U.S. ELECTORAL SYSTEM IN NEED OF CHANGE?

A COMPUTATIONAL STUDY OF TWO SIMPLE ALTERNATIVES TO PLURALITY VOTING

An Honors Thesis for the Department of Mathematics

Author: Nuo Yuan

Advisor: Prof. Christoph Börgers

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Abstract

This study was motivated by FairVote's consistent effort to advocate for the nationwide adoption of instant run-off (IRV) in place of time-honored plurality voting. Since the election of Condorcet candidate is considered to be an acceptable outcome for the entire electorate, and both IRV and plurality voting can fail to elect the Condorcet candidate, the first question that this study attempts to provide an answer to is how likely Condorcet breakdown is to take place in an election using IRV or plurality voting? Moreover, in light of the fact that there are voting methods completely free from Condorcet breakdown (i.e., Condorcet-fair methods), but that they incentivize voters to bury the likely winner, the second question that this study seeks to answer is how likely a bloc of voters can get their favorite candidate elected, after the tie-breaking round using Borda count or IRV, by burying the Condorcet candidate?

The computational setup for this study is a reasonable simplification of reality. Specifically, the author assumes that the voters' judgments about the candidates are single-issue oriented, giving rise to the one-dimensional left-right random election model. For the candidates, their distributions were generated by one of the three following distributions: continuous uniform distribution in [-1,1], standard Gaussian distribution, and the Cauchy distribution. For the voters, their distribution along the left-right spectrum was also determined by one of the three distributions from which the candidates' positions were drawn. Consequently, voters would rank the candidates according to their spatial proximities to them, giving rise to a preference schedule at the end.

The computational results of the study indicate that although the probability of Condorcet breakdown is, in the majority of cases for three, four, and five candidates, lower in an election conducted in IRV than one conducted in plurality voting, IRV still admits a nonnegligible level of Condorcet breakdown, and the probability seems to increase as the candidate pool gets larger. On the contrary, the probability of successful burying associated with pairwise comparison with Borda count as the tie-breaker is reasonably low both in a relative sense, when compared to that associated with pairwise comparison with IRV as the tie-breaker, and in an absolute sense. Moreover, the probability seems to decrease as the number of candidates increases.

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Introduction

Motivation and Purpose

The primary motivation for this study stems from efforts to replace the traditional plurality voting method commonly used in elections at various levels in the United States with the instant run-off method taken by a group called FairVote.[1] Founded in 1992, the organization has demonstrated a sustained commitment to the nationwide adoption of IRV due to its belief that IRV promotes fairer elections and increases voters' representation in and access to elections. Moreover, within the United States, IRV has been used in statewide elections in Maine and citywide elections in places such as Minneapolis, San Francisco, New Mexico, and Burlington, suggesting that IRV is not an entirely new idea.

Being informed by FairVote's determination to realize nationwide adoption of IRV, I want to test the desirability of this possible change by studying the likelihood of one important drawback associated with IRV— "Condorcet breakdown." In an election, the Condorcet candidate is someone who would prevail in all one-on-one contests against the other candidates. Therefore, Condorcet breakdown presents itself in situations where the Condorcet candidate fails to be elected given that candidate exists. In fact, Condorcet breakdown occurred in the 2009 mayoral election in Burlington, Vermont when candidate Bob Kiss claimed the victory after three run-off rounds but candidate Andy Montroll was the Condorcet candidate.

However, it is not the case that all voting methods allow Condorcet breakdown: voting methods such as pairwise comparison guarantee the Condorcet candidate to be the election winner, provided that there is a Condorcet candidate. Collectively, these voting methods are called Condorcet-fair methods. In fact, pairwise comparison has received vehement support from Nobel laureates Eric Maskin and Amartya Sen as an election method capable of electing candidates "who truly command majority support" ("How Majority Rule Might Have Stopped Donald Trump," *The New York Times*).

Nevertheless, Condorcet-fair methods are not perfect, and one oft-cited reason against their use in real-life elections is that they "invite" strategic voting.[2] In an election with candidate A being the likely Condorcet candidate and candidate B the second strongest candidate, voters who favor candidate B the most can, contrary to their genuine judgment about the candidates, "bury" candidate A by placing them at the bottom of their ballots. Consequently, candidate A may no longer be the Condorcet candidate: Specifically, in situations where there is a tie (i.e., there are multiple winners), a tie-breaker is needed, which might hand the victory to candidate B. In such case, voters who most favor candidate B enabled their favorite candidate to win the tie-breaking election by committing what I call "successful burying." Therefore, I will investigate the probability of successful burying associated with pairwise comparison. Since the question of which alternative should be used to select one candidate out of the candidate pool is trivial in the case where the candidate pool only contains two candidates, the computational analysis for both IRV and plurality voting will be conducted for election simulations with three, four, and five candidates.

Central Questions

1. How likely is Condorcet breakdown to happen in a real-life election using instant run-off?
2. How likely is it that a bloc of voters succeeds in getting their favorite candidate elected, after the tie-breaking election using Borda count or IRV, by burying the Condorcet candidate?

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[1] Instant run-off, and all the other voting methods discussed in this study, will be clearly explained in the Methods section.

[2] By "invite," I mean that voters' temptation for strategic voting is much greater in an election using some Condorcet-fair method than one using IRV.

Methods

Plurality Voting

In an election using the plurality voting method, voters are only required to indicate their most favored candidates on the ballot. The winner of the election using plurality voting is simply the candidate receiving the most first-preference votes.

Instant Run-Off

In an election utilizing the instant run-off method, voters are asked to rank the candidates, resulting in a preference schedule consisting of different rankings of candidates together with the number of ballots corresponding to each ranking. If a candidate wins a majority in the first-preference votes, he or she is declared the winner of the election. Otherwise, the candidate with the fewest first-preference votes is eliminated, and the resulting gaps in the preference schedule are filled by the candidate ranked immediately below. A new tally is conducted to determine whether any candidate has won a majority of the adjusted votes. The process is repeated until a candidate wins at an outright majority.

Borda Count

Similar to IRV, the Borda count voting method requires voters to rank all candidates. On each voter's ballot, the first-preference candidate receives as many points as there are candidates (e.g., if there are in total four candidates running for an election, the first-place candidate for each voter earns four points). For all the remaining candidates, each candidate receives one fewer point than the candidate immediately above them does (following the above example, the second-preference candidate receives three points, the third-preference candidate two points, and the last-preference candidate one point). Once this process is done for all voters, the candidate with the most points is declared the winner of the election.

