let us change the integration variable from θ to t as follows.

$$t = \cosh^{-1}(\ \cosh\lambda\ \cosh\mu + \sinh\lambda\ \sinh\mu\ \cos(\theta)\ )$$

$$\sinh t\, dt = -\sinh\lambda\ \sinh\mu\ \sin(\theta)\, d\theta$$

$$\theta = 0 \Rightarrow \cosh t = \cosh(\lambda+\mu)$$

$$\theta = \pi \Rightarrow \cosh t = \cosh(\lambda-\mu) = \cosh(-\lambda+\mu)$$

The function $\sin\theta$, $0 \le \theta \le \pi$, can be expressed in terms of t as below.

$$\begin{aligned}
\sin\theta &= \sqrt{1-\cos^2\theta} \\
&= \left\{1 - \left(\frac{\cosh t - \cosh\lambda\ \cosh\mu}{\sinh\lambda\ \sinh\mu}\right)^2\right\}^{1/2} \\
&= \frac{\left\{\sinh^2\lambda\ \sinh^2\mu - (\cosh t - \cosh\lambda\ \cosh\mu)^2\right\}^{1/2}}{\sinh\lambda\ \sinh\mu} \\
&= \frac{\left\{(\cosh^2\lambda - 1)\,(\cosh^2\mu - 1) - (\cosh t - \cosh\lambda\ \cosh\mu)^2\right\}^{1/2}}{\sinh\lambda\ \sinh\mu} \\
&= \frac{\left\{1 - \cosh^2\lambda - \cosh^2\mu - \cosh^2 t + 2\cosh\lambda\ \cosh\mu\ \cosh t\right\}^{1/2}}{\sinh\lambda\ \sinh\mu}
\end{aligned}$$

So with h defined in (4.2), we have

$$\begin{aligned}
M(o,\lambda,\mu) &= \frac{\Omega_{n-1}}{\Omega_n\,(\sinh\lambda\ \sinh\mu)^{n-2}} \int_{|\lambda-\mu|}^{\lambda+\mu} h(n;\lambda,\mu,t)\ \sinh t\, F(t)\, dt \\
&= \frac{\Omega_{n-1}}{\Omega_n\,(\sinh\lambda\ \sinh\mu)^{n-2}} \int_{|\lambda-\mu|}^{\lambda+\mu} h(n;\lambda,\mu,t)\ \sinh t\, M^t f(o)\, dt \qquad (4.12)
\end{aligned}$$

By a straightforward shifting argument, we can show the following result for any $x \in \mathbb{H}^n$.

$$M(x,\lambda,\mu) = \frac{\Omega_{n-1}}{\Omega_n\,(\sinh\lambda\ \sinh\mu)^{n-2}} \int_{|\lambda-\mu|}^{\lambda+\mu} h(n;\lambda,\mu,t)\ \sinh t\, M^t f(x)\, dt \qquad (4.13)$$

Now it remains to show that (4.2) and (4.3) are equivalent. This is done by arithmetic operations using some of the basic identities about hyperbolic functions.

4.2 Determination of a function with vanishing mean values by its values on a ball

In this section we consider functions on $\mathbb{R}^n$, $\mathbb{H}^n$, or S^n whose mean values over all spheres with a fixed radius $r > 0$ are known. In [15, Chapter VI], John proved for such $f \in C(\mathbb{R}^3)$ that $f(x)$ is uniquely determined by its values in $B(0, r+\epsilon)$, $\epsilon > 0$. Moreover, he proved for such $f \in C^\infty(\mathbb{R}^3)$ that $f(x)$ is uniquely determined by its values in $\bar{B}(0, r)$. We explain that a generalization of the former claim can be obtained directly from a result by Quinto. The generalization of the latter claim to $\mathbb{R}^n$ was recently done by M. Agranovsky and P. Kuchment ([1, Theorem 10]), but we give a different proof in $\mathbb{R}^5$ that utilizes John's iterated mean value theorem. We also prove a variation of the latter claim in $\mathbb{R}^3$: a function is determined by its values in (instead of $\bar{B}(0, r)$) two split regions: a certain ball and a certain annulus, or certain two annuli ($B(0, a) \cup B(0, 2r-a) \backslash \bar{B}(0, r)$, or $B(0, r) \backslash \bar{B}(0, a) \cup B(0, 2r+a) \backslash \bar{B}(0, 2r)$, $0 < a < r$). The results we obtain are in Corollary 4.4 and Theorems 4.5, 4.6 below. The proofs of our corollaries and theorems are in Sections 4.2.1 – 4.2.3. Prior to that we will provide some comments about our proofs.

Corollary 4.4. *(A corollary of Quinto's [21, Theorem 1.1].) Let M be $\mathbb{R}^n$, $\mathbb{H}^n$, or S^n. Fix $r \in (0, \infty)$ in the case M is $\mathbb{R}^n$ or $\mathbb{H}^n$, fix $r \in (0, \pi)$ in the case M is S^n. Let $f \in C(M)$. Suppose $M^r f(x) = 0$ for any $x \in M$. If there is an $\epsilon > 0$ such that $f(x) = 0$ whenever $x \in B(0, r+\epsilon)$, then $f(x) = 0$ for any $x \in M$.*

Theorem 4.5. *(The case $n = 3$ is in [15, Chapter VI], and the general n case is in [1, Theorem 10].) Let $n = 3, 5$. Fix $r > 0$. Let $f \in C^\infty(\mathbb{R}^n)$. Suppose $M^r f(x) = 0$ for any*

$x \in \mathbb{R}^n$. If $f(x) = 0$ whenever $x \in \bar{B}(0, r)$, then $f(x) = 0$ for any $x \in \mathbb{R}^n$.

Theorem 4.6. *Fix $r > 0$, and $a \in (0, r)$. Let $f \in C^\infty(\mathbb{R}^3)$. Suppose that $M^r f(x) = 0$ for any $x \in \mathbb{R}^n$, and that one of the following two conditions holds.*

(i) $f(x) = 0$ for $x \in B(0, a) \cup B(0, 2r - a) \backslash \bar{B}(0, r)$

(ii) $f(x) = 0$ for $x \in B(0, r) \backslash \bar{B}(0, a) \cup B(0, 2r + a) \backslash \bar{B}(0, 2r)$

Then $f(x) = 0$ for any $x \in \mathbb{R}^n$.

Our proof of Corollary 4.4 utilizes a theorem by Quinto obtained by microlocal analysis ([21]). Though we stated it as Lemma 2.27, let us partly state it here again.

> Let M be a real analytic Riemannian manifold, and let $\mathcal{A} \subset M$ be open and connected. ...... Let P_μ be a Pompeiu transform on geodesic spheres in M of radius r with nowhere zero real analytic weight μ. Assume $f \in \mathcal{D}'(M)$ with $P_\mu f(y) = 0$ for all $y \in \mathcal{A}$ and assume, for some $y_0 \in \mathcal{A}$, the ball $\bar{B}(y_0, r)$ is disjoint from $\operatorname{supp} f$. Then for all $y \in \mathcal{A}$, $\bar{B}(y, r)$ is disjoint from $\operatorname{supp} f$.

Corollary 4.4 is a direct consequence of it.

Now let us comment about our proofs of Theorem 4.5 and Theorem 4.6. The regions where the value of the function f is zero in the hypothesis of Theorem 4.5 and the hypotheses (i) and (ii) in Theorem 4.6 are plotted in Figure 4.4. Whereas we used the microlocal analysis result in [21] for proving Corollary 4.4, we cannot use it for proving Theorem 4.5 and Theorem 4.6 since the hypothesis about antipodal points in [21, Proposition 2.1] is not satisfied.

We prove them by adapting John's proof in $\mathbb{R}^3$ ([15, Chapter VI]) which can be roughly summarized as follows. In his proof he fixes $r = 1$.

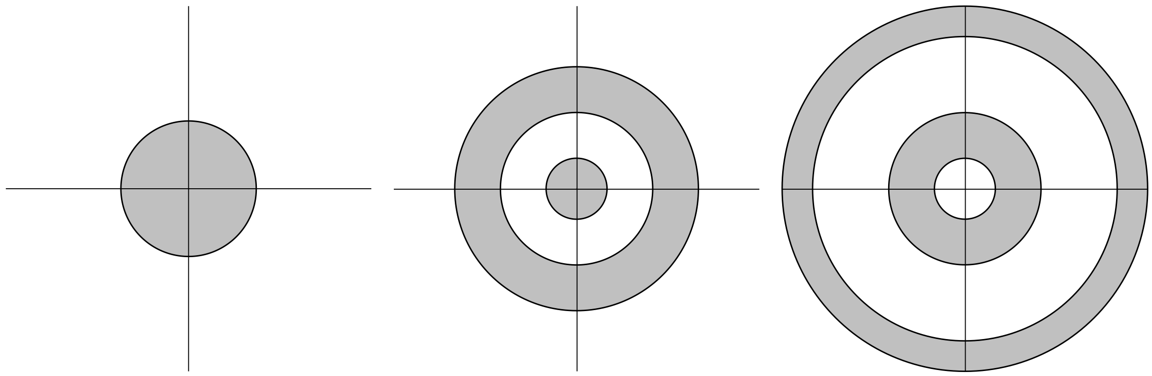

Figure 4.4: The shaded regions in the three figures from left to right are the regions where the $f(x) = 0$ in the hypothesis of Theorem 4.5, in the hypotheses (i) in Theorem 4.6, and in the hypotheses (ii) in Theorem 4.6, respectively.

(1) By using the iterated mean value theorem (Theorem 4.1) and the condition that $f(x) = 0$ whenever $x \in \bar{B}(0,1)$, John showed that $M^t(D^\alpha f)(0) = 0$ for any multi-index α, and any $t \in \mathbb{R}$.

(2) By using the divergence theorem, he showed that $M^t(Pf)(0) = 0$ for any polynomial P and any $t \in \mathbb{R}$.

(3) From (2), he concluded that $f(x) = 0$ on $\mathbb{R}^3$.

To explain an aspect of the adaptation of the step (1) above in $\mathbb{R}^n$, $n = 3, 5$, we present a differentiation step with respect to λ below.

$$\begin{aligned} 0 &= \int_{\lambda-1}^{\lambda+1} Q(\lambda, s)\, s\, F(s)\, ds \\ &\Downarrow \\ 0 &= Q(\lambda, \lambda+1)\,(\lambda+1)\,F(\lambda+1) - Q(\lambda, \lambda-1)\,(\lambda-1)\,F(\lambda-1) \\ &\quad + \int_{\lambda-1}^{\lambda+1} \frac{\partial}{\partial\lambda} Q(\lambda, s)\, s\, F(s)\, dr\,, \qquad (4.14) \end{aligned}$$

where $Q(\lambda, s)$ is the kernel in the iterated mean value formula, $F(r) = M^1 f(0)$, and f is a function satisfying the hypothesis in Theorem 4.5 with $r = 1$. Note that $Q(\lambda, s) = [\, p(\lambda, s)\,]^{(n-3)/2}$ where $p(\lambda, s)$ is a polynomial in λ and s (See (4.1) for the explicit formula. $Q(\lambda, s)$ here corresponds to $k(n; \lambda, 1, s)$ in (4.1).)

By (4.14) or by further differentiating (4.14), we obtain a delay differential equation

$$D_2 F(s+2) - D_0 F(s) = 0\,,$$

where D_2 and D_0 are certain linear ordinary differential operator. Thus if we know that $F(s) = 0$ for s in an interval $[a, b]$, then we have linear homogeneous ordinary differential equation

$$D_2 F(s) = 0$$

on $[a+2, b+2]$. In $\mathbb{R}^3$, D_2 and D_0 are so simple that we have a simple delay equation

$$F(s+2) - F(s) = 0\,.$$

Thus if we know that $F(s) = 0$ for s in an interval $[a, b]$, then we have

$$F(s) = 0$$

on $[a+2, b+2]$.

4.2.1 Proof of Corollary 4.4

In this section we explain that Corollary 4.4 is a direct consequence of Quinto's [21, Theorem 1.1].

Lemma 4.7. *(A special case of Quinto's [21, Theorem 1.1].) Let M be a real analytic Riemannian manifold, I_y denote the injectivity radius of the exponential map at $y \in M$. Fix $r > 0$. Suppose $I_y > r$ for any $y \in M$. Let $f \in \mathcal{D}'(M)$. Suppose $M^r f(x) = 0$ for any $x \in M$. Fix $o \in M$. If there is an $\epsilon > 0$ such that $f(x) = 0$ whenever $d(x, o) < r + \epsilon$, then $f(x) = 0$ for any $x \in M$.*

Proof. We will use Lemma 2.27. Let $\mathcal{A} = M$. We can let the weight μ be an appropriate nonzero constant function, and let the Pompeiu transform P_μ be M^r. Then $P_\mu(y) = 0$ for any $y \in \mathcal{A} = M$.

Let $y_0 = o$. Since $f(x) = 0$ whenever $d(x, o) < r + \epsilon$, the ball $\bar{B}(y_0, r)$ is disjoint from $\operatorname{supp} f$. Thus by Lemma 2.27, we can conclude that for any $y \in \mathcal{A}$ the ball $\bar{B}(y, r)$ is disjoint from $\operatorname{supp} f$. Thus $f(y) = 0$ for any y in $\mathcal{A} = M$. □

For convenience, let us repeat the statement of Corollary 4.4 here.

> Let M be $\mathbb{R}^n$, $\mathbb{H}^n$, or S^n. Fix $r \in (0, \infty)$ in the case M is $\mathbb{R}^n$ or $\mathbb{H}^n$, fix $r \in (0, \pi)$ in the case M is S^n. Let $f \in C(M)$. Suppose $M^r f(x) = 0$ for any $x \in M$. If there is an $\epsilon > 0$ such that $f(x) = 0$ whenever $x \in B(0, r + \epsilon)$, then $f(x) = 0$ for any $x \in M$.

