Hooke' Law

Hooke's law: linear relations between ϵ_{ij} & σ_{ij}

Isotropic material (same properties in all directions):

$$\begin{cases} \epsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{22}+\sigma_{33}) \\ \epsilon_{22} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11}+\sigma_{33}) \\ \epsilon_{33} = \frac{1}{E}\sigma_{33} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) \end{cases} \qquad \begin{cases} \epsilon_{12} = \frac{1}{2G}\sigma_{12} \\ \epsilon_{23} = \frac{1}{2G}\sigma_{23} \\ \epsilon_{31} = \frac{1}{2G}\sigma_{31} \end{cases}$$

Of 3 elastic constants E, ν, G only 2 are independent

$$G = \frac{E}{2(1+\nu)}$$

In tensor form: $\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$

Inverting:

$$\begin{cases} \sigma_{11} = \lambda(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) + 2G\,\epsilon_{11} \\ \sigma_{22} \text{ — — — — — } \epsilon_{22} \\ \sigma_{33} \text{ — — — — — } \epsilon_{33} \end{cases} \qquad \begin{cases} \sigma_{12} = 2G\,\epsilon_{12} \\ \text{— — —} \\ \text{— — —} \end{cases}$$

where $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

In tensor form: $\sigma_{ij} = \lambda\,\epsilon_{kk}\delta_{ij} + 2G\,\epsilon_{ij}$

verify the inversion

Verify the inversion:

$$\sigma_{ij} = \lambda\, \epsilon_{kk}\, \delta_{ij} + 2G\, \epsilon_{ij}$$

Substitute strains: $\quad \epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\,\delta_{ij}$

$$\Rightarrow \quad \epsilon_{mm} = \frac{1+\nu}{E}\sigma_{mm} - 3\cdot\frac{\nu}{E}\sigma_{kk} =$$

$$= \frac{1-2\nu}{E}\sigma_{kk}$$

obtain

$$\sigma_{ij} = \lambda\left(\frac{1-2\nu}{E}\sigma_{kk}\right)\delta_{ij} + $$

$$+ 2G\left(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\,\delta_{ij}\right)$$

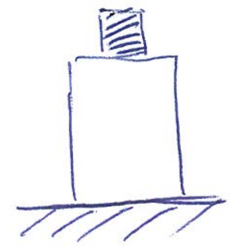
$$\frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$= \sigma_{ij} \qquad \checkmark$$

Comments on elastic constants

- Young's modulus E must be positive

  otherwise: thermodynamics violated
  (get work for free)



- Relative volume change

$$\epsilon_{ii} = \frac{1-2\nu}{E}\underbrace{(\sigma_{11}+\sigma_{22}+\sigma_{33})}_{3p}$$

p: ave. hydro. stress

$$= \frac{1}{K}p$$

K: bulk modulus $\quad K = \frac{E}{3(1-2\nu)}$

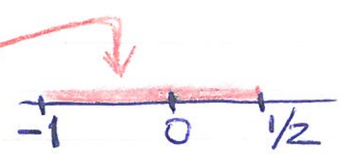
Since $K > 0$:
(same argument)

$$\Rightarrow \boxed{\nu \leq \tfrac{1}{2}}$$

($\nu = \frac{1}{2}$: incompressible material)

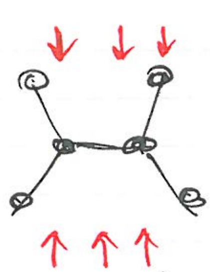
- Shear modulus $\quad G = \frac{E}{2(1+\nu)} \geq 0 \;\Rightarrow\; \boxed{\nu \geq -1}$

Thus, Poisson's ratio ν



Note: negative ν not ruled out!