Pairwise Comparison

Similar to instant run-off and Borda count, the pairwise comparison voting method requires voters to rank all candidates to generate a preference schedule. Subsequently, each possible pair of candidates is placed in a one-on-one contest. For instance, in the one-on-one battle between candidate A and candidate B, ballots listing candidate A higher than candidate B are counted, and the number is compared with the number of ballots listing candidate B higher than candidate A. If the first number is higher than the second number, then candidate A gets one point and candidate B zero point. If the first number equals the second number, both candidate A and candidate B receive half a point. If the first number is smaller than the second number, candidate B receives one point and candidate A zero point. This process is repeated for every possible pair of candidates, and the candidate with the most points wins the election.

Further Illustration

To further illustrate the differences among the voting methods aforementioned, let's take a look at the following preference schedule:

Table 1. A Preference Schedule that Gives Different Outcomes for Dfferent Voting Methods

6	4	3	2
A	B	C	D
C	D	D	B
B	C	B	C
D	A	A	A

In plurality voting, candidate A should be declared the winner. However, if the election was conducted in IRV, the following would be the outcome:

Therefore, under instant run-off, candidate B would be the winner. If the voting method was Borda count, the result would look like this:

6	4	3	2
A	B	C	D
C	D	D	B
B	C	B	C
D	A	A	A

→

6	4	3	2
A	B	C	B
C	C	B	C
B	A	A	A

→

6	4	3	2
A	B	B	B
B	A	A	A

→

6	4	3	2
B	B	B	B

# of points candidate A gets = $6\times 4 + 4\times 1 + 3\times 1 + 2\times 1 = 33$
# of points candidate B gets = $6\times 2 + 4\times 4 + 3\times 2 + 2\times 3 = 40$
# of points candidate C gets = $6\times 3 + 4\times 2 + 3\times 4 + 2\times 2 = 42$
# of points candidate D gets = $6\times 1 + 4\times 3 + 3\times 3 + 2\times 4 = 35$

According to the scoreboard above, candidate C would be the winner of Borda count. Similarly, in pairwise comparison:

<u>A vs. B</u>
Nine voters prefer candidate B to candidate A. Therefore, candidate B would win the one-on-one contest against candidate A, giving candidate A zero point and candidate B one point.

<u>A vs. C</u>
Nine voters prefer candidate C to candidate A. Therefore, candidate C would win the one-on-one contest against candidate A, giving candidate A zero point and candidate C one point.

<u>A vs. D</u>
Nine voters prefer candidate D to candidate A. Therefore, candidate D would win the one-on-one contest against candidate A, giving candidate A zero point and candidate D one point.

<u>B vs. C</u>
Nine voters prefer candidate C to candidate B. Therefore, candidate C would win the one-on-one contest against candidate B, giving candidate B zero point and candidate C one point.

<u>B vs. D</u>
Ten voters prefer candidate B to candidate D. Therefore, candidate B would win the one-on-one contest against candidate D, giving candidate B one point and candidate D zero point.

<u>C vs. D</u>
Nine voters prefer candidate C to candidate D. Therefore, candidate C would win the one-on-one contest against candidate D, giving candidate C one point and candidate D zero point.

Cumulatively, after all the possible one-on-one contests were completed, candidate A would have zero point, candidate B two points, candidate C three points, and candidate D one point, making candidate C the winner of pairwise comparison.

Criteria

Condorcet-fairness

A voting method satisfies the Condorcet fairness criterion if its winner is guaranteed to be the Condorcet candidate, given that such candidate exists. The notion of Condorcet candidate, first introduced by Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet, refers to the candidate that would beat every other candidate in a one-on-one contest.

Non-Manipulability

A voting method is susceptible to manipulation if voters can alter the election outcome by not revealing their true judgments about the candidates: For example, assume that we use plurality voting, with ties resolved alphabetically. Now consider the following situation:

Table 2. An Example from *Mathematics of Social Choice-Voting, Compensation, and Division* (2010)

4	3	2
B	A	C
A	B	A
C	C	B

According to table 2, candidate B is the winner. However, if one of the two voters who most prefer C and least prefer candidate B decided to dishonestly change their ballots by swapping candidate C and candidate A, the new preference schedule would look like:

Table 3. An Example from *Mathematics of Social Choice-Voting, Compensation, and Division* (2010)

4	3	1	1
B	A	C	A
A	B	A	C
C	C	B	B

In table 3, there is now a tie between candidate A and candidate B. Following the rule that whoever has the first letter of their last name come earliest in the alphabet wins the election, candidate A should be declared the winner of the election. In short, the move by one of the voters who most favor candidate C to place candidate A atop and candidate C second, an act that doesn't reflect his/her genuine preferences for the candidates, could change the election result from candidate B winning the election to candidate A claiming the victory. Therefore, the election method discussed above is manipulable. In fact, according to the Gibbard-Satterthwaite theorem, the non-manipulability criterion is unrealizable by voting methods except dictatorship.

Gibbard-Satterthwaite Theorem

Laying out the substance of this theorem requires knowing what Pareto-efficiency means. Specifically, a voting method is Pareto-efficient if it guarantees that candidate Y is the sole winner if he/she is ranked first by every voter (Börgers 2010, p.62). With Pareto-efficiency clearly defined, the Gibbard-Satterthwaite theorem states that for more than two candidates, the only single-winner method that is Pareto-efficient and non-manipulable is dictatorship (Börgers 2010, p.69).