Now let us prove this corollary. Let I_y denote the injectivity radius of the exponential map at $y \in M$. In the case M is $\mathbb{R}^n$ or $\mathbb{H}^n$, $I_y = \infty$ for any $y \in M$. In the case M is S^n, $I_y = \pi$ for any $y \in M$. So we have $I_y > r$ for any $y \in M$. Since $f \in C(M)$, $f \in \mathcal{D}'(M)$. Hence by Lemma 4.7, $f(x) = 0$ for any $x \in M$.

4.2.2 Proof of Theorem 4.5

In this section we prove Theorem 4.5.

Let us explain which lemmas are used in the steps (1), (2), and (3) of Johns proof in $\mathbb{R}^3$ which we mentioned in section 4.2. Lemmas 4.9 – 4.12 are used in the step (1). Lemmas 4.13 and 4.14 are used in the step (2). Lemma 4.15 is used in the step (3).

Lemma 4.8. *Let $n \geq 2$, and $R > 1$. Let $g \in C^\infty(B(0,R))$ and α be a multi-index. Let $g_\alpha(x)$ denote $D^\alpha g(x)$. If $M^1 g(x) = 0$ for any $x \in \mathbb{R}^n$ with $S(x,1) \subset B(0,R)$ and $g(x) = 0$ for any $x \in \bar{B}(0,1)$, then $M^1 g_\alpha(x) = 0$ for any $x \in \mathbb{R}^n$ and $g_\alpha(x) = 0$ for any $x \in \bar{B}(0,1)$.*

Proof. Since $M^1 g(x) = 0$ for any $x \in \mathbb{R}^n$ with $S(x,1) \subset B(0,R)$, $D^\alpha M^1 g(x) = 0$ for any $x \in \mathbb{R}^n$. On the other hand, since we can interchange the order of integration and differentiation, $D^\alpha M^1 g(x) = M^1 g_\alpha(x)$. Thus we have $M^1 g_\alpha(x) = 0$ for any $x \in \mathbb{R}^n$ with $S(x,1) \subset B(0,R)$.

Since $g(x) = 0$ for any $x \in \bar{B}(0,1)$, $g_\alpha(x) = 0$ for any $x \in \bar{B}(0,1)$. □

The lemma below shows the difference between $\mathbb{R}^5$ and $\mathbb{R}^3$ in John's approach using the iterated mean value formula ([15]). In $\mathbb{R}^3$ we obtain a delay equation, but in $\mathbb{R}^5$ we obtain a delay differential equation as shown in the lemma.

Lemma 4.9. *Let $k(n;\mu,\lambda,r)$ be as in Theorem 4.1. Let $a < b$, and $F(r) \in C^3((a-1,b+1))$. Let*

$$\begin{aligned} G_p(\lambda) &= 8\lambda(\lambda+1)^2\, F^{(3)}(\lambda+1) + 8(-\lambda^3+9\lambda^2+14\lambda+4)\, F''(\lambda+1) \\ &\quad +8(-9\lambda^2+15\lambda+16)\, F'(\lambda+1) - 120\lambda\, F(\lambda+1)\,, \end{aligned} \tag{4.15}$$

and

$$G_m(\lambda) = 8\lambda(\lambda-1)^2\, F^{(3)}(\lambda-1) + 8(\lambda^3+9\lambda^2-14\lambda+4)\, F''(\lambda-1)$$
$$+8(9\lambda^2+15\lambda-16)\, F'(\lambda-1) + 120\lambda\, F(\lambda-1)\,.$$

If

$$\int_{\lambda-1}^{\lambda+1} k(5;\lambda,1,r)\, r\, F(r)\, dr = 0 \quad \textit{for } \lambda \in (a,b), \tag{4.16}$$

then

$$G_p(\lambda) + G_m(\lambda) = 0 \quad \textit{for } \lambda \in (a,b). \tag{4.17}$$

Proof. We will introduce functions $B_1(\lambda)$, $B_2(\lambda)$, ..., $B_5(\lambda)$, and $(\frac{\partial}{\partial\lambda})^j\, k(5;\lambda,1,r)$ below. For readability, we will present their explicit expressions in Section C.1.

By differentiating each side of (4.16) with respect to λ, we obtain

$$B_1(\lambda) + \int_{\lambda-1}^{\lambda+1} (\frac{\partial}{\partial\lambda})^1\, k(5;\lambda,1,r)\, r\, F(r)\, dr = 0\,, \tag{4.18}$$

where $B_1(\lambda) = [k(5;\lambda,1,r)\, r\, F(r)]_{r=\lambda+1} - [k(5;\lambda,1,r)\, r\, F(r)]_{r=\lambda-1}$. By differentiating each side of (4.18), we obtain

$$B_2(\lambda) + \int_{\lambda-1}^{\lambda+1} (\frac{\partial}{\partial\lambda})^2\, k(5;\lambda,1,r)\, r\, F(r)\, dr = 0\,,$$

where $B_2(\lambda) = \frac{d}{d\lambda}B_1(\lambda) + \left[(\frac{\partial}{\partial\lambda})^1\, k(5;\lambda,1,r)\, r\, F(r)\right]_{r=\lambda+1} - \left[(\frac{\partial}{\partial\lambda})^1\, k(5;\lambda,1,r)\, r\, F(r)\right]_{r=\lambda-1}$. By repeating the differentiation, we obtain the following equations.

$$B_3(\lambda) + \int_{\lambda-1}^{\lambda+1} (\frac{\partial}{\partial\lambda})^3\, k(5;\lambda,1,r)\, r\, F(r)\, dr = 0 \tag{4.19}$$

$$B_4(\lambda) + \int_{\lambda-1}^{\lambda+1} (\frac{\partial}{\partial\lambda})^4\, k(5;\lambda,1,r)\, r\, F(r)\, dr = 0$$

Since $(\frac{\partial}{\partial\lambda})^4\, k(5;\lambda,1,r)$ is a constant, we have an equation without integral.

$$B_5(\lambda) = 0 \tag{4.20}$$

By looking at the explicit formula for B_5 in (C.1) in Section C.1, we can see that $B_5(\lambda) = G_p(\lambda) + G_m(\lambda)$. Thus (4.20) means (4.17). □

Lemma 4.10. *Let $F(t) \in C^3((1,\infty))$. Let G_p be as defined in (4.15). The general solution to the differential equation $G_p(t-1) = 0$ is*

$$\begin{aligned} F(t) &= C_1\, t^{-3} + C_2\,(t-1)\, e^t\, t^{-3} \\ &\quad + C_3\,\{\mathrm{Ei}(1,t-1)e^{t-1}(t-1)^3 - t^2 + 3t - 1\}(t-1)^{-2}t^{-3}\,, \end{aligned} \tag{4.21}$$

where $\mathrm{Ei}(a,z)$ denotes the exponential integral $\int_1^\infty e^{-zk}k^{-a}dk$ for $\mathrm{Re}(z) > 0$, and C_1, C_2, and C_3 are arbitrary constants.

Proof. $G_p(t-1) = 0$ is a linear third order ordinary differential equation in $F(t)$ with polynomial coefficients. By straightforward differentiation, we can verify that each of the three terms in (4.21) is a solution. □

In the lemma below, the number $3/2$ is chosen as a number slightly larger than 1.

Lemma 4.11. *Let $f \in C^3(B(0,3/2))$. If $M^1 f(x) = 0$ for any $x \in \mathbb{R}^5$ with $S(x,1) \subset B(0,3/2)$ and $f(y) = 0$ for any $y \in \bar{B}(0,1)$, then $M^s f(0) = 0$ for any $s \in [0,3/2)$.*

Proof. Let $I(x,r) = M^r f(x)$ and $M(x,\lambda,\mu) = \frac{1}{\Omega_n}\int_{\zeta\in S^{n-1}} I(x+\lambda\zeta,\mu)\, dw(\zeta)$, where dw denotes the standard surface area measure in $\mathbb{R}^n$.

By the assumption that $f(x) = 0$ for $x \in \bar{B}(0,1)$, we have

$$I(0, r) = 0, \text{ for any } |r| \leq 1. \tag{4.22}$$

On the other hand, since $M^1 f(x) = I(x,1) = 0$ for any $x \in \mathbb{R}^5$ with $S(x,1) \subset B(0,3/2)$, $M(x, \lambda, 1) = 0$ for any x and λ with $d(0,x) + |\lambda| + 1 < 3/2$. Thus by Theorem 4.1,

$$\int_{\lambda-1}^{\lambda+1} k(5; \lambda, 1, r)\, r\, I(x,r)\, dr = 0 \quad d(0,x) + |\lambda| + 1 < 3/2. \tag{4.23}$$

In addition, at any fixed $x \in \mathbb{R}^5$, $I(x,r)$ is in $C^3(\mathbb{R})$ as a function of r, since $f \in C^3(\mathbb{R}^5)$.[4] For convenience, let

$$F(r) = I(0,r)\,.$$

Then by (4.23) and Lemma 4.9, we have

$$G_p(\lambda) + G_m(\lambda) = 0 \qquad \text{for any } \lambda \in (-1/2, 1/2), \tag{4.24}$$

where G_p is as defined in Lemma 4.9.

From (4.22), we have $F(t) = 0$ for $t \in [-1, 1]$. So by (4.24), we have

$$G_p(t-1) = 0 \qquad \text{for any } t \in [1, 3/2)\,. \tag{4.25}$$

We will show that $F(t) = 0$ for $t \in [1, 3/2)$

Due to (4.25) and Lemma 4.10, $F(t)$ is of the form in (4.21) for $t \in [1, 3/2)$. Since $F(t) = I(0,t)$ is continuous at $t = 1$ while the factor $\{\mathrm{Ei}(1, t-1)e^{t-1}(t-1)^3 - t^2 + 3t - 1\}$ in the third term in (4.21) goes to 1 as $t \to 1$ and the factor $(t-1)^{-2}t^{-3}$ in the same term blows up as $t \to 1$, $C_3 = 0$. Hence we have $F(t) = C_1\, t^{-3} + C_2\,(t-1)\, e^t\, t^{-3}$ for $t \in [1, 3/2)$.

[4] $\frac{\partial}{\partial r} I(x,r) = \lim_{h\to 0} \frac{I(x,r+h) - I(x,r)}{h} = \int_{y \in S^{n-1}} \nabla f(x + ry) \cdot y\, dw(y)$

By the assumption $F(t) = 0$ for $t \in [-1,1]$, $F(1) = 0$ and $F'(1) = 0$. With these initial conditions, we obtain $F(t) = 0$ for $t \in [1, 3/2)$.

Now by the evenness of $I(x,r)$ as a function of r, we have $F(t) = 0$ for $t \in (-3/2, 3/2)$. Since $F(t) = I(0,t) = M^t f(0)$, the proof is done. □

Lemma 4.12. *Let $R > 1$, and $B(0,R)$ be a ball in $\mathbb{R}^5$. Let $g \in C^\infty(B(0,R))$ and α be a multi-index. Let $g_\alpha(x)$ denote $D^\alpha g(x)$.*

If $M^1 g(x) = 0$ for all $x \in \mathbb{R}^5$ with $S(x,1) \subset B(0,R)$ and $g(x) = 0$ for all $x \in \bar{B}(0,1)$, then $M^s g_\alpha(0) = 0$ for any $s \in (-R,R)$.

Proof. By Lemma 4.8, $M^1 g_\alpha(x) = 0$ for any $x \in \mathbb{R}^5$ with $S(x,1) \subset B(0,R)$ and $g_\alpha(x) = 0$ for any $x \in \bar{B}(0,1)$. So by Lemma 4.11, $M^s g_\alpha(0) = 0$ for any $s \in \mathbb{R}$. □

Lemma 4.13. *Let $n \geq 2$, $R > 1$, $B(0,R)$ be a ball in $\mathbb{R}^n$, and $g \in C^1(B(0,R))$. Let $j \in \{1,2,\ldots,n\}$, $g_j(x) = \frac{\partial}{\partial x_j} g(x)$ and $h_j(x) = x_j g(x)$. If $M^s g(0) = 0$ and $M^s g_j(0) = 0$ for all $s \in (-R,R)$, then $M^s h_j(0) = 0$ for all $s \in (-R,R)$.*

Proof. Fix $r \in (0,R)$ and $x \in \mathbb{R}^n$. By applying the divergence theorem to the vector field having g as jth component and zero function as other components over the ball of radius r centered at x, we have

$$\int_{\xi \in S^{n-1}} g(x + r\xi)\, \xi_j\, r^{n-1}\, dw(\xi) = \int_0^r \int_{\xi \in S^{n-1}} g_j(x + t\xi)\, t^{n-1}\, dw(\xi)\, dt\,,$$

where dw denotes the standard surface area measure on $S^{n-1} \subset \mathbb{R}^n$. Thus we have

$$
\begin{aligned}
r^{n-1}\Omega_n\, M^r h_j(x) &= \int_{\xi\in S^{n-1}} (x_j + r\xi_j)\, g(x+r\xi)\ r^{n-1}\, dw(\xi) \\
&= x_j \int_{\xi\in S^{n-1}} g(x+r\xi)\ r^{n-1}\, dw(\xi) \\
&\quad + r\int_0^r \int_{\xi\in S^{n-1}} g_j(x+t\xi)\, t^{n-1}\, dw(\xi)\, dt \\
&= x_j r^{n-1}\Omega_n\, M^r g(x) + r\int_0^r t^{n-1}\Omega_n\, M^t g_j(x)\, dt \qquad (4.26)
\end{aligned}
$$

By letting $x = 0$ and utilizing the assumption that $M^s g(0) = 0$ and $M^s g_j(0) = 0$ for any $s \in (-R, R)$ in (4.26), we obtain $M^r h_j(0) = 0$. Since this argument holds for any $r \in (0, R)$, we have $M^r h_j(0) = 0$ for any $r \in (-R, R)$ □

Lemma 4.14. *Let $n \geq 2$, and $R > 1$. Let $f \in C^\infty(B(0,R))$. For a multi-index α, let $f_\alpha(x)$ denote $D^\alpha f(x)$. For a polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$, let Pf denote the function $P(x)f(x)$. If $M^s f_\alpha(0) = 0$ for any multi-index α and any $s \in (-R, R)$, then $M^t(Pf_\alpha)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$, any multi-index α, and any $t \in (-R, R)$.*

Proof. For a polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$, let $\deg P$ be the degree of P.

