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The Schur multiplier, fields, roots of unity, and a natural splitting

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1. Introduction.

Let G be a finite group and let Ω be a subgroup of the multiplicative group $\mathbb{C}^\times$ of complex numbers; assume that Ω contains a root $\zeta_{|G|}$ of unity of order $|G|$. I shall consider relationships between $H^2(G,\Omega)$ and the Schur multiplier $H^2(G,\mathbb{C}^\times)$. If also $\Omega = K^\times$ for a field K, there is a connection with the problem of writing projective representations of G in K.

By the universal coefficient theorem of group cohomology, it is easy to write a split exact sequence

$$1 \longrightarrow \operatorname{Ext}(G/G',\Omega) \longrightarrow H^2(G,\Omega) \longrightarrow H^2(G,\mathbb{C}^\times) \longrightarrow 1, \qquad (1.1)$$

which gives an isomorphism

$$H^2(G,\Omega) \cong \operatorname{Ext}(G/G',\Omega) \times H^2(G,\mathbb{C}^\times).$$

The splitting is known to be natural in Ω. One main result (Theorem 4.1) is that (1.1) splits naturally in G as well as Ω by a certain group-homomorphism

$$\sigma_{G,\Omega}\colon H^2(G,\mathbb{C}^\times) \to H^2(G,\Omega).$$

The map $\sigma_{G,\Omega}$ can be described very simply in terms of orders of cocycles (see (3.4)); the problem is to show that it is well-defined (Theorem 3.3).

As to fields, a well-known theorem of Brauer states that every representation of a finite group G in $\mathbb{C}$ is equivalent to one in the cyclotomic field $\mathbb{Q}(\zeta_{\exp G})$. In [**16**] a corresponding theorem for projective representations and $\mathbb{Q}(\zeta_{|G|})$ was given. More explicitly, let $f \in Z^2(G,\mathbb{C}^\times)$ and let K contain $\zeta_{|G|}$. It was shown that there exists an element e of the cohomology class of f in $H^2(G,\mathbb{C}^\times)$ such that for every subgroup H of G, every projective representation of H in $\mathbb{C}$ whose 2-cocycle is the restriction of e to H is linearly equivalent to one in K. It is clear that $e \in Z^2(G,K^\times)$ and that e can be replaced by any element of its cohomology class in $H^2(G,K^\times)$.

Theorem 5.1 states that the set of all cocycles by which e can be replaced is precisely one such class, namely $\sigma_{G,K^\times}\left(eB^2(G,\mathbb{C}^x)\right)$. This improves some of the results of [**16**]; in particular, the naturality of $\sigma_{G,K^\times}$ answers a question asked there in connection with Clifford theory (see Section 6). This paper includes a treatment of these results based on the theorem of Alperin and Kuo [**1**] discussed in Section 2.

As in [**16**], there are corresponding results for prime characteristic. Large parts of Sections 3, 4, and 5 can be read independently of each other.

I want to thank the American Mathematical Society for supporting my attendance at the 1986 Summer Institute at Arcata, where I presented a version of this paper. Especially I want to thank Karl Gruenberg, whose conversations at Arcata about the universal coefficient theorem put things in their cohomological context and led to Section 4.

Notation. E is always an algebraically closed field of characteristic $p \geq 0$, $E^\times$ its multiplicative group, and Ω a subgroup of $E^\times$. For a positive integer n, $n_{p'}$ denotes the p-regular part of n; $n_{0'} = n$. If p does not divide m, ζ_m is a root of unity in E of order m. G is a finite group, with center $Z(G)$ and commutator subgroup G'. Cohomology will be with respect to trivial G-action, with cochains normalized at the identity (cf. [**16**], for example).

2. Exponents and Orders.

This section deals with a result (Theorem 2.4) of Alperin and Kuo that is used in both Sections 3 and 5.

By the *coexponent* of a finite group G I shall mean the integer

$$\operatorname{coexp} G = \frac{|G|}{\exp G}$$

(see [**1**, p. 412]), which may be thought of as a measure of the noncyclicity of G. I have given a name to this quotient since it has the following basic properties:

Lemma 2.1. (a) *If $P_1, \ldots, P_m$ are Sylow subgroups of G, one for each prime divisor of $|G|$, then* $\operatorname{coexp} G = \prod_{i=1}^m \operatorname{coexp} P_i$.

(b) *If $H \leq G$, then* $\operatorname{coexp} H$ *divides* $\operatorname{coexp} G$.

(c) *If N is a normal subgroup of G, then* $\operatorname{coexp} N \operatorname{coexp} G/N$ *divides* $\operatorname{coexp} G$.

Proof. Statements (a) and (c) are equivalent to the simple facts that $\exp G = \prod \exp P_i$ and $\exp G \mid \exp N \exp G/N$. In proving (b), we can suppose by (a) that G is a p-group, and by induction on $|G : H|$ that H is maximal in G. Then H is normal in G, so that (c) implies (b).