Negative ν:

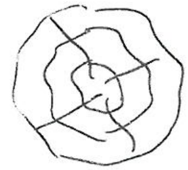


man-made material

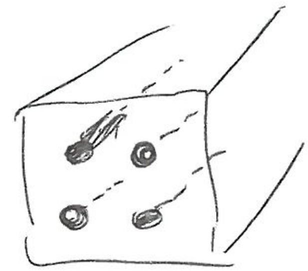
when compressed, buckles in not out

Anisotropic Materials: properties different in different directions

- wood

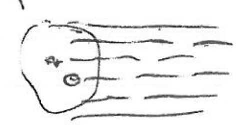


- steel-reinforced concrete



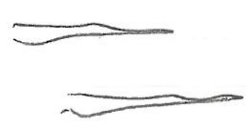
- various composites
  - layered

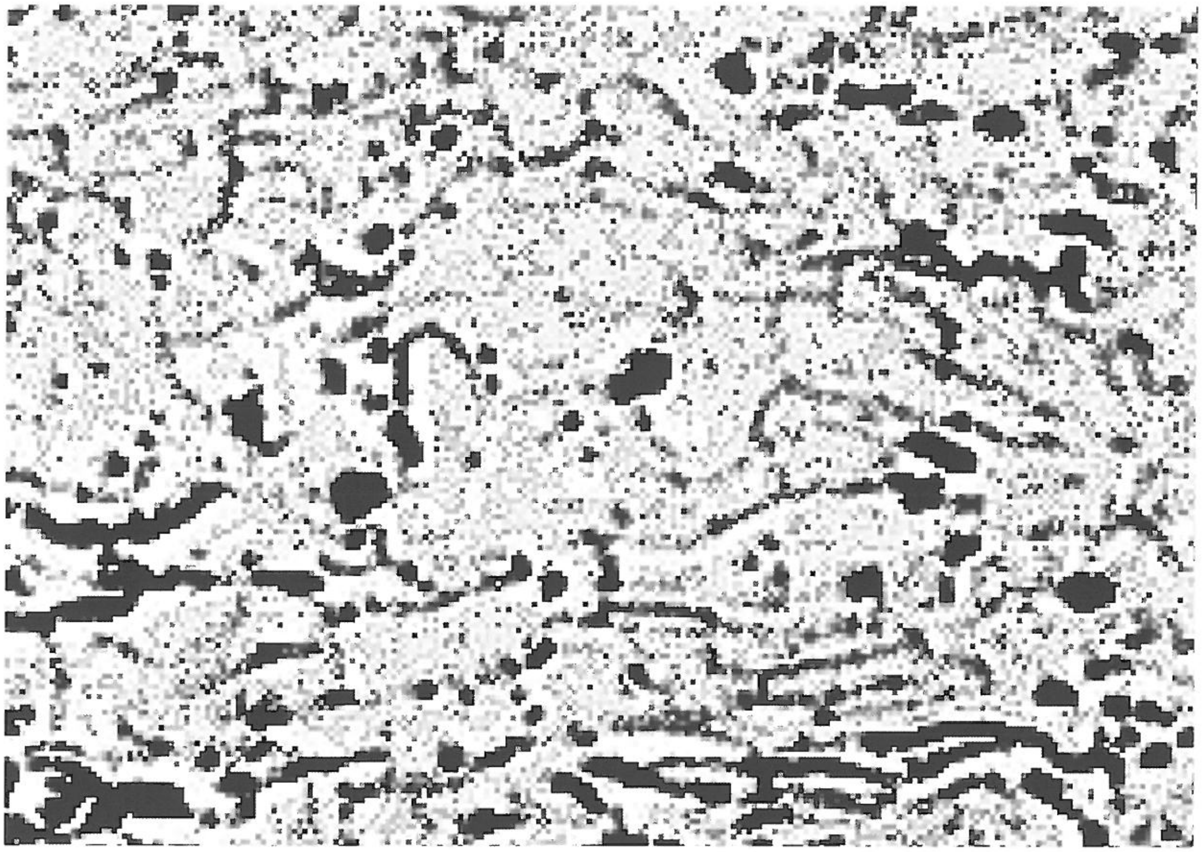
  - fiber-reinforced

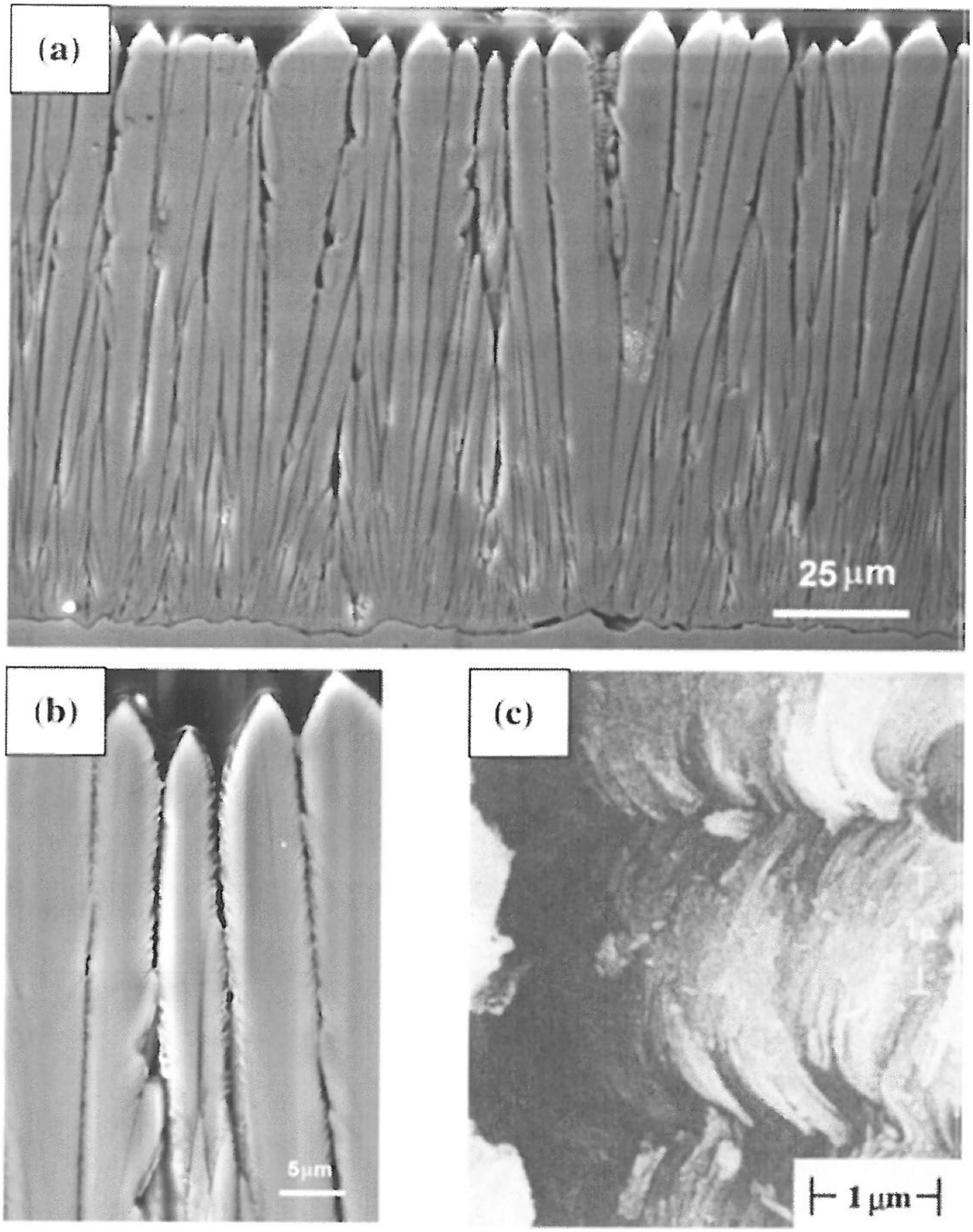


- crystals

- Rocks
  - at micro-level, due to crystallites having preferential orientations
  - at large scale: due to large fractures that are approximately parallel






(a)
25 μm
(b)
5μm
(c)
1 μm

pores filled with electrically conductive biological fluids and soft tissue—blood, lymph, nerve tissue, etc.

Our analysis is based on description of the bone microstructure given by Martin and Burr [341], Currey [101], and Fung [145] sketched in Fig. 7.11. We model the bone as porous, elastically transversely isotropic material of low electric conductivity containing three systems of pores filled with elastically soft and electrically highly conductive tissue:

- Parallel cylindrical pores (Haversian canals) modeled as strongly prolate spheroidal inhomogeneities, their axes coinciding with the transverse isotropy axis x_3 of the material symmetry of the matrix. Their aspect ratio is $\gamma = 3h/4R$ where h is the length of the osteon (we used the average value of 4 mm) and R is the radius of the canal (we assumed 125 μm). This implies modeling of the canals by spheroids with aspect ratio $\gamma = 120$ that is needed to preserve the volume of canals and their radii.

