# *The Equivariant de Rham Theorem in Equivariant Cohomology*

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## Abstract

Equivariant cohomology is a cohomology theory for a topological space $M$ with a continuous group action $G$. Two models for equivariant cohomology are the *Borel model* and the *Cartan model*. Let $M$ be a manifold. In cohomology, a theorem known as the *de Rham theorem* states that singular cohomology with real coefficients $H^*(M;\mathbb{R})$ and de Rham cohomology $H^*_{DR}(M)$ are isomorphic rings. The equivariant de Rham theorem states that when $M$ is a manifold and $G$ is a compact, connected Lie group acting smoothly on $M$, the equivariant cohomology of $M$ in both models is the same. Henri Cartan proved the equivariant de Rham theorem in [5]. In this Master's thesis, we describe both models of equivariant cohomolgy and provide an alternate proof as outlined in [7] of the equivariant de Rham theorem in the case that $M$ has a finite good cover.

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To my parents

# Contents

# 1
# The Borel Model

In this section, we define the Borel model of equivariant cohomology of a $G$-space $M$ and compute the equivariant cohomology of a point and a homogeneous space $M = G/H$. For proofs of the theory behind the Borel model, we refer the reader to [4, Ch. 1]

A ***left action*** of a topological group $G$ acting on a topological space $M$ is a continuous map

$$G \times M \to M \tag{1.1}$$

$$(g, x) \mapsto g \cdot x \tag{1.2}$$

such that for all $x \in M$ and $g, h \in G$,

(i) $1 \cdot x = x$, where $1$ is the identity element in $G$,

(ii) $(gh) \cdot x = g \cdot (h \cdot x)$.

A ***right action*** is defined analogously. Every left action can be turned into a right action and vice versa via $g \cdot x = x \cdot g^{-1}$. We say that $M$ is a ***left G-space*** or a ***right G-space*** depending on whether $G$ acts on the left or on the right. By a construction due to Milnor [10], for any topological group $G$, there exists a contractible space $EG$ on which $G$ acts freely (we can assume that $G$ acts on the right). We can construct a new left $G$-space $EG \times M$, where the left action is the ***diagonal action***:

$$\text{for } (e,x) \in EG \times M, g \in G, g \cdot (e,x) = (e \cdot g^{-1}, g \cdot x)\,.$$

Since $G$ acts freely on $EG$, the diagonal action of $G$ on $EG \times M$ is free. In addition, $EG \times M$ is clearly homotopically equivalent to $M$. Since $G$ acts freely on $EG \times M$, we can define the ***homotopy quotient*** of $M$ by $G$ to be:

$$M_G = EG \times_G M := (EG \times M)/G\,.$$

The Borel model of equivariant cohomology defines the equivariant cohomology $H^*_G(M)$ of the $G$-space $M$ to be the singular cohomology of its homotopy quotient $H^*(M_G)$. Cohomology can be taken with any coefficients, but we will denote $H^*()$ to be singular cohomology with real coefficients. It turns out that the homotopy quotient is well defined up to *G-homotopy equivalence* and so the Borel model of equivariant cohomology is a well-defined topological invariant of $G$-spaces.

## 1.1 The space EG

Before we can compute the equivariant cohomology of a point and a homogeneous space, we need to construct an equivalent description of the contractible space $EG$ on which $G$ acts freely. The theory of $G$-bundles provides such a description.

Let $X$ and $Y$ be right $G$-spaces. A map $f : X \to Y$ is said to be ***G-equivariant*** if for all $x \in X$ and for all $g \in G, f(x \cdot g) = f(x) \cdot g$. Let $f_0, f_1 : X \to Y$ be two $G$-equivariant maps of left $G$-spaces. If $I$ denotes the unit interval $[0,1]$, then $G$

acts on $X \times I$ by $g \cdot (x,t) = (g \cdot x, t)$ . A *G-**homotopy*** from $f_0$ to $f_1$ is a $G$-equivaraint map $F : X \times I \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

If such a $G$-homotopy exists from $f_0$ to $f_1$ , we say that $f_0$ and $f_1$ are ***G-homotopic***. Let $f : X \to Y$ be a $G$-equivariant map. A ***G-homotopy inverse*** of $f : X \to Y$ is a $G$-equivariant map $h : X \to Y$ such that are $h \circ f$ and $f \circ h$ are $G$-homotopic to $\not\Vdash_X$ and $\not\Vdash_Y$ respectively. A $G$-equivariant map $f : X \to Y$ is a ***G-homotopy equivalence*** if it has a $G$-homotopy inverse. In this case, $X$ and $Y$ are said to be ***G-homotopy equivalent*** or have the same ***G-homotopy type***. A ***G-bundle*** or principal G-bundle over a topological space $B$ is a fiber bundle $\pi : P \to B$ with fiber $G$ and a local trivialization $\{U, \varphi_U\}$ , such that:

(i) $G$ acts freely on $P$ , and

(ii) each fiber-preserving homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times G$ is $G$-equivariant.

Note that the base space $B$ can be identified with the orbit space $P/G$ . When $P$ is weakly contractible, the $G$-bundle $\pi : P \to B$ is said to be a ***universal** G-**bundle*** (recall from homotopy theory that a space $X$ is said to be ***weakly contractible*** if all of its homotopy groups $\pi_k(X)$ are trivial). The usual notation for a universal $G$-bundle is $EG \to BG$ or $\pi : EG \to BG$. The space $EG$ is called the ***total space*** and the space $BG$ is called the ***base space*** or a ***classifying space*** for $G$.

From a construction due to Milnor, every topological group $G$ has a well-defined universal $G$-bundle. Specifically, suppose that $E \to B$ and $E^{'} \to B^{'}$ are universal $G$-bundles over CW complexes $B$ and $B^{'}$ respectively. Then $B$ and $B^{'}$ are homotopy equivalent and $E$ and $E^{'}$ are $G$-homotopy equivalent. Thus, the universal bundle of a topological group is unique up to $G$-homotopy. In addition, we can assume that the universal bundle $EG \to BG$ admits the following decomposition: $BG = \cup_n BG_n$ is a $CW$-complex, $EG = \cup_n EG_n$, and $EG_n \to BG_n$ is a principal $G$-bundle (for more details, see [1, Sec. 1.1] and [4, Ch. 7]).

**Example 1.1.** *The Universal Bundle of* $G = S^1$ *:*

Let $g = e^{i\theta}$ . The group $G = S^1$ acts on $\mathbb{C}^n$ by rotations as follows:

$$g \cdot (z_1, ..., z_n) = (g \cdot z_1, ..., g \cdot z_n) = (e^{i\theta} \cdot z_1, ..., e^{i\theta} \cdot z_n) \text{, where } z_i \in \mathbb{C}.$$

Note that this action preserves norms. Therefore, $G$ also acts on the unit sphere $S^{2n-1}$ in $\mathbb{C}^n$ . The following argument shows that the action on $S^{2n-1}$ is free. Suppose that $g \cdot z = z$ for some $g \in G$ , where $z \in \mathbb{C}^*$ . Then

$$0 = g \cdot z - z = g \cdot z - 1 \cdot z = (g - 1) \cdot z.$$

Since $z \neq 0$ , $g = 1$ .

We have that $S^1 \subset S^3 \subset S^5 \subset \ldots$ and $G$ acts freely on $S^{2n-1}$ for all $n$ . Therefore, $G$ acts freely on the space $S^\infty = \cup_{n=1}^{\infty} S^{2n-1}$ . We now will show that $S^\infty$ is weakly contractible.

Let $a \in \pi_k(S^\infty)$ . By definition, $a$ is a continuous map $a : S^k \to S^\infty$ . Since $a$ is continuous and $S^k$ is compact, the image $a(S^k)$ is compact. Thus, $a(S^k)$ lies inside $S^n$ for some $n$ so we can think of $a$ as being a continuous map from $S^k$ to $S^n$. Without loss of generality, we can assume that $k < n$. Since $\pi_k(S^n) = 0$ for $k < n$, $a$ must be null-homotopic. Thus, $\pi_k(S^\infty) = 0$ for all $k$ .

We can take $ES^1$ to be $S^\infty$ . The quotient of $S^{2n-1}$ by the above action of $S^1$ is the complex projective space $\mathbb{C}P^{n-1}$ . Therefore, the classifying space $BS^1$ for $G$ is:

$$BS^1 = \cup_{n=1}^{\infty} (S^{2n-1}/S^1) = \cup_{n=0}^{\infty} CP^n = \mathbb{C}P^\infty .$$

So $S^\infty \to \mathbb{C}P^\infty$ is a universal bundle for $S^1$ .

## 1.2 Computations of equivariant cohomology in the Borel model

Now that we have formally defined the Borel model of equivariant cohomology, we can compute the equivariant cohomology of a point and a homogeneous space under this model. Let $EG \to BG$ be a universal bundle of the compact, connected Lie group $G$ .

**Example 1.2.** *The equivariant cohomology of a point:*

Let $M = \{pt\}$ . Any group acts trivially on $M$ so the homotopy quotient is:

$$M_G = EG \times_G M \simeq EG/G \simeq BG\,.$$

Thus, the equivariant cohomology $H_G^*(M)$ of $M$ under the group $G$ is

$$H_G^*(M) = H^*(M_G) = H^*(BG)\,.$$

The ***rank*** of a compact Lie group $G$ is the dimension of a maximal torus $T$ in $G$. Let $T = S^1 \times \ldots \times S^1$ be a maximal torus of $G$ of rank $n$. The ***Weyl group*** $W$ of $T$ in $G$ is

$$W := N_G(T)/T,$$

where $N_G(T)$ is the normalizer of the torus $T$. For a compact, connected Lie group, it is known that the Weyl group is a finite reflection group.