Later-No-Harm Criterion

A voting method satisfies the later-no-harm criterion if ranking a voter's less-preferred candidate further down the list can't harm a voter's more-preferred candidate. Regarding the focus of this study, IRV is a complying method, whereas pairwise comparison fails to comply with the later-no-harm criterion. IRV satisfies the later-no-harm criterion because a voter's less-preferred candidate would not even be taken into consideration until a more-preferred candidate was eliminated. The reason why pairwise comparison fails the later-no-harm criterion is made clear by the following example:

Table 4. A Preference Schedule that Illustrates Pairwise Comparison Being a Noncomplying Method for the Later-no-Harm Criterion

ranking \ percentage	30%	25%	20%	25%
1st	A	B	B	C
2nd	B	A	C	B
3rd	C	C	A	A

According to table 4, candidate B would win the pairwise comparison with candidate A as 70% of the ballots list candidate B ahead of candidate A, conferring on candidate B one point and candidate A no point. Moreover, candidate B would also prevail over candidate C as 75% of the voters prefer the candidate B to candidate C, giving candidate B one point and candidate C zero point. Finally, in a potential one-on-one contest between candidate A and candidate C, candidate A would win against candidate C as 55% of the voters placed candidate A ahead of candidate C, thereby handing over one point to candidate A and no point to candidate C. Therefore, candidate B is the Condorcet candidate and, therefore, the pairwise comparison winner. This election outcome is no good news for the first category of voters (i.e., voters whose candidate ranking is A→B→C), especially when they realize that by revealing their honest opinion that candidate B is their second-preferred candidate, they have indirectly done harm to their most-favored candidate by enabling candidate B to win pairwise comparison. To see why this is the case, let's assume that the first category of voters had recognized the harm honestly ranking candidate B as their second-preferred candidate could have done to their most-favored candidate, and thereupon decided to remove candidate B from their ballots and move candidate C one slot up the ranking, resulting in the following preference schedule:

Table 5. The Preference Schedule after the Strategic Move by the First Category of Voters

ranking \ percentage	30%	25%	20%	25%
1st	A	B	B	C
2nd	C	A	C	B
3rd	blank	C	A	A

According to table 5, after the strategic move by the first category of voters, candidate B would lose the battle against candidate C as 55% of the ballots rank candidate C above candidate B. Since the outcomes of the other two one-on-one comparisons remain unchanged, the preference schedule as shown in table 5 would result in a three-way tie. For the first category of voters, this tie represents a better outcome than candidate B straight winning the election as some tie-breaking mechanism could make candidate A the ultimate winner.

A Special Kind of Strategic Voting: Burying the Likely Winner

For the purpose of this research, I will focus on a special type of strategic voting: voters try to prevent the candidate that they predict to be the winner of the election from claiming the victory by placing that candidate last on their ballots, or "burying the likely winner." By successful burying, I refer to situations where voters succeed in getting their most favored candidate elected by burying the likely winner.

The Condorcet Candidate isn't Always Present in a Real-life Election

Table 6. A Preference Schedule that Results in no Condorcet Candidate

20%	11%	18%	20%	20%	11%
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

According to table 6, candidate A would prevail in a one-on-one contest against candidate B by a positive margin of 2% (51% of the voters ranking candidate A ahead of candidate B versus 49% of them ranking candidate B ahead of candidate A), conferring on candidate A one point and candidate B zero point. In a one-on-one competition between candidate B and candidate C, candidate B would claim the victory by a positive margin of 16% (58% of the voters ranking candidate B ahead of candidate C versus 42% of them having the reverse ranking), giving candidate B one point and candidate C zero point. Finally, regarding the relative popularity of candidate C and candidate A, candidate C would be more popular than candidate A by a positive margin of 2% (51% of the voters ranking candidate C ahead of candidate A and 49% of them the other way around), handing one point to candidate C and zero point to candidate A. Therefore, after all the one-on-one comparisons, each candidate would have one point, giving rise to no Condorcet candidate.

Can the Pairwise Comparison Winner Fail to be the Condorcet Candidate?

First, if the Condorcet candidate existed and if pairwise comparison produced a unique winner, the winner of pairwise comparison would necessarily be the Condorcet candidate since the number of points required for a candidate to be the Condorcet candidate after one-on-one contests among all the candidates is the maximum number of points any candidate could get.

However, if the assumption about the existence of the Condorcet candidate was dropped, the picture would be different: As the example below shows, the pairwise comparison winner can fail to be the Condorcet candidate when the number of candidates is greater than two.

Table 7. A Preference Schedule Whose Pairwise Comparison Winner is not the Condorcet Candidate

12.5%	25%	12.5%	25%	20%	5%
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

According to table 7, candidate A would emerge victorious in a one-on-one contest against candidate B because the percentage of voters preferring candidate A to candidate B equals 57.5% (12.5%+25%+20%), enabling candidate A to get one point and candidate B zero point. Regarding the relative popularity between candidate A and candidate C, there is a tie between the two candidates for half of the voters prefer candidate A to candidate C and the other half the other way around, resulting in both candidate A and candidate C gaining one half point each. Lastly, the result of a tournament between candidate B and candidate C would also be a tie, granting both candidate B and candidate C one half point. Cumulatively, the final score board looks like this: candidate A one and a half point, candidate B half point, and candidate C one point. Although candidate A is the winner of pairwise comparison, he/she isn't the Condorcet candidate because he/she doesn't have two points, the number of points required for the Condorcet candidacy in the case of three candidates.

In cases where any one-on-one contest between two candidates cannot turn out to be a tie, the situation regarding the relations between pairwise comparison winner and the Condorcet candidate is different from the situation where a tie is allowed.

For two candidates, the reason for the equivalence between the pairwise comparison winner and the Condorcet winner is trivial (i.e., in a pairwise comparison election with only candidate A and candidate B competing for which a tie is not allowed, either candidate A or candidate B needs to win the one-on-one contest between them, giving the winner one point and simultaneously making the winner the Condorcet candidate).

For three candidates, the pairwise comparison winner and the Condorcet winner are also equivalent. Specifically, two points are required for someone to claim the Condorcet status. Moreover, assume that the pairwise comparison winner only had one point, then there would be two points left to be distributed between the two other candidates (the total points available to be distributed are three). If one of them had two points, this would contradict the assumption that the pairwise comparison winner only had one point. If each of them had one point, then there would be a tie among the three candidates, contradicting the implicit assumption that pairwise comparison only gives rise to a unique winner. Therefore, whoever wins the pairwise comparison election needs to have two points, making him/her simultaneously the Condorcet winner.