Let $m = 0$. By the hypothesis of this lemma, $M^t(Pf_\alpha)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$ with $\deg P \leq m$, any multi-index α, and any $t \in (-R, R)$.

Let $m = 0, 1, 2, \ldots$. Suppose $M^t(Pf_\alpha)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$ with $\deg P \leq m$, any multi-index α, and any $t \in (-R, R)$. Let Q be a polynomial in the components of $x \in \mathbb{R}^n$ with $\deg Q \leq m + 1$. We can write Q as follows.

$$Q(x) = \sum_{j=1}^{n} x_j P_j(x)$$

Here $P_j(x)$ is a polynomial in the components of $x \in \mathbb{R}^n$ with $\deg P_j \leq m$. Let β be a multi-index. We can apply Lemma 4.13 with $g(x) = P_j(x) f_\beta(x)$ and obtain $M^s h_j(0) = 0$ for all $s \in \mathbb{R}$, where $h_j(x) = x_j P_j(x) f_\beta(x)$. This argument holds for any $j \in \{1, 2, \ldots, n\}$. Thus we have $M^s(Qf_\beta)(0) = 0$ for all $s \in (-R, R)$. Thus we have $M^t(Pf_\alpha)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$ with $\deg P \leq m + 1$, any multi-index α, and any $t \in (-R, R)$. $\square$

Lemma 4.15. *Let $g \in C(\mathbb{R}^n)$. For a polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$, let Pg denote the function $P(x)g(x)$. Fix $s > 0$. If $M^s(Pg)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$, then $g(y) = 0$ for any $y \in S(0, s)$.*

Proof. Consider all the polynomials in the components of $x \in \mathbb{R}^n$ with coefficients in $\mathbb{C}$. Let A be the class of these polynomials restricted to $S(0, s)$. Then by the complex version of the Stone-Weierstrass theorem, A is dense in $C(S(0, s))$.

Let g be a function satisfying the hypothesis of this Lemma. Suppose there is an $x \in S(0, s)$ satisfying $g(x) \neq 0$. Then we can show a contradiction to the condition that $M^s(Pg)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$, by using a bump function supported on a certain neighborhood of $x_o \in S(0, s)$ with $g(x_o) > 0$ and a polynomial which is very close to that bump function in the sense of supremum norm.Thus we have $g(x) = 0$ for any $x \in S(0, s)$. $\square$

Lemma 4.5 below is Theorem 4.5 specialized to $r = 1$.

Lemma 4.16. *[The case $n = 3$ is in [15, Chapter VI]] Let $n = 3, 5$. Let $f \in C^\infty(\mathbb{R}^n)$. Suppose $M^1 f(x) = 0$ for all $x \in \mathbb{R}^n$. If $f(x) = 0$ for all $x \in \bar{B}(0, 1)$, then $f(y) = 0$ for any*

$y \in \mathbb{R}^n$.

Proof. The proof for the case $n = 3$ is in [15, Chapter VI].

Now consider the case $n = 5$. By the hypothesis of this lemma, we are able to apply Lemma 4.11 and Lemma 4.12, and obtain $M^s f_\alpha(0) = 0$ for any multi-index α and for any $s \in (-3/2, 3/2)$, where $f_\alpha(x) = D^\alpha f(x)$. Now we can apply Lemma 4.14, and obtain $M^t(Pf)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$ and for any $t \in (-3/2, 3/2)$. Now we apply Lemma 4.15 and obtain $f(y) = 0$ for any $y \in B(0, 3/2)$. Finally we apply Corollary 4.4 and obtain $f(y) = 0$ for any $y \in \mathbb{R}^n$. □

From Lemma 4.16 above, we obtain Theorem 4.5 by scaling: $h(x) = f(rx)$.

4.2.3 Proof of Theorem 4.6

In this section we prove Theorem 4.6.

Lemma 4.17. *Let $n \geq 2$, $a \in (0,1)$, and $R > 1$. Let $f \in C^\infty(B(0,R))$ and α be a multi-index. Let $f_\alpha(x)$ denote $D^\alpha f(x)$. Suppose that $M^1 f(x) = 0$ for any $x \in \mathbb{R}^n$ with $S(x,1) \subset B(0,R)$, and that one of the following two conditions holds.*

(i) $f(x) = 0$ for $x \in B(0,a) \cup B(0, 2-a) \backslash \bar{B}(0,1)$

(ii) $f(x) = 0$ for $x \in B(0,1) \backslash \bar{B}(0,a) \cup B(0, 2+a) \backslash \bar{B}(0,2)$

Then $M^1 f_\alpha(x) = 0$ for any $x \in \mathbb{R}^n$ with $S(x,1) \subset B(0,R)$. Moreover, (i–1) below holds in the case of (i), and (ii-2) below holds in the case of (ii).

(i–1) $f_\alpha(x) = 0$ for any $x \in B(0,a) \cup B(0,2-a) \backslash \bar{B}(0,1)$

(ii–2) $f_\alpha(x) = 0$ *for any* $x \in B(0,1)\backslash\bar{B}(0,a) \;\cup\; B(0,2+a)\backslash\bar{B}(0,2)$

Proof. Straightforward. □

Lemma 4.18. *Let* $a \in (0,1)$*, and* $R = 2+a$*. Let* $f \in C(B(0,R))$*. Suppose that* $M^1 f(x) = 0$ *for any* $x \in \mathbb{R}^3$ *with* $S(x,1) \subset B(0,R)$*, and that one of the following two conditions holds.*

(i) $f(x) = 0$ *for* $x \in B(0,a) \;\cup\; B(0,2-a)\backslash\bar{B}(0,1)$

(ii) $f(x) = 0$ *for* $x \in B(0,1)\backslash\bar{B}(0,a) \;\cup\; B(0,2+a)\backslash\bar{B}(0,2)$

then $M^s f(0) = 0$ *for any* $s \in [0,R)$*.*

Proof. Let $I(x,r) = M^r f(x)$ and $M(x,\lambda,\mu) = \frac{1}{\Omega_n}\int_{\zeta\in S^{n-1}} I(x+\lambda\zeta,\mu)\,dw(\zeta)$, where dw denotes the standard surface area measure in $\mathbb{R}^n$.

Since $M^1 f(x) = I(x,1) = 0$ for any $x \in \mathbb{R}^3$ with $S(x,1) \subset B(0,R)$, $M(x,\lambda,1) = 0$ for any x and λ with $d(0,x) + |\lambda| + 1 < R$. Thus by Theorem 4.1,

$$\int_{\lambda-1}^{\lambda+1} r\, I(x,r)\, dr = 0 \quad d(0,x) + |\lambda| + 1 < R. \tag{4.27}$$

For convenience, let

$$F(r) = I(0,r)\,.$$

Then by (4.27), we have

$$(\lambda+1)\, F(\lambda+1) - (\lambda-1)\, F(\lambda-1) = 0 \qquad \text{for any } \lambda \in (-R+1, R-1). \tag{4.28}$$

First consider the case (i). By condition (i), we have $F(t) = 0$ for $t \in (-a,a) \cup (1, 2-a)$. Thus by (4.28), we additionally have

$$F(t) = 0 \qquad \text{for any } t \in (2-a, 2+a) \cup (-1, -a)\,.$$

Using the evenness of F, we additionally have $F(t) = 0$ for $t \in (a, 1)$. Hence we have $F(t) = 0$ for $t \in (-1, 2 + a)$.

Next consider the case (ii). By condition (ii), we have $F(t) = 0$ for $t \in (-1, -a) \cup (a, 1) \cup (2, 2 + a)$. Thus by (4.28), we additionally have

$$F(t) = 0 \quad \text{for any } t \in (0, a) \cup (1, 2 - a)\ .$$

Using the evenness of F, we additionally have $F(t) = 0$ for $t \in (-a, 0)$. Using (4.28), we additionally have $F(t) = 0$ for $t \in (2 - a, 2)$. Hence we have $F(t) = 0$ for $t \in (-1, 2 + a)$. □

Lemma 4.19. *Let $a \in (0, 1)$, and $R = 2 + a$. Let $f \in C^\infty(B(0, R))$ and α be a multi-index. Let $f_\alpha(x)$ denote $D^\alpha f(x)$. Suppose that $M^1 f(x) = 0$ for any $x \in \mathbb{R}^3$ with $S(x, 1) \subset B(0, R)$, and that one of the following two conditions holds.*

(i) $f(x) = 0$ for $x \in B(0, a) \ \cup \ B(0, 2 - a) \backslash \bar{B}(0, 1)$

(ii) $f(x) = 0$ for $x \in B(0, 1) \backslash \bar{B}(0, a) \ \cup \ B(0, 2 + a) \backslash \bar{B}(0, 2)$

Then $M^s f_\alpha(0) = 0$ for any $s \in (-R, R)$.

Proof. From Lemma 4.17 and Lemma 4.18, we have the conclusion of this lemma. □

Lemma 4.20. *Fix $a \in (0, 1)$. Let $f \in C^\infty(\mathbb{R}^3)$. Suppose that $M^1 f(x) = 0$ for any $x \in \mathbb{R}^n$, and that one of the following two conditions holds.*

(i) $f(x) = 0$ for $x \in B(0, a) \ \cup \ B(0, 2 - a) \backslash \bar{B}(0, 1)$

(ii) $f(x) = 0$ for $x \in B(0, 1) \backslash \bar{B}(0, a) \ \cup \ B(0, 2 + a) \backslash \bar{B}(0, 2)$

Then $f(x) = 0$ for any $x \in \mathbb{R}^n$.

Proof. By Lemma 4.19, we have $M^s f_\alpha(0) = 0$ for any multi-index α and any $s \in (-R, R)$, $R = 2 + a$. Thus by Lemma 4.14, we obtain $M^t(P f_\alpha)(0) = 0$ for any polynomial $P(x)$ in the components of $x \in \mathbb{R}^n$, any multi-index α, and any $t \in (-R, R)$. So by Lemma 4.15, $f(x) = 0$ for any $x \in B(0, R)$. Finally, we apply Corollary 4.4 and obtain $f(x) = 0$ for any $x \in \mathbb{R}^n$. □

From Leamma 4.20, Theorem 4.6 by scaling: $h(x) = f(rx)$.

4.3 A solution to inhomogeneous spherical mean value equation in $\mathbb{H}^3$

John found a solution to the inhomogeneous spherical mean value equation $M^1 f(x) = g(x)$ with given function g in $\mathbb{R}^3$ ([15]). By adapting his proof, we obtain a solution in $\mathbb{H}^3$. Our result is in Theorem 4.24 below.

Let us first state Johns result.

Theorem 4.21. *(John, [15, Chapter VI, equation (6.35)]) Let $g \in C^3(\mathbb{R}^3)$. Suppose $g(x)$ and its derivatives of order ≤ 3 are $O(|x|^{-\alpha})$ with $\alpha > 2$.*[5] *Let L denote the Laplace-Beltrami operator in $\mathbb{R}^3$. The function f below is a solution to the equation $M^1 f(x) = g(x)$ of class $C^1(\mathbb{R}^3)$.*

$$\begin{aligned} f(x) &= -2\sum_{j=0}^{\infty}(2j+1)\, L\, M^{2j+1}g(x) \\ &= -2\sum_{j=0}^{\infty}(2j+1)\, M^{2j+1}(Lg)(x) \end{aligned}$$

[5]In [15], it reads that the function and its derivatives "go to 0 at least like $|x|^{-3}$" as $|x| \to \infty$. Here we want to allow g to be in a little bit larger class of functions. We find that the proof in [15] works when the functions and its derivatives are $O(|x|^{-\alpha})$ with $\alpha > 2$.

Our Theorem 4.24 is a hyperbolic space version of the above result by John. We will provide lemmas needed for Theorem 4.24, and then the theorem.