By a lemma in [11, P. 189], the cohomology of $BG$ is the subring of $W$-invariants:

$$H^*(BG) = H^*(BT)^W.$$

Note that if $EG \to BG$ and $EG' \to BG'$ are universal bundles of CW complexes for two topological groups $G$ and $G'$ respectively, then $EG \times EG' \to BG \times BG'$ is a universal bundle of CW complexes for $G \times G'$. Therefore,

$$\begin{aligned} H^*(BT) &= H^*(B(S^1 \times \ldots \times S^1)) = H^*(BS^1 \times \ldots \times BS^1) \\ &= H^*(BS^1) \otimes \ldots \otimes H^*(BS^1)(bytheKunnethformula) \\ &= H^*(\mathbb{C}P^\infty) \otimes \ldots \otimes H^*(\mathbb{C}P^\infty). \end{aligned}$$

A spectral sequence argument shows that the cohomology of $\mathbb{C}P^\infty$ is $\mathbb{R}[u]$, where $u$ is a polynomial of degree 2. Therefore,

$$H_G^*(\mathrm{pt}) = H^*(BG) = H^*(BT)^W \simeq (\mathbb{R}[u_1] \otimes \ldots \otimes \mathbb{R}[u_n])^W \quad = \mathbb{R}[u_1, \ldots, u_n]^W = S(\mathfrak{t}^\vee)^W,$$

where $\mathfrak{t}^\vee$ is the dual space to the Lie algebra $\mathfrak{t}^\vee$ of the torus $T$ and $S(\mathfrak{t}^\vee)$ is the algebra of symmetric polynomials on $\mathfrak{t}^\vee$. Finally, the Chevalley restriction theorem [6, P. 200] states that the restriction of a maximal subalgebra $\mathfrak{t} \subset \mathfrak{g}$ gives rise to an isomorphism of algebras

$$S(\mathfrak{g}^\vee)^G \to S(\mathfrak{t}^\vee)^W.$$

Thus, $H_G(\mathrm{pt}) = S(\mathfrak{g}^\vee)^G$.

*Remark:* The group $G$ acts on the algebra of symmetric polynomials $S(\mathfrak{g}^\vee)$ by the adjoint representation. So $S(\mathfrak{g}^\vee)^G$ consists of polynomials invariant under the adjoint representation.

**Example 1.3.** *The equivariant cohomology of a homogeneous space:*

Let $H$ be a closed subgroup of $G$. The group $G$ acts on the homogeneous space $M = G/H$ by left multiplication: for $g \in G$ and for $aH$ in $G/H$, the group action of $G$ on $G/H$ is $g \cdot aH = gaH$.

To describe the homotopy quotient of a homogeneous space, we mention some relationships between principal $G$-bundles and subgroups $H$ of $G$ (for proofs of these results, see [4, sec. 4 and 8]).

**Proposition 1.4.** *Let $\pi : P \to B$ be a principal $G$-bundle and $H$ a subgroup of $G$. Let $G$ act on $G/H$ by left multiplication. Then there is a bundle isomorphism*

$$P \times_G (G/H) \xrightarrow{\sim} P/H$$

*over $B$.*

**Proposition 1.5.** *Let $H$ be a closed subgroup of the Lie group $G$. If $\pi : P \to B$ is a principal $G$-bundle, then the projection $P \to P/H$ is a principal $H$-bundle. As a corollary, if $EG \to BG$ is a universal $G$-bundle, then $EG \to EG/H$ is a universal $H$-bundle.*

By proposition 1.4, the homotopy quotient $M_G$ of the homogeneous space $G/H$ is

$$M_G = EG \times_G (G/H) \simeq EG/H.$$

By proposition 1.5, the bundle $EG \to EG/H$ is a universal $H$-bundle. Since $EH \to BH$ is a universal $H$-bundle, $EG/H$ is homotopic to the base space $BH$. Therefore, the equivariant cohomology of a homogeneous space $M = G/H$ is

$$H^*_G(G/H) = H^*(M_G) = H^*(EG/H) = H^*(BH)\,.$$

Suppose that $H$ is connected. Let $S \subset H$ a maximal torus of $H$ and let $W_H$ denote the Weyl group of $S$ in $H$. By repeating the argument used for computing the cohomology $H^*(BG)$, we have that

$$H^*(BH) \simeq S(\mathfrak{s}^\vee)^{W_H} \simeq S(\mathfrak{h}^\vee)^H.$$

Now suppose that $H$ is a closed subgroup of $G$ and let $H_0$ be the connected component of $H$ containing the identity element in $e \in G$. For $g \in G$, let $c_g : G \to G$ be the conjugation map. Since $H_0$ is connected, $c_g(H_0)$ is connected in $G$. Since $e \in c_g(H_0)$ and $c_g(H_0)$ is connected for all $g \in G$. This argument shows that $H_0$ is a normal subgroup of $H$. Since $G$ is compact, $H$ and $H_0$ are closed, the group $R := H/H_0$ is a finite group. In addition, the covering space $BH_0 \to BH$ is a finitely-sheeted covering space and $R$ acts on this covering space. By a proposition in [9, Prop. 3G.1], the cohomology $H^*(BH)$ is

$$H^*(BH) \simeq (H^*(BH_0))^R \simeq (S(\mathfrak{h}_0)^{H_0})^R$$

(since $H_0$ is a closed and connected subgroup of $G$). Since $\mathfrak{h} = T_eH \simeq T_eH_0 = \mathfrak{h}_0$, $(S(\mathfrak{h}_0)^{H_0})^R$ is isomorphic to $(S(\mathfrak{h})^{H_0})^R$.

To compute $(S(\mathfrak{h})^{H_0})^R$, it suffices to prove the following lemma.

**Lemma 1.6.** *Suppose that a group $H$ acts on a set $X$ and $H_0$ is a normal subgroup of $H$. Let $R = H/H_0$ and let $X^H$ denote the set of elements in $X$ invariant under this $H$-action, i.e. $X^H := \{x \in X | h \cdot x = x, \forall h \in H\}$. Then*

$$X^H = (X^{H_0})^R$$

*Proof.* Let $x \in X^H$. Then $h \cdot x = x$ for all $h \in H$ and so $x \in X^{H_0}$. But every $h \in H$ has the form $h = r \cdot h_0$ for some $h_0 \in H_0$ and $r \in R$. Since $x \in X^H$, $x = h \cdot x = (r \cdot h_0) \cdot x = r \cdot (h_0 \cdot x) = r \cdot x$ for all $r \in R$. Hence, $x \in (X^{H_0})^R$. Conversely, let $x \in (X^{H_0})^R$ and consider $h \in H$. Since $h = r \cdot h_0$ for some $h_0 \in H_0$ and $r \in R$, $h \cdot x = (r \cdot h_0) \cdot x = r \cdot (h_0 \cdot x) = r \cdot x = x$ (since $x \in (X^{H_0})^R$). Hence, $x \in X^H$. □

As a consequence of this lemma, we have that $H^*(BH) = (S(\mathfrak{h})^{H_0})^R = S(\mathfrak{h})^H$.

# 2

# The Cartan Model

In this section, we summarize the Cartan model of equivariant cohomology and compute the equivariant cohomology of a homogeneous space. For more details about the Cartan model, see [4, 8]. Recall that a ***representation*** of a group $G$ on a vector space $V$ is a homomorphism $\rho : G \to GL(V)$. We can think of a representation $\rho$ as an action of $G$ on $V$ and write $g \cdot v$ for $\rho(g)(v)$. When $G$ is a Lie group, we require the homomorphism to be smooth. The ***dual representation*** of the representation $\rho : G \to GL(V)$ is the map

$$\rho^\vee : G \to GL(V^\vee)$$

defined by

$$\rho^\vee(g)(\alpha)(v) = \alpha(\rho(g^{-1})(v)) \text{ or } (g \cdot \alpha)(v) = \alpha(g^{-1} \cdot v),$$

for $\alpha \in V^\vee$ and $v \in V$ (it is necessary to take the inverse of $g$ so that $\rho^\vee$ will be a group homomorphism). Suppressing $v \in V$, we can write the dual representation as

$$\rho^\vee(g)(\alpha) = (\alpha \circ \rho)(g^{-1}) = \rho(g^{-1})^\vee(\alpha).$$

We can therefore think of the dual representation $\rho^\vee$ as an action of $G$ on $V^\vee$ and write $g \cdot \alpha$ for $\rho(g^{-1})^\vee(\alpha)$.

For each $g \in G$, the differential at the identity of the conjugation map $c_g := l_g \circ r_{g^{-1}} : G \to G$ is a linear isomorphism $c_{g*} : \mathfrak{g} \to \mathfrak{g}$, i.e. $c_{g*} \in \mathrm{GL}(\mathfrak{g})$. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ defined by $\mathrm{Ad}(g) = c_{g*}$ is a representation called the ***adjoint representation*** of the Lie group $G$. The dual representation of the adjoint representation of a Lie group, $\mathrm{Ad}^\vee : G \to \mathrm{GL}(\mathfrak{g}^\vee)$, is called the ***coadjoint representation***:

$$\begin{aligned} (g \cdot \alpha)(X) &= ((\mathrm{Ad}^\vee g)\alpha)(X) \\ &= \alpha((\mathrm{Ad}\, g^{-1})(X)) = (\mathrm{Ad}\, g^{-1})^* \alpha(X). \end{aligned}$$

Let $X_1, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$ and let $\theta_1, \ldots, \theta_n$ be the corresponding dual basis for $\mathfrak{g}^\vee$. From the dual space $\mathfrak{g}^\vee$, we construct the algebra of symmetric polynomials $S(\mathfrak{g}^\vee)$. The symmetric algebra $S(\mathfrak{g}^\vee)$ is generated by the set of polynomials $u_1, \ldots, u_n$ and is dual to the basis $\mathfrak{g}^\vee$, i.e. $u_i(X_j) = \theta_i(X_j) = \delta_{ij}$. Each $u_i$ has degree 2. Since the coadjoint representation defines an action of $G$ on $\mathfrak{g}^\vee$, the coadjoint representation induces an action on $S(\mathfrak{g}^\vee)$. In addition, the coadjoint representation induces an action on the exterior algebra $\wedge(\mathfrak{g}^\vee)$ by the pullback map: for $\omega \in \wedge^k(\mathfrak{g}^\vee)$ and $g \in G$

$$g \cdot \omega = (\mathrm{Ad}^\vee g)\omega = (\mathrm{Ad}\, g^{-1})^* \omega.$$

## 2.1 The Cartan complex

Let $M$ be a $G$-space for a connected Lie group $G$. Each $g \in G$ induces a diffeomorphism under left multiplication by $g$:

$$l_g : M \to M,$$
$$p \mapsto g \cdot p.$$

The group $G$ acts linearly on the de Rham complex $\Omega(M)$ of $M$ by the pullback of forms:

$$g \cdot \omega = l^*_{g^{-1}} \omega.$$

Note that for $g, h \in G$ and $\omega \in \Omega(M)$, $g \cdot (h \cdot \omega) = (gh) \cdot \omega$. We say that a form $\omega$ on $M$ is ***left-invariant*** if $l^*_g \omega = \omega$ for all $g \in G$.