For four candidates, the equivalence between the pairwise comparison winner and the Condorcet winner is equally valid. First, note that the total points available to be distributed among the four candidates are six. Next, for someone to be the Condorcet winner, he/she needs to prevail over all the other three candidates, giving him/her three points. Now, assume that the winner of pairwise comparison only had two points. Consequently, there would be four points left to be distributed among the three other candidates. According to the pigeonhole principle, in the case when a tie is not allowed, at least one of the three remaining candidates would have more than one point. In the case when someone out of the three unassigned candidates has two points, it contradicts the implicit assumption that pairwise comparison only gives rise to a unique winner. When someone among the three unassigned candidates gets hold of all the three points, it contradicts the assumption that the pairwise comparison winner only has two points. Therefore, the pairwise comparison winner needs to obtain three points, making him/her simultaneously the Condorcet winner.

For five candidates, the equivalence between the pairwise comparison winner and the Condorcet winner breaks down. More precisely, someone can win pairwise comparison without being the Condorcet winner. For example, in a pairwise comparison election involving candidate A, candidate B, candidate C, candidate D, and candidate E. The pairwise comparison diagram can look something like this:

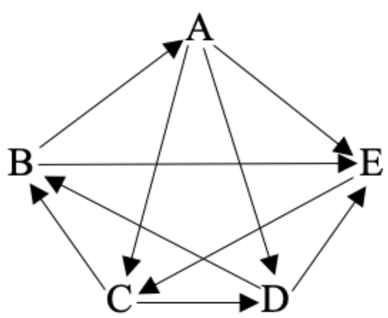

Figure 1. A Pairwise Comparison Diagram Whose Unique Winner Fails to be the Condorcet Candidate

According to figure 1, candidate A has three points, candidate B, candidate C, and candidate D each have two points, and candidate E has one point. Therefore, candidate A is the winner of pairwise comparison, but is one point short of being the Condorcet candidate. However, is the pairwise comparison diagram in figure 3 realizable? That is, does the diagram correspond to some preference schedule? The answer is yes, and the way to produce one corresponding preference schedule is fairly straightforward. Specifically, I can break down the pairwise comparison diagram in figure 1 into the following table:

Table 8. Pairwise Comparison Outcomes by Candidates[3]

Candidate	A	B	C	D	E
	A → C A → D A → E	B → A B → E	C → B C → D	D → B D → E	E → C

According to table 8, the electorate collectively prefers candidate A to candidate C. Now, if I could come up with some arrangement of ballots that rank candidate A ahead of candidate C more often than otherwise while ensuring a tie in all the other nine one-on-one comparisons (e.g., candidate A against D, candidate A against candidate E, and etc.), I could repeat this process of ballot design for each of the nine remaining one-on-one contests according to the contest results documented in table 6. Now, let's consider how the ballots should be designed to have the previously mentioned feature for the one-on-one tournament between candidate A and candidate C. Let the first ballot have the following candidate ranking: A→C→B→D→E. Let the second ballot be the exact reverse of the first one, except keeping candidate A immediately above candidate C, resulting in the following candidate ranking: E→D→B→A →C. Now, candidate A is ranked ahead of candidate C on both ballots. However, for each of the other candidate pairs, the one-on-one competition turns out to be a tie.

To summarize, if the Condorcet candidate existed and if pairwise comparison produced a unique winner, the pairwise comparison winner would also be the Condorcet candidate regardless of the size of the candidate pool. However, when the assumption about the existence of the Condorcet candidate was dropped, the pairwise comparison winner could fail to be the Condorcet candidate in an election with as few as three candidates if ties were allowed. If ties were not allowed, the pairwise comparison winner would necessarily be the Condorcet candidate with three and four candidates, but could fail to attain the Condorcet candidacy starting from five candidates.

Random Election Outcomes: Left-Right Model of Politics

For the purpose of this study, it is assumed that voters' judgments about the candidates are single-issue oriented, giving rise to the one-dimensional left-right random election model.

Specifically, given N candidates, their positions on the left-right spectrum will be generated by one of the following three candidate distributions: continuous uniform distribution in [-1,1], normal distribution with mean 0 and standard deviation 1, and the Cauchy distribution, listed in order of an increasingly polarized candidate pool.

Regarding the voters, their positions will also be assumed to be randomly distributed according to each of the following three voter distributions: uniform distribution in [-1,1], normal distribution with mean 0 and standard deviation 1, and the Cauchy distribution, listed

[3] All the "→" in the table signify that the candidate coming before the arrow is preferred to the candidate coming after the arrow.

in order of an increasingly polarized electorate. Then, voters will rank the N candidates according to their spatial proximity to candidates' positions on the left-right spectrum, generating a preference schedule given some pair of the candidate and the voter distributions.

For example, in an election with candidate A, candidate B, and candidate C whose positions are generated by the standard normal distribution, what would the left-right politics model and the final preference schedule look like if voters' positions are also distributed according to the standard normal distribution?

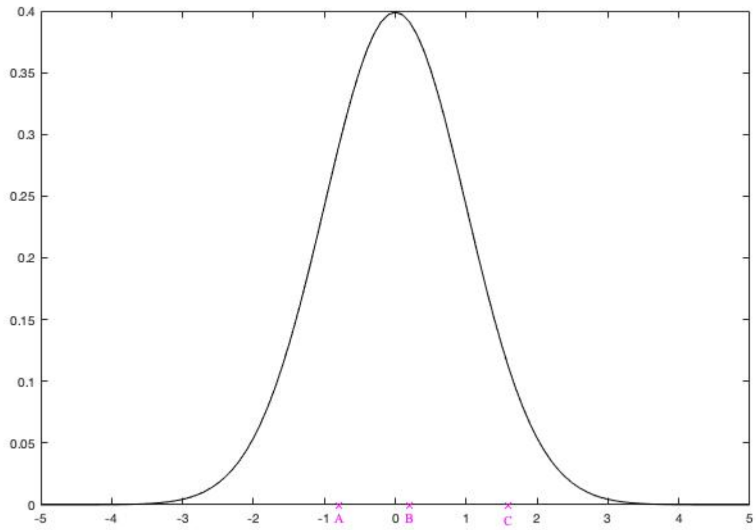

Figure 2. A Left-Right Politics Model with both the candidate and the voter distributions being N (0,1)