Brandis [**5**] has given a proof of Theorem 2.4 using the simplest facts about transfer; I give a variant that uses characters instead. The original cohomological proof [**1**] is also of interest. (The reason that the argument of [**16**] worked, although awkwardly, is that Proposition 2.2 served as a substitute for Theorem 2.4.)

Proposition 2.2. *Let U and H be subgroups of G such that $U \leq G' \cap Z(G)$ and $U \leq H$. Then for any linear character λ of H, the multiplicative order of $\operatorname{res}_{H\to U}\lambda$ divides $|G : H|$. Furthermore $\exp\big(U/(H' \cap U)\big)$ divides $|G : H|$.*

Proof. The first conclusion is [**16**, Theorem 2]; for convenience I repeat the proof. Let $\omega = \operatorname{res}_{H\to U}\lambda$. There exists $u \in U$ such that $\omega(u)$ is a root of unity whose order equals that of ω. Let T be the representation of G induced by λ (as a representation of H). Since $u \in Z(G)$, $\det T(u) = \det\big(\omega(u)I\big) = \omega(u)^{|G:H|}$ where I is the identity transformation. Since $u \in G'$, $\det T(u) = 1$; the first conclusion follows. Since the restrictions $\operatorname{res}\lambda$ are just those linear characters of U whose kernels contain $H' \cap U$, the second conclusion is a dualization of the first.

Corollary 2.3. *For every abelian subgroup A of G, $\exp\big(G' \cap Z(G)\big)$ divides $|G : A|$* (cf. [**5**]).

Proof. Apply Proposition 2.2 with $U = G' \cap Z(G)$ and $H = UA$.

Theorem 2.4 (Alperin-Kuo)**.** *For any group G:*
(a) *if $U = G' \cap Z(G)$, then* $\exp U$ *divides* $\operatorname{coexp} G/U$ [**1**, Theorem 1].
(b) $\exp H^2(G, E^\times)$ *divides* $(\operatorname{coexp} G)_{p'}$ [**1**, p. 412].

Proof. For (a), every cyclic subgroup of G/U has form A/U with A abelian. By Corollary 2.3, $\exp U$ divides $|G : A| = |G/U|/|A/U|$. Then (a) follows since the exponent of G/U is the l.c.m. of the orders of its cyclic subgroups.

Karpilovsky [**11**, Proposition 4.1.14] has pointed out that (a) implies (b) by applying (a) to a representation group of G over E (cf. the proof of Theorem 3.4). This can be simplified a bit by using a suitable f-covering group (f-representation group) [**11**, pp. 98-99] instead of a representation group. It is also true that (b) implies (a); this is proved in [**1**]. (Alperin and Kuo stated (b) only for $\mathbb{C}$, but their proof can be adapted easily for E. This is connected with the fact that $H^2(G, E^\times)$ is isomorphic to the p-regular part of $H^2(G, \mathbb{C}^\times)$ [**2**, §1], [**19**, Proposition 3.2], which is an immediate consequence of a fact stated after (4.3) below.)

Theorem 2.4(b) improves Schur's result that $\exp H^2(G, \mathbb{C}^\times)$ divides $|G|$ [**10**, Corollary VI.16.5]. Together with Lemma 2.1, it gives a uniform bound for exponents of second cohomology groups of subgroups and quotient groups of G, as follows.

Theorem 2.5. (a) *If $H \leq G$, then* $\exp H^2(H, E^\times)$ *divides* $(\operatorname{coexp} G)_{p'}$.
(b) *If N is normal in G, then* $\exp H^2(G/N, E^\times)$ *divides* $(\operatorname{coexp} G)_{p'}$.

3. Subgroups of $E^\times$.

This section is devoted to the existence and some properties of $\sigma_{G,\Omega}$. I shall call an element f of $Z^2(G, E^\times)$ *order-normalized* if its order $o(f)$ is equal to the (finite) order of the cohomology class $fB^2(G, E^\times) \in H^2(G, E^\times)$. (Others have called such cocycles simply "normalized".) These exist by a well-known result of Schur [**7**, p. 360]:

Proposition 3.1. *Every class in $H^2(G, E^\times)$ contains an order-normalized 2-cocycle.*

I now state a uniqueness theorem for these cocycles.

Theorem 3.2. *Let Ω be any subgroup of $E^\times$ such that*

$$\zeta_{|G|_{p'}} \in \Omega. \tag{3.1}$$

Then any two order-normalized cocycles in the same element of $H^2(G, E^\times)$ are in the same element of $H^2(G, \Omega)$.