We now construct a subcomplex of $S(\mathfrak{g}^\vee) \otimes \Omega(M)$ called the Cartan complex. An element $\alpha \in S(\mathfrak{g}^\vee) \otimes \Omega(M)$ is a finite sum

$$\alpha := \sum u^I \omega_I, \text{ where } u^I = u_1^{i_1} ... u_n^{i_n} \text{ and } \omega_I \in \Omega(M),$$

i.e. $\alpha$ is a polynomial in $u_1, ..., u_n$ with coefficients in $\Omega(M)$. An element $\alpha$ of the complex can be interpreted as a polynomial function on $\mathfrak{g}$ with values in $\Omega(M)$ as follows: define $\bar{\alpha} : \mathfrak{g} \to \Omega(M)$ by

$$\bar{\alpha}(X) = \sum u^I(X)\omega_I = \sum u_1(X)^{i_1} \cdots u_n(X)^{i_n} \omega_I \in \Omega(M).$$

We say that a function $\beta : \mathfrak{g} \to \Omega(M)$ is ***polynomial*** if $\beta = \bar{\alpha}$ for some $\alpha \in S(\mathfrak{g}^\vee) \otimes \Omega(M)$.

Since $G$ acts linearly on $S(\mathfrak{g}^\vee)$ by the induced action of the coadjoint representation and on $\Omega(M)$ by the pullback $l^*_{g^{-1}}$, $G$ acts linearly on $S(\mathfrak{g}^\vee) \otimes \Omega(M)$ by

$$g \cdot \alpha = g \cdot (u^I \otimes \omega) = (g \cdot u^I) \otimes (g \cdot \omega).$$

An element $\alpha \in S(\mathfrak{g}^\vee) \otimes \Omega(M)$ is said to be ***G-invariant*** if the corresponding polynomial map $\bar{\alpha} : \mathfrak{g} \to \Omega(M)$ is $G$-equivariant: for all $g \in G$ and $X \in \mathfrak{g}$,

$$\bar{\alpha}(g \cdot X) = \bar{\alpha}((\mathrm{Ad}\, g)X) = l^*_{g^{-1}}(\bar{\alpha}(X)) = g \cdot (\bar{\alpha}(X)).$$

The ***Cartan complex*** is defined to be the subcomplex

$$\Omega_G(M) := (S(\mathfrak{g}^\vee) \otimes \Omega(M))^G \subseteq S(\mathfrak{g}^\vee) \otimes \Omega(M)$$

consisting of elements of $S(\mathfrak{g}^\vee) \otimes \Omega(M)$ that are $G$-invariant.

There is a differential operator $d_G$ on $\Omega_G(M)$ called the ***Cartan differential*** defined as follows: for $\alpha \in \Omega_G(M)$ and $X \in \mathfrak{g}$

$$(d_G\alpha)(X) := d(\alpha(X)) - \iota_{\bar{X}}(\alpha(X)),$$

where $d$ is the exterior derivative and $\iota_{\bar{X}}$ denotes interior multiplication by $\bar{X}$, the fundamental vector field $\bar{X}$ on $M$ associated to $X \in \mathfrak{g}$. The Cartan differential is *nilsquare*, i.e. $d_G^2 = 0$.

The Cartan model of equivariant cohomology of a $G$-space $M$ is defined to be the cohomology of the differential complex: $H^*\{\Omega_G(M), d_G\}$. Using the Cartan model, we will soon compute the equivariant cohomology of a point and a homogeneous space.

## 2.2 The Cartan complex of a homogeneous space

Let $H$ be a closed, connected subgroup of a compact, connected Lie group $G$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$ respectively. The Cartan complex of the homogeneous space $G/H$ is

$$\Omega_G(G/H) = (S(g^\vee) \otimes \Omega(G/H))^G.$$

We will show that the Cartan complex is isomorphic to another complex and compute the cohomology of the new complex in the next section. To construct this isomorphic complex, we first will show that the exterior algebra $\Omega^*(G/H)^G$

of left-invariant forms on the homogeneous space $G/H$ is isomorphic to a subcomplex of the exterior algebra $\Omega^*(G)^G$ of left-invariant forms on $G$.

Let $H$ be a closed subgroup of a Lie group $G$. A $k$-form $\omega$ on $G$ is said to be $\mathrm{Ad}(H)$-***invariant*** if

$$h \cdot \omega_e = (\mathrm{Ad}^\vee h)\omega_e = (\mathrm{Ad}\, h^{-1})^* \omega_e = \omega_e \in \wedge^k(\mathfrak{g}^\vee)$$

for all $h \in H$. A $k$-form $\omega$ on $G$ ***annihilates*** $\mathfrak{h}$ if $\omega_e(v_1, \ldots, v_k) = 0$ and some $v_i$ is in $\mathfrak{h}$. Suppose that $\omega$ and $\tau$ are two $\mathrm{Ad}(H)$-invariant, left-invariant forms on $G$ that annihilate $H$ of degrees $k$ and $l$ respectively. Since the wedge product commutes with the pullback, the wedge product $\omega \wedge \tau$ is a $(k+l)$ left-invariant and $\mathrm{Ad}(H)$-invariant form. In addition, the wedge product $\omega \wedge \tau$ is a form that annihilates $\mathfrak{h}$ from the very definition of the wedge product. Specifically, let $v_1, \ldots, v_{k+l} \in \mathfrak{g}$ with $v_i \in \mathfrak{h}$ for some $i = 1, \ldots, k+l$. By definition of the wedge product,

$$\begin{aligned}
\omega \wedge \tau(v_1, \ldots, v_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\mathrm{sgn}\, \sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\mathrm{sgn}\, \sigma)0 \text{ (since } \omega \text{ and } \tau \text{ are } \mathrm{Ad}(H)\text{-invariant)} \\
&= 0.
\end{aligned}$$

Finally, since the exterior derivative commutes with the pullback, $d\omega$ is an $\mathrm{Ad}(H)$-invariant, left-invariant forms on $G$ that annihilate $H$. Therefore, we have shown that the set of left-invariant, $\mathrm{Ad}(H)$-invariant forms that annihilate $\mathfrak{h}$ is a subalgebra of the exterior algebra $\wedge(\mathfrak{g}^\vee)$.

**Theorem 2.1.** *: Let $\pi : G \to G/H$ be the natural projection map. The pullback map*

$$\pi^* : \Omega(G/H) \to \Omega(G) \tag{2.1}$$

$$\omega \mapsto \pi^* \omega \tag{2.2}$$

*gives rise to the following one-to-one correspondence:*

*{left-invariant k-forms on G/H}* $\leftrightarrow$ *{left-invariant* $\mathrm{Ad}(H)$*-invariant k-forms on G that annihilate* $\mathfrak{h}$*}.*

*Proof.* ($\Rightarrow$) Let $\omega$ be a left-invariant $k$-form on $G/H$. We have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\;l_g\;} & G \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G/H & \xrightarrow[\;l_{\bar{g}}\;]{} & G/H \end{array}$$

where $l_{\bar{g}}$ is left-multiplication by the left coset $gH$, i.e. $l_{\bar{g}}(aH) = g \cdot aH = gaH$. For convenience, we will sometimes write $l_{\bar{g}}$ as $l_g$.

(*$\pi^*\omega$ is left-invariant:*) Since $\pi \circ l_g = l_{\bar{g}} \circ \pi$, the pullback maps also commute: $(\pi \circ l_g)^* = (l_{\bar{g}} \circ \pi)^*$. Therefore,

$$\begin{aligned} l_g^*(\pi^*\omega) &= (\pi \circ l_g)^*(\omega) \\ &= (l_{\bar{g}} \circ \pi)^*\omega = \pi^* l_{\bar{g}}^* \omega \\ &= \pi^*\omega \text{ (since } \omega \text{ is left-invariant).} \end{aligned}$$

(*$\pi^*\omega$ is* $\mathrm{Ad}(H)$*-invariant on G:*) Let $h \in H$.

$$\begin{aligned} (Adh)^*\pi^*\omega &= (l_h \circ r_{h^{-1}})^*\pi^*\omega \\ = r_{h^{-1}}^*(l_h^*\pi^*\omega) &= r_{h^{-1}}^*\pi^*\omega \\ &= \pi^*\omega \text{ (since } \pi = \pi \circ r_{h^{-1}} \text{ on } G/H). \end{aligned}$$

(*$\pi^*\omega$ annihilates $\mathfrak{h}$:*) Let $v_1, \ldots, v_i, \ldots, v_k \in \mathfrak{g}$ with $v_i \in \mathfrak{h}$.

$$\begin{aligned} (\pi^*\omega)_e(v_1, \ldots, v_i, \ldots, v_k) &= \omega_{eH}(\pi_* v_1, \ldots, \pi_* v_i, \ldots, \pi_* v_k) \\ &= \omega_{eH}(\pi_* v_1, \ldots, 0, \ldots, \pi_* v_k) = 0 \end{aligned}$$

(if $\pi : G \to G/H$ is the projection of $G/H$, then the induced map $\pi_* : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ sends $\mathfrak{h}$ to $0$).

($\Leftarrow$) Conversely, let $\eta \in \Omega^k(G)$ be a left-invariant $\mathrm{Ad}(H)$-invariant $k$-form on $G$ that annihilates $\mathfrak{h}$. We want to construct a left-invariant $k$-form $\omega$ on $G/H$ such that $\pi^*\omega = \eta$. Since $\eta$ is a left-invariant form on $G$, it is generated by a $k$-form $\eta_e \in \wedge^k(\mathfrak{g}^\vee)$ on the Lie algebra $\mathfrak{g}$. Since $\eta$ annihilates $\mathfrak{h}$, clearly $\eta_e$ also annihilates $\mathfrak{h}$. Similarly, $\eta_e$ is $\mathrm{Ad}(H)$-invariant since $\eta$ is $\mathrm{Ad}(H)$-invariant. Thus, the $k$-form $\eta_e : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R}$ induces a $k$-form $\eta_0 \in \wedge^k(\mathfrak{g}/\mathfrak{h})^\vee$ in the sense that $\eta_e(v_1, \ldots, v_k) = \eta_0(\pi_* v_1, \ldots, \pi_* v_k) = \pi^*\eta_0(v_1, \ldots, v_k)$ for $v_1, \ldots, v_k \in \mathfrak{g}$. Therefore, we define $\omega_{eH}$ on $G/H$ to be $\eta_0$, and for $a \in G$, we define $\omega_{aH}$ to be $l^*_{a^{-1}}\omega_{eH}$.

We need to show that this construction of the form $\omega \in \Omega(G/H)$ is independent of our choice of coset representatives, i.e. if $aH = bH$, then $\omega_{aH} = \omega_{bH}$. Suppose that $aH = bH$. Then $ab^{-1} \in H$. Let $c = ab^{-1}$.

By the commutative diagram, $\pi^* l^*_c \omega_{eH} = l^*_c \pi^* \omega_{eH}$.