Lemma 4.22. *(Helgason, [10, Chapter II, Exercises and Further Results, F.2]) Let n be odd, and L be the Laplace-Beltrami operator on $\mathbb{H}^n$. Let $u_0(x), u_1(x) \in C^2(\mathbb{H}^n)$. A function $u(x,s) \in C^2(\mathbb{H}^n \times \mathbb{R})$ satisfies*

$$\left(L + \left(\frac{n-1}{2}\right)^2\right) u(x,s) = \frac{\partial^2}{\partial s^2} u(x,s), \quad u(x,0) = u_0(x), \text{ and } \left.\frac{\partial}{\partial s} u(x,s)\right|_{s=0} = u_1(x),$$

if and only if

$$\begin{aligned} u(x,s) &= \frac{1}{2\left(\frac{1}{2}(n-3)\right)!} \frac{\Omega_n}{\Omega_{n-1}} \left[\frac{\partial}{\partial s}\left(\frac{\partial}{\partial(2\cosh s)}\right)^{(n-3)/2} \left[\sinh^{n-2} s\, (M^s u_0)(x)\right]\right. \\ &\quad \left. + \left(\frac{\partial}{\partial(2\cosh s)}\right)^{(n-3)/2} \left[\sinh^{n-2} s\, (M^s u_1)(x)\right]\right]. \end{aligned} \tag{4.29}$$

Lemma 4.23. *(Helgason, [10, Chapter II, Proposition 4.12]) Let G/K be a symmetric space of rank one, L be the Laplace-Beltrami operator. Then for any $f \in C^\infty(G/K)$,*

$$M^r(Lf)(x) = L(M^r f)(x)\,.$$

Theorem 4.24. *Let L be the Laplace-Beltrami operator on $\mathbb{H}^3$. Let $g \in C^3(\mathbb{H}^3)$. Suppose $g(x)$ and its derivatives of order ≤ 3 go to 0 at least like $e^{-d(o.x)}\, d(o,x)^{-\alpha}$ with $\alpha > 2$ as $d(o,x) \to \infty$. The function f defined below is a soloution to the equation $M^1 f(x) = g(x)$.*

$$\begin{aligned} f(x) &= -2\sinh(1) \sum_{j=0}^{\infty} \sinh(2j+1)\, (L+1) M^{2j+1} g(x) \\ &= -2\sinh(1) \sum_{j=0}^{\infty} \sinh(2j+1)\, M^{2j+1}\left((L+1)g\right)(x) \end{aligned}$$

Proof. Let us define functions $\overline{I}$ and $\overline{u}$ on $\mathbb{H}^3 \times \mathbb{R}$ as follows.

$$\overline{I}(x,t) = M^t g(x)$$

$$\overline{u}(x,t) = \sinh t\, \overline{I}(x,t)$$

Then we can rewrite $\overline{u}(x,t)$ as follows, where $v_0(x)$ denotes the zero function ($v_0(x) = 0$ for any $x \in \mathbb{H}^3$), and $v_1(x) = \frac{\partial}{\partial t}\overline{u}(x,t)\big|_{t=0} = g(x)$.

$$\begin{aligned} \overline{u}(x,t) &= \sinh t\, \overline{I}(x,t) \\ &= \frac{\partial}{\partial t}\left[\sinh t\, M^t v_0(x)\right] + \sinh t\, \overline{I}(x,t) \\ &= \frac{\partial}{\partial t}\left[\sinh t\, M^t v_0(x)\right] + \sinh t\, M^t(v_1)(x) \\ &= \frac{\Omega_3}{2\left(\frac{1}{2}0\right)!\Omega_2}\left[\frac{\partial}{\partial t}\left[\sinh t\, M^t v_0(x)\right] + \sinh t\, M^t(v_1)(x)\right] \end{aligned}$$

Thus $\overline{u}$ is of the form (4.29) with $n = 3$. Thus by Lemma 4.22, $\overline{u}$ satisfies

$$(L_x + 1)\overline{u}(x,t) = \frac{\partial^2}{\partial t^2}\overline{u}(x,t), \quad \overline{u}(x,0) = 0, \text{ and } \frac{\partial}{\partial t}\overline{u}(x,t)\bigg|_{t=0} = g(x). \tag{4.30}$$

For convenience, let us use subscript notation for partial derivatives: $u_t(x,t) = \frac{\partial}{\partial t}u(x,t)$, etc. Because of the decay property of g, we can define a function u as follows.

$$u(x,t) = -\sum_{j=0}^{\infty}\left[\overline{u}_t(x, 2j+1+t) - \overline{u}_t(x, 2j+1-t)\right] \tag{4.31}$$

In the next steps, interchanges of the order of differentiation and summation will occur. They are legitimate, because the series are uniformly convergent on a neighborhood of t or on a neighborhood of x.[6]

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) &= -\frac{\partial^2}{\partial t^2}\sum_{j=0}^{\infty}\left[\overline{u}_t(x,2j+1+t)-\overline{u}_t(x,2j+1-t)\right] \\
&= -\frac{\partial}{\partial t}\sum_{j=0}^{\infty}\left[\overline{u}_{tt}(x,2j+1+t)-\overline{u}_{tt}(x,2j+1-t)\right] \\
&= -\frac{\partial}{\partial t}\sum_{j=0}^{\infty}\left[(L_x+1)\overline{u}(x,2j+1+t)-(L_x+1)\overline{u}(x,2j+1-t)\right] \quad \text{(by (4.30))} \\
&= (L_x+1)\left[(-1)\sum_{j=0}^{\infty}\left[\overline{u}_t(x,2j+1+t)-(L_x+1)\overline{u}_t(x,2j+1-t)\right]\right] \\
&= (L_x+1)u(x,t)\,. \qquad (4.32)
\end{aligned}
$$

Now let

$$f(x)=\sinh(1)\,u_t(x,0). \qquad (4.33)$$

With (4.32), (4.33), and Lemma 4.22, we can see that u satisfies

$$(L_x+1)u(x,t)=\frac{\partial^2}{\partial t^2}u(x,t),\quad u(x,0)=0,\ \text{and } u_t(x,0)=\frac{1}{\sinh 1}f(x). \qquad (4.34)$$

By the construction of u by the series in (4.31), $u(x,1)=-\sum_{j=0}^{\infty}\left[\overline{u}_t(x,2j+2)-\overline{u}_t(x,2j)\right]=\overline{u}_t(x,0)$. By (4.30), $\overline{u}_t(x,0)=g(x)$. So $u(x,1)=g(x)$. On the other hand, $u(x,1)=\sinh(1)$ $M^1(\frac{1}{\sinh 1}f)(x)$ by (4.34) and Lemma 4.22. Thus we have $M^1f(x)=g(x)$.

[6]For example, we can show the uniform convergence by using the decay condition of g and Weierstrass M test (For Weirstrass M test, see for example [17].). By the uniform convergence, we can interchange the order of the summation and differentiation.

Now let us rewrite f as follows.

$$\begin{aligned}
f(x) &= \sinh(1)\, u_t(x,0) \\
&= -\sinh(1)\Big[\frac{\partial}{\partial t}\sum_{j=0}^{\infty}\left[\overline{u}_t(x,2j+1+t) - \overline{u}_t(x,2j+1-t)\right]\Big]_{t=0} \\
&= -2\sinh(1)\sum_{j=0}^{\infty}\overline{u}_{tt}(x,2j+1) \\
&= -2\sinh(1)\sum_{j=0}^{\infty}(L_x+1)\overline{u}(x,2j+1) \\
&= -2\sinh(1)\sum_{j=0}^{\infty}(L_x+1)\left[\sinh(2j+1)\overline{I}(x,2j+1)\right] \\
&= -2\sinh(1)\sum_{j=0}^{\infty}\sinh(2j+1)\;(L+1)M^{2j+1}g(x) \\
&= -2\sinh(1)\sum_{j=0}^{\infty}\sinh(2j+1)\;M^{2j+1}\left((L+1)g\right)(x)
\end{aligned}$$

In the above series of equations, the last equation holds by Lemma 4.23. □

Appendix A

Supplements to Chapter 2

A.1 Proof of Lemma 2.24

This section is only for readers unfamiliar with Lie theory. Even though Lemma 2.24 is a known result and is easy to prove, we present its proof for the completeness of our argument.

Let $\xi = (\lambda, 0, 0 \ldots) \in M$. It suffices to show that for any $\tilde{\xi} \in M$ there is a $g \in SO(n, \mathbb{C})$ satisfying $g\,\xi = \tilde{\xi}$. Let us write $\tilde{\xi} = \tilde{\zeta} + i\tilde{\eta}$, where $\tilde{\zeta}, \tilde{\eta} \in \mathbb{R}^n$. Recall that $SO(n)$ acts transitively on $\mathbb{R}^n$ and $SO(n) \subset SO(n, \mathbb{C})$. Let g_1 be an $SO(n)$ matrix satisfying $g_1\tilde{\zeta} = (\tilde{x}_1, 0, 0, \ldots)^T$, and $g_1\tilde{\eta} = (\tilde{y}_1, \tilde{y}_2, 0, 0, \ldots)^T$.

Since $\tilde{\xi} \in M$, $\tilde{\xi}_1^2 + \tilde{\xi}_2^2 + \ldots + \tilde{\xi}_n^2 = \tilde{\zeta}^T\tilde{\zeta} - \tilde{\eta}^T\tilde{\eta} + 2i\tilde{\zeta}^T\tilde{\eta} = \lambda^2$. So $\tilde{x}_1\tilde{y}_1 = 0$, and $\tilde{x}_1^2 - (\tilde{y}_1^2 + \tilde{y}_2^2) = \lambda^2$. Thus we have $\tilde{y}_1 = 0$, $\tilde{x}_1 \neq 0$, and $\tilde{x}_1^2 - \tilde{y}_2^2 = \lambda^2$.

Let us define an n by n matrix h by

$$h_{2X2} := \frac{1}{\lambda}\begin{pmatrix} \tilde{x}_1 & -i\tilde{y}_2 \\ i\tilde{y}_2 & \tilde{x}_1 \end{pmatrix}\ , \text{ and } h := \left(\begin{array}{c|c} h_{2X2} & 0 \\ \hline 0 & I_{n-2} \end{array}\right)\ ,$$

where I_{n-2} is the $(n-2)$ by $(n-2)$ identity block matrix. Then we have $\det h = 1$, $h^T h = h\, h^T = I$, and $h\,\xi = g_1\,\tilde{\xi} = (\tilde{x}_1, i\tilde{y}_2, 0, \ldots, 0)^T$.

Let $g = g_1^T h$. Then $g \in SO(n, \mathbb{C})$ and $g\,\xi = \tilde{\xi}$.

Appendix B

Supplements to Chapter 3

B.1 A proof of a lemma about Paley-Wiener estimate of the ratio of holomorphic functions with an additional parameter

This section we present a proof of Lemma B.3 below. It is a refinement of the proof of (iv) $\Rightarrow$ (v) in [14, Theorem 16.3.10]. It shows that the estimate in [14, (16.3.15)] holds uniformly with an additional parameter. For the uninitiated reader, this section may serve as a supplementary note for reading just the proof of (iv) $\Rightarrow$ (v) in [14, Theorem 16.3.10].

Lemma B.1. *([16, Chapter 2, Theorem 1]) Suppose functions $f : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$ satisfy the conditions* (B.1) - (B.4) *below.*

$$\text{The functions } f \text{ and } g \text{ are entire.} \tag{B.1}$$

$$\exists (C,R) \in \mathbb{R}_+ \times \mathbb{R}_+ \left[\forall (\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n \left[|f(\lambda_1,\ldots,\lambda_n)| \le C e^{R(|\lambda_1|+\cdots+|\lambda_n|)} \right] \right] \tag{B.2}$$

$$\exists (C',R') \in \mathbb{R}_+ \times \mathbb{R}_+ \left[\forall (\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n \left[|g(\lambda_1,\ldots,\lambda_n)| \le C' e^{R'(|\lambda_1|+\cdots+|\lambda_n|)} \right] \right] \tag{B.3}$$

$$\frac{f(\lambda_1,\ldots,\lambda_n)}{g(\lambda_1,\ldots,\lambda_n)} \text{ is an entire function.} \tag{B.4}$$

Then there exists a pair of constants $(C'',R'') \in \mathbb{R}_+ \times \mathbb{R}_+$ *depending only on* C, C', R, *and* R' *such that*

$$\forall (\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n \left[\left| \frac{f(\lambda_1,\ldots,\lambda_n)}{g(\lambda_1,\ldots,\lambda_n)} \right| \le C'' e^{R''(|\lambda_1|+\cdots+|\lambda_n|)} \right].$$

Lemma B.2. *([14, Lemma 16.3.11]) Let* v *be a subharmonic function in* $\mathbb{C}$ *such that* $v \le 0$ *on* $\mathbb{R}$, $v \le C$ *on the positive imaginary axis and* $v(z) \le C + A|z|$ *when* $\operatorname{Im} z \ge 0$. *Then* $v(z) \le 0$ *when* $\operatorname{Im} z \ge 0$.

Lemma B.3. *Let* B *be a set. Let* $d\lambda$ *denote the Lebesgue measure in* $\mathbb{C}^n$. *Suppose functions* $W : \mathbb{C}^n \times B \to \mathbb{C}$ *and* $U : \mathbb{C}^n \times B \to \mathbb{C}$ *satisfy the conditions* (B.5) - (B.9) *below.*

$$\text{For each fixed } b \in B, \text{ the functions } W(\zeta,b) \text{ and } U(\zeta,b) \text{ are holomorphic in } \zeta. \tag{B.5}$$

$$\exists (C_W,N_W,R_W) \in \mathbb{R}^3_+ \ \left[\forall (\zeta,b) \in \mathbb{C}^n \times B \left[\, |W(\zeta,b)| \le C_W (1+|\zeta|)^{N_W} e^{R_W |\operatorname{Im}\zeta|} \, \right] \right] \tag{B.6}$$

$$\exists (C_U,N_U,R_U) \in \mathbb{R}^3_+ \ \left[\forall (\zeta,b) \in \mathbb{C}^n \times B \left[\, |U(\zeta,b)| \le C_U (1+|\zeta|)^{N_U} e^{R_U |\operatorname{Im}\zeta|} \, \right] \right] \tag{B.7}$$

$$\forall a > 0 \quad \exists A_a > 0 \quad \left[\forall b \in B \left[\forall \xi \in \mathbb{R}^n \left[|\xi| > 2 \Rightarrow \int_{|\zeta|<a} \ln |U(\xi + \zeta \, \ln|\xi|, b)| \, d\lambda(\zeta) \ > \ - A_a \, \ln |\xi| \right] \right] \right] \tag{B.8}$$

$$\text{For each fixed } b \in B, \frac{W(\zeta,b)}{U(\zeta,b)} \text{ is holomorphic in } \zeta. \tag{B.9}$$

Then there exists a triple of constants $(C,N,R) \in \mathbb{R}^3_+$ *such that*

$$\forall (\zeta,b) \in \mathbb{C}^n \times B \ \left[\left| \frac{W(\zeta,b)}{U(\zeta,b)} \right| \le C \, (1+|\zeta|)^N e^{R \, |\operatorname{Im}\zeta|} \right]. \tag{B.10}$$

Proof. For the proof, we follow part of the proof of [14, Theorem 16.3.10], but with an extra variable $b \in B$.