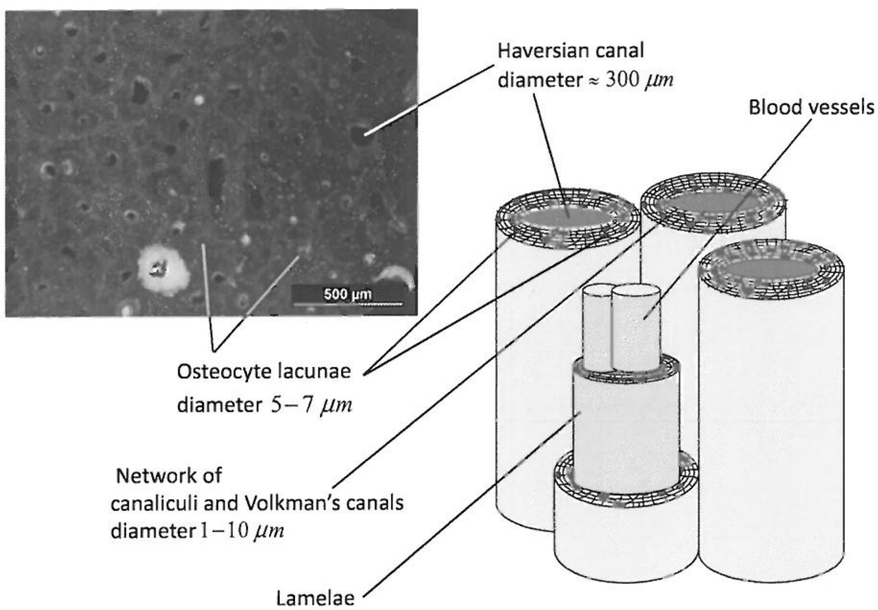
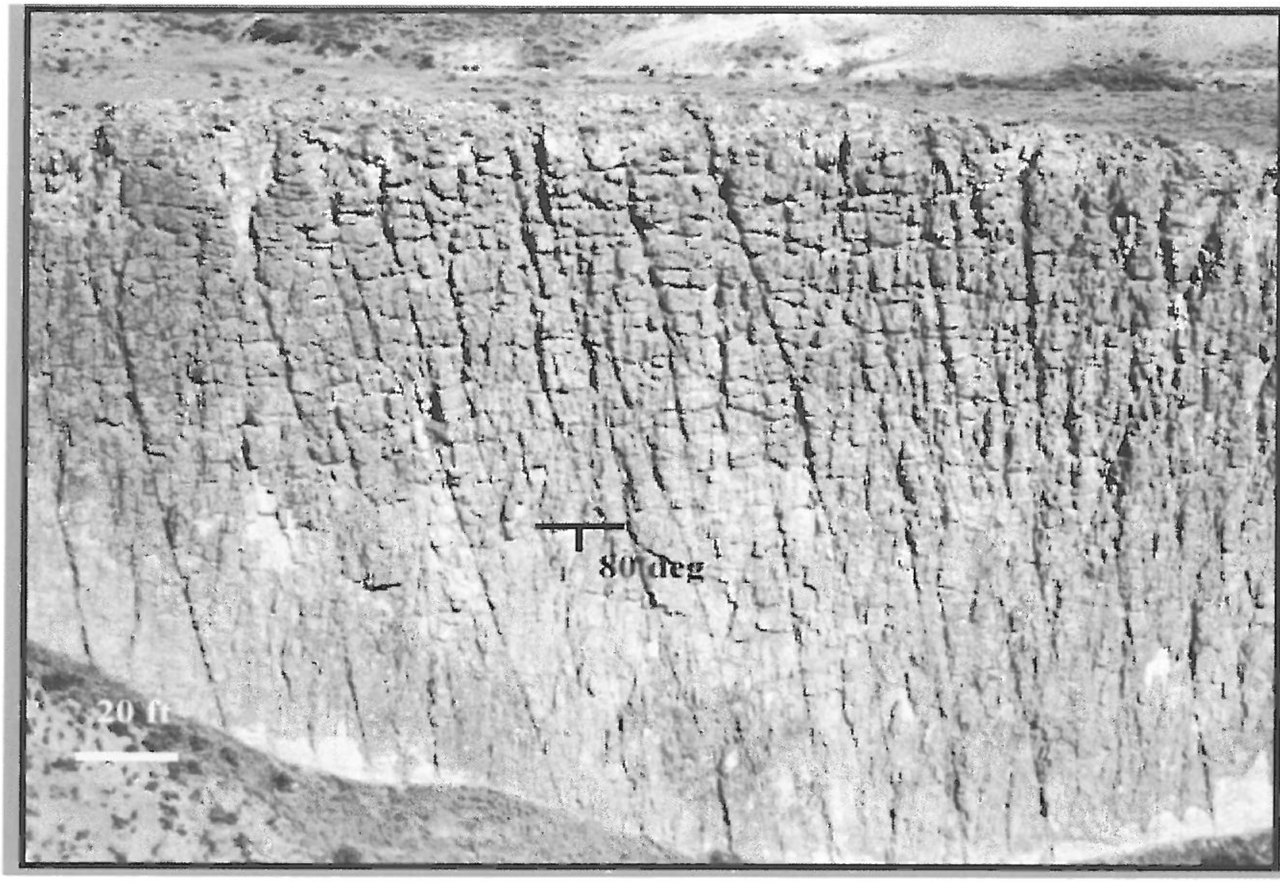