Let $r_g$ denote right multiplication by $g \in G$ on $G/H$. Since $\pi \circ r_{h^{-1}} = \pi$ for all $h \in H$, $r^*_{c^{-1}}\pi^* = \pi^*$. Thus,

$$\begin{aligned} l^*_c \pi^* \omega_{eH} &= l^*_c r^*_{c^{-1}} \pi^* \omega_{eH} \\ Adc)^* \pi^* \omega_{eH} &= (\mathrm{Ad}\, c)^* \eta_e \\ &= \eta_e \text{ (since } \eta_e \text{ is } \mathrm{Ad}(H)\text{-invariant)} \\ &= \pi^* \omega_{eH}. \end{aligned}$$

Since the pullback map $\pi^*$ is injective, $l^*_{cH}\omega_{eH} = \omega_{eH}$. Finally,

$$l^*_c \omega_{eH} = l^*_{ab^{-1}} \omega_{eH} = l^*_{b^{-1}} l^*_a \omega_{eH} = \omega_{eH}.$$

Thus,

$$l^*_a \omega_{eH} = l^*_b \omega_{eH}.$$

In addition, since $\eta$ is left-invariant, $\omega$ must also be left-invariant. Finally, we need to show that $\eta = \pi^*\omega$. But this follows from the fact that $\eta_e = \pi^*\omega_{eH}$ and both $\eta$

and $\pi^*\omega$ are left-invariant. $\square$

Now that we have established a one-to-one correspondence between left-invariant $k$-forms on $G/H$ and left-invariant $\mathrm{Ad}(H)$-invariant $k$-forms on $G$ that annihilate $\mathfrak{h}$, we can construct a complex isomorphic to the Cartan complex $\Omega_G(G/H)$.

The Cartan complex $\Omega_G(G/H)$ is $(S(\mathfrak{g}^\vee) \otimes \Omega(G/H))^G$. Recall that another way of defining this complex is that it is the set of $G$-equivariant polynomial maps: for all $\alpha \in \Omega^*(G/H)$, $g \in G$, and $X \in \mathfrak{g}$,

$$\alpha((\mathrm{Ad}\, g)X) = l^*_{g^{-1}}(\alpha(X)).$$

This condition can be rewritten as

$$\alpha(X) = l^*_{g^{-1}}(\alpha((\mathrm{Ad}\, g^{-1})X)).$$

Consider the complex $S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee$. As in our discussion of the Cartan complex, an element $\beta \in S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee$ is a finite sum

$$\alpha := \sum u^I \omega_I, \text{ where } u^I = u_1^{i_1} ... u_n^{i_n} \text{ an } \omega_I \in \wedge(\mathfrak{g}/\mathfrak{h})^\vee.$$

We can interpret an element of this complex as a polynomial functions on $\mathfrak{g}$ with values in $\wedge(\mathfrak{g}/\mathfrak{h})^\vee$. An element $\alpha \in S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee$ is said to be $\mathrm{Ad}(H)$-***invariant*** if the corresponding polynomial map $\beta : \mathfrak{g} \to \wedge(\mathfrak{g}/\mathfrak{h})^\vee$ is $\mathrm{Ad}(H)$-equivariant: for all $X \in \mathfrak{g}$ and $h \in H$,

$$\begin{aligned}\beta(h \cdot X) &= \beta(\mathrm{Ad}(H)X) \\ &= (\mathrm{Ad}\, h)^*(\beta(X)) = (\mathrm{Ad}^\vee h^{-1})(\beta(X)) = h^{-1} \cdot \beta(X).\end{aligned}$$

Let $(S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H$ denote the subalgebra of forms in $S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee$ that are $\mathrm{Ad}(H)$-invariant. For simplicity, we will write $\wedge_{\mathrm{Ad}\, H}(\mathfrak{g}/\mathfrak{h})^\vee$ to mean $(S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H$. We now will show that the Cartan complex is isomorphic to the complex

$$\wedge_{\mathrm{Ad}\, H}(\mathfrak{g}/\mathfrak{h})^\vee := (S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H.$$

One way of describing this new complex $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$ is that it is equal to the set of $\mathrm{Ad}(H)$-equivariant polynomial maps $\alpha : \mathfrak{g} \to \wedge(\mathfrak{g}/\mathfrak{h})^\vee$.

**Theorem 2.2.** *The Cartan complex* $\Omega_G(G/H) = (S(\mathfrak{g}^\vee) \otimes \Omega(G/H))^G$ *is isomorphic to the complex* $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee = (S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H$.

*Proof.* Consider the maps

$$\begin{aligned}\varphi : \Omega_G(G/H) &\to \wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee \\ \alpha(X) &\mapsto \tilde{\alpha}(X) := \alpha(X)_{eH}\end{aligned}$$

and

$$\begin{aligned}\psi : \wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee &\to \Omega_G(G/H) \\ \beta(X)_{aH} &\mapsto \bar{\beta}(X) := l^*_{a^{-1}}\beta((\mathrm{Ad}\,a^{-1})X).\end{aligned}$$

The motivation behind the map $\psi$ is that it should send an element $\beta$ to a $G$-equivariant polynoimial map since an element $\alpha \in S(\mathfrak{g}^\vee) \otimes \Omega(G/H)$ is $G$-invariant if for all $X \in \mathfrak{g}$ and for all $g \in G$,

$$\alpha(X) = l^*_{g^{-1}}(\alpha((\mathrm{Ad}\,g^{-1})X)) = g \cdot \alpha(g^{-1} \cdot x).$$

It suffices to show that for all $X \in \mathfrak{g}$, $a \in G$, $\alpha \in \Omega_G(G/H)$, and $\beta \in \wedge(\mathfrak{g}/\mathfrak{h})^\vee$, $\varphi\psi(\alpha(X)) = \alpha(X)$ and $\psi\varphi(\beta(X)) = \beta(X)$.

Before proving that $\varphi\psi = \mathbb{1}_{\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee}$ and $\psi\varphi = \mathbb{1}_{\Omega_G(G/H)}$, we first need to show that $\tilde{\alpha}(X)$ is an $\mathrm{Ad}(H)$-invariant form in $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$. Then we need to show that $\psi$ is a well-defined map. Since a $G$-equivariant polynomial satisfies $\alpha((\mathrm{Ad}\,g)X) = l^*_{g^{-1}}(\alpha(X))$ and $\beta$ is a left-invariant and $\mathrm{Ad}(H)$-invariant form, we can immediately see that $\bar{\beta}(X) = l^*_{a^{-1}}\beta((\mathrm{Ad}\,a^{-1})X) = \beta(X)$ is a $G$-equivariant polynomial, i.e. $\bar{\beta}$ is a form in the Cartan complex.

($\tilde{\alpha}$ *is* $\mathrm{Ad}(H)$-*invariant*:) Let $h \in H$, $X \in \mathfrak{g}$. Then

$$\begin{aligned}\tilde{\alpha}((\mathrm{Ad}\,h)X) &= \alpha((\mathrm{Ad}\,h)X)_{eH} = (l^*_{h^{-1}}\alpha(X))_{eH} \text{ (since } \alpha \text{ is } G\text{-equivariant)} \\ &= (l^*_{h^{-1}}r^*_h\alpha(X))_{eH} = (c^*_{h^{-1}}\alpha(X))_{eH}\end{aligned}$$

(since $\alpha(X)$ is a form on $G/H$ and $r_h = \mathbb{1}_{G/H}$ on $G/H$ so $\alpha(X) = r_h^* \alpha(X)$)

$$\begin{aligned} &= \operatorname{Ad}(h^{-1})^*(\alpha(X)_{eH}) = (\operatorname{Ad} h^{-1})^*(\tilde{\alpha}(X)) \\ &= (\operatorname{Ad}^\vee h)(\tilde{\alpha}(X)). \end{aligned}$$

($\psi$ *is well-defined*:) Let $\beta \in \wedge_{\operatorname{Ad} H}(\mathfrak{g}/\mathfrak{h})^\vee$ and $X \in \mathfrak{g}/\mathfrak{h}$. Consider $\beta(X)_{aH}, \beta(X)_{bH} \in \wedge(\mathfrak{g}/\mathfrak{h})^\vee$ and suppose that $aH = bH$. Let $c = ab^{-1} \in H$. To prove that $\psi$ is well-defined, it suffices to show that

$$\psi(\beta(X)_{cH}) = l_{c^{-1}}^*((\operatorname{Ad} c^{-1})\beta)(X) = l_e^* \beta((\operatorname{Ad} c)X) = \beta(X).$$

But by definition, a form $\beta(X)$ in $\Omega(G/H)$ is $G$-equivariant if for all $g \in G$, $l_g^*(\beta((\operatorname{Ad} g)X)) = \beta(X)$. So $\psi(\beta(X)_{cH}) = \beta(X)$ and the map $\psi$ is well-defined. Note that we also have shown that $\psi\varphi = \mathbb{1}_{\Omega(G/H)}$.

$$\begin{aligned} (\,\varphi\psi = \mathbb{1}_{\wedge_{\operatorname{Ad} H}(\mathfrak{g}/\mathfrak{h})^\vee}:)\varphi\psi(\beta(X)_{aH}) &= \varphi(l_{a^{-1}}^*((\operatorname{Ad} a^{-1})\beta)(X)) \\ &= l_{a^{-1}}^*((\operatorname{Ad} a^{-1})\beta)(X)_{eH} = ((\operatorname{Ad} a^{-1})\beta)(X)_{aH} \\ &= \beta(X)_{aH} \text{ (since } \beta(X)_{aH} \text{ is } \operatorname{Ad}(H)\text{-invariant, by theorem 2.1).} \end{aligned}$$

□

## 2.3 The Cohomology of the Complex $\wedge_{\operatorname{Ad} H}(\mathfrak{g}/\mathfrak{h})^\vee$

The Cartan complex has a Cartan differential $d_G$ and the Cartan model of equivariant cohomology states that $H_G^*(M) = H^*\{\Omega_G(M), d_G\}$. By theorem 2.2, the Cartan complex $\Omega_G(G/H)$ of the homogeneous space $G/H$ is isomorphic to $\wedge_{\operatorname{Ad} H}(\mathfrak{g}/\mathfrak{h})^\vee$. Therefore, our strategy for computing the equivariant cohomology of the homogeneous space $G/H$ is to construct the differential operator $\tilde{d}_G$ in $\wedge_{\operatorname{Ad} H}(\mathfrak{g}/\mathfrak{h})^\vee$ corresponding to the Cartan differential $d_G$ (i.e. construct the differential operator that makes the diagram below commute) and compute the cohomology of the differential complex $\{\wedge_{\operatorname{Ad} H}(\mathfrak{g}/\mathfrak{h})^\vee, \tilde{d}_G\}$.