Since we have the condition (B.9), define F as follows for convenience.

$$F(\zeta, b) = \frac{W(\zeta, b)}{U(\zeta, b)}$$

Since W and U satisfy conditions (B.6) and (B.7), there exist constants $C_{M,W} > 0$, $C_{M,U} > 0$, $R_{M,W} \geq 0$, and $R_{M,U} \geq 0$ satisfying (B.11) and (B.12) below.

$$\forall b \in B \left[\forall \zeta \in \mathbb{C}^n \left[\, |W(\zeta, b)| \leq C_{M,W} \; e^{R_{M,W}|\zeta|} \, \right] \right] \tag{B.11}$$

$$\forall b \in B \left[\forall \zeta \in \mathbb{C}^n \left[\, |U(\zeta, b)| \leq C_{M,U} \; e^{R_{M,U}|\zeta|} \, \right] \right] \tag{B.12}$$

Due to the above two conditions, the holomorphicity conditions (B.5) and (B.9), we can apply Lemma B.1. Thus there exist constants $C_M > 0$ and $R \geq 0$ depending only on $C_{M,W}$, $C_{M,U}$, $R_{M,W}$, and $R_{M,U}$, and thus independent of $b \in B$, satisfying

$$\forall (\zeta, b) \in \mathbb{C}^n \times B \left[\, |F(\zeta, b)| \leq C_M \; e^{R|\zeta|} \, \right]. \tag{B.13}$$

Recall that $F(\zeta, b)$ is holomorphic in $\zeta \in \mathbb{C}^n$ by (B.9). So for each fixed $b \in B$, $\ln(|F(\zeta, b)|)$ is plurisubharmonic on $\mathbb{C}^n$ (See for example [13, Chapter IV, Example below Definition 4.1.1]). Thus for any $\xi \in \mathbb{C}^n$ and $r > 0$, we have the following estimate of $\ln |F(\xi, b)|$ by the integration over the ball in $\mathbb{C}^n$ ([13, Theorem 4.1.3]), where $V(r)$ is the volume of the ball of radius r in $\mathbb{C}^n$.

$$V(r) \; \ln |F(\xi, b)| \leq \int_{|\zeta|<r} \ln |F(\xi + \zeta, b)| \, d\lambda(\zeta) \tag{B.14}$$

Now fix $a > 0$ and let $\xi \in \mathbb{R}^n$ with $|\xi| > 2$. Then we have the following inequality by (B.14).

$$\begin{aligned} V(a \ln |\xi|) \ln |F(\xi, b)| &\leq \int_{|\zeta'| < a \ln |\xi|} \ln |F(\xi + \zeta', b)| \, d\lambda(\zeta') \\ &= \frac{V(a \ln |\xi|)}{V(a)} \int_{|\zeta| < a} \ln |F(\xi + \zeta \ln |\xi|, b)| \, d\lambda(\zeta) \end{aligned} \tag{B.15}$$

The integrals $\int_{|\zeta|<a} \ln |W(\xi + \zeta \ln |\xi|, b)| \; d\lambda(\zeta)$ and $\int_{|\zeta|<a} \ln |U(\xi + \zeta \ln |\xi|, b)| \; d\lambda(\zeta)$ exist, because at each fixed $b \in B$ $W(\zeta, b)$ and $U(\zeta, b)$ are holomorphic in $\zeta \in \mathbb{C}^n$, and thus $\ln |W(\zeta, b)|$ and $\ln |U(\zeta, b)|$ are plurisubharmonic in $\zeta \in \mathbb{C}^n$. Moreover, $F(\zeta, b) = \frac{W(\zeta,b)}{U(\zeta,b)}$. Thus we can split the integral in (B.15) as follows.

$$\begin{aligned} V(a) \ln |F(\xi, b)| &\leq \int_{|\zeta| < a} \ln |F(\xi + \zeta \ln |\xi|, b)| \, d\lambda(\zeta) \\ &= \int_{|\zeta| < a} \ln |W(\xi + \zeta \ln |\xi|, b)| \, d\lambda(\zeta) \\ &\quad - \int_{|\zeta| < a} \ln |U(\xi + \zeta \ln |\xi|, b)| \, d\lambda(\zeta) \end{aligned}$$

Now by using (B.6), we have

$$\begin{aligned}
V(a)\ln|F(\xi,b)| &\leq \int_{|\zeta|<a} \ln|W(\xi+\zeta\ln|\xi|,b)|\,d\lambda(\zeta) \\
&\qquad - \int_{|\zeta|<a} \ln|U(\xi+\zeta\ln|\xi|,b)|\,d\lambda(\zeta) \\
&\leq \int_{|\zeta|<a} \ln(C_W(1+|\,\xi+\zeta\ln|\xi|\,|)^{N_W} e^{R_W|\operatorname{Im}(\xi+\zeta\ln|\xi|)|})\,d\lambda(\zeta) \\
&\qquad - \int_{|\zeta|<a} \ln|U(\xi+\zeta\ln|\xi|,b)|\,d\lambda(\zeta) \\
&= \int_{|\zeta|<a} \ln(C_W(1+|\,\xi+\zeta\ln|\xi|\,|)^{N_W} e^{R_W\ln|\xi|\,|\operatorname{Im}\zeta|})\,d\lambda(\zeta) \\
&\qquad - \int_{|\zeta|<a} \ln|U(\xi+\zeta\ln|\xi|,b)|\,d\lambda(\zeta) \\
&\leq V(a)\ln(C_W(1+|\,\xi+a\ln|\xi|\,|)^{N_W} e^{a\,R_W\ln|\xi|}) \\
&\qquad - \int_{|\zeta|<a} \ln|U(\xi+\zeta\ln|\xi|,b)|\,d\lambda(\zeta) \\
&\leq V(a)\ln(C_W(2+a)^{N_W}|\xi|^{N_W+a\,R_W}) \\
&\qquad - \int_{|\zeta|<a} \ln|U(\xi+\zeta\ln|\xi|,b)|\,d\lambda(\zeta).
\end{aligned}$$

Now let A_a be a positive number whose existence i.e. guaranteed by (B.8). Then we have

$$V(a)\ln|F(\xi,b)| \leq V(a)\ln(C_W(2+a)^{N_W}|\xi|^{N_W+a\,R_W}) \;+\; A_a\ln|\xi|\;.$$

Thus we have the following inequality.

$$\forall b\in B\;\;\forall\xi\in\mathbb{R}^n\;\big[\;|\xi|>2 \Rightarrow |F(\xi,b)| \leq \{C_W(2+a)^{N_W}\}\;(1+|\xi|)^{N_W+aR_W+A_a/V(a)}\;\big]$$

Now by (B.13), we also have the following inequality.

$$\forall b\in B\;\;\forall\xi\in\mathbb{R}^n\;\big[\;|\xi|\leq 2 \Rightarrow |F(\xi,b)| \leq C_M e^{2R}\;\big]$$

Let

$$C_{\mathbb{R}^n} = \max\{C_W(2+a)^{N_W}, C_M e^{2R}\}\text{ and}$$

$$N = N_W + a\,R_W + A_a/V(a)\;.$$

Then we have the following inequality.

$$\forall (\xi, b) \in \mathbb{R}^n \times B \left[\, |F(\xi, b)| \le C_{\mathbb{R}^n} (1 + |\xi|)^N \,\right] \tag{B.16}$$

Let us summarize our key results so far. Let

$$C' = \max\{C_M, C_{\mathbb{R}^n}\}.$$

Then from (B.13) and (B.16), we have

$$\forall (\zeta, b) \in \mathbb{C}^n \times B \left[\, |F(\zeta, b)| \le C' e^{R|\zeta|} \,\right], \text{ and} \tag{B.17}$$

$$\forall (\xi, b) \in \mathbb{R}^n \times B \left[\, |F(\xi, b)| \le C' (1 + |\xi|)^N \,\right]. \tag{B.18}$$

Now we will use Lemma B.2 to obtain (B.22) below from (B.17) and (B.18). Let $\xi, \eta \in \mathbb{R}^n$ and consider a subharmonic function v defined as follows ([14, Section 16.3, in "End of proof of Theorem 16.3.10"]).

$$\begin{aligned} v : \mathbb{C} &\to [-\infty, \infty) \\ z &\mapsto \ln |F(\xi + z\eta, b)| - N \ln |\, 2 + |2\xi| - iz|2\eta| \,| - R|\eta| \operatorname{Im} z - \ln C' \end{aligned}$$

In the case $z = x + i0$, $x \in \mathbb{R}$, we have the following inequality from (B.18).

$$\begin{aligned}
v(z) &= \ln|F(\xi + x\eta, b)| - N\ln|\,2 + |2\xi| - ix|2\eta|\,| - R|\eta|\operatorname{Im}(x + i0) - \ln C' \\
&\leq \ln(C'(1 + |\xi + x\eta|)^N) - N\ln|\,2 + |2\xi| - ix|2\eta|\,| - \ln C' \\
&\leq N_{\mathbb{R}}\ln(1 + |\xi + x\eta|) - N\ln|\,2 + |2\xi| - ix|2\eta|\,| \\
&\leq N\ln\frac{1 + |\xi + x\eta|}{|\,2 + |2\xi| - ix|2\eta|\,|} \\
&\leq N\ln\frac{1 + |\xi| + |x\eta|}{|\,2 + |2\xi| - ix|2\eta|\,|} \\
&\leq N\ln\frac{\sqrt(2)|\,1 + |\xi| + i|x\eta|\,|}{|\,2 + |2\xi| - ix|2\eta|\,|} \\
&= N\ln(\frac{\sqrt{2}}{2}\,\frac{|\,1 + |\xi| + i|x\eta|\,|}{|\,1 + |\xi| - ix|\eta|\,|}) \\
&= -\frac{\ln 2}{2}N \\
&\leq 0 \qquad \text{(B.19)}
\end{aligned}$$

In the case $z = 0 + iy$, $y \geq 0$, we have the following inequality from (B.17).

$$\begin{aligned}
v(z) &= \ln|F(\xi + iy\eta, b)| - N\ln|\,2 + |2\xi| + y|2\eta|\,| - R|\eta|\operatorname{Im}(0 + iy) - \ln C' \\
&\leq \ln(C'e^{R|\xi + iy\eta|}) - N\ln|\,2 + |2\xi| + y|2\eta|\,| - R|\eta|\operatorname{Im}(0 + iy) - \ln C' \\
&= R|\xi + iy\eta| - N\ln|\,2 + |2\xi| + y|2\eta|\,| - Ry|\eta| \\
&\leq R(|\xi + iy\eta| - y|\eta|) \\
&= R(|y\eta - i\xi| - |y\eta|) \\
&\leq R|-i\xi| \\
&= R|\xi| \qquad \text{(B.20)}
\end{aligned}$$

Now consider the case $z = x + iy$, $y \geq 0$. From (B.17) we have the inequality below.

$$\begin{aligned} v(z) &= \ln|F(\xi + x\eta + iy\eta, b)| - N\ln|\,2 + |2\xi| + (-ix + y)|2\eta|\,| \\ &\quad -R|\eta|\,\mathrm{Im}(x+iy) - \ln C' \\ &\leq \ln(C' e^{R|\xi + x\eta + iy\eta|}) - N\ln|\,2 + |2\xi| + (-ix + y)|2\eta|\,| \\ &\quad -R|\eta|\,\mathrm{Im}(x+iy) - \ln C' \\ &\leq R|\xi + x\eta + iy\eta| - N\ln|\,2 + |2\xi| + (-ix + y)|2\eta|\,| - Ry|\eta| \\ &= R(|\xi + x\eta + iy\eta| - y|\eta|) - N\ln|\,2 + |2\xi| + (-ix + y)|2\eta|\,| \\ &\leq R(|\xi + x\eta + iy\eta| - y|\eta|) \\ &= R(|y\eta - i(\xi + x\eta)| - |y\eta|) \\ &\leq R|\xi + x\eta| \\ &\leq R(|\xi| + |x\eta|) \\ &= R\,|\xi| + R\,|\eta|\,|x| \\ &\leq R\,|\xi| + R\,|\eta|\,|z| \end{aligned} \tag{B.21}$$

The inequilities (B.19), (B.20), and (B.21) show that $v(z)$ satisfies the condition in Lemma B.2. So $v(z) \leq 0$ when $\mathrm{Im}\, z \geq 0$. So $v(i) \leq 0$, that is

$$\ln|F(\xi + i\eta, b)| \;\leq\; \ln C' + N\ln|\,2 + |2\xi| + |2\eta|\,| + R|\eta|.$$

Thus we have

$$\begin{aligned}
|F(\xi + i\eta, b)| &\leq C' \mid 2 + |2\xi| + |2\eta| \mid^N e^{R|\operatorname{Im}(\xi + \mathrm{i}\eta)|} \\
&\leq C' \mid 2 + \sqrt{2}\ |2\xi + 2i\eta| \mid^N e^{R|\operatorname{Im}(\xi + \mathrm{i}\eta)|} \\
&\leq C' \mid 2\sqrt{2}\ + 2\sqrt{2}\ |\xi + i\eta| \mid^N e^{R|\operatorname{Im}(\xi + \mathrm{i}\eta)|} \\
&= C'\ (2\sqrt{2})^N\ (1 + |\xi + i\eta|)^N\ e^{R|\operatorname{Im}(\xi + \mathrm{i}\eta)|}.
\end{aligned}$$

Since this inequality holds for any $\xi, \eta \in \mathbb{R}^n$ and $b \in B$, we have

$$\forall (\zeta, b) \in \mathbb{C}^n \times B \left[\ |F(\zeta, b)| \leq C(1 + |\zeta|)^N e^{R|\operatorname{Im}\zeta|}\ \right], \tag{B.22}$$

where $C = C'\ (2\sqrt{2})^N$. This proves (B.10). □

Appendix C

Supplements to Chapter 4

C.1 Explicit expressions of some functions in the proof of Lemma 4.9

This section is for those who want to examine the proof of Lemma 4.9 more carefully. In the proof, we introduced some functions, but kept their explicit expressions hidden for the sake of readability. Here we give the explicit expressions of those functions.