Proof. Let e and f be order-normalized elements of the same $u \in H^2(G, E^\times)$. Then

$$o(e) = o(u) \bigm| \exp H^2(G, E^\times) \bigm| |G|_{p'}$$

by Theorem 2.4(b) (or Schur's weaker result); hence $e \in Z^2(G, \langle \zeta_{|G|_{p'}} \rangle) \leq Z^2(G, \Omega)$, and similarly for f. Now $f = (\delta c)e$ for some $c \in C^1(G, E^\times)$. Then

$1 = f^{o(u)} = \delta(c^{o(u)})e^{o(u)} = \delta(c^{o(u)})$, so that $c^{o(u)}$ is a homomorphism of G to $E^\times$, whence $o(c^{o(u)}) \mid (\exp G/G')_{p'}$ and $o(c) \mid \exp H^2(G, E^\times)(\exp G/G')_{p'}$. By Theorem 2.4(b), $o(c) \bigm| |G|_{p'}$; thus $c \in C^1(G, \Omega)$, whence $f \in eB^2(G, \Omega)$ as required.

The above existence and uniqueness theorems together give a main result:

Theorem 3.3. *Let Ω be any subgroup of $E^\times$ that satisfies* (3.1)*. Then for each class $u \in H^2(G, E^\times)$, the order-normalized cocycles in u are all contained in a single element*

$$v = \sigma_{G,\Omega}(u) \tag{3.2}$$

of $H^2(G, \Omega)$. Hence there is a well-defined injection

$$\sigma_{G,\Omega}\colon H^2(G, E^\times) \to H^2(G, \Omega). \tag{3.3}$$

Furthermore we can write

$$\sigma_{G,\Omega}(u) = \bigl(u \cap Z^2(G, \langle\zeta_{o(u)}\rangle)\bigr)B^2(G, \Omega). \tag{3.4}$$

Here order-normalized cocycles e in u exist by Theorem 3.1, and $v = eB^2(G, \Omega)$ for any such e. Since the set of all such e is just $u \cap Z^2(G, \langle\zeta_{o(u)}\rangle)$, (3.4) holds, with the subset of $Z^2(G, \Omega)$ on the right being a single coset of $B^2(G, \Omega)$.

Theorems 3.2 and 3.3 follow immediately from the special case where Ω is replaced by $\langle\zeta\rangle = \langle\zeta_{|G|_{p'}}\rangle$, with $\sigma_{G,\Omega}(u) = \sigma_{G,\langle\zeta\rangle}(u)B^2(G, \Omega)$.

Now for some properties of $\sigma_{G,\Omega}$.

Theorem 3.4. *Under the assumptions of Theorem 3.3, the injection $\sigma_{G,\Omega}$ is a homomorphism.*

Proof. (Cf. [**11**, Proposition 7.2.4].) By [**3**, §1] G possesses at least one representation group R over E, describable as follows: $1 \to A \to R \to G \to 1$ is a central extension with 2-cocycle $\beta \in Z^2(G, A)$, such that if we set $f_\lambda(g_1, g_2) = \lambda(\beta(g_1, g_2))$ for all $\lambda \in \mathrm{Hom}(A, E^\times)$, the transgression map $\lambda \mapsto u_\lambda = f_\lambda B^2(G, E^\times)$ of $\mathrm{Hom}(A, E^\times)$ to $H^2(G, E^\times)$ is an isomorphism. Then f_λ is order-normalized in $Z^2(G, E^\times)$, whence $f_\lambda \in Z^2(G, \Omega)$ and $f_\lambda B^2(G, \Omega) = \sigma_{G,\Omega}(u_\lambda)$. Since the maps $u_\lambda \mapsto \lambda$ and $\lambda \mapsto f_\lambda$ are homomorphisms, so is $\sigma_{G,\Omega}$.

This proof implies that $\{f_\lambda\}$ is a group consisting of order-normalized cocycles, not unique in general. Equivalently, the exact sequence

$$1 \longrightarrow B^2(G,\Omega) \longrightarrow Z^2(G,\Omega) \longrightarrow H^2(G,\Omega) \longrightarrow 1$$

splits, but not naturally (cf. [**19**, Proposition 3.1]).

This nonuniqueness is partially compensated for as follows. Let us call the elements of $\sigma_{G,\Omega}(u)$, for all u, Ω-*normalized*; thus an element f of $Z^2(G,E^\times)$ is Ω-normalized if and only if $f \in \sigma_{G,\Omega}\big(fB^2(G,E^\times)\big)$. (Remember that we have assumed (3.1)). Then we have:

Corollary 3.5. *All the Ω-normalized 2-cocycles in $Z^2(G,E^\times)$ form a subgroup of $Z^2(G,\Omega)$.*

Now I show that the mapping $\sigma_{G,\Omega}$ has "commuting" properties that relate it to the restriction, corestriction (transfer), inflation, and conjugation mappings of 2-cohomology groups. The definitions and notations follow Weiss [**18**, Chapter II, especially Section 2-5].