$$
\begin{array}{ccc}
\Omega_G^k(G/H) & \xrightarrow{\varphi} & \wedge_{\mathrm{Ad}\,H}^k(\mathfrak{g}/\mathfrak{h})^\vee \\
{\scriptstyle d_G}\downarrow & & \downarrow{\scriptstyle \tilde{d}_G} \\
\Omega_G^{k+1}(G/H) & \xrightarrow[\varphi]{} & \wedge_{\mathrm{Ad}\,H}^{k+1}(\mathfrak{g}/\mathfrak{h})^\vee
\end{array}
$$

So to describe the differential operator $\tilde{d}_G$, it suffices to determine what the Cartan differential does on generators for $\Omega_G(G/H) = (S(\mathfrak{g}^\vee) \otimes \Omega(G/H))^G$. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$ with $X_1, \ldots, X_k$ a basis for $\mathfrak{g}/\mathfrak{h}$ and $X_{k+1}, \ldots, X_n$ a basis for $\mathfrak{h}$. Let $\theta_1, \ldots, \theta_n$ be the dual basis of $X_1, \ldots, X_n$ for $\mathfrak{g}^\vee$ and let $u_1, \ldots, u_n$ be generators for the algebra of symmetric polynomials $S(\mathfrak{g}^\vee)$ such that the polynomials $u_1, \ldots, u_n$ are dual to $X_1, \ldots, X_n$, i.e. $u_i(X_j) = \theta_i(X_j) = \delta_{ij}$. As an $\mathbb{R}$-algebra, the Cartan complex $\Omega_G(G/H)$ is generated by $\{u_i \otimes 1, 1 \otimes \omega(G/H) : \omega \in \Omega(G/H)\}$. Therefore, we just have to compute $d_G(u_i \otimes 1)$ and $d_G(1 \otimes \omega)$.

Recall that for $X \in \mathfrak{g}$, $\alpha \in \Omega_G(G/H)$, $(d_G\alpha)(X) = d(\alpha(X)) - \iota_{\bar{X}}(\alpha(X))$, where $\bar{X}$ is the fundamental vector field on $G/H$ associated to $X$. Therefore,

$$
\begin{aligned}
(d_G(u_i \otimes 1))(X_j) &= d((u_i \otimes 1)(X_j)) - \iota_{\bar{X}_j}((u_i \otimes 1)(X_j)) \\
(u_i \otimes 1)(X_j) &= u_i(X_j) \cdot 1 = \delta_{ij}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(d_G(u_i \otimes 1))(X_j) &= 0. \\
(d_G(1 \otimes \omega))(X_j) &= d(1(X_j)\omega) - \iota_{\bar{X}_j}(1(X_j)\omega) \\
&= d\omega - \iota_{\bar{X}_j}\omega.
\end{aligned}
$$

Let us now consider the complex $\wedge_H(\mathfrak{g}/\mathfrak{h})^\vee = (S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H$. This complex is generated by $\{u_i \otimes 1, 1 \otimes \theta_1, \ldots, 1 \otimes \theta_k\}$ (since $X_1, \ldots, X_k$ is a basis for $\mathfrak{g}/\mathfrak{h}$). For notational convenience, we will occaisonally drop the tensor product so that $\theta_i = 1 \otimes \theta_i$ and $u_i = u_i \otimes 1$. Let $\omega_i$ be the left-invariant form on

$\Omega(G/H)$ corresponding to $\theta_i$ (the form $\omega_i$ is obtained simply by the pullback of left-multiplication) so that $\varphi(\omega_i) = (\omega_i)_{eH} = \theta_i$.

$$\begin{aligned}((\tilde{d}_G \circ \varphi)(u_i \otimes 1))(X_j) &= ((\varphi \circ d_G)(u_i \otimes 1))(X_j)\\ &= \varphi(0) = 0.\end{aligned}$$

$$\begin{aligned}((\tilde{d}_G)(1 \otimes \theta_i))(X_j) = ((\tilde{d}_G \circ \varphi)(1 \otimes \omega_i))(X_j) &= ((\varphi \circ d_G)(1 \otimes \omega_i))(X_j)\\ = \varphi(d\omega_i - \iota_{\bar{X}_j}\omega_i) &= (d(1(X_j)\omega_i) - \iota_{\bar{X}_j}(1(X_j)\omega_i))_{eH}\\ = (d\omega_i)_{eH} - (\iota_{\bar{X}_j}\omega_i)_{eH} &= d\theta_i - \delta_{ij}.\end{aligned}$$

Therefore,

$$\tilde{d}_G(1 \otimes \theta_i) = 1 \otimes d\theta_i - u_i \otimes 1.$$

We now provide an alternate algebraic description of the complex $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$ by introducing the *Koszul complex*. By describing $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$ in terms of a Koszul complex, we obtain a simpler description of the differential operator $\tilde{d}_G$.

## 2.4 The Koszul Complex

Let $V$ be an $n$-dimensional vector space. Let $\wedge(V^\vee)$ be the exterior algebra of $V^\vee$, and let $S(V^\vee)$ be the algebra of symmetric polynomials. All of the elements of the symmetric algebra $S(V^\vee)$ are even. The ***Koszul algebra*** $K(V)$ of $V$ is the tensor product $S(V^\vee) \otimes \wedge(V^\vee)$. Suppose that $u_1, \ldots, u_n$ generate $S(V^\vee)$ and that $\theta_1, \ldots, \theta_n$ generate $\wedge(V^\vee)$. Then $u_i \otimes 1$ and $1 \otimes \theta_i$ generate the Koszul algebra. Note that $u_i \otimes 1$ has degree 2 and $1 \otimes \theta_i$ has degree 1. The ***Koszul operator*** $d_K$ is the anti-derivation defined on the generators by:

$$\begin{aligned}d_K(u_i \otimes 1) &= 0,\\ d_K(1 \otimes \theta_i) &= u_i \otimes 1\end{aligned}$$

and extended to $K(V)$ as an antiderivation. Since $d_K^2 = 0$ on the generators, it follows that $d_K^2 = 0$ for all $\alpha \in K(V)$. The ***Koszul complex*** of $V$ is the Koszul algebra $K(V)$ of $V$ with the Koszul operator $d_K$. The cohomology of the Koszul complex is ***acyclic***, i.e.

$$H^n(K(V)) = \begin{cases} \mathbb{R} & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

We can describe the complex $\{\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee, \tilde{d}_G\}$ in terms of a Koszul complex. Let us decompose $\mathfrak{g}^\vee$ into $\mathfrak{h}^\vee$ and $\mathfrak{g}/\mathfrak{h}^\vee$ such that the decomposition is $\mathrm{Ad}(H)$-invariant. We have that $\mathfrak{g}^\vee = \mathfrak{h}^\vee \oplus \mathfrak{g}/\mathfrak{h}^\vee$ and $S(\mathfrak{g}^\vee) = S(\mathfrak{h}^\vee) \otimes S(\mathfrak{g}/\mathfrak{h})^\vee$. Therefore, we can decompose $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$ as follows:

$\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee = (S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H = (S(\mathfrak{h}^\vee) \otimes S(\mathfrak{g}/\mathfrak{h})^\vee \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H.$

Note that $S(\mathfrak{g}/\mathfrak{h})^\vee \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee$ is the Koszul algebra $K(\mathfrak{g}/\mathfrak{h})$ of $\mathfrak{g}/\mathfrak{h}$. Let $\tilde{d}_K$ be the extension of the Koszul operator $d_K$ on $S(\mathfrak{g}/\mathfrak{h})^\vee \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee$ to

$$S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee = (S(\mathfrak{h}^\vee) \otimes S(\mathfrak{g}/\mathfrak{h})^\vee \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee)^H$$

by setting $\tilde{d}_K(u_i \otimes 1 \otimes 1)$ to be zero. Then we have that $\tilde{d}_G = 1 \otimes d - \tilde{d}_K$ (or, more simply, $d - \tilde{d}_K$) since

$$\begin{aligned} d((u_i \otimes 1)(X_j)) - (\tilde{d}_K(u_i \otimes 1))(X_j) &= 0 - 0 \\ &= (\tilde{d}_G(u_i \otimes 1))(X_j) \end{aligned}$$

and

$$\begin{aligned} d((1 \otimes \theta_i)(X_j)) - (\tilde{d}_K(1 \otimes \theta_i))(X_j) &= d\theta_i - \delta_{ij} \\ &= (\tilde{d}_G(1 \otimes \theta_i))(X_j). \end{aligned}$$

## 2.5 The Cohomology of $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$

We have a differential complex $\{\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee, \tilde{d}_G\}$ and by the previous section, this differential complex is the same as the differential complex $\{(S(\mathfrak{h}^\vee) \otimes K(\mathfrak{g}/\mathfrak{h}))^H, d - \tilde{d}_K\}$. We now show that $\tilde{d}_G$ is $\mathrm{Ad}(H)$-equivariant. As a consequence, we have that

$$H^*\{(S(\mathfrak{h}^\vee)\otimes K(\mathfrak{g}/\mathfrak{h}))^H, \tilde{d}_G\} = (H^*\{S(\mathfrak{h}^\vee)\otimes K(\mathfrak{g}/\mathfrak{h}), \tilde{d}_G\})^H$$

**Lemma 2.3.** *The differential operator $\tilde{d}_G$ on $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$ is $\mathrm{Ad}(H)$-equivariant.*

*Proof.* To prove that $\tilde{d}_G$ is $\mathrm{Ad}(H)$-equivariant it suffices to show that $d$ and $\tilde{d}_K$ are $\mathrm{Ad}(H)$-equivariant maps on the generators $u_i\otimes 1$ and $1\otimes\theta_i$ of $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$. Let $h\in H$ and $X\in\mathfrak{h}$ and $X\in\mathfrak{g}$.
(*$d$ is $\mathrm{Ad}(H)$-equivariant on $1\otimes\theta_i$:*)

$$d((\mathrm{Ad}^\vee h)(1\otimes\theta_i)(X)) = d((\mathrm{Ad}\,h^{-1})^*(1\otimes\theta_i)(X))$$
$$= d((1\otimes\theta_i)((\mathrm{Ad}\,h^{-1})X)) = d(1((\mathrm{Ad}\,h^{-1})X)\theta_i) = d\theta_i$$

and

$$(\mathrm{Ad}^\vee h)d((1\otimes\theta_i)(X)) = (\mathrm{Ad}^\vee h)d\theta_i = (\mathrm{Ad}\,h)^*d\theta_i.$$