Fig. 7.11 Microstructure of cortical bone used in the present model: it is formed by osteons surrounding Haversian canals that contain blood and lymph vessels and nerves. Volkman's canals and canaliculi are randomly oriented in the planes orthogonal to the Haversian canals. The lamellae in osteons contain osteocytes located in oblate spheroidal pores (lacunae) (from Casas and Sevostianov [70], with permission)


80 deg
20 ft

Anisotropic Materials : Hooke's law

Each ϵ_{ij} is a linear f-n of all σ_{ij}:

$$\epsilon_{ij} = \underbrace{S_{ijkl}}_{\text{compliance tensor, } 4^{th} \text{ rank}} \sigma_{kl}$$

same summation convention

Isotropic case:

$$S_{1111} = \frac{1}{E}, \quad S_{1122} = -\frac{\nu}{E}, \quad S_{1212} = \frac{1}{4G}$$

--- etc. ---

[note: $\epsilon_{12} = S_{1212}\sigma_{12} + S_{1221}\sigma_{21}$]

Number of independent compliances:

$$9^2 = 81$$

However: → Since $\epsilon_{ij} = \epsilon_{ji}$, $\sigma_{kl} = \sigma_{lk}$

$$6^2 = 36$$

→ Since $S_{ijkl} = S_{klij}$ $\Rightarrow$ 21

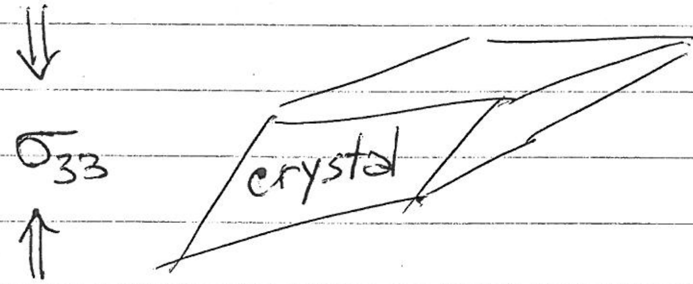
number of elastic constants

Inverse form:

$$\sigma_{ij} = \underbrace{C_{ijkl}}_{\text{stiffnesses}} \epsilon_{kl}$$

In anisotropic materials, <u>normal</u> & <u>shear</u> modes may be <u>coupled</u>:

$$S_{1233} \neq 0 \quad \Rightarrow \quad \epsilon_{12} \text{ depends on } \sigma_{33}$$



produces $\rightleftarrows \epsilon_{12}$

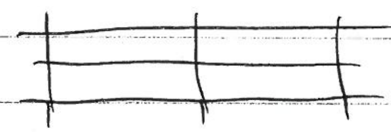
$$\epsilon_{ij} = S_{ijkl}\,\sigma_{kl}$$

21 constants (compliances)

Material symmetries greatly reduce the number of constants

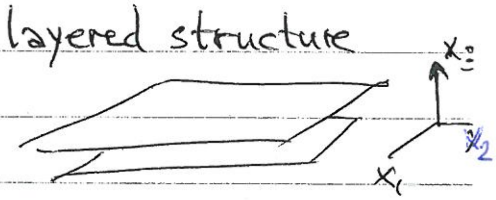
important cases:

- Orthotropy
  (rectangular symm.)