Theorem 3.6. *Under the assumptions of Theorem 3.2, we have:*
(a) *if $H \le G$ and $u \in H^2(G,E^\times)$, then*

$$\mathrm{res}_{G\to H}(\sigma_{G,\Omega}(u)) = \sigma_{H,\Omega}(\mathrm{res}_{G\to H}\, u).$$

(b) *if $H \le G$ and $u \in H^2(H,E^\times)$, then*

$$\mathrm{cor}_{H\to G}(\sigma_{H,\Omega}(u)) = \sigma_{G,\Omega}(\mathrm{cor}_{H\to G}\, u).$$

(c) *if $Q = G/N$ is a quotient group of G and $u \in H^2(Q,E^\times)$, then*

$$\inf_{Q\to G}(\sigma_{Q,\Omega}(u)) = \sigma_{G,\Omega}(\inf_{Q\to G} u).$$

(d) *if $H \le G$, $g \in G$, and $u \in H^2(H,E^\times)$, then*

$$\mathrm{con}_g(\sigma_{H,\Omega}(u)) = \sigma_{g^{-1}Hg,\Omega}(\mathrm{con}_g\, u).$$

Proof. Note that (3.1) is assumed for the group G; this implies the corresponding conditions for H and Q. I prove (c). The maps $\sigma_{Q,\Omega}$ and $\sigma_{G,\Omega}$ are defined, and we can choose order-normalized cocycles $e \in \sigma_{Q,\Omega}(u)$ and $f \in \sigma_{G,\Omega}(\inf u)$. For the cochain map inf defined by $(\inf e)(g_1,g_2) =$

$e(g_1N, g_2N)$, we have $\inf e \in \inf \sigma_{Q,\Omega}(u) \subseteq \inf u$. If $m = (\operatorname{coexp} G)_{p'}$, Theorem 2.5 yields

$$o(\inf e) = o(e) \mid \exp H^2(Q, E^\times) \mid (\operatorname{coexp} G)_{p'},$$

or $(\inf e)^m = 1$; similarly $f^m = 1$. Since $\inf e$ and f are both in $\inf u$, $\inf e = f(\delta c)$ for some $c \in C^1(G, E^\times)$. Then $1 = (\inf e)^m = f^m(\delta c)^m = \delta(c^m)$, $c^m \in \operatorname{Hom}(G, E^\times)$, $o(c^m) \mid (\exp G)_{p'}$, and $o(c) \mid m(\exp G)_{p'} = |G|_{p'}$, whence $c \in C^1(G, \Omega)$ by (3.1), and (c) follows.

The proofs of the other statements are very similar; in the proof of (b) the fact that $o(\operatorname{cor} e) \mid o(e)$ for $e \in Z^2(H, E^\times)$ follows from [**18**, p. 81]. Later we shall use this theorem to prove Proposition 4.2, whose statement subsumes it (except (b)).

Corollary 3.7. *The cochain maps of restriction, corestriction, inflation, and conjugation all carry* Ω*-normalized cocycles to* Ω*-normalized cocycles.*

The corresponding statements for restriction and inflation of order-normalized cocycles are easily seen to be false.

4. A Natural Splitting.

This section originated in the sequence (4.4), which Karl Gruenberg kindly pointed out to me.

Let G, E, and Ω be as before. Assume (3.1), so that

$$\langle \zeta_{|G|_{p'}} \rangle \le \Omega \le E^\times. \tag{4.1}$$

By the universal coefficient theorem for cohomology [**10**, Theorems V.3.3 and VI.15.1] (see also [**9**, §3.7] and [**4**, I.5,8]), there is an exact sequence

$$1 \longrightarrow \operatorname{Ext}\big(H_1(G), \Omega\big) \longrightarrow H^2(G, \Omega) \overset{\alpha_{G,\Omega}}{\longrightarrow} \operatorname{Hom}\big(H_2(G), \Omega\big) \longrightarrow 1 \tag{4.2}$$

where $H_i(G) = H_i(G, \mathbb{Z})$. This sequence is natural in both G and Ω; it splits by a homomorphism

$$\theta_{G,\Omega}\colon \operatorname{Hom}\big(H_2(G), \Omega\big) \to H^2(G, \Omega)$$

that is natural in Ω but not in G. Here saying that a diagram is *natural* means that each group in the diagram is given by a functor from a suitable category

$\mathcal{C}$ and that each arrow in the diagram is given by a natural transformation between the functors that give its ends.