Since $\theta_i$ is in $\wedge(\mathfrak{g}/\mathfrak{h})^\vee$, $\theta_i$ corresponds to a left-invariant form on $G/H$. Therefore, by theorem 2.1, $\theta_i$ also corresponds to a left-invariant $\mathrm{Ad}(H)$-invariant form on $G$ that annihilates $\mathfrak{h}$. Since $\theta_i$ is $\mathrm{Ad}(H)$-invariant, $d\theta_i$ is $\mathrm{Ad}(H)$-invariant. Thus,

$$d((\mathrm{Ad}^\vee h)(1\otimes\theta_i)(X)) = (\mathrm{Ad}^\vee h)d((1\otimes\theta_i)(X)) = d\theta_i.$$

(*$\tilde{d}_K$ is $\mathrm{Ad}(H)$-equivariant on $1\otimes\theta_i$:*)

$$\tilde{d}_K((\mathrm{Ad}^\vee h)(1\otimes\theta_i)(X)) = \tilde{d}_K(1((\mathrm{Ad}\,h^{-1})X)\theta_i) = u_i$$
$$(\mathrm{Ad}^\vee h)\tilde{d}_K((1\otimes\theta_i)(X)) = (\mathrm{Ad}^\vee h)u_i.$$

Recall that the Cartan complex $\Omega_G(G/H)$ is $(S(\mathfrak{g}^\vee)\otimes\Omega(G/H))^G$ where $G$ acts on $S(\mathfrak{g}^\vee)$ by the coadjoint representation. Therefore, by definition, a polynomial $u^I\otimes 1$ is in $\Omega_G(G/H)$ if $u^I$ is invariant under the coadjoint representation. Since $u_i$ is invariant in the Cartan complex, $u_i$ must also be invariant in $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$.
In addition, since

$$\tilde{d}_K(u_i\otimes 1) = d(u_i\otimes 1) = 0$$

$\tilde{d}_G$ is clearly equivariant on $u_i\otimes 1$. Thus, $\tilde{d}_G$ is $\mathrm{Ad}(H)$-equivariant. □

We now compute the cohomology of the complex $\{S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}/\mathfrak{h})^\vee, \tilde{d}_G\}$ by a spectral sequence argument and by treating $S(\mathfrak{h}^\vee) \otimes K(\mathfrak{g}/\mathfrak{h})$ as a double complex. Let

$$E_0^{p,q} = \begin{cases} S^{p/2}(\mathfrak{h}^\vee) \otimes K^q(\mathfrak{g}/\mathfrak{h}) & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases}$$

where $K^q(\mathfrak{g}/\mathfrak{h}) = \oplus_{q=2i+j} S^i(\mathfrak{g}/\mathfrak{h})^\vee \otimes \wedge^j(\mathfrak{g}/\mathfrak{h})^\vee$. This construction reflects the fact that $S(\mathfrak{g}^\vee)$ consists only of elements of even degree, so $S^p(\mathfrak{h}^\vee)$ is $0$ for $p$ odd. We equip the complex $E_0^{p,q}$ with the extended Koszul differential $\tilde{d}_K$.

Since the Koszul complex $K(\mathfrak{g}/\mathfrak{h})$ is acyclic with respect to the Koszul differential $d_K$, $E_0^{p,q} = S^p(\mathfrak{g}^\vee) \otimes K(\mathfrak{g}/\mathfrak{h})$ is acyclic with respect to the extended Koszul differential $\tilde{d}_K$. Therefore, we have that the cohomology of $E_0$ with respect to the extended Koszul differential $\tilde{d}_K$ is:

$$E_1^{p,q} := H(E_0^{p,q}, \tilde{d}_K) = \begin{cases} S^{p/2}(\mathfrak{h}^\vee) \otimes \mathbb{R} & \text{if } p \text{ is even } q = 0 \\ 0 & \text{if } q > 0 \text{ or } p \text{ is odd.} \end{cases}$$

Note that the cohomology of the double complex is concentrated along the bottom row. In addition, since the tensor product is over $\mathbb{R}$, $S^p(\mathfrak{h}^\vee) \otimes \mathbb{R}$ is isomorphic to $S^p(\mathfrak{h}^\vee)$.

Let $E_1 = \oplus E_1^{p,q} = \oplus E_1^{2p,0} = S^p(\mathfrak{h}^\vee)$ and equip $E_1$ with the exterior derivative $d$. Since $d$ sends every element in $S(\mathfrak{g}^\vee)$ to $0$,

$$E_2^{p,q} := H(E_1^{p,q}, d) = \begin{cases} S^{p/2}(\mathfrak{h}^\vee) & \text{for } p \text{ even and } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

So $E_2 = \oplus_{p,q} E_2^{p,q} \simeq \oplus_{p=0}^{\infty} E_2^{2p,0} = S(\mathfrak{h}^\vee)$. Inductively, if $d_r = d_1 = d$ and $E_r = H(E_{r-1}, d_r)$ for $r \geq 2$, we have:

$$E_1^{p,q} = E_2^{p,q} = \ldots$$

The stationary value of $E_1^{p,q}$ is denoted $E_\infty^{p,q}$.

From the theory of spectral sequences, the associated graded complex $GH^n_{\tilde{d}_C}(\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee)$ of the total cohomology is given by

$$GH^n_{\tilde{d}_G}(\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee) = \oplus_{2p+q=n} E^{p,q}_\infty(\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee)$$

$$\simeq E^{n/2,0}_\infty(\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee) = \begin{cases} S^{n/2}(\mathfrak{h}^\vee) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

(for a proof, see [3, Thm. 14.14]). In addition, since the complex $\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee$ is a vector space, we have a *vector space isomorphism*:

$$H^{2n}_{\tilde{d}_G}(\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee) \simeq GH^{2n}_{\tilde{d}_G}(\wedge_{\mathrm{Ad}\,H}(\mathfrak{g}/\mathfrak{h})^\vee) \simeq S^n(\mathfrak{h}^\vee)$$

(see [3, Rem. 14.17]). In the next subsection, we will show that there is a ring homomorphism from the Cartan model to the Borel model. Since a ring homomorphism and a vector space isomorphism between two algebras is an algebra isomorphism, the two models of equivariant cohomology are isomorphic as *algebras*:

$$H^*(\Omega_G(G/H), d_G) \to H^*(EG \times_G G/H) \simeq S(\mathfrak{h}^\vee)^H.$$

Note also that when $H = G$, we have the equivariant cohomology of a point under the Cartan model:

$$S^*(\mathfrak{h}^\vee)^H = S^*(\mathfrak{g}^\vee)^G,$$

which is the same as the equivariant cohomology of a point under the Borel model. Therefore, we have shown that the Borel and Cartan models compute the same equivariant cohomology of $M$ when $M$ is a homogeneous space and when $M$ is a point.

## 2.6 A Ring Homomorphism from the Cartan model to the Borel model

To construct the ring homomorphism from the Cartan model to the Borel model, we will describe a third model of equivariant cohomology called the *Weil* *model* that is isomorphic to the Cartan model and construct a ring homomorphism from the Weil model to the Borel model (for more details on the Weil model, see [8, Ch. 4]). The ***Weil algebra*** $W(G)$ of a Lie group $G$ with Lie algebra $\mathfrak{g}$ is the Koszul algebra of $\mathfrak{g}^\vee$:

$$W(G) := S(\mathfrak{g}^\vee) \otimes \wedge(\mathfrak{g}^\vee).$$

Suppose that $X_1, \ldots, X_n$ is a basis for the Lie algebra $\mathfrak{g}$. If $u_1, \ldots, u_n$ is the dual basis for $S(\mathfrak{g})$ and $\theta_1, \ldots, \theta_n$ is the dual basis for $\bigwedge(\mathfrak{g}^\vee)$, then $u_i := u_i \otimes 1$ and $\theta_j := 1 \otimes \theta_j$ generate the Weil algebra. Since $[X_i, X_j]$ is in $\mathfrak{g}$, there exist constants $c_{ij}^k$ such that

$$[X_i, X_j] = \textstyle\sum_k c_{ij}^k X_k.$$

We can construct the exterior derivative $d$ on the Weil algebra by defining $d$ on generators and extending it to $W(G)$ as an antiderivation:

$$d\theta_i = u_i - \tfrac{1}{2} \textstyle\sum c_{jk}^i \theta_j \theta_k$$

$$du_i = \textstyle\sum c_{jk}^i u_j \theta_k.$$

For $X \in \mathfrak{g}$, we define the ***contraction*** with $X$ on the Weil algebra to be the following map: set

$$\iota_X \theta_i = \theta_i(X), \iota_X u_j = 0,$$

and extend it to $W(G)$ as an antiderivation of degree -1. The Lie derivative $\mathcal{L}_X$ is

$$\mathcal{L}_X = \iota_X d + d\iota_X.$$

An element $a$ of the Weil model $W(G)$ is said to be ***basic*** if $\iota_X a = 0$ and $\mathcal{L}_X a = 0$ for all $X \in \mathfrak{g}$. The set $W(G)_{\text{bas}}$ of basic elements of $W(G)$ form a subalgebra of the Weil algebra. The set $W(G)_{\text{bas}}$ is also closed under the exterior derivative $d$. Thus, $W(G)_{\text{bas}}$ is a subcomplex of $W(G)$.

For a $G$-space $M$, the de Rham complex $\Omega(M)$ is equipped with the exterior derivative $d$ and interior multiplication $\iota_X := \iota_{\bar{X}}$, where $\bar{X}$ is the fundamental vector field on $M$ associated to $X \in \mathfrak{g}$. The complex $W(G) \otimes \Omega(M)$ is a differential graded algebra with antiderivations $d$ and $\iota_X$ for $X \in \mathfrak{g}$. A basic element in $W(G) \otimes \Omega(M)$ is defined in the same way as for the Weil algebra $W(G)$. The set $(W(G) \otimes \Omega(M))_{\text{bas}}$ of basic elements of $W(G) \otimes \Omega(M)$ is closed under the exterior derivative $d$. The ***Weil model*** of equivariant cohomology defines the equivariant cohomology of a $G$-space $M$ to be the cohomology of the basic complex $\{(W(G) \otimes \Omega(M))_{\text{bas}}, d\}$.