Figure 2 shows a left-right politics model when both the candidate and the voter distributions are N (0,1). Based on their spatial proximities to the positions of candidate A, candidate B, and candidate C, the voters can be grouped into the following four camps: the left camp, the center-left camp, the center-right camp, and the right camp, each corresponding to one of the four ballot arrangements: A→B→C, B→A→C, B→C→A, and C→B→A. Specifically, 38.08% of the voters are in the left camp, 27.15% in the center-left camp, 16.17% of them in the center-right camp, and 18.6% of them in the right camp, giving rise to the preference schedule in table 9:

Table 9. The Preference Schedule Corresponding to the Left-Right Politics Model in Figure 2

Left camp	Center-left camp	Center-right camp	Right camp
38.08% of the ballots	27.15% of the ballots	16.17% of the ballots	18.6% of the ballots
A	B	B	C
B	A	C	B
C	C	A	A

Now, what would happen if I scaled and/or shifted both the candidate and the voter distributions the same way. To be more precise, given some candidate distribution $\rho(x)$ that generates two candidates' positions x_1 and x_2, and some voter distribution $\phi(x)$ that generates some voter's position y. Assume, without any loss of generality, that x_1 was closer to y than x_2 was. What would happen to the relations between the distance between x_1 and y and the distance between x_2 and y if $\tilde{\rho}(x) = \frac{1}{c}\rho(\frac{x-m}{c})$ and $\tilde{\phi}(x) = \frac{1}{c}\phi(\frac{x-m}{c})$, $c > 0$ and $m \in \mathbb{R}$? After scaling and/or shifting both the candidate and the voter distributions the same way, x_1 and x_2 would become $cx_1 + m$ and $cx_2 + m$, and y would become $cy + m$. Therefore, if x_1 was closer to y than x_2 before the transformation, this would still be the case after it. The significance of this observation is that as long as both the candidate and the voter distributions have undergone the same transformation (i.e., scaling and/or shifting), the transformation would not change the resulting preference schedule. However, when the transformation applied to the candidate distribution is different from the transformation applied to the voter distribution, the post-transformation preference schedule would be different from the pre-transformation preference schedule.

There Always Exists a Condorcet Candidate in the Left-Right Model

To be more mathematically rigorous, the purpose of this section is to prove the following theorem:

Theorem 1. *In an election conducted in the left-right model with N candidates whose positions are generated by some candidate distribution and an odd number of voters generated by some voter distribution, there necessarily exists a Condorcet candidate.*

To prove theorem 1, it is useful to prove the following lemma first:

Lemma 1. *In an election with an odd number of voters, if there is no Condorcet candidate, then there exists at least one preference cycle. For N candidates, a preference cycle exists if there is a preference chain in which one candidate appears two times.*

Proof of Lemma 1

Without loss of generality, let pick candidate X_1 from the candidate pool of size N. Since X_1 is not the Condorcet candidate, there necessarily exists candidate X_2 who beats candidate X_1. Again, since candidate X_2 cannot be the Condorcet candidate, there should be candidate X_3 whom is preferred to X_2. Since there are finitely many candidates, there must be candidate X_z that appears at least two times in this preference chain. Therefore, the chain segment between the first two appearances of candidate X_z constitutes a preference cycle.

Accounting for the lemma proven above, I only need to prove the following theorem:

Theorem 2. *In an election conducted in the left-right model with N candidates whose positions are generated by some candidate distribution and an odd number of voters generated by some voter distribution, there doesn't exist any preference cycle. In other words, in an election conducted in the left-right model, for candidates A, B, C whose positions are generated by some candidate distribution,* $A \rightarrow B$ *and* $B \rightarrow C$ *imply* $A \rightarrow C$.

Proof of Theorem 2

Denote the cumulative density function of the voter distribution by $F_X(x)$. Note that there are six possible spatial arrangements for candidates A , B, and C on the left-right spectrum: $a<b<c$, $a<c<b$, $b<a<c$, $b<c<a$, $c<a<b$, and $c<b<a$. Therefore, I will attempt to prove theorem 2 in all the six possible cases.

1. $a<b<c$

To start with, it is known that $\frac{a+b}{2} < \frac{a+c}{2} < \frac{b+c}{2}$.

A→B implies 1-$F_X(\frac{a+b}{2}) < F_X(\frac{a+b}{2})$, or $F_X(\frac{a+b}{2})>\frac{1}{2}$.

B→C implies 1-$F_X(\frac{b+c}{2})< F_X(\frac{b+c}{2})$, or $F_X(\frac{b+c}{2})>\frac{1}{2}$.

For A→C, I need 1-$F_X(\frac{a+c}{2})< F_X(\frac{a+c}{2})$, or $F_X(\frac{a+c}{2})>\frac{1}{2}$.

Since $F_X(x)$ is monotonically increasing, $\frac{a+b}{2} < \frac{a+c}{2}$ and $F_X(\frac{a+b}{2})>\frac{1}{2}$ imply $F_X(\frac{a+c}{2})>\frac{1}{2}$, or A→C.

2. a<c<b

To start with, it is known that $\frac{a+c}{2} < \frac{a+b}{2} < \frac{b+c}{2}$.

A→B implies 1-$F_X(\frac{a+b}{2}) < F_X(\frac{a+b}{2})$, or $F_X(\frac{a+b}{2})>\frac{1}{2}$.

B→C implies 1-$F_X(\frac{b+c}{2})>F_X(\frac{b+c}{2})$, or $F_X(\frac{b+c}{2})<\frac{1}{2}$.

However, since $F_X(x)$ is monotonically increasing, $\frac{a+b}{2} < \frac{b+c}{2}$ and $F_X(\frac{a+b}{2})>\frac{1}{2}$ imply $F_X(\frac{b+c}{2})>\frac{1}{2}$, constituting a contradiction on the a priori assumption that $F_X(\frac{b+c}{2})<\frac{1}{2}$. Therefore, this scenario is not possible.

3. b<a<c

To start with, it is known that $\frac{a+b}{2} < \frac{b+c}{2} < \frac{a+c}{2}$.

A→B implies 1-$F_X(\frac{a+b}{2}) > F_X(\frac{a+b}{2})$, or $F_X(\frac{a+b}{2})<\frac{1}{2}$.

B→C implies 1-$F_X\frac{b+c}{2}$ ()<$F_X(\frac{b+c}{2})$, or $F_X(\frac{b+c}{2})>\frac{1}{2}$.

For A→C, I need 1-$F_X(\frac{a+c}{2})< F_X(\frac{a+c}{2})$, or $F_X(\frac{a+c}{2})>\frac{1}{2}$.