We can modify (4.2) to get a sequence that splits naturally in both arguments, as follows. $H_1(G)$ can be replaced by the isomorphic group G/G' [**10**, (VI.4.4)]; since $E^\times$ is divisible, $\mathrm{Ext}(G/G', E^\times) = 1$ [**10**, Theorem I.7.1 and Proposition III.2.6]. Then (4.2) for Ω and for $E^\times$, together with the naturality in Ω, gives a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Ext}(G/G', \Omega) & \longrightarrow & H^2(G, \Omega) & \overset{\alpha_{G,\Omega}}{\longrightarrow} & \mathrm{Hom}\big(H_2(G), \Omega\big) & \longrightarrow & 1 \\ & & & & \Big\downarrow{\scriptstyle \pi_{G,\Omega}} & & \Big\downarrow{\scriptstyle \rho_{G,\Omega}} & & \\ & & 1 & \longrightarrow & H^2(G, E^\times) & \overset{\alpha_{G,E^\times}}{\longrightarrow} & \mathrm{Hom}\big(H_2(G), E^\times\big) & \longrightarrow & 1 \end{array} \tag{4.3}$$

where the vertical maps are induced by the inclusion map of Ω to $E^\times$. By definition $H_2(G)$ is finitely-generated abelian, and by the second row of (4.3) $\mathrm{Hom}(H_2(G), E^\times)$ is finite; these imply that $H_2(G)$ is torsion and hence finite, whence $H^2(G, E^\times)$ is isomorphic to the p-regular part of $H_2(G)$ (this fact is well known, at least for characteristic 0). Then $\exp H_2(G)$ divides $|G|$, and (4.1) implies that $\rho_{G,\Omega}$ is an isomorphism. Since $\alpha_{G,E^\times}$ is also an isomorphism, this yields an exact sequence

$$1 \longrightarrow \mathrm{Ext}(G/G', \Omega) \longrightarrow H^2(G, \Omega) \overset{\pi_{G,\Omega}}{\longrightarrow} H^2(G, E^\times) \longrightarrow 1. \tag{4.4}$$

(Karpilovsky [**11**, Theorem 2.2.9] comes close to stating this, using more elementary arguments.) From now on, the only maps between groups Ω that I shall allow are inclusion maps

$$\iota\colon \Omega_2 \to \Omega_1 \qquad \text{for } \Omega_2 \le \Omega_1 \le E^\times. \tag{4.5}$$

With this restriction, (4.4) is natural in both G and Ω and splits naturally in Ω (by $\theta_{G,\Omega} \circ \rho_{G,\Omega}^{-1} \circ \alpha_{G,E^\times}$).

Now we can apply the results of Section 3. Since the mapping $\sigma_{G,\Omega}$ of (3.3) satisfies $\pi_{G,\Omega} \circ \sigma_{G,\Omega} = 1$, $\sigma_{G,\Omega}$ splits the sequence (4.4). Furthermore we have:

Theorem 4.1. *Under the assumption* (4.1)*, the split exact sequence*

$$1 \longrightarrow \mathrm{Ext}(G/G', \Omega) \longrightarrow H^2(G, \Omega) \underset{\sigma_{G,\Omega}}{\overset{\pi_{G,\Omega}}{\rightleftarrows}} H^2(G, E^\times) \longrightarrow 1 \tag{4.6}$$

is natural in both G *and* Ω *in the sense determined by* (4.5)*. Hence the direct decomposition*

$$H^2(G,\Omega) = \ker \pi_{G,\Omega} \times \operatorname{im} \sigma_{G,\Omega} \tag{4.7}$$

is natural in the same sense. Here

$$\ker \pi_{G,\Omega} = \frac{Z^2(G,\Omega) \cap B^2(G,E^\times)}{B^2(G,\Omega)} \cong \operatorname{Ext}(G/G',\Omega),$$

$$\operatorname{im} \sigma_{G,\Omega} \cong H^2(G,E^\times).$$

Explicitly, the naturality of (4.6) can be described as follows. The objects of $\mathcal{C}$ are the pairs (G,Ω) where G is a finite group and Ω satisfies (4.1). The morphisms of $\mathcal{C}$ from (G_1,Ω_1) to (G_2,Ω_2) are as follows: if $\Omega_2 \leq \Omega_1$ they are the pairs (h,ι) for all homomorphisms $h\colon G_1 \to G_2$ and for the inclusion map ι of Ω_2 to Ω_1, otherwise there are none. They are multiplied by $(h',\iota')(h,\iota) = (h' \circ h, \iota \circ \iota')$.

I shall give further details only for $\sigma_{G,\Omega}$. There is a natural contravariant functor $F = H^2(-,-)$ from $\mathcal{C}$ to the category $\mathcal{G}$ of groups, for which $F(G,\Omega) = H^2(G,\Omega)$ and

$$F(h,\iota)\big(fB^2(G_2,\Omega_2)\big) = (\iota \circ f \circ (h \times h))\, B^2(G_1,\Omega_1)$$

(cf. [**10**, p. 190]). Similarly there is $F_0 = H^2(-,E^\times)$ from $\mathcal{C}$ to $\mathcal{G}$, which ignores second arguments, defined by $F_0(G,\Omega) = H^2(G,E^\times)$ and

$$F_0(h,\iota)\big(fB^2(G_2,E^\times)\big) = (f \circ (h \times h))\, B^2(G_1,E^\times).$$

With these definitions the naturality of the splitting is given by:

Proposition 4.2. *The mapping*

$$\sigma_{-,-}\colon (G,\Omega) \mapsto \sigma_{G,\Omega}$$

is a natural transformation of F_0 *to* F*.*

Proof. Since $\sigma_{G,\Omega}$ is a homomorphism by Theorem 3.4, the proposition just asserts that

$$F(h,\iota) \circ \sigma_{G_2,\Omega_2} = \sigma_{G_1,\Omega_1} \circ F_0(h,\iota). \tag{4.8}$$

In the factorization $h = h''' \circ h'' \circ h'$,

$$G_1 \xrightarrow{h'} G_1/\ker h \xrightarrow{h''} \operatorname{im} h \xrightarrow{h'''} G_2,$$

it is enough to prove (4.8) for the factors. But since all pairs occurring are in $\mathcal{C}$, for h' (4.8) follows from (c) of Theorem 3.6 and for h''' it follows from (a), while for the isomorphism h'' it is easy (cf. (d)). This proves Proposition 4.2 and Theorem 4.1.

If $G = G'$, then $\pi_{G,\Omega}$ is an isomorphism and $\sigma_{G,\Omega}$ is merely $\pi_{G,\Omega}^{-1}$; in this case every cocycle in $Z^2(G, E^\times)$ is Ω-normalized. But this is false in general: for example, if G is the Klein four-group, $E = \mathbb{C}$, and $\Omega = \langle i \rangle$, $H^2(G, \Omega)$ is elementary abelian of order 8 but $|H^2(G, \mathbb{C}^\times)| = 2$. The assumption (4.1) is essential; for example, if we replace Ω here by $\{\pm 1\}$, the conclusion of Theorem 3.2 becomes false and, although (4.4) splits, there appears to be no natural splitting.

In [**16**] I asked, for $p = 0$, whether $\zeta_{|G|}$ can be replaced by $\zeta_{\exp G}$ in the existence part of Theorem 5.1. Opolka [**14**] has answered this question in the negative; also see [**17**].

5. Subfields of E.

The rest of the paper will deal with realizability of projective representations in fields. (Some related results in a different direction can be found in [**13**].)

Let K be a subfield of E. I shall say that K *splits* the element f of $Z^2(G, E^\times)$ if every irreducible f-representation of G in E (i. e., projective representation of G in E with 2-cocycle f) is linearly equivalent in E to an f-representation of G in K. (Recall that if T is an f-representation, M an invertible linear transformation, and c a 1-cochain, then the $(\delta c)f$-representation $g \mapsto c(g)M^{-1}T(g)M$ is called *projectively equivalent* to T; if $c = 1$ it is *linearly equivalent* [**16**]. The present use of "split" is unrelated to the term "split exact sequence".) Then the same holds for all completely reducible f-representations of G in E; if p does not divide $|G|$ this means *all* f-representations [**11**, Theorem 3.2.10]. If K splits f, then $f \in Z^2(G, K^\times)$ and K splits every cocycle in the cohomology class $fB^2(G, K^\times)$. Furthermore, I shall say that K *splits f on subgroups* if K splits the restriction $\operatorname{res}_{G\to H} f$ for every subgroup H of G.

Recall that the *twisted group algebra* $K[G,f]$ is the K-algebra with basis $\{b_g | g \in G\}$ such that $b_{g_1} b_{g_2} = f(g_1, g_2) b_{g_1 g_2}.$. The f-representations T of G correspond bijectively to the representations $\mathcal{T}$ of $K[G,f]$ by $T(g) = \mathcal{T}(b_g)$. Then K splits f if and only if K is a splitting field for $K[G,f]$ by [**11**, Proposition 1.6.4] (cf. [**7**, pp. 292-294]).

The next result is the main existence and uniqueness theorem for cocycles that are split on subgroups. Except for the assertion about $\sigma_{G,K^\times}$, it will follow at once from Theorems 5.3 and 5.4. Those two theorems are independent of each other, and of Section 3.

Theorem 5.1. *Let K be a subfield of E such that*

$$\zeta_{|G|_{p'}} \in K. \tag{5.1}$$

Then for each $u \in H^2(G, E^\times)$, there exist cocycles $f \in u$ such that K splits f on subgroups. The set of all such $f \in u$ is exactly one class in $H^2(G, K^\times)$. In fact, it is the class $\sigma_{G,K^\times}(u)$ in the notation of (3.2)*; thus for $f \in Z^2(G, E^\times)$, K splits f on subgroups if and only if f is $K^\times$-normalized.*

Proposition 5.2. *If f has finite order in $Z^2(G, E^\times)$ and if K contains a root of unity of order $m = o(f)(\exp G)_{p'}$, then K splits f on subgroups.*

This result (essentially [**11**, Theorem 6.5.15], cf. [**8**, §4]) is obtained at once by applying Brauer's theorem on splitting fields [**7**, (41.1)] to the f-covering group of G (see references in Section 2), since the p-regular part of the exponent of that group divides m. For $f = 1$, it reduces to Brauer's theorem. (For Brauer's theorem in prime characteristic, see [**6**, §2] or [**7**, (83.7)].)