$$B_1(\lambda) = 0$$

$$(\frac{\partial}{\partial\lambda})^1 \, k(5; \lambda, 1, r) = 4\lambda - 4\lambda^3 + 4r^2\lambda$$

$$B_2(\lambda) = 8\lambda(\lambda+1)^2 F(\lambda+1) + 8\lambda(\lambda-1)^2 F(\lambda-1)$$

$$(\frac{\partial}{\partial\lambda})^2 \, k(5; \lambda, 1, r) = 4 + 4r^2 - 12\lambda^2$$

$$
\begin{aligned}
B_3(\lambda) &= 8\lambda(\lambda+1)^2F_\lambda(\lambda+1)+8(-\lambda^3+3\lambda^2+6\lambda+2)F(\lambda+1)\\
&\quad +8\lambda(\lambda-1)^2F_\lambda(\lambda-1)+8(\lambda^3+3\lambda^2-6\lambda+2)F(\lambda-1)
\end{aligned}
$$

$$
(\frac{\partial}{\partial\lambda})^3\,k(5;\lambda,1,r)=-24\lambda
$$

$$
\begin{aligned}
B_4(\lambda) &= 8\lambda(\lambda+1)^2F_{\lambda\lambda}(\lambda+1)+8(-\lambda^3+6\lambda^2+10\lambda+3)F_\lambda(\lambda+1)\\
&\quad +24(-2\lambda^2+\lambda+2)F(\lambda+1)\\
&\quad +8\lambda(\lambda-1)^2F_{\lambda\lambda}(\lambda-1)+8(\lambda^3+6\lambda^2-10\lambda+3)F_\lambda(\lambda-1)\\
&\quad +(8(6\lambda^2+3\lambda-6))F(\lambda-1)
\end{aligned}
$$

$$
(\frac{\partial}{\partial\lambda})^4\,k(5;\lambda,1,r)=-24
$$

$$
\begin{aligned}
B_5(\lambda) &= 8\lambda(\lambda+1)^2F_{\lambda\lambda\lambda}(\lambda+1)+8(-\lambda^3+9\lambda^2+14\lambda+4)F_{\lambda\lambda}(\lambda+1)\\
&\quad +8(-9\lambda^2+15\lambda+16)F_\lambda(\lambda+1)-120\lambda F(\lambda+1)\\
&\quad 8\lambda(\lambda-1)^2F_{\lambda\lambda\lambda}(\lambda-1)+8(\lambda^3+9\lambda^2-14\lambda+4)F_{\lambda\lambda}(\lambda-1)\\
&\quad +8(9\lambda^2+15\lambda-16)F_\lambda(\lambda-1)+120\lambda F(\lambda-1) \qquad \text{(C.1)}
\end{aligned}
$$

$$
(\frac{\partial}{\partial\lambda})^5\,k(5;\lambda,1,r)=0
$$

Appendix D

Support theorems for the single radius spherical mean value operator from delay differential equations

In this chapter, we prove some support theorems for the single radius spherical mean value operator using delay differential equations.

Section D.1, we prove support theorems for the single radius spherical mean value operator for radial functions in $\mathbb{R}^n$, $n = 3, 5$. In the case $n = 3$, we have delay equations in the proof, for which we use telescoping series. In the case $n = 5$, we have delay differential equations in the proof, for which we use Wright's theorem about delay differential equation ([27]). Our support theorems are different from the support theorems published by by V. V. Volchkov ([24, Corollary 3.3]) and by M. Agranovsky and P. Kuchment ([1, Theorem 1]), because our support theorems are for different classes of functions. Some functions in our classes blow up at infinity. In the next section, Section D.2, we do not have the restriction that the functions are radial. We use the iterated mean value theorem in $\mathbb{H}^n$ obtained in Section 4.1 and a support theorem by Helgason in $\mathbb{H}^n$ to prove the support theorem for the single radius spherical mean value operator in $\mathbb{H}^3$. We expect that the methods shown in this chapter work in higher odd dimensional Euclidean or hyperbolic space. We also expect that we can have better result in the case the dimension of the space is higher than 3, if we have more optimized tool for delay differential equations.

D.1 Supplementary support theorem for the single radius spherical mean value operator for radial functions in $\mathbb{R}^n$, $n = 3, 5$

In this section we consider the following type of support theorem, where $\mathcal{B}$ is a class of functions in $\mathbb{R}^n$.

> Fix $A > 0$ and $r > 0$. Suppose $f \in \mathcal{B}$. If $M^r f(x) = 0$ for $d(0, x) > A + r$, then $f(x) = 0$ for $d(0, x) > A$.

M. Agranovsky and P. Kuchment (2011, [1, Theorem 1]), and V. V. Volchkov (2003, [24, Corollary 3.3]) proved this type of support theorem where $\mathcal{B} = L^p(\mathbb{R}^n)$, $1 \leq p \leq 2n/(n-1)$, $n > 1$. We restrict our interest in radial functions, and prove the support theorem of this type in $\mathbb{R}^n$, $n = 3, 5$, for $\mathcal{B}$ different from the class of radial functions in $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2n/(n-1)$.

Our results are Theorem D.2, Corollary D.3, and Theorem D.9 below. They are in Sections D.1.1 and D.1.2 with their proofs.

Our $\mathcal{B}$ in the case $n = 3$ is defined by $f \in \mathcal{B}$ as follows.

> Fix $m \in \{0, 1, 2, \ldots\}$. Let $f \in C(\mathbb{R}^n \backslash \bar{B}(0, A))$ be radial. Define the normalized radial function $\phi(w) = f(x)$ with $x \in \mathbb{R}^n$ satisfying $\|x\| = rw$. Suppose $\phi \in C^m((A/r, \infty))$ and $\lim_{w\to\infty} \left[m\phi^{(m-1)}(w) + w\phi^{(m)}(w)\right] = 0$.

The condition for the functions in our $\mathcal{B}$ with $m = 0$ is easy to understand: $\lim_{w\to\infty} [w\phi(w)] = 0$. The description of our $\mathcal{B}$ in the case $n = 5$ is complicated and looks far more restricting. Therefore, even though it is clear that our $\mathcal{B}$ is different from the class of radial functions in $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2n/(n-1)$, $n = 5$, we are unsure of the usefulness our support theorem in the case $n = 5$. This might be because there is some intrinsic difference between $\mathbb{R}^3$ and $\mathbb{R}^n$, $n = 5, 7, \ldots$, or because our tool for the delay differential equations is not well optimized for the purpose of extending the class $\mathcal{B}$ in the support theorem. When we prove our theorem in the case $n = 3$, we deal with delay equations, a very special case of delay differential equations.

Let us present some examples of functions in our $\mathcal{B}$ for $n = 3$, but not in $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2n/(n-1)$. Suppose f is a radial function on $\mathbb{R}^3$ such that $\phi(w) = O(|w(\ln(w))^{1/3}|)$. Since $\int_e^\infty \left|\frac{1}{w(\ln(w))^{1/3}}\right|^{2\cdot 3/(3-1)} w^{3-1}dw = \int_1^\infty \frac{1}{w\ln(w)}dw = \infty$, such an f is not in $L^p(\mathbb{R}^3)$ with $1 \leq p \leq 2 \cdot 3/(3-1)$. However, such a function is in our $\mathcal{A}$, since $\lim_{w\to\infty} w\frac{1}{w(\ln(w))^{1/3}} = 0$. Other examples are radial functions f on $\mathbb{R}^3$ such that $\phi(w) = 1/\ln(\ln(w+1))$, $\phi(w) = \ln(w)$, $\phi(w) = w$, $\phi(w) = w^2$, or, $\phi(w) = w^3$. Such functions are not in $L^p(\mathbb{R}^3)$ with

$1 \le p \le 2 \cdot 3/(3-1)$, because they increase or decrease too slowly as $\|x\|$ increases. However, they satisfy $\lim_{w\to\infty} j\phi^{(j-1)}(w) + w\phi^{(j)}(w) = 0$ with some $j \in \{1, 2, \ldots, 5\}$.

Now let us present examples of functions that are in our $\mathcal{B}$ for $n = 5$, but not necessarily in $L^p(\mathbb{R}^n)$ with $1 \le p \le 2n/(n-1)$. From (D.26) and (D.26), we can see that the following examples belong to the class of functions in Theorem D.9.

$$\begin{aligned} f(x) &= \left(\sum_{j=0}^{5} c_j (\frac{\|x\|}{r})^j\right) + \frac{c_6}{\frac{\|x\|}{r}} + \frac{c_7}{(\frac{\|x\|}{r})^3} \\ &+ \frac{1}{184320\, q^6} \{ e^{-q^2(\frac{\|x\|}{r})^2} (\frac{30}{q^2(\frac{\|x\|}{r})^2} + 292 + 152 q^2 (\frac{\|x\|}{r})^2 + 16 q^4 (\frac{\|x\|}{r})^4) \\ &+ \sqrt{\pi}\ \mathrm{erf}(q\frac{\|x\|}{r})(-\frac{15}{q^3(\frac{\|x\|}{r})^3} + \frac{120}{q(\frac{\|x\|}{r})} + 360 q (\frac{\|x\|}{r}) + 160 q^3 (\frac{\|x\|}{r})^3 + 16 q^5 (\frac{\|x\|}{r})^5)\}\ , \\ & q > 0 \end{aligned} \tag{D.1}$$

$$f(x) = \left(\sum_{j=0}^{14} c_j (\frac{\|x\|}{r})^j\right) + \frac{c_{15}}{\frac{\|x\|}{r}} + \frac{c_{16}}{(\frac{\|x\|}{r})^3} \tag{D.2}$$

With certain choice of costants c_js, f satisfies $\lim_{\|x\|\to\infty} |f(x)| = \infty$, and thus $f \notin L^p(\mathbb{R}^n)$ with $p \le 2n/(n-1)$.

Now let us clarify some of notation we use in the following two sections. By $\|x\|$ for $x \in \mathbb{R}^n$, we mean $d(0, x)$. If $B \subset \mathbb{R}^n$ is compact and $x \in \mathbb{R}^n$, $d(x, B) = \min\{d(x, y) \mid y \in B\}$.

D.1.1 Supplementary support theorem in $\mathbb{R}^3$

We prove Theorem D.2 and Corollary D.3 below after proving a lemma.

Lemma D.1. *Let $n \in \{2, 3, 4, \ldots\}$, $A > 0$, $r > 0$, and $f \in C(\mathbb{R}^n \backslash \bar{B}(0, A))$ be radial. Let $\phi(w) = f(x)$ with $x \in \mathbb{R}^n$ satisfying $\|x\| = rw$. If $\operatorname{supp} M^r f \subset \bar{B}(0, A + r)$, then*

$$\forall\, y > A/r + 1 \qquad \int_{y-1}^{y+1} \phi(w) \left[-\left\{ w^2 - (y^2+1) \right\}^2 + 4y^2 \right]^{(n-3)/2} w\, dw = 0\ . \tag{D.3}$$

Proof. The mean of f over a sphere of radius r centered at $Y \in \mathbb{R}^n$ with $d(Y, \bar{B}(0, A)) > r$ can be written as follows using the idea described in Figure D.1.

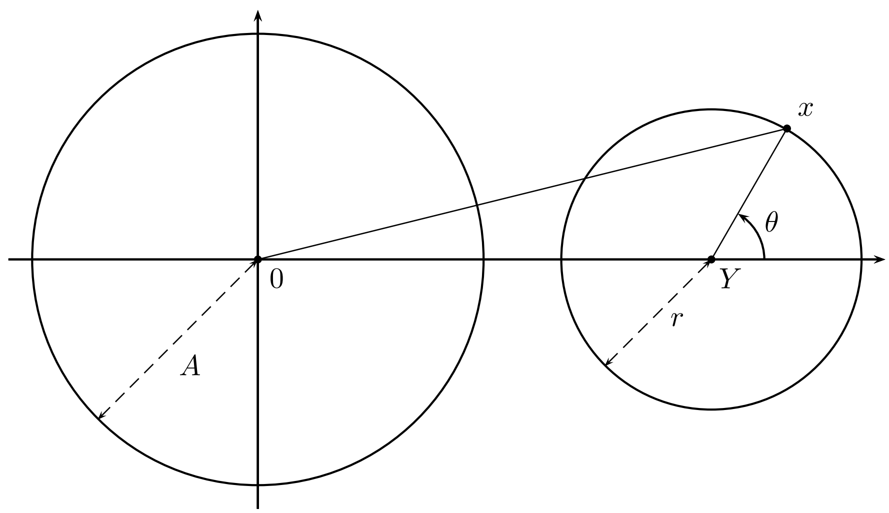

Figure D.1: Evaluation of spherical mean of a radial function using an azimuth angle.

$$\begin{aligned} M^r f(Y) &:= c_r \int_{x \in S(Y,r)} f(x) dm_S(x) \\ &= c_r \int_0^{\pi} \phi(w) \frac{2\pi^{(n-1)/2} (r \sin\theta)^{n-2}}{\Gamma((n-1)/2)} r d\theta \end{aligned} \tag{D.4}$$

Let $y := \|Y\| / r$. Then we can see that

$$\begin{aligned} w &= \|x\| / r \\ &= \sqrt{\{\|Y\| + r\cos\theta\}^2 + \{r \sin\theta\}^2} / r \\ &= \sqrt{y^2 + 2y\cos\theta + 1} \,, \\ d\theta &= -\frac{w}{y \sin\theta} dw \,, \text{ and} \\ \sin\theta &= \sqrt{-\{w^2 - (y^2+1)\}^2 + 4y^2} / (2y) \end{aligned}$$

for $\theta \in [0, \pi]$. So (D.4) can be written as follows.

$$\begin{aligned} M^r f(Y) &= c_r \frac{2\pi^{(n-1)/2} r^{n-1}}{\Gamma((n-1)/2)} \int_{y-1}^{y+1} \phi(w) (\sin\theta)^{n-2} \frac{w}{y \sin\theta} \, dw \\ &= \frac{2^{3-n} \pi^{-1/2} y^{2-n} \Gamma(n/2)}{\Gamma((n-1)/2)} \int_{y-1}^{y+1} \phi(w) \left[- \left\{ w^2 - (y^2+1) \right\}^2 + 4y^2 \right]^{(n-3)/2} w \, dw \end{aligned} \tag{D.5}$$