Since $F_X(x)$ is monotonically increasing, $\frac{b+c}{2} < \frac{a+c}{2}$ and or $F_X(\frac{b+c}{2})>\frac{1}{2}$ imply $F_X(\frac{a+c}{2})>\frac{1}{2}$, or A→C.

4. b<c<a

To start with, it is known that $\frac{b+c}{2} < \frac{a+b}{2} < \frac{a+c}{2}$.

A→B implies 1-$F_X(\frac{a+b}{2}) > F_X(\frac{a+b}{2})$, or $F_X(\frac{a+b}{2})<\frac{1}{2}$.

B→C implies 1-$F_X(\frac{b+c}{2})<F_X(\frac{b+c}{2})$, or $F_X(\frac{b+c}{2})>\frac{1}{2}$.

However, since $F_X(x)$ is monotonically increasing, $\frac{b+c}{2} < \frac{a+b}{2}$ and $F_X(\frac{a+b}{2})<\frac{1}{2}$ imply $F_X(\frac{b+c}{2})<\frac{1}{2}$, constituting a contradiction on the a priori assumption that $F_X(\frac{b+c}{2})>\frac{1}{2}$. Therefore, this scenario is not possible.

5. c<a<b

To start with, it is known that $\frac{a+c}{2}<\frac{b+c}{2}<\frac{a+b}{2}$.

A→B implies 1-$F_X(\frac{a+b}{2}) < F_X(\frac{a+b}{2})$, or $F_X(\frac{a+b}{2})>\frac{1}{2}$.

B→C implies 1-$F_X(\frac{b+c}{2})>F_X(\frac{b+c}{2})$, or $F_X(\frac{b+c}{2})<\frac{1}{2}$.

For A→C, I need 1-$F_X(\frac{a+c}{2})> F_X(\frac{a+c}{2})$, or $F_X(\frac{a+c}{2})<\frac{1}{2}$.

Since $F_X(x)$ is monotonically increasing, $\frac{a+c}{2}<\frac{b+c}{2}$ and $F_X(\frac{b+c}{2})<\frac{1}{2}$. imply $F_X(\frac{a+c}{2})<\frac{1}{2}$, or A→C.

6. c<b<a

To start with, it is known that $\frac{b+c}{2}<\frac{a+c}{2}<\frac{a+b}{2}$.

A→B implies 1-$F_X(\frac{a+b}{2}) >F_X(\frac{a+b}{2})$, or $F_X(\frac{a+b}{2})<\frac{1}{2}$.

B→C implies 1-$F_X(\frac{b+c}{2})>F_X(\frac{b+c}{2})$, or $F_X(\frac{b+c}{2})<\frac{1}{2}$.

For A→C, I need 1-$F_X(\frac{a+c}{2})> F_X(\frac{a+c}{2})$, or $F_X(\frac{a+c}{2})<\frac{1}{2}$.

Since $F_X(x)$ is monotonically increasing, $\frac{a+c}{2}< \frac{a+b}{2}$ and or $F_X(\frac{a+b}{2})<\frac{1}{2}$. imply $F_X(\frac{a+c}{2})<\frac{1}{2}$, or A→C. □

Condorcet Breakdown in Instant Run-Off and Plurality Voting

In this section, I will use computational simulations to study the first central question: How likely is Condorcet breakdown to happen in a real-life election using instant run-off? Since this study is aimed at forming a rigorous mathematical basis for/against the replacement of plurality voting by IRV in the United States, a change that FairVote has committed itself to for years, I will also conduct computational simulations to calculate the probability of Condorcet breakdown associated with plurality voting for comparison between the two voting methods in terms of susceptibility to Condorcet breakdown.

Regarding the simulation design, I have translated plurality voting and IRV into executable MATLAB codes that accept an arbitrary number of candidates. For the purpose of this study, I will only examine the likelihoods of Condorcet breakdown associated with plurality voting and IRV for three, four, and five candidates. Specifically, for some number of candidates and some pair of the candidate and the voter distributions, one million simulations were run for both plurality voting and IRV to generate the ratios between the number of Condorcet breakdowns and the total number of simulations, which was done twice to ensure statistical significance. The results are documented in table 10, table 11, and table 12.

Case 1: Three Candidates

Table 10. The Probabilities of Condorcet Breakdown in Different Combinations of the Candidate and the Voter Distributions in Elections Using IRV and plurality voting (note: the first entry in any grid represents the value corresponding to IRV and the second to plurality voting) for Three Candidates.

Voter distribution / Candidate distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0.194, 0.39	0.37, 0.49	0.50, 0.50
N (0,1)	0.27, 0.17	0.15, 0.29	0.28, 0.41
Cauchy distribution	0.70, 0.068	0.13, 0.12	0.11, 0.21

According to table 10, when the candidate distribution is U (-1,1), Condorcet breakdown becomes more likely for both IRV and plurality voting as the electorate becomes more politically extreme. Moreover, Condorcet breakdown associated with IRV is less common than that associated with plurality voting, despite by a smaller margin as political extremism increases within the constituency, partially confirming the ground taken by FairVote. However, judging by itself, IRV still allows for a fair chance of Condorcet breakdown, ranging from approximately one in every five elections to one in every two as the electorate is filled with more and more extremists.

When the candidate distribution is the standard normal distribution, the relations between the likelihood of Condorcet breakdown associated with IRV and that associated with plurality voting are less clear-cut than when the candidate distribution is U (-1,1). Specifically, when the voter distribution is U (-1,1), Condorcet breakdown is more likely to happen in an election using IRV than plurality voting. However, the opposite is the case when the voter distribution is standard normal or Cauchy. Similar to what would happen with U (-1,1) being the candidate distribution, IRV still admits a decent level of Condorcet breakdown.

Lastly, when the candidate distribution is the Cauchy distribution, one noticeable simulation outcome is the probability of Condorcet down associated with IRV when the voter distribution is U (-1,1): it suggests that Condorcet breakdown would happen in seven of every ten elections, which is a powerful piece of evidence against the adoption of IRV in place of plurality voting.

Summing up, although IRV presents a more promising picture of allowing for a lower chance of Condorcet breakdown than plurality voting in the majority of cases for three candidates, IRV is still not satisfactorily immune to Condorcet breakdown and, under certain assumptions, can become extremely vulnerable. Therefore, there is no sufficient ground for FairVote's call to replace plurality voting with IRV.