*Remark:* The intuition behind the Weil model of equivariant cohomology is that the Weil algebra $W(G)$ should serve as an algebraic model for the total space $EG$ of a universal $G$-bundle. For a principal $G$-bundle $\pi : P \to M$, a differential form $\omega$ on $P$ is said to be basic if it is the pullback of a form on $M$ (for more details, see [4, Sec. 14]). A form $\omega \in \Omega(P)$ is basic if and only if $\iota_X \omega = 0$ and $\mathcal{L}_X \alpha = 0$ for all $X \in \mathfrak{g}$ (for a proof, see [4, Prop. 14.12]). Since $\pi_* : T_p P \to T_{\pi(p)} M$ is surjective for any $p \in P$, the pullback map $\pi^* : \Omega(M) \to \Omega(P)$ is injective. Thus, there is a one-to-one correspondence between the forms on $M$ and the basic forms on $P$. The isomorphism

$$\pi^* : \Omega^*(M) \xrightarrow{\sim} (P)_{\text{bas}}$$

induces an isomorphism in cohomology

$$H^*(M) \xrightarrow{\sim} H^*\{\Omega^*(P)_{\text{bas}}\}.$$

There is an isomorphism between the Cartan model and the Weil model of equivariant cohomology called the ***Mathai-Quillen isomorphsim*** (for a proof, see [8, Ch. 4]). Therefore, it suffices to construct a homomorphism between the Weil model and the Borel model of equivariant cohomology. Recall that $EG = \cup_n EG_n$, $BG = \cup_n BG_n$, and that $EG_n \to BG_n$ is a principal $G$-bundle. There is a connection form $\omega$ on $EG_n$; it is a $\mathfrak{g}$-valued 1-form on $EG_n$, i.e.

$$\omega = \sum \omega^i X_i.$$

The connection form $\omega$ on $EG_n$ gives rise to a curvature form $\Omega$ on $EG_n$:

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega].$$

The algebra homomorphism

$$\varphi : W(G) \to \Omega^*(EG_n) \tag{2.3}$$

$$\theta^i \mapsto \omega^i \tag{2.4}$$

$$u^j \mapsto \Omega^j \tag{2.5}$$

is a well-defined homomorphism that commutes with the exterior derivative $d$ and the contraction $\iota_X$ for any $X \in \mathfrak{g}$ (for a proof, see [4, Sec. 18.5]). Since the homomorphism $\varphi$ commutes with $d$ and $\iota_X$, it also commutes with the Lie derivative $\mathcal{L}_X = \iota_X d + d\iota_X$. Therefore, $\varphi$ induces a ring homomorphism

$$\bar{\varphi} : W(G)_{\text{bas}} \to \Omega^*(EG_n)_{\text{bas}} = \Omega^*(BG_n)$$

of basic complexes (this induced homomorphism is called the ***Chern-Weil homomorphism***). The Chern-Weil homomorphism can be naturally extended from a ring homomorphism $W(G)_{\text{bas}} \to \Omega^*(EG_n)_{\text{bas}}$ to another ring homomorphism $W(G)_{\text{bas}} \otimes \Omega(M) \to \Omega(EG_n)_{\text{bas}} \otimes \Omega(M)$. Therefore, in cohomology, we have the following ring homomorphism ($H^*_G(M)$ denotes the equivariant cohomology in the Cartan model):

$$H^*_G(M) \simeq H^*((W(G) \otimes \Omega(M))_{\text{bas}}) \to H^*((\Omega^*(EG_n) \otimes \Omega(M))_{\text{bas}}).$$

By the Künneth formula and the remark on the previous page,

$$H^*((\Omega^*(EG_n) \otimes \Omega(M))_{\text{bas}}) = H^*(\Omega^*(EG_n \times M)_{\text{bas}}) = H^*((EG_n \times M)/G).$$

Let $M_n$ denote the homotopy quotient $(EG_n \times M)/G$. Note that since $EG = \cup_n EG_n$, $M_G = \cup_n M_n$. We have shown that there is a ring homomorphism from the Cartan model of equivariant cohomology to $H^*(M_n)$ for all $n$. Therefore, there is a ring homomorphism

$$H^*_G(M) \to \varprojlim H^*(M_n).$$

Since $H^*(M_n)$ is the real singular cohomology of $M_n$, $\varprojlim H^*(M_n) \simeq H^*(M_G)$ [9, Prop. 3F.5]. Thus, there is a ring homomorphism from the Cartan model to the Borel model.

# 3
# The Equivariant de Rham Theorem

## 3.1 Properties of G-orbits

In this secton, we study some properites of $G$-orbits on a $G$-space $M$. We show that when $G$ is compact, every $G$-orbit is a regular submanifold of $M$. The compactness condition on $G$ produces two other results about $G$-spaces that have analogues in the theory of manifolds:

1) Every $G$-orbit has a *$G$-invariant tubular neighborhood*.

2) Every $G$-space has an *invariant good cover*.

These results have the following analogues in the theory of manifolds:

1) Every regular submanifold of a manifold has a tubular neighborhood.

2) Every manifold has a good cover.

(An open cover $\mathfrak{U} = \{U_\alpha\}$ of a manifold $M$ of dimension $n$ is called a ***good cover*** if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ are diffeomorphic to

$\mathbb{R}^n$. A manifold which has a finite good cover is said to be of ***finite type***). Finally, the existence of an invariant good cover allows us to prove the equivariant de Rham theorem when $G$ is a compact, connected Lie group and $M$ is a $G$-space of finite type.

Let $M$ be a left $G$-space, where $G$ is a Lie group. A subset $A$ of $M$ is said to be $G$-***invariant*** if $l_g(A) \subset A$ for all $g \in G$. For $x \in M$, let $G \cdot x$ denote the $G$-orbit of $x$:

$$G \cdot x = \{y \in M | y = g \cdot x \text{ for some } g \in G\}.$$

Let $G_x$ denote the stabilizer of $x$:

$$G_x = \{g \in G | g \cdot x = x\}.$$

**Lemma 3.1.** *For $x \in M$, $G_x$ is a closed subgroup of $G$.*

*Proof.* Suppose that $\{g_k\}$ is a sequence in $G_x$ such that $g_k \to g$.

Since $g_k \in G_x$, $g_k \cdot x = x$ for all $k$. Therefore,

$$g \cdot x = (\lim_{k\to\infty} g_k) \cdot x = \lim_{k\to\infty}(g_k \cdot x) = x.$$

□

By the orbit-stabilizer theorem, $G \cdot x \simeq G/G_x$. Since $G_x$ is a closed subgroup of $G$, $G \cdot x$ is a submanifold of $G$ [13]. When $G$ is compact, $G/G_x$ is a regular submanifold of $G$. We will soon prove that $G/G_x$ is also a regular submanifold of $M$ by introducing *fundamental vector fields*.

Every Lie group $G$ has an exponential map exp: $\mathfrak{g} \to G$ defined on its Lie algebra $\mathfrak{g}$. Each element $Y \in \mathfrak{g}$ generates a curve $e^{-tY}$ :=exp$(-tY)$ in $G$, where $t$ is a real variable. We define the ***fundamental vector field*** $\bar{Y}$ associated to $Y$ by

$$\bar{Y}_x := \frac{d}{dt}\Big|_{t=0} e^{-tY} \cdot x \text{ for } x \in M.$$

We have the following theorem about the zeros of fundamental vector fields (for a proof of this theorem and the corollary that immediately follow, see [4, Prop. 2.4]):

**Theorem.** *Suppose a Lie group $G$ acts smoothly on a manifold $M$. For any element $A$ in the Lie algebra $\mathfrak{g}$ of a Lie group $G$, a point $p \in M$ is a zero of the fundamental vector field $\bar{A}$ on $M$ if and only if $p$ is a fixed point of the action of the one-parameter subgroup $\{e^{tA} \in G | t \in \mathbb{R}\}$ in $G$ on $M$.*

**Corollary:** *For $A \in \mathfrak{g}$, the fundamental vector field $\bar{A}$ vanishes at $p$ if and only if $A$ is in the Lie algebra of the stabilizer $G_p$.*

If we fix $x \in M$, the ***orbit map***

$$f_x : G \to M \tag{3.1}$$

$$g \mapsto g \cdot x \tag{3.2}$$

is a smooth map whose image is the orbit $G \cdot x$. Its differential at the identity is the linear map

$$(f_x)_{*,e} : \mathfrak{g} \to T_x M$$

that sends $Y \in \mathfrak{g}$ to its fundamental vector field $\bar{Y}_x$.

**Proposition 3.2.** *For $x \in M$, the orbit map $f_x : G/G_x \to M$ is an injective immersion. Furthermore, when $G$ is compact, the orbit $G \cdot x$ is a regular submanifold of $M$.*

*Proof.* Let $H = G/G_x$. It suffices to show that the map

$$f_x : H \to M \tag{3.3}$$

$$g \mapsto g \cdot x \tag{3.4}$$

is a injective immersion and that when $G$ is compact, the map is also a proper map. The map $f_x$ is injective because $H$ acts freely on $x$. Since $H$ acts freely on $x$, $H$ also acts freely for every $y \in H \cdot x$. This is a consequence of the fact that if $y = g \cdot x$ for some $g \in H$, then $G_{y \cdot x} = g^{-1} G_x g$. So if the stabilizer $H_x$ of $x$ is trivial, then $H_y$ is trivial too. To show that $f_x$ is an immersion, we need to compute the kernel of the differential map

$$(f_x)_{*,g} : T_gH \to T_{g\cdot x}M.$$

It suffices to consider the kernel when $g = e$, since $\ker \pi_{*,g} = l_{g*}(\ker \pi_{*,e})$. By the previous corollary, for $A \in \mathfrak{h}$, if the fundamental vector field $\bar{A}$ vanishes at $g \cdot x$, then $A$ is in the Lie algebra of the stabilizer $H_{g\cdot x}$. But since the stabilizer $H_{g\cdot x}$ is trivial, $A = 0$. Therefore, $f_x$ is an injective immersion.

Suppose that $G$ is compact. Recall that a map $f : X \to Y$ is ***proper*** if for every compact set $V \subseteq Y$, its preimage $f^{-1}(V)$ is compact in $X$. Since any continuous map from a compact space to a Hausdorff space is proper, the map $f_x : H \to M$ is proper.

By a theorem in manifolds, if a manifold $N$ is compact, then an injective immersion $f : N \to M$ is an embedding and the image $f(N)$ is a regular submanifold of $M$ [12, Th. 11.13 and Ex. 11.5]. Thus, if $G$ is compact, $f_x(G/G_x) = G \cdot x$ is a regular submanifold of $M$. □

## 3.2 The Tangent Space of a G-orbit

Suppose that a compact Lie group $G$ acts on a manifold $M$ on the left (in this section, $G$ is always assumed to be compact). By proposition 3.2, every $G$-orbit is a regular submanifold of $M$. Let $x \in M$. Then since $G \cdot x$ is a regular submanifold of $M$, for each $p \in G \cdot x$, the tangent space $T_p(G \cdot x)$ of $G \cdot x$ is well defined. Therefore, the tangent space $T_pM$ can be decomposed into a direct sum:

$$T_pM = T_p(G \cdot x) \oplus N_x,$$

where $N_x$ is the quotient vector space $T_pM/T_p(G \cdot x)$. Note that if $y$ is in the orbit $G \cdot x$, then $N_y$ and $N_x$ are isomorphic vector spaces.

