B. Theorem 1: Kepler's Area Rule -- The First Key

1. The first key to the other results is the realization that a body orbiting in a plane under the influence of a centripetal force automatically satisfies Kepler's area rule
   a. I.e. the bodies describe, by radii drawn to the center in question, areas proportional to the times
   b. So that these areas can be used in what follows to represent times (geometrically)!
   c. Answers the question, which, if any, measure of time in uniform circular motion carries over to centripetally governed motion generally? Answer: not arc length or angle, but area
2. The proof proceeds in two steps, the first of which replaces the action of the continuous centripetal force by a series of discrete force-impulses
   a. By the parallelogram rule, such a force impulse at B will produce a motion BC compounded from Bc and Cc, the latter parallel to BS since the force is acting in the direction of BS
   b. But then triangles SCB and SBA are equal in area, since SCB is equal in area to ScB, and the latter is equal in area to SBA (same base, same heights)
3. The second step then says that the proposition follows from taking the triangles to be infinitely small and infinite in number, so that the force-impulses approach the action of a continuous force
   a. Whiteside has argued that this proof holds rigorously only so long as arc BF is infinitesimal, which in fact is the only application of the theorem in De Motu; Pourciau has singled out the missing assumption, which can be proved by modern methods: "Every motion generated by a centripetal force acting uninterruptedly for a given time is the limit of motions generated by centripetal impulses." (*Archive for History of Exact Sciences*, vol. 58, no. 4, 2004, p. 320)
   b. Newton and others generally took the proof to have shown that Kepler's area rule holds automatically under centripetal forces, which in fact is true
   c. Undoubtedly, neither Newton nor anyone else realized that the proof may be problematic -- that the action of a continuous force perhaps cannot be represented over non-infinitesimal arcs by the action of an infinity of force-impulses
4. The main thing to notice is how easy the first part of the proof is -- raising questions about why no one had noticed it before
   a. Only suggestion I can offer is that no one had ever put the two ideas -- centrally directed force and Kepler's area rule -- together
   b. Streete and Wing did not adopt Kepler's area rule, so maybe it never occurred to Hooke or even to Wren to put the two together, and Newton had no earlier reason to do so since he was thinking in terms of *conatus a centro* and not centripetal *vim*
   c. Newton probably did so precisely because he was looking for a suitable geometric measure of time in order to determine the exact trajectory in the problem Hooke had set him (see Appendix on question of how to generalize from uniform circular motion)
      (1) Need some geometric representation of time, and cannot use arc length or angle

(2) Natural, given Kepler Horrocks,and Mercator to look to see if area will work

5. The result, taken at face value, is of course of profound significance
   a. No physical mystery to Kepler's area rule, for it holds under a comparatively weak condition: all forces -- all departures from uniform motion in a straight line -- directed toward a common point throughout the trajectory, with the magnitude of the force irrelevant
   b. Though the converse not proved here, the result does suggest that the area rule holds exactly only if this condition is satisfied, for forces acting in other directions will clearly alter the areas
   c. A further implication: if area rule holds regardless of the force rule, then the latter must determine the particular trajectory -- i.e. given the trajectory, infer the force rule

C. Theorem 2: Forces in Uniform Circular Motion

1. This theorem simply gives Newton's earlier "*conatus a centro*" result for uniform circular motion, but now in a slightly different form: centripetal force varies as arc $\text{length}^2/r = (\theta * r)^2/r$ in a given time
2. Of course, in the case of a circle obeying the area rule, the force is constant -- i.e. the force does not vary around the circumference
3. The theorem gives a determination of the relative magnitude of the force for different uniform motions in different circles, not of the variation of force along a single trajectory
4. The proof is along the same lines as that given in the ca. 1669 tract, though rephrased for an arc of the circle, comparing two circles in the manner of Huygens
   a. Given two circles with arcs BD and bd in the same time, then by their own internal force would instead describe straight lines BC = arc BD and bc = arc bd in this time
   b. Then, for this time the centripetal forces are as CD:cd
   c. (Note peculiar orientations of CD and cd – see DTW note, p. 38, comparing Newton’s choice of orientation of CD and cd with Huygens’s at the time unpublished choice)
   d. From Euclid, Proposition 36, Book 3, latter ratio is equal to $BC^2/CF : bc^2/cf = BD^2/CF/2 : bd^2/cf/2$
   e. Result then follows when time taken to be infinitesimal
5. Five corollaries are then given, the first four of which are essentially the same as results announced, without proof, at the end of Huygens's *Horologium Oscillatorium*, while the fifth uses these results to tie Kepler's 3/2 power rule applied to uniform circular motion to inverse-square centripetal forces
   a. E.g. force as v^2/r, as r/P^2 etc.
   b. Corollary 5 now states that inverse-square force is both a necessary and sufficient condition for the 3/2 power rule to hold for uniform circular motion
6. In the Scholium that follows Newton simply asserts that Corollary 5 holds for the planets and for the satellites of Jupiter and Saturn
   a. He subsequently withdraws the claim about Saturn when Flamsteed expresses an inability to detect the two additional satellites that Cassini had recently announced

b. And notice that he does not assert it for the moon, for the same reason that he subsequently withdraws it for Saturn: need multiple orbiting bodies to establish it empirically

c. Of course, in stating it here, he is treating the orbits as circular, and hence he must mean that it holds to the extent that their orbits approximate circles and their motions uniformity

d. While the trajectories do closely approximate circles, the motions are far from uniform: ratio of max v to min v = $(1+e)/(1-e)$, which amounts to 1.2/0.8 for Mercury and nearly 1.1/0.9 for Mars