Theorem 5.3. *If $e \in Z^2(G, E^\times)$ is order-normalized and if K contains $\zeta_{|G|_{p'}}$, then K splits e on subgroups.*

Proof. Theorem 2.4(b) implies that

$$o(e)(\exp G)_{p'} \bigm| \exp H^2(G, E^\times)(\exp G)_{p'} \bigm| |G|_{p'};$$

then the result follows from Proposition 5.2. It is curious that here the Alperin-Kuo result is used to prove existence, whereas in Theorem 3.2 it is used to prove uniqueness.

This existence theorem is essentially contained in [**16**, Theorem 5 and Corollary]. Now for a corresponding uniqueness theorem, the simple result from which this paper grew.

Theorem 5.4. *Let K be any subfield of E. Then each cohomology class in $H^2(G, E^\times)$ contains at most one class in $H^2(G, K^\times)$ whose elements are split on subgroups by K.*

Proof. Suppose that f and f' are elements of $Z^2(G, K^\times)$ that are cohomologous over E. Choose $c \in C^1(G, E^\times)$ such that $f' = (\delta c)f$. Construct a configuration of twisted group algebras as follows: embed $K[G, f]$ in $E \otimes_K K[G, f] = E[G, f]$, which has $\{b_g | g \in G\}$ as an E-basis; set $b'_g = c(g)b_g \in E[G, f]$. Then $b'_{g_1} b'_{g_2} = f'(g_1, g_2) b'_{g_1 g_2}$, so that the K-space spanned by $\{b'_g\}$ is a twisted group algebra $K[G, f']$ for f'; we can identify the corresponding algebra $E[G, f']$ with $E[G, f]$, using the E-basis $\{b'_g\}$.

Now assuming that K splits both f and f' on subgroups, we shall see that $c(g) \in K^\times$ for all $g \in G$. Since our hypotheses carry over to $K[H, \mathrm{res}_{G\to H} f]$ for subgroups H, we can reduce to the case that G is a cyclic group $\langle g \rangle$. In this case $E[G, f]$, being generated by one element, is commutative, so that all its irreducible representations $\mathcal{T}$ are one-dimensional. Since $\mathcal{T}(b_g)$ and $\mathcal{T}(b'_g) = c(g)\mathcal{T}(b_g)$ are both in $K^\times$, so is $c(g)$ as required. Observe that in the configuration $K[G, f] = K[G, f']$.

I do not know whether the words "on subgroups" are required for the truth of Theorem 5.4.

The following example shows that we must deal carefully with isomorphic algebras in the above configuration. Let $G = \langle g \rangle$ be of order 3; let $K = \mathbb{Q}$, $E = \mathbb{C}$, $f = 1$, and $c(g^i) = {\zeta_3}^i$. Then $f' = 1$ also, but $\mathbb{Q}[G, f]$ and $\mathbb{Q}[G, f']$ are two distinct copies of the group algebra $\mathbb{Q}[G] = \mathbb{Q}[G, 1]$ within one copy of $\mathbb{C}[G]$. Of course $\mathbb{Q}$ does not split f.

Now we can quickly complete the proof of Theorem 5.1. By Proposition 3.1, u contains an order-normalized e. By Theorem 5.3, K splits e on subgroups; the same holds for all elements of $eB^2(G, K^\times)$. By Theorem 5.4, these are all the elements of u that K splits on subgroups. Finally Theorem 3.2 implies that $e \in \sigma_{G,K^\times}(u)$, whence $\sigma_{G,K^\times}(u) = eB^2(G, K^\times)$.

Theorem 5.1 has an interpretation in terms of algebras, loosely stated as follows:

Theorem 5.5. *If* (5.1) *holds and if Δ is any twisted group algebra for G over E, there is essentially one twisted group algebra Γ for G over K such that $\Delta = E \otimes_K \Gamma$ while K is a splitting field for the restrictions of Γ to all the subgroups H of G.*

If also $G = G'$, then since $\pi_{G,K^\times}$ is an isomorphism, K is a splitting field for all the restrictions of every twisted group algebra Γ for G over K; thus the last clause of Theorem 5.5 can be omitted in this case.

6. Applications to Clifford Theory.

The following theorem implies the results of Section 2 of [**16**], especially Theorem 5. Statement (a) gives an affirmative answer to a question asked on p. 196 there, and (c) improves Theorem 6.

Theorem 6.1. *Let N be a normal subgroup of G and $Q = G/N$. Then every cohomology class u in $H^2(Q, E^\times)$ contains a 2-cocycle e such that:*

(a) *e is order-normalized.*

(b) *if K contains the $|G|_{p'}$th roots of unity, then $\inf_{Q\to G} e^i$ is $K^\times$-normalized and hence is split on subgroups by K for all $i \in \mathbb{Z}$.*

(c) *$o(e)$ divides $(\operatorname{coexp} Q)_{p'}$, which in turn divides $(\operatorname{coexp} G)_{p'}$.*

Proof. By Proposition 3.1, u contains an order-normalized e; I show that any such e satisfies (b) and (c). Clearly every power e^i is order-normalized, hence $K^\times$-normalized. Then Corollary 3.7 and Theorem 5.1 imply (b). Finally, (c) follows by Section 2.