Note that $d(Y, \bar{B}(0, A)) > r \;\Leftrightarrow\; \|Y\| > A + r \;\Leftrightarrow\; y > A/r + 1$. Thus with the assumtion that $M^r f(Y) = 0$ for all $Y \in \mathbb{R}^n$ with $d(Y, \bar{B}(0, A)) > r$, we obtain (D.3) from (D.5). □

Theorem D.2. *Let $A > 0$ and $r > 0$. Let $n = 3$. Let $m \in \{0, 1, 2, \ldots\}$. Let $f \in C(\mathbb{R}^n \backslash \bar{B}(0, A))$ be radial. Let $\phi(w) := f(x)$ with $x \in \mathbb{R}^n$ satisfying $\|x\| = rw$. Suppose $\phi \in C^m((A/r, \infty))$ and $\lim_{w\to\infty} \left[m\phi^{(m-1)}(w) + w\phi^{(m)}(w) \right] = 0$. If $M^r f(x) = 0$ for $d(0, x) > A + r$, then $f(x) = 0$ for $d(0, x) > A$.*

Proof. Let $g(w) = w\phi(w)$. By Lemma D.1, we have

$$\forall\, y > A/r + 1 \qquad \int_{y-1}^{y+1} g(w) \, dw = 0 \,. \tag{D.6}$$

By taking the derivative of (D.6) with respect to y $(m+1)$ times, we obtain

$$\forall\, y > A/r + 1 \qquad g^{(m)}(y+1) - g^{(m)}(y-1) = 0 \,.$$

Thus we have the telescoping sum $g^{(m)}(y+1) - g^{(m)}(y-1) + g^{(m)}(y+3) - g^{(m)}(y+1) + g^{(m)}(y+5) - g^{(m)}(y+3) \ldots + g^{(m)}(y+2N+1) - g^{(m)}(y+2N-1) = -g^{(m)}(y-1) + g^{(m)}(y+2N+1) = 0$ for $y > A/r + 1$. Since $g^{(m)}(w) = m\phi^{(m-1)}(w) + w\phi^{(m)}(w)$, we have $\lim_{w\to\infty} g(w) = 0$ from the hypothesis of this lemma. So by taking the limit as $N \to \infty$, we have $g^{(m)}(w) = 0$ for $w > A/r$. This implies $g(w) = c_{m-1}w^{m-1} + \ldots + c_2 w^2 + c_1 w + c_0$ for $w > A/r$, where $c_{m-1}, \ldots, c_2, c_1, c_0$ are arbitrary constants.

First differentiate (D.6) with respect to y $(m-1)$ times. Then we have $g^{(m-2)}(y+1) - g^{(m-2)}(y-1) = 0$, which implies $c_{m-1}(y+1) - c_{m-1}(y-1) + c_{m-2} - c_{m-2} = 0$. Thus we have $c_{m-1} = 0$. Now we know that $g(w) = c_{m-2}w^{m-2} + \ldots + c_2 w^2 + c_1 w + c_0$ for $w > A/r$, where $c_{m-2}, \ldots, c_2, c_1, c_0$ are arbitrary constants. By repeating this procedure, we obtain $g(w) = 0$ for $w > A/r$.

Therefore $\phi(w) = 0$ for $w > A/r$. This implies $f(x) = 0$ for $\|x\| > A$. $\square$

Corollary D.3. *Let $A > 0$ and $r > 0$. Let $n = 3$. Let $m \in \{0, 1, 2, \ldots\}$. Let $f \in C^\infty(\mathbb{R}^n)$ be radial. Let $\phi(w) := f(x)$ with $x \in \mathbb{R}^n$ satisfying $\|x\| = rw$. Suppose there exists $m \in \{0, 1, 2, \ldots\}$ satisfying $\lim_{w\to\infty} \left[m\phi^{(m-1)}(w) + w\phi^{(m)}(w)\right] = 0$. If $M^r f(x) = 0$ for $d(0, x) > A + r$, then $f(x) = 0$ for $d(0, x) > A$.*

Proof. This corollary follows from Theorem D.2. $\square$

D.1.2 Supplementary support theorem in $\mathbb{R}^5$

We prove Theorem D.9 below via a series of lemmas.

Lemma D.4. *Let $A > 0$, $r > 0$, and $\phi \in C((A/r, \infty))$. If ϕ satisfies*

$$\forall\, y > A/r + 1 \qquad \int_{y-1}^{y+1} \phi(w) \left[-\left\{ w^2 - (y^2+1) \right\}^2 + 4y^2 \right]^{\frac{5-3}{2}} w \, dw = 0 \,, \tag{D.7}$$

then ϕ satisfies

$$\forall\, y > A/r + 1 \qquad \int_{y-1}^{y+1} \phi(w)\, (y^2 - w^2 - 1) w \, dw = 0 \,. \tag{D.8}$$

Proof. Note the following fact which can be verified by expanding each side of the equation.

$$-\left\{ w^2 - (y^2+1) \right\}^2 + 4y^2 = -\left\{ y^2 - (w^2+1) \right\}^2 + 4w^2$$

By differentiating each side of (D.7) with respect to y and using the above fact, we have

$$\begin{aligned}
0 &= \phi(y+1)\cdot 0 - \phi(y-1)\cdot 0 \\
&\quad + \int_{y-1}^{y+1} \frac{d}{dy}\left[\phi(w)\ \left[-\left\{w^2-(y^2+1)\right\}^2+4y^2\right]^{\frac{5-3}{2}} w\right] dw \\
&= \int_{y-1}^{y+1} \frac{d}{dy}\left[\phi(w)\ \left[-\left\{y^2-(w^2+1)\right\}^2+4w^2\right]^{\frac{5-3}{2}} w\right] dw \\
&= \int_{y-1}^{y+1} \phi(w)\ \left[-4y\left\{y^2-(w^2+1)\right\}\right] w\, dw \\
&= -4y\int_{y-1}^{y+1} \phi(w)\,(y^2-w^2-1)\, w\, dw\ .
\end{aligned}$$

So we obtain the result (D.8). □

The class of functions $W_{p,b}$ defined below is used to define the class of functions in our support theorem in $\mathbb{R}^5$.

Definition D.5. *For $p \in \{1,2,3,\ldots\}$, and $b \in \mathbb{R}$, we define a class of functions $W_{p,b}$ as follows.*

$h \in W_{p,b}$ if and only if $h \in C^p([b,\infty))$ and h satisfies conditions (D.9) – (D.11).

$$\text{For any } y \geq b \text{ and } k \in \mathbb{R}\ , \text{ we have } \int_y^\infty \left|h^{(p)}(t)\right| e^{kt}\, dt\ <\ \infty\ . \tag{D.9}$$

$$\lim_{y\to\infty} |h(y)| \neq \infty \tag{D.10}$$

$$\text{There is no nonzero constant } l \text{ satisfying } \lim_{y\to\infty} |h(y)| = l\ . \tag{D.11}$$

Since we often work with a class of functions that is a vector space, we define here a subset of $W_{p,b}$ that is manifestly a vector space of complex valued functions.

Definition D.6. *For $p \in \{1,2,3,\ldots\}$, and $b \in \mathbb{R}$, we define a class of functions $W^0_{p,b} \subset W_{p,b}$ as follows.*

$h \in W^0_{p,b}$ if and only if $h \in C^p([b,\infty))$ and h satisfies conditions (D.9) and (D.12).

$$\lim_{y\to\infty} |h(y)| = 0 \tag{D.12}$$

Let us see some examples of functions in $W^0_{1,b} \subset W_{1,b}$. If we choose a function f in (D.13) – (D.16) , and a function g in (D.17) – (D.19), then $h(y) = f(y)g(y)$ is in $W^0_{1,b}$.

$$f(y) = e^{zy},\ \ z \in \mathbb{C}\ . \tag{D.13}$$

$$f(y) = (y - b + s)^q,\ \ q \geq 0, s > 0\ . \tag{D.14}$$

$$f(y) = \frac{1}{(y - b + s)^q},\ \ q \geq 0, s > 0\ . \tag{D.15}$$

$$f(y) = \ln(y - b + s),\ \ s > 0\ . \tag{D.16}$$

$$g(y) = e^{-q(y-b+s)^p},\ \ p > 1, q > 0, s > 0\ . \tag{D.17}$$

$$g(y) = e^{-q(y-b+s)\ln(y-b+s)},\ \ q > 0, s > 0\ . \tag{D.18}$$

$$g(y) = e^{-q(y-b+s)\ln(\ln(y-b+s))},\ \ q > 0, s > 1\ . \tag{D.19}$$

We can verify that such $h(y) = f(y)g(y)$ satisfies the condition (D.9) as follows. For such an h, $\lim_{t\to\infty} h'(t)e^{kt}(t-b+s)^2 = 0$ for any $k \in \mathbb{R}$. So, there is an $M > b$ such that $0 \leq |h'(t)|\, e^{kt} < 1/(t - b + s)^2$ for any $t > M$. Since $\int_y^\infty 1/(t - b + s)^2 dt < \infty$ for $y > M$, $\int_y^\infty |h'(t)|\, e^{kt} dt < \infty$ for $y > M$. In addition, such an h is bounded on $[b, M]$. So $\int_y^\infty |h'(t)|\, e^{kt} dt < \infty$ for $y \geq b$.

The following theorem by Wright plays a key role in the proof of our support theorem in $\mathbb{R}^5$.

Lemma D.7. *(From [27, Theorem 4].) Let* $k, p \in \{1, 2, 3, \ldots\}$*, and* $b \in \mathbb{R}$*. Let* $h \in W_{p,b}$*, where* $W_{p,b}$ *is the class of functions defined in Definition D.5. If* h *is a solution to the difference-differential equation*

$$\forall y \geq b \quad \sum_{\mu=0}^{k} \sum_{\nu=0}^{p} A_{\mu\nu}(y)\, h^{(\nu)}(y + a_\mu) = 0 \ , \tag{D.20}$$

where

$$0 = a_0 < a_1 < a_2 < \ldots < a_k \ ,$$

$$\forall y \geq b \ \ A_{0p}(y) = 1 \ , \ \text{and}$$

$$\exists C \in \mathbb{R} \quad \forall \mu, \nu \ \ \forall y \geq b \ \ |A_{\mu\nu}(y)| < C \ ,$$

then

$$\forall y \geq b \ \ h(y) = 0 \ .$$

Lemma D.7 is a slightly restricted version of [27, Theorem 4] by E. M. Wright. In [27], (D.20) is vied as an integral equation in $h^{(p)}(y)$, and $h^{(\nu)}(y)$ for $\nu \leq p - 1$ is defined by $h^{(\nu)}(y) = h^{(\nu)}(b) + \int_b^y h^{(\nu+1)}(t)dt$, where the integral is Lebesgue integral. However, we limit our interest to the cases where $h \in W_{p,b} \subset C^p([b, \infty))$, and view (D.20) as a differential equation in $h(y)$.

In Lemma D.7, however, we limit our interest to the cases where $h \in W_{p,b} \subset C^p([b, \infty))$, and view (D.20) as a differential equation in $h(y)$.

A condition on the function ϕ on $(A/r, \infty)$, $A, r > 0$, in Lemma D.8 below is concisely described in terms of $\left(\frac{d}{dw}\right)^m (w \int w\, \phi(w)\, dw)$. However, we sometimes prefer more explicit expanded formulae for it. So we provide some of them here.

$$\left(\frac{d}{dw}\right)^m (w \int w\,\phi(w)\,dw) =$$
$$\begin{cases} w \int w\,\phi(w)\,dw & \text{if } m = 0 \\ \int w\,\phi(w)\,dw + w^2\phi(w) & \text{if } m = 1 \\ 3w\,\phi(w) + w^2\phi'(w) & \text{if } m = 2 \\ (m^2 - 2m)\,\phi^{(m-3)}(w) + (2m-1)\,w\,\phi^{(m-2)}(w) + w^2\,\phi^{(m-1)}(w) & \text{if } m \geq 3 \end{cases}$$

Lemma D.8. *Let $A > 0$, $r > 0$, $b = A/r + 2$, and $\phi(w) \in C((A/r, \infty))$. Suppose there exists $m \in \{0, 1, 2, \dots\}$ such that $\left(\frac{d}{dw}\right)^m (w \int w\,\phi(w)\,dw) \in C^1((A/r, \infty))$ and $\left(\frac{d}{dw}\right)^m (w \int w\,\phi(w)\,dw) \in W_{1,b}$. If*

$$\forall\, y > A/r + 1 \qquad \int_{y-1}^{y+1} \phi(w)\,(y^2 - w^2 - 1)\,w\,dw = 0\,, \tag{D.21}$$

then

$$\forall\, w > A/r \quad \phi(w) = \begin{cases} 0\,, & m = 0\,, \\ c/w^3\,, & m \in \{1, 2, 3, \dots\}\,, \end{cases}$$

where c is an arbitrary constant.