Case 2: Four Candidates

Table 11. The Probabilities of Condorcet Breakdown in Different Combinations of the Candidate and the Voter Distributions in Elections Using IRV and Plurality Voting (note: the first entry in any grid represents the value corresponding to IRV and the second to plurality voting) for Four Candidates.

Candidate distribution \ Voter distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0.35, 0.56	0.55,0.70	0.75, 0.75
N (0,1)	0.395, 0.329	0.30,0.50	0.42, 0.62
Cauchy distribution	0.81,0.17	0.24,0.26	0.23. 0.36

According to table 11, all the observations made in the case of three candidates are applicable when there are four candidates. Moreover, it is worth noting that in a case-by-case sense, Condorcet breakdown becomes more common for both IRV and plurality voting in the case of four candidates than three.

Case 3: Five Candidates

Table 12. The Probabilities of Condorcet Breakdown in Different Combinations of the Candidate and the Voter Distributions in Elections Using IRV and Plurality Voting (note: the first entry in any grid represents the value corresponding to IRV and the second to plurality voting) for Five Candidates

Candidate distribution \ Voter distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0.46, 0.66	0.64,0.82	0.88, 0.87
N (0,1)	0.51, 0.46	0.41, 0.57	0.50, 0.74
Cauchy distribution	0.89, 0.28	0.35, 0.38	0.34, 0.48

According to table 12, all the trends present in three and four candidates are also present in five candidates, with Condorcet breakdown becoming even more likely for both IRV and plurality voting in a case-by-case sense.

Conclusion

In light of the foregoing observations, it is reasonable for me to arrive at the following conclusion regarding the first central question of the study: IRV does reduce the probability of Condorcet breakdown except for a few cases. However, with the existence of Condorcet-fair methods in mind, the lower likelihood of Condorcet breakdown associated with IRV in comparison with that of plurality voting doesn't suggest that IRV should take the place of plurality voting because IRV still permits a nonnegligible chance of Condorcet breakdown that, based on an extrapolation from the trend discovered in the cases of three, four, and five candidates, increases as the candidate pool gets more numerous.

Successful Burying is Impossible in Instant Run-Off

In an election using IRV, voters cannot achieve successful burying for the simple reason that their other choices don't matter until their first choice is eliminated.

Burying can Turn the Condorcet Candidate into a Non-Condorcet Candidate

Table 13. A Preference Schedule that Allows the Deprivation of Condorcet Status by Burying the Likely Winner in an Election Using any Condorcet-fair Method in the Left-Right Model of Politics

20%	31%	18%	31%
A	B	B	C
B	A	C	B
C	C	A	A

According to table 13, candidate B would prevail over both candidate A and candidate C in a potential one-on-one contest, making candidate B the Condorcet candidate (in a one-one-one comparison between candidate A and candidate B, B would win since 80% of the voters placed candidate B ahead of candidate A; in a one-on-one tournament between candidate B and candidate C, candidate B would also win since 69% of the voters placed candidate B ahead of candidate C). Now, let's assume that the voters who most favor candidate A, the second most-favored candidate in the election (in a potential one-on-one contest between candidate A and candidate C, candidate A would emerge victorious since 51% voters ranked candidate A ahead of candidate C), decided to alter the election result by burying candidate B. Would they succeed in depriving candidate A of his/her Condorcet status with their burying attempt? The answer is yes, as shown by table 6 below.

Table 14. The Resulting Preference Schedule After Burying

20%	31%	18%	31%
A	B	B	C
C	A	C	B
B	C	A	A

In table 14, by swapping candidate B with candidate C, the strategic voters, or those whose most-favored candidate is candidate A, would manage to change the one-on-one comparison outcome between candidate B and candidate C, with the results of other one-on-one contests unchanged. Now, candidate B would lose the battle against candidate C by a negative margin of 2%, resulting in a tie among the three candidates with no Condorcet winner (the one point lost by candidate B goes to candidate C). In short, the burying move taken by the strategic voters succeed in depriving candidate B of his/her Condorcet status.

Successful Burying in Pairwise Comparison with Borda Count as the Tie-Breaker

In this section, I will try to answer the first half of the second central question of the study: How likely are the strategic voters to get their favorite candidate elected by voting to bury in a tie-breaking election using Borda count after pairwise comparison resulted in a tie?

To achieve the section objective, I incorporated Borda count into my pairwise comparison MATLAB program and set it to run in situations where there is a tie after pairwise comparison. As before, I ran one million simulations for each combination of the candidate and the voter distributions to get the ratio of the number of the instances of successful burying and the total number of simulations, which was repeated twice to ensure that the output is statistically significant. All the outputs are documented in table 15, table 16, and table 17.

Case 1: Three Candidates

Table 15. The Probabilities that Strategic Voters' First-Preference Candidate Wins the Tie-Breaking Election Using Borda Count in Different Combinations of the Candidate and the Voter Distributions for Three Candidates

Voter distribution / Candidate distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0.028	0.0020	0
N (0,1)	0.035	0.054	0.041
Cauchy distribution	0.014	0.031	0.057

According to table 15, when the candidate distribution is U (-1,1), successful burying with Borda count as the tie-breaker becomes less and less likely as the electorate gets more and more politically extreme. Specifically, when the electorate is distributed according to the Cauchy distribution, successful burying associated with Borda count as the tie-breaker becomes strictly impossible, or so rare that MATLAB cannot capture it.[4] However, a consistent trend is absent from the situation where the candidate distribution is standard Gaussian. When the candidate distribution is Cauchy, a rising trend is present regarding the probability of successful burying with Borda count as the tie-breaker as the electorate becomes more and more polarized.

[4] Regarding all the zero outputs documented in the tables hereinafter, I will regard them as truly zero.

Despite the trends noted above, successful burying nevertheless remains fairly unlikely in an election using pairwise comparison with Borda count as the tie-breaker for three candidates, ranging from approximately six in every one hundred elections to strictly zero.