7. No applications of the Theorem here of the sort in the earlier "Moon test": just a bare statement of a result that he knew from Halley had become widely recognized, even though no proof had been published for the v^2/r result Huygens had announced

D. Theorem 3: Force Variation along Arbitrary Trajectories: The Fundamental Enabling Result

1. Issue: what is the force rule, given a body moving along a curvilinear trajectory, with the force always acting in the direction of a single point S

   a. In contrast to theorem 2, which compared forces in two separate uniform circular motions, Theorem 3 concerns how the centripetal force varies in magnitude along a single trajectory

   b. In effect, then, the areal velocity is being treated as a given in this theorem

   c. Force here entirely in terms of change in velocity -- a kinematic result, in the manner of Galileo and Huygens, without reference to mass or bulk

2. The theorem states that the force varies as $1/SP^2 * \lim(QR/QT^2)$ as Q approaches P

   a. This theorem thus gives a geometric representation of the magnitude of the centripetal force, just as Theorem 1 gives a geometrical representation of time

   b. This for a single body moving along a curvilinear path under a centripetal force

3. The proof uses Theorem 1 and Hypothesis 4, with the former taken only over an infinitesimal arc:

   a. Given the time -- i.e. for a fixed time interval -- QR varies as the magnitude of the force (displacement over a given time is proportional to the magnitude of the force, just as in Huygens's solution for uniform circular motion)

   b. Given the force -- that is, for a given value of the force -- QR varies as time squared, via Hypothesis 4

   c. Therefore QR varies as $\text{force} * \text{time}^2$

   d. But time^2 as $(\text{area } QPS)^2$, which approaches $(QT*SP/2)^2$

   e. Therefore force varies as $1/SP^2 * \lim (QR/QT^2)$

4. A non-geometric formulation of this result was developed by Johann Bernoulli after 1710, using what we now call Taylor's Series (which was already known to Newton at the time of De Motu)

   a. Assume polar coordinates: r,θ

   b. Then $\lim(QR/QT^2) = 1/2 * (1/r + d^2/d\theta^2(1/r))$

   c. But then the force varies as $1/r^2$ times this limiting value

   d. More precisely the force = $(k/r)^2$ times this, where k is Kepler's constant ($r^2 * d\theta/dt$), the areal velocity
5. The question this theorem answers is how the force -- i.e. the change of motion -- varies along non-circular trajectories dictated by centripetal forces -- i.e. when all changes of motion are directed toward a single point
   a. In the case of a circle so dictated, the force does not vary since equal areas entail uniform motion
   b. The key to obtaining it is the realization that time can be represented by the area swept out
   c. This is why I suspect that the question answered in Theorem 3 was posed first, leading to the question answered in Theorem 1; which then unlocked the door to everything else
6. Theorem 3 should be seen as a generalization of Theorem 2, employing essentially the same geometric construction to infer a measure of force from departures from inertial motion
   a. Old approach, using curvatures and Theorem 2 directly, had not yielded a well-behaved measure of force: two unknowns, the normal "force "and the tangential "force"
   b. Added constraint of directed toward a center reduces the number of separate unknowns
7. The corollary to Theorem 3 simply states that the result gives a means for determining the rule of force for any point along a given trajectory governed by centripetal forces
8. A subtle, but radical step has been taken with Theorem 3, for force is now being treated in the abstract, as a mere magnitude, divorced from any question of mechanism!
   a. I.e. in contrast to Huygens's treatment of the static centrifugal force in a string or on a wall
   b. Now just a mathematically characterizable force taken to be acting on the moving object
   c. In other words, Newton's talk of forces in the abstract here parallels his talk of rays in the abstract in his earlier work on optics (which people had objected to)

E. Problems 1 & 2: Two Applications of Theorem 3
1. The precise reason unclear why Newton included Problems 1 and 2, both of which involve direct applications of Theorem 3, but to questions that there is no immediately obvious reason to be asking
   a. At first glance mere mathematical curiosities, serving to illustrate the thrust of Theorem 3 before turning to its key application (but only at first glance)
   b. Notice in manuscript here and elsewhere, "*gravitas*" was replaced by "*vis centripeta*"
2. Problem 1: the rule of force for a body going through a circular arc under centripetal forces directed to a point on that arc
   a. Akin to Galileo's rejected claim in *Two New Sciences* that a circular arc all the way to the center of the earth
   b. Symbolically, this case has $1/r = 1/D * \sec(\theta)$
   c. Of course, a singularity at the center of force, so solution only up to that point
3. Solution: force varies as $1/r^5$, where r is distance from the force center