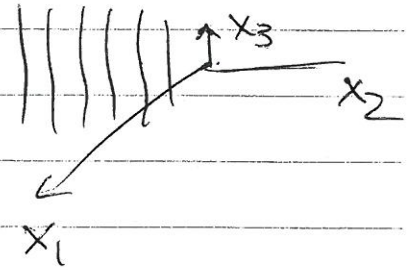


- Transverse isotropy (TI)
  (isotropy within $x_1 x_2$ plane)

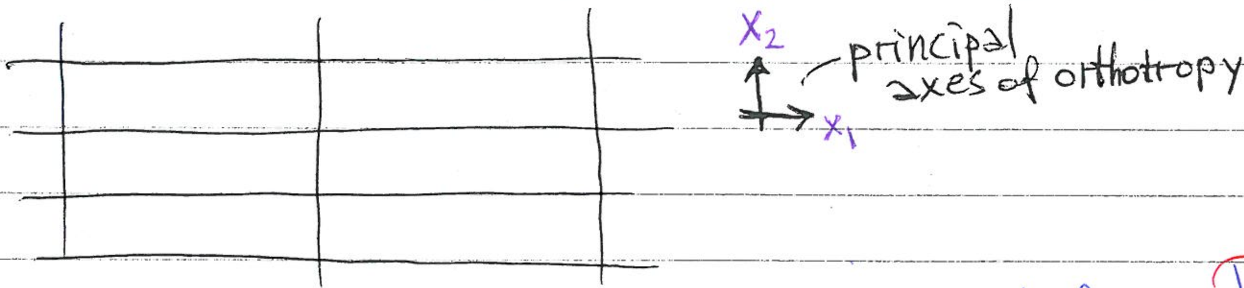
  - layered structure



  - parallel fibers



Orthotropy in 2-D



represent in the form $-\dfrac{\nu_{12}}{E_1}$ — Poisson's ratio (one of)

$$\begin{cases} \epsilon_{11} = \underbrace{S_{1111}}_{1/E_1}\sigma_{11} + S_{1122}\cdot\sigma_{22} \\[2ex] \epsilon_{22} = \underbrace{S_{2211}}_{-\nu_{21}/E_2}\sigma_{11} + \underbrace{S_{2222}}_{1/E_2}\sigma_{22} \\[2ex] \epsilon_{12} = S_{1212}\sigma_{12} + S_{1221}\sigma_{21} \end{cases}$$

$$= \underbrace{2S_{1212}}_{1/2G_{12}}\sigma_{12}$$

Note: shear modulus G_{12} is independent of E's & ν's [unlike isotropy $G = E/2(1+\nu)$]

$$S_{1122} = S_{2211} \quad \left(\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}\right)$$

$\Rightarrow$ $\cancel{5}$ 4 independent elastic constants

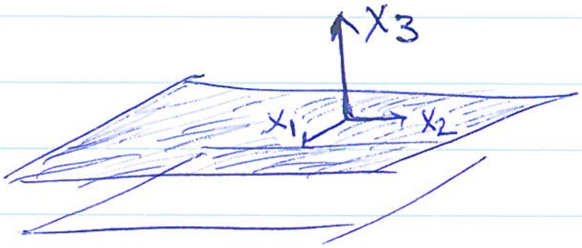
3-D orthotropy :

$$\begin{cases} \epsilon_{11} = \dfrac{1}{E_1}\sigma_{11} - \dfrac{\nu_{21}}{E_2}\sigma_{22} - \dfrac{\nu_{31}}{E_3}\sigma_{33} \\ \epsilon_{22} = -\dfrac{\nu_{12}}{E_1}\sigma_{11} + \dfrac{1}{E_2}\sigma_{22} - \dfrac{\nu_{32}}{E_3}\sigma_{33} \\ \epsilon_{33} = -\dfrac{\nu_{13}}{E_1}\sigma_{11} - \dfrac{\nu_{23}}{E_2}\sigma_{22} + \dfrac{1}{E_3}\sigma_{33} \end{cases}$$

$$\begin{cases} \epsilon_{12} = \dfrac{1}{2G_{12}}\sigma_{12} \\ \epsilon_{23} = \dfrac{1}{2G_{23}}\sigma_{23} \\ \epsilon_{31} = \dfrac{1}{2G_{31}}\sigma_{31} \end{cases}$$