Case 2: Four Candidates

Table 16. The Probabilities that Strategic Voters' First-Preference Candidate Wins the Tie-Breaking Election Using Borda Count in Different Combinations of the Candidate and the Voter Distributions for Four Candidates

Candidate distribution \ Voter distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0	0	0
N (0,1)	0.0090	0.0044	0.0014
Cauchy distribution	0.012	0.018	0.024

According to table 16, when the candidate distribution is U (-1,1), an interesting set of results emerges: successful burying is impossible for all three types of the voter distribution. For situations where the candidate distributions are standard Gaussian and Cauchy, the observations regarding trend are also applicable to the case of four candidates.

For four candidates, successful burying remains unlikely for all combinations of the candidate and the voter distributions. Moreover, on a case-by-case basis, the likelihood of successful burying deceases as the number of candidates increases from three to four.

Case 3: Five Candidates

Table 17. The Probabilities that Strategic Voters' First-Preference Candidate Wins the Tie-Breaking Election Using Borda Count in Different Combinations of the Candidate and the Voter Distributions for Five Candidates

Candidate distribution \ Voter distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0	0	0
N (0,1)	0.00094	0.00012	0.000012
Cauchy distribution	0.0050	0.0056	0.0052

According to table 17, Borda count again proves itself to be perfectly immune to successful burying for all three types of the voter distribution when candidates' positions are drawn from U (-1,1). When the candidate distribution is the standard Gaussian distribution, the probability of successful burying first decreases, by a significant percentage, when the voter distribution changes from U (-1,1) to standard normal, and then stays unchanged when the voter distribution is Cauchy. When the candidate distribution is Cauchy, there is a lack of clear trend in how the likelihood of successful burying changes as the electorate becomes more polarized.

On a case-by case basis, the probability of successful burying reduces further for five candidates compared to four candidates.

Conclusion

In light of the foregoing observations on table 15, table 16, and table 17, pairwise comparison with Borda count as the tie-breaker makes successful burying reasonable unlikely. Moreover, based on an extrapolation of the trend seen in the cases of three, four and five candidates, the bigger the candidate pool is, the more unlikely successful burying gets in an election using pairwise comparison with Borda count as the tie-breaker.

Successful Burying in Pairwise Comparison with IRV as the Tie-Breaker

In this section, I will answer the second half of the second central question of this study: How likely are the strategic voters to succeed in getting their favorite candidate elected in a tie-breaking election using IRV after pairwise comparison gave rise to a tie?

For the purpose of this section, I combined my IRV MATLAB program with the pairwise comparison MATLAB program so that the IRV program would run in situations where pairwise comparison resulted in a tie. As always, one-million simulations were run for each combination of the candidate and the voter distributions to get the ratio between the number of cases of successful burying and the total number of simulations, which was repeated twice to ensure the statistical significance of the output. All the outputs are documented in table 17, table 18, and table 19.

Case 1: Three Candidates

Table 18. The Probabilities that Strategic Voters' First-Preference Candidate Wins the Tie-Breaking Election Using IRV in different Combinations of the Candidate and the Voter Distributions for Three Candidates

Candidate distribution \ Voter distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0.098	0.18	0.25
N (0,1)	0.039	0.073	0.14
Cauchy distribution	0.017	0.030	0.054

According to table 18, when the candidate distribution is U (-1,1), successful burying becomes more likely as the electorate gets more polarized. Similarly, when the candidate distribution is the standard Gaussian, successful burying becomes more common as the percentage of extreme voters increases in the electorate. An increasing trend is also observed when the candidate distribution is Cauchy.

In a case-by-case basis, compared to the probabilities of successful burying associated with Borda count, those associated with IRV are markedly higher.

Case 2: Four Candidates

Table 19. The Probabilities that Strategic Voters' First-Preference Candidate Wins the Tie-Breaking Election Using IRV in Different Bombinations of the Candidate and the Voter Distributions for Four Candidates

Candidate distribution \ Voter distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0.11	0.19	0.25
N (0,1)	0.052	0.089	0.14
Cauchy distribution	0.0170	0.042	0.066

According to table 19, all the trend observations noted in the case of three candidates are also applicable to four candidates.

On a case-by-case basis, successful burying is much less likely with Borda count as the tie-breaker than with IRV.

Case 3: Five Candidates

Table 20. The Probabilities that Strategic Voters' First-Preference Candidate Wins the Tie-Breaking Election Using IRV in Different Combinations of the Candidate and the Voter Distributions for Five Candidates

Candidate distribution \ Voter distribution	U (-1,1)	N (0,1)	Cauchy distribution
U (-1,1)	0.067	0.13	0.19
N (0,1)	0.038	0.06	0.088
Cauchy distribution	0.010	0.035	0.049

According to table 20, all the trend observations noted in the cases of three and four candidates are also applicable to five candidates.

On a case-by-case basis, successful burying is much less likely with Borda count as the tie-breaker than with IRV.

Summary

When judged on its own, pairwise comparison with IRV as the tie-breaker doesn't do a sufficiently good job of defending against successful burying, allowing it to happen as frequently as one in every four elections under certain assumptions. Moreover, compared to pairwise comparison with Borda count as the tie-breaker, pairwise comparison with IRV as the tie-breaker is evidently an inferior Condorcet-fair method in preventing the undesirable outcome of successful burying.

How Should You Vote on the IRV Ballot Question in November

Based on the research outcomes, people should vote in favor of the adoption of IRV in place of plurality voting as the voting method for future state-level and local elections held in Massachusetts in November due to the lower likelihood of Condorcet breakdown associated with IRV relative to that associated with plurality voting. However, IRV isn't the best election method that people living in Massachusetts can hope for because it is quite susceptible to Condorcet breakdown in an absolute sense under certain assumptions. Fortunately, this research shows that there might be an even better alternative: pairwise comparison coupled with Borda count as the tie-breaking mechanism. Specifically, through computational analysis, this research found out that for pairwise comparison with Borda count as the tie-breaker, the likelihood of successful burying, a special case of strategic voting which pairwise comparison especially incentivizes voters to commit, is reasonably low both in a relative (when compared with pairwise comparison with IRV as the tie-breaker) and absolute sense. However, to more thoroughly investigate the probability of successful burying associated with pairwise comparison with Borda count as the tie-breaker, further research could focus on studying how probable successful burying is in pairwise comparison when Borda count is used as a tie-breaker not for the entire candidate pool but only for the pairwise comparison winners.

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