Proof. For convenience, let us define $\psi(w) = w\,\phi(w)$. Then we have

$$\forall\, y > A/r + 1 \qquad \int_{y-1}^{y+1} \psi(w)\,(y^2 - w^2 - 1)\,dw = 0\,.$$

We can rewrite this integral equation as

$$-\int_{y-1}^{y+1} \psi(w)\,w^2\,dw \;+\; (y^2 - 1)\int_{y-1}^{y+1} \psi(w)\,dw = 0\,. \tag{D.22}$$

By differentiating each side of (D.22) with respect to y, we have

$$-(y+1)^2\psi(y+1)+(y-1)^2\psi(y-1)+(y^2-1)\left\{\psi(y+1)-\psi(y-1)\right\}+2y\left\{\bar{\psi}(y+1)-\bar{\psi}(y-1)\right\}=0\ , \tag{D.23}$$

where $\bar{\psi}$ is an antiderivative of ψ. The equation (D.23) can be rewritten as follows.

$$\begin{aligned}0 &= -2(y+1)\psi(y+1)-2(y-1)\psi(y-1)+2y\left\{\bar{\psi}(y+1)-\bar{\psi}(y-1)\right\}\\ &= 2(y+1)\bar{\psi}(y+1)-2(y-1)\bar{\psi}(y-1)\\ &\quad -2\bar{\psi}(y+1)-2\bar{\psi}(y-1)\\ &\quad -2(y+1)\psi(y+1)-2(y-1)\psi(y-1)\end{aligned}$$

By defining $f(y) := y\,\bar{\psi}(y)$, we have $f(y+1)-f(y-1)-f'(y+1)-f'(y-1)=0$ for $y > A/r+1$. Since $f^{(m)} \in C^1((A/r,\infty))$ by the hypothesis of the lemma, we have

$$\forall\, y > A/r \qquad -f^{(m)}(y+2)+f^{(m)}(y)+f^{(m+1)}(y+2)+f^{(m+1)}(y)=0\ . \tag{D.24}$$

By the hypothesis of this lemma, $f^{(m)}|_{[b,\infty)} \in W_{1,b}$. So we will check whether other conditions in Lemma D.7 are satisfied. From (D.24), we have

$$\forall y\ \geq b\ \ \sum_{\mu=0}^{1}\sum_{\nu=0}^{1} A_{\mu\nu}(y)(f^{(m)})^{(\nu)}(y+a_\mu)=0\ ,$$

where

$$0 = a_0 < a_1 = 2\ ,$$

$$\forall y \geq b\ \ A_{01}(y) = 1\ ,\ \text{and}$$

$$|A_{00}(y)| = |A_{10}(y)| = |A_{11}(y)| = 1\ .$$

So, by Lemma D.7,

$$\forall y \geq b \;\; f^{(m)}(y) = 0 \; . \tag{D.25}$$

Now let us consider $y \in (A/r, b)$. Because of (D.25) and (D.24), we have

$$\forall y \in (A/r, b) \;\; f^{(m)}(y) + f^{(m+1)}(y) = 0 \; .$$

So we have $f^{(m)}(y) = Ce^{-y}$, where C is an arbitrary constant. Since $f^{(m)}(y) \in C^1((A/r, \infty))$ and $f^{(m)}(b) = 0$, we have $C = 0$. So we have

$$\forall \; y > A/r \;\; f^{(m)}(y) = 0 \; .$$

If $m \geq 3$, $f^{(m)}(y) = (m^2 - 2m)\, \phi^{(m-3)}(y) + (2m-1)\, y\, \phi^{(m-2)}(y) + y^2 \; \phi^{(m-1)}(y) \equiv 0$, and thus $\phi^{(m-3)}(y) = C_1\, y^{-m+2} + C_2\, y^{-m}$, where C_1 and C_2 are arbitrary constants. In case $m = 4, 5, 6, \ldots$, $\forall \; w > A/r \;\; \phi(w) = c_1/w + c_2/w^3 + b_1 + b_2 w + \ldots + b_{m-3} w^{m-4}$, where c_1, c_2, b_1, b_2, . . ., and b_{m-3} are arbitrary constants. Since ϕ satisfies (D.21), we have for any $y > A/r + 1$

$$\begin{aligned} 0 &= \int_{y-1}^{y+1} (c_1/w + c_2/w^3 + \sum_{j=1}^{m-3} b_j w^{j-1})\, (y^2 - w^2 - 1) w \, dw \\ &= -\frac{8}{3} c_1 - 4 \sum_{j=1}^{m-3} b_j y^j \; . \end{aligned}$$

So we have $c_1 = d_1 = d_2 = \ldots = d_{m-3} = 0$, and thus $\forall \; w > A/r \;\; \phi(w) = c_2/w^3$. In case $m = 3$, $\forall \; w > A/r \;\; \phi(w) = C_1/w + C_2/w^3$, where C_1 and C_2 are arbitrary constants. Since ϕ satisfies (D.21), we have for any $y > A/r + 1$

$$\begin{aligned} 0 &= \int_{y-1}^{y+1} (C_1/w + C_2/w^3)\, (y^2 - w^2 - 1) w \, dw \\ &= -\frac{8}{3} C_1 \; . \end{aligned}$$

So we have $C_1 = 0$, and thus $\forall\, w > A/r \;\; \phi(w) = C_2/w^3$.

If $m = 2$, $f^{(m)}(y) = 3y\,\phi(y) + y^2\phi'(y) \equiv 0$. Thus we obtain $\forall\, y > A/r \;\; \phi(y) = c/y^3$, where c is an arbitrary constant.

If $m = 1$, we have $f^{(m)}(y) = \int y\,\phi(y)\,dy + y^2\phi(y) \equiv 0$, which implies $3\phi(y) + y\,\phi'(y) \equiv 0$. Thus we have $\forall\, y > A/r \;\; \phi(y) = c/y^3$, where c is an arbitrary constant.

In case $m = 0$, we have $f^{(m)}(y) = f(y) = y\int y\,\phi(y)\,dy \equiv 0$, which implies $\int y\,\phi(y)\,dy \equiv 0$, and thus $y\,\phi(y) \equiv 0$. So we have $\forall\, y > A/r \;\; \phi(y) = 0$. □

Let us present examples of ϕ satisfying the conditions in Lemma D.8. In the seven examples below, $\phi(y)$ satisfies $\frac{d^3}{dy^3}(y\int y\,\phi(y)\,dy) \in W_{1,b}$

$$\phi(y) = 1/y$$

$$\phi(y) = 1/y^3$$

$$\phi(y) = \frac{1}{y}\cos(ze^{-qy\ln(y)}),\;\; z \in \mathbb{C}, q > 0$$

$$\phi(y) = \frac{1}{y}\,\mathrm{erf}(qy),\;\; q > 0$$

$$\phi(y) = y^{-\cos(ze^{-qy\ln(y)})},\;\; z \in \mathbb{C}, q > 0$$

$$\phi(y) = y^{-\cos(ze^{-qy\ln(\ln(y-A/r+s))})},\;\; z \in \mathbb{C}, q > 0, s > 1$$

$$\phi(y) = e^{q/(y-A/r)}e^{-y\ln(\ln(y-A/r+s))},\;\; q > 0, s > 1$$

In the examples (D.26) and (D.27) below, with certain choice of constants $c_j \in \mathbb{C}$, $\phi(y)$ can satisfy $\lim_{y\to\infty}|\phi(y)| = \infty$. In the example (D.26), $\phi(y)$ satisfies $\frac{d^9}{dy^9}(y\int y\phi(y)\,dy) = e^{-q^2y^2} \in$

$W_{1,b}$. In the example (D.27), $\phi(y)$ satisfies $\frac{d^{18}}{dy^{18}}(y\int y\,\phi(y)\,dy) = 0 \in W_{1,b}$.

$$\begin{aligned}
\phi(y) &= \left(\sum_{j=0}^{5} c_j y^j\right) + \frac{c_6}{y} + \frac{c_7}{y^3} \\
&\quad + \frac{1}{184320 q^6}\{e^{-q^2y^2}(\frac{30}{q^2y^2} + 292 + 152q^2y^2 + 16q^4y^4) \\
&\quad + \sqrt{\pi}\ \mathrm{erf}(qy)(-\frac{15}{q^3y^3} + \frac{120}{qy} + 360qy + 160q^3y^3 + 16q^5y^5)\}\ ,\ q > 0\ . \qquad \text{(D.26)}
\end{aligned}$$

$$\phi(y) = \left(\sum_{j=0}^{14} c_j y^j\right) + \frac{c_{15}}{y} + \frac{c_{16}}{y^3}\ . \qquad \text{(D.27)}$$

Notice that, if $\phi(y)$ is a polynomial in y, then $\frac{d^m}{dy^m}(y\int y\,\phi(y)\,dy) = 0 \in W_{1,b}$ for large enough m.

Theorem D.9. *Let $n = 5$, $A > 0$, $r > 0$, $b = A/r + 2$, and $f \in C(\mathbb{R}^n \backslash \bar{B}(0, A))$ be radial. Let $\phi(w) = f(x)$ with $x \in \mathbb{R}^n$ satisfying $\|x\| = rw$. Suppose there exists $m \in \{0, 1, 2, \ldots\}$ such that $\left(\frac{d}{dw}\right)^m (w\int w\,\phi(w)\,dw) \in C^1((A/r, \infty))$ and $\left(\frac{d}{dw}\right)^m (w\int w\,\phi(w)\,dw) \in W_{1,b}$. If $M^r f(x) = 0$ for $d(0, x) > A + r$, then $f(x) = 0$ for $d(0, x) > A$.*

Proof. Since f is radial on $\mathbb{R}^n \backslash \bar{B}(0, A)$, we can define a function $\phi : (A/r, \infty) \to \mathbb{C}$ by $\phi(w) := f(x)$ with $x \in \mathbb{R}^n \backslash \bar{B}(0, A)$ satisfying $\|x\| = rw$. Since $f \in C(\mathbb{R}^n \backslash \bar{B}(0, A))$, $\phi(w) \in C((A/r, \infty))$. By Lemma D.1, we have (D.7). So by Lemma D.4, we have (D.8), i.e. (D.21). So by Lemma D.8, we have $\phi(w) = c/w^3$ for $w > A/r$, where c is a complex constant. Since ϕ satisfies (D.7), we have the following equation for $y > A/r + 1$.

$$\begin{aligned}
0 &= \int_{y-1}^{y+1} \left\{\frac{c}{w^3}\right\} \left[-\left\{w^2 - (y^2+1)\right\}^2 + 4y^2\right]^{\frac{5-3}{2}} w\,dw \\
&= \frac{16}{3} c
\end{aligned}$$

Hence $c = 0$. So $\phi(w) = 0$ for $w > A/r$, and thus $f(x) = 0$ for $\|x\| > A$. ∎

D.2 Support theorem for the single radius spherical mean value operator in $\mathbb{H}^3$

We prove Theorem D.12 below, which is a support theorem for the single radius spherical mean value operator for functions in $C^1(\mathbb{H}^3)$ with a certain decay condition. To prove it, we use the iterated mean value theorem in $\mathbb{H}^n$ in Section 4.1 and Lemma D.11 below, which is a support theorem in $\mathbb{H}^n$ by Helgason.

Lemma D.10. *Let $R_0 > 0$. Suppose $f \in C(\mathbb{H}^3)$ satisfies $\lim_{t\to\infty} \sinh(t) M^t f(x) = 0$ for any $x \in \mathbb{H}^3$. If $\operatorname{supp} M^1 f \subset \bar{B}(o, R_0 + 1)$, then $M^\lambda f(x) = 0$ for any $\lambda \in \mathbb{R}$ and $x \in \mathbb{H}^3$ satisfying $\lambda > d(o,x) + R_0$.*

Proof. Let $M(x, \lambda, \mu)$ be he iterated mean value of f at x with radii λ and μ. (Let $I(x, r) = M^r f(x)$ and $M(x, \lambda, \mu) = \int_K I(gk \cdot y, \mu)\, dk$, where y is a point in $\mathbb{H}^n$ with $d(o, y) = \lambda$ and g is an element in G with $g \cdot o = x$.)

Fix $x_o \in \mathbb{H}^3$. Since $\operatorname{supp} M^1 f \subset \bar{B}(o, R_0 + 1)$, $M(x_o, \lambda, 1) = 0$ whenever $\lambda > d(o, x_o) + R_0 + 1$. Thus from Theorem 4.2, $\int_{\lambda-1}^{\lambda+1} \sinh(s) M^s f(x_o) ds = 0$ for any $\lambda > d(o, x_o) + R_0 + 1$. By differentiating with respect to λ, we have

$$\sinh(\lambda + 1) M^{\lambda+1} f(x_o) - \sinh(\lambda - 1) M^{\lambda-1} f(x_o) = 0\,.$$

Thus we have

$$
\begin{aligned}
0 &= \sinh(\lambda+2m+1)M^{\lambda+2m+1}f(x_o) - \sinh(\lambda+2m-1)M^{\lambda+2m-1}f(x_o) \\
&\quad \vdots \\
&\quad + \sinh(\lambda+5)M^{\lambda+5}f(x_o) - \sinh(\lambda+3)M^{\lambda+3}f(x_o) \\
&\quad + \sinh(\lambda+3)M^{\lambda+3}f(x_o) - \sinh(\lambda+1)M^{\lambda+1}f(x_o) \\
&\quad + \sinh(\lambda+1)M^{\lambda+1}f(x_o) - \sinh(\lambda-1)M^{\lambda-1}f(x_o) \\
&= \sinh(\lambda+1+2m)M^{\lambda+1+2m}f(x_o) - \sinh(\lambda-1)M^{\lambda-1}f(x_o)\,.
\end{aligned}
$$

Due to the condition that $\lim_{t\to\infty}\sinh(t)M^t f(x_o) = 0$, we can take the limit as $m \to \infty$ and obtain

$$M^{\lambda-1}f(x_o) = 0\,, \text{ for any } \lambda > d(o,x_o) + R_0 + 1.$$

□

Lemma D.11. *(Helgason, [11, Chapter III, Lemma 1.8])*

Let $f \in C^1(\mathbb{H}^n)$ satisfy the conditions:

(i) For each integer $m > 0$, $f(x)\,e^{m\,d(o,x)}$ is bounded.

(ii) There exists a number $R > 0$ such that $M^\lambda f(x) = 0$ whenever $\lambda > d(o,x) + R$.

Then $f(x) = 0$ for $d(o,x) > R$.

Theorem D.12. *Let $R_0 > 0$. Let $f \in C^1(\mathbb{H}^3)$ satisfy the conditions:*

(i) For each integer $m > 0$, $f(x)\,e^{m\,d(o,x)}$ is bounded.

(ii) $M^1 f(x) = 0$ for $d(0,x) > R_0 + 1$.

Then $f(x) = 0$ for $d(o,x) > R_0$.

Proof. We have $M^\lambda f(x) = 0$ whenever $\lambda > d(o,x) + R_0$ by Lemma D.10. Thus by Lemma D.11, $f(x) = 0$ for $d(o,x) > R_0$. □

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