In conclusion, some of the above results can be applied to the situation of Clifford's theorems and Mackey's generalization of them (see [**12**] and [**11**, Theorem 6.6.4]) as follows so as to improve [**16**, §3]. Given a normal subgroup N of G and an irreducible f-representation T of G over E, Mackey's results state that T is linearly equivalent to the f-representation induced by a projective representation T' of a certain group S, $N \le S \le G$, while T' is a tensor product $Y \otimes (\inf_{S/N\to S} Z)$ of two projective representations of S; here $\operatorname{res}_{S\to N} Y$ is an irreducible constituent of $\operatorname{res}_{G\to N} T$ and Z is an e^{-1}-representation of S/N for some e. Let K be the subfield generated by the $|G|_{p'}$th roots of unity. Given Mackey's results, we can, after a projective equivalence, take f order-normalized; or, if f happens to be inflated from G/N, we can instead take f inflated from an order-normalized cocycle on G/N. In either case f is $K^\times$-normalized by Theorem 6.1; by Corollary 3.7, so is the cocycle $\operatorname{res}_{G\to S} f$ of T'. After another projective equivalence we can take e order-normalized; then $\inf_{S/N\to S} e$ is $K^\times$-normalized and, by Corollary 3.5, so is the cocycle $\inf e \operatorname{res} f$ of Y; so Y and Z are both linearly equivalent to projective representations over K, whence so are T' and T.

(When we change Z we must also change Y so as to preserve the cocycle of T', in order to induce to T.) In Clifford's case $f = 1$ we can use $|S|_{p'}$th roots of unity; I do not know whether this is true in general, despite a vague and inaccurate statement I made in [**16**, §3].

Theorem 6.1 can also be applied in the Addendum of [**15**] to avoid an increase of a group order in a construction.

References.

1. J. L. Alperin and T.-N. Kuo, The exponent and the projective representations of a finite group, Illinois J. Math. **11** (1967), 410-413.
2. K. Asano, M. Osima, and M. Takahasi, Über die Darstellung von Gruppen durch Kollineationen im Körper der Charakteristik p, Proc. Phys.-Math. Soc. Japan (3) **19** (1937), 199-209.
3. K. Asano and K. Shoda, Zur Theorie der Darstellungen einer endlichen Gruppe durch Kollineationen, Comp. Math. **2** (1935), 230-240.
4. F. R. Beyl and J. Tappe, "Group Extensions, Representations, and the Schur Multiplicator," Lecture Notes in Math. **958**, Springer, Berlin/Heidelberg/New York, 1982.
5. A. Brandis, Beweis eines Satzes von Alperin und Kuo Tzee-Nan, Illinois J. Math. **13** (1969), 275.
6. R. Brauer, On the representation of a group of order g in the field of the g-th roots of unity, Amer. J. Math. **67** (1945), 461-471.
7. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York/London, 1962.
8. B. Fein, The Schur index for projective representations of finite groups, Pacific J. Math. **28** (1969), 87-100.
9. K. W. Gruenberg, "Cohomological Topics in Group Theory," Lecture Notes in Math. **143**, Springer, Berlin/Heidelberg/New York, 1970.
10. P. J. Hilton and U. Stammbach, "A Course in Homological Algebra," Springer, New York/Heidelberg/Berlin, 1971.
11. G. Karpilovsky, "Projective Representations of Finite Groups," Dekker, New York/Basel, 1985.
12. G. W. Mackey, Unitary representations of group extensions I, Acta Math. **99** (1958), 265-311.
13. E. Nauwelaerts and F. Van Oystaeyen, The Brauer splitting theorem and projective representations of finite groups over rings, J. Algebra **112** (1988), 49-57.

14. H. Opolka, Projective representations of finite groups in cyclotomic fields, Pacific J. Math. **94** (1981), 207-210.
15. W. F. Reynolds, Blocks and normal subgroups of finite groups, Nagoya Math. J. **22** (1963), 15-32.
16. ———, Projective representations of finite groups in cyclotomic fields, Illinois J. Math. **9** (1965), 191-198.
17. ———, Noncommutators and the number of projective characters of a finite group, Proc. Symp. Pure Math. **47** (1987), part 2, 71-74.
18. E. Weiss, "Cohomology of Groups," Academic Press, New York/San Francisco/London, 1969.
19. K. Yamazaki, On projective representations and ring extensions of finite groups, J. Fac. Sci. Univ. Tokyo Sec. 1 **10** (1964), 147-195.