9 independent constants

Transverse Isotropy (TI)



$X_1 X_2$ – plane of isotropy

Hooke's law $\epsilon_{ij} = S_{ijkl}\,\sigma_{kl}$ takes form:

plane of isotropy

$$\left.\begin{aligned}
\epsilon_{11} &= \frac{1}{E_0}\sigma_{11} - \frac{\nu_0}{E_0}\sigma_{22} - \frac{\nu_{31}}{E_3}\sigma_{33} \\
\epsilon_{22} &= -\frac{\nu_0}{E_0}\sigma_{11} + \frac{1}{E_0}\sigma_{22} - \frac{\nu_{32}}{E_3}\sigma_{33} \\
\epsilon_{33} &= -\frac{\nu_{13}}{E_0}\sigma_{11} - \frac{\nu_{23}}{E_0}\sigma_{22} + \frac{1}{E_3}\sigma_{33}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\epsilon_{12} &= \frac{1}{2G_0}\sigma_{12} \\
\epsilon_{23} &= \frac{1}{2G_{23}}\sigma_{23} \\
\epsilon_{31} &= \frac{1}{2G_{31}}\sigma_{31}
\end{aligned}\right\}$$

5 independ. constants:

$E_0\,,\ \nu_0 \qquad (G_0 = E_0/2(1+\nu_0))$

$E_3\,,\ \nu_{13}\,,\ G_{23}$

Important consequence of anisotropy:

Hydrostatic loading produces shear strains

Example: 2-D orthotropy

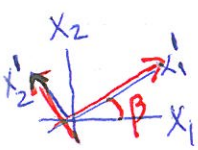
$$\begin{cases} \epsilon_{11} = \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22} \\ \epsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{1}{E_2}\sigma_{22} \\ \epsilon_{12} = \frac{1}{2G_{12}}\sigma_{12} \end{cases}$$

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

Apply: $\sigma_{11} = \sigma_{22} = p$; $\sigma_{12} = 0$ (hydro. loading)

$$\begin{cases} \epsilon_{11} = \left(\frac{1}{E_1} - \frac{\nu_{21}}{E_2}\right) p \\ \epsilon_{22} = \left(\frac{1}{E_2} - \frac{\nu_{12}}{E_1}\right) p \neq \epsilon_{11} \; ! \\ \epsilon_{12} = 0 \end{cases}$$

Since $\epsilon_{11} \neq \epsilon_{22}$, shear strains induced on some orient's:



$$\epsilon'_{12} = (\epsilon_{22} - \epsilon_{11}) \sin\beta\cos\beta + \underbrace{\epsilon_{12}}_{0} (\cos^2\beta - \sin^2\beta)$$

Used for: restructuring of crystals (graphite → diamond) by high hydro. pressures

Stress-strain relations beyond Hooke's law (inelastic)

Experim. observations :

- Under hydrostatic loading, volume change remains elastic : $\epsilon_{ii} = \frac{1}{K} \sigma_{jj}$
- Shear loading may produce inelastic response

Separate applied stress, into the hydrostatic and shear parts:

$$\sigma_{ij} = \underbrace{\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)}_{\text{stress deviator, } \sigma'_{ij}} + \underbrace{\frac{1}{3}\sigma_{kk}\delta_{ij}}_{\text{hydrost. part}}$$

Note: $\sigma'_{ii} = 0$

Same for strains:

$$\epsilon_{ij} = \underbrace{\left(\epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij}\right)}_{\text{strain deviator } \epsilon'_{ij}} + \underbrace{\frac{1}{3}\epsilon_{kk}\delta_{ij}}_{\text{volume change}}$$

plastic solid : $d\epsilon'_{ij}$ in terms of σ'_{ij}

Creep of metals (high temp.) : $\dot{\epsilon}'_{ij} = B(T)\, \sigma'^{\,m}_{ij}$

plus : elastic response to the hydrostatic part of σ_{ij}