For $g \in G_x$, the map

$$l_g : M \to M \tag{3.5}$$

$$y \mapsto g \cdot y \tag{3.6}$$

fixes the orbit $G \cdot x$ set-wise. The differential map

$$(l_g)_{*,x} : T_xM \to T_{g\cdot x}M = T_xM$$

is an isomorphism that sends the tangent space of the orbit into itself. Thus, the differential map also induces a linear map from $N_x$ to itself. This argument shows that to each $x \in M$ is associated a group homomorphism

$$G_x \to GL(N_x),$$

i.e. we have a linear representation of the stabilizer $G_x$ on $N_x$. As a result, the stabilizer $G_x$ acts on $G \times N_x$ by left multiplication on $G$ and by the linear representation on $N_x$.

In general, if $H$ is a closed subgroup of the Lie group $G$ and if $V$ is a vector space with an $H$-linear action (that is, every element $h \in H$ induces a linear map on $V$), then there is a free action of $H$ on $G \times V$:

$$h \cdot (g, v) = (gh^{-1}, h \cdot v).$$

We can obtain a quotient space $G \times_H V$ by declaring $(g, v)$ and $(g_0, v_0)$ to be equivalent if there exists an $h \in H$ such that $h \cdot (g, v) = (g_0, v_0)$ (the notation $[g, v]$ will denote the equivalence calss of $(g, v)$ in $G \times_H V$).

From this construction, $G \times_H V$ is a vector bundle on $G/H$ with fiber $V$. The vector bundle $G \times_H V$ has a well-defined $G$-action:

$$g_0 \cdot [g, v] = [g_0g, v].$$

We can treat $G/H$ as the zero section of the vector bundle $G \times_H V$, i.e.

$$G/H = \{[g, 0] | g \in G\} \subseteq G \times_H V.$$

Taking $H$ to be $G_x$ and $V$ to be $N_x$, we have constructed a vector bundle $G \times_{G_x} N_x$ with fiber $N_x$ and whose base space is $G \cdot x = G/G_x$. The following theorem states that the orbit map $f_x$ can be extended to some neighborhood of the zero section $G/G_x$ (for an outline of the proof, see [2, Thm. 1.2.1]):

**Theorem 3.3.** *("The slice theorem"): Suppose a compact Lie group $G$ acts on a manifold $M$. Then there exists a $G$-equivariant diffeomorphism from a $G$-invariant* *open neighborhood of the zero section in $G \times_{G_x} N_x$ to a $G$-invariant open neighborhood of $G \cdot x$ in $M$, which sends the zero section $G/G_x$ to the orbit $G \cdot x$ by the orbit map $f_x$.*

By the slice theorem, every $G$-orbit in $M$ has a $G$-invariant tubular neighborhood. For each $x \in M$, let $U_x$ denote such a $G$-invariant tubular neighborhood of $G \cdot x$.

**Proposition 3.4.** *The open cover $\mathfrak{U} = \{U_x\}_{x \in M}$ is an **invariant good cover**, i.e. any nonempty finite intersection of elements of $\mathfrak{U}$ is a $G$-invariant tubular neighborhood of some $G$-orbit.*

*Proof.* It suffices to prove this result for the intersection of two elements in the open cover $\mathfrak{U}$. Let $U_x$ and $U_y$ be elements in $\mathfrak{U}$ and suppose that $U_{x,y} := U_x \cap U_y \neq \varnothing$. Let $z \in U_{x,y}$. Since the tubular neighborhood $U_x$ is $G$-invariant, $g \cdot z \in U_x$ for all $g \in G$. Similarly, $g \cdot z \in U_y$ for all $g \in G$, so $G \cdot z \subseteq U_{x,y}$. Since our choice of $z \in U_{x,y}$ was arbitrary, for all $z \in U_{x,y}$, $g \cdot z \in U_{x,y}$. Thus, $U_{x,y}$ is a $G$-invariant tubular neighborhood of any $G$-orbit contained in $U_{x,y}$. □

## 3.3 The Equivariant de Rham Theorem

We are now able to prove the equivariant de Rham theorem for the case that $M$ is a manifold of finite type.

**Theorem 3.5.** *Suppose that a compact, connected Lie group $G$ acts smoothly on a manifold $M$ of dimension $n$ and of finite type. Then the equivariant cohomology of $M$ in the Borel model and the equivariant cohomology of $M$ in the Cartan model are isomorphic.*

*Proof.* We have already examined the case when $M = \{pt\}$. By the previous proposition, $M$ admits an invariant good cover $\mathfrak{U} = \{U_{x_i}\}_{x_i \in M}$, where $U_{x_i}$ is an invariant tubular neighborhood about some $G$-orbit in $M$. Since $M$ is of finite type, we can assume that $\mathfrak{U}$ is finite. Let $H^*_{G,\mathrm{Bor}}(X)$ and $H^*_{G,\mathrm{Car}}(X)$ denote the equivariant cohomology in the Borel model and the Cartan model respectively of a $G$-space $X$.

(Case 1: $\mathfrak{U} = \{U\}$) Suppose that $\mathfrak{U} = \{U\}$ is an invariant good cover of $M$. By definition, $U$ is diffeomorphic to $\mathbb{R}^n$. Equivariant cohomology is a $G$-homotopy invariant of a $G$-space. Since $\mathbb{R}^n$ is $G$-homotopy equivalent to a point, the equivariant cohomology of $M$ (in either model) is isomorphic to the equivariant cohomology of a point and, as we showed earlier, $H^*_{G,\mathrm{Bor}}(\{pt\}) \simeq H^*_{G,\mathrm{Car}}(\{pt\})$.

(Case 2: $\mathfrak{U} = \{U_0, \ldots, U_p\}$) We prove this case by an induction argument on the cardinality of an invariant good cover. We will now prove the theorem for the case when $\mathfrak{U} = \{U_0, U_1\}$ because this proof uses an argument that we can apply in the more general case. Suppose that $\mathfrak{U} = \{U_0, U_1\}$ is an invariant good cover for $M$, where $U_i$ is a $G$-invariant tubular neighborhood about the $G$-orbit $G \cdot x_i$. In ordinary cohomology, the Mayer-Vietoris sequence

$$0 \to \Omega^*(M) \to \Omega^*(U_0) \oplus \Omega^*(U_1) \to \Omega^*(U_0 \cap U_1) \to 0$$

induces a long exact sequence in cohomology, also called a Mayer-Vietoris sequence:

$$\cdots \to H^*(M) \to H^*(U_0) \oplus H^*(U_1) \to H^*(U_0 \cap U_1) \to H^{*+1}(M) \to \cdots.$$

In equivariant cohomology (in either model), there is a Mayer-Vietoris sequence:

$$\cdots \to H^*_G(M) \to H^*_G(U_0) \oplus H^*_G(U_1) \to H^*_G(U_0 \cap U_1) \to H^{*+1}_G(M) \to \cdots.$$

The $G$-invariant open sets $U_0, U_1$, and $U_0 \cap U_1$ are each $G$-homotopy equivalent to some $G$-orbit in $M$. Since $G$ is compact, every $G$-orbit $G \cdot x$o is homeomorphic to the homogeneous space $G/G_x$. Therefore, the equivariant cohomology of each open set in either model is isomorphic to the equivariant cohomlogy of a homogeneous space. We showed earlier that for any homogeneous space $G/H$, the equivariant cohomology of $G/H$ in Borel model and in the Cartan model are isomorphic. Let $\alpha : H^*_{G,\mathrm{Bor}}(U_0 \cap U_1) \to H^*_{G,\mathrm{Car}}(U_0 \cap U_1)$ and $\beta : H^*_{G,\mathrm{Bor}}(U_0) \oplus H^*_{G,\mathrm{Bor}}(U_1) \to H^*_{G,\mathrm{Car}}(U_0) \oplus H^*_{G,\mathrm{Car}}(U_1)$ be the isomorphisms between the two models. We have the following commutative diagram of exact rows:

$$\begin{array}{ccccccccccc}
\cdots \to H^k_{G,\mathrm{Bor}}(U_0) \oplus H^k_{G,\mathrm{Bor}}(U_1) & \to & H^k_{G,\mathrm{Bor}}(U_0 \cap U_1) & \to & H^k_{G,\mathrm{Bor}}(M) & \to & H^{k+1}_{G,\mathrm{Bor}}(U_0) \oplus H^{k+1}_{G,\mathrm{Bor}}(U_1) & \to & H^{k+1}_{G,\mathrm{Bor}}(U_0 \cap U_1) & \to \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta & \\
\cdots \to H^k_{G,\mathrm{Car}}(U_0) \oplus H^k_{G,\mathrm{Car}}(U_1) & \to & H^k_{G,\mathrm{Car}}(U_0 \cap U_1) & \to & H^k_{G,\mathrm{Car}}(M) & \to & H^k_{G,\mathrm{Car}}(U_0) \oplus H^{k+1}_{G,\mathrm{Car}}(U_1) & \to & H^{k+1}_{G,\mathrm{Car}}(U_0 \cap U_1) & \to
\end{array}$$

Since the maps $\alpha$ and $\beta$ are isomorphisms, by the five lemma, the map in the middle column $\gamma : H^k_{G,\mathrm{Bor}}(U_0) \cup U_1) \to H^k_{G,\mathrm{Car}}(U_0 \cup U_1)$ is an isomorphism as well. Finally, we now proceed by induction on the cardinality of an invariant good cover. Suppose $H^*_{G,\mathrm{Bor}}(M)$ and $H^*_{G,\mathrm{Car}}(M)$ are isomorphic for any $G$-space $M$ having an invariant good cover with at most $p$ open sets. Consider a $G$-space having an invariant good cover $\mathfrak{U} = \{U_0, \ldots, U_p\}$ with $p+1$ open sets. Now $(U_1 \cup \ldots \cup U_{p-1}) \cap U_p$ has a good cover with $p$ open sets, i.e. $\{U_{0,p}, U_{1,p}, \ldots, U_{p-1,p}\}$, where $U_{i,p} = U_i \cap U_p$. By the induction hypothesis, the equivariant cohomology in the Borel and Cartan models of $U_0 \cup \ldots \cup U_{p-1}$, $U_p$, and $(U_0 \cup \ldots \cup U_{p-1}) \cap U_p$ are all isomorphic. By the five lemma, the equivariant cohomology in the Borel and Cartan models of $U_0 \cup \ldots \cup U_p$ are also isomorphic. This completes the induction. $\square$

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