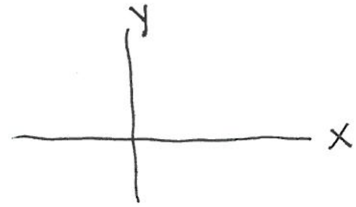
Plane problem - 1

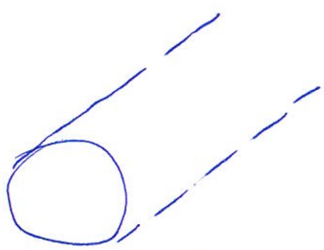
Plane (2-D) problems



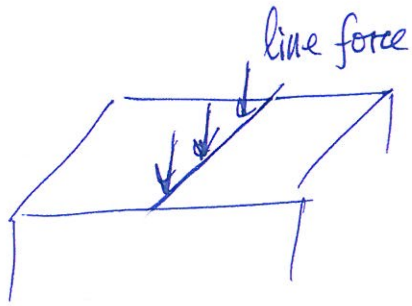
$(\epsilon_{xx}, \epsilon_{xy}, \epsilon_{yy})$ related to $(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$

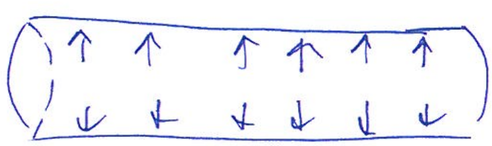
Plane strain: infinite extension in Z-direction

Any XY cross-sect. is representative



tunnel





pressurized pipe
far from edges

$$\left.\begin{array}{l} u_z = 0 \\ u_x = u_x(x,y) \\ u_y = u_y(x,y) \end{array}\right\} \Rightarrow \epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$$

Plane stress: thin plates under in-plane tension

Hooke's law for plane strain:

$$\epsilon_{zz} = \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}) \quad = 0$$

$$\Rightarrow \quad \boxed{\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})}$$

Substitute into other components of strain:

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{yy}+\sigma_{zz})$$

$$\epsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{zz})$$

substitute

$$\left.\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left[(1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy}\right] \\
\epsilon_{yy} &= \frac{1}{E}\left[-\nu(1+\nu)\sigma_{xx} + (1-\nu^2)\sigma_{yy}\right] \\
\epsilon_{xy} &= \frac{1}{2G}\sigma_{xy}
\end{aligned}\right\} \quad \text{Hooke's law for plane } \epsilon$$

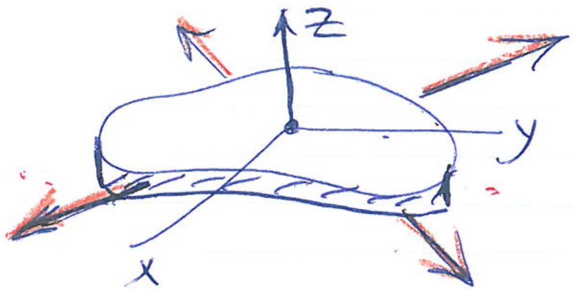
equilibrium eq-ns : only two of the three $\partial\sigma_{ij}/\partial x_j = 0$ are non-trivial:

$$\left.\begin{aligned}
\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} &= 0 \\
\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} &= 0
\end{aligned}\right\}$$

Compatibility :

$$\frac{\partial^2\epsilon_{xx}}{\partial y^2} + \frac{\partial^2\epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2\epsilon_{xy}}{\partial x\partial y}$$

Plane stress : thin plate loaded by tractions parallel to plate



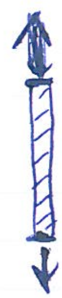


Plate is thin $\Rightarrow$ stresses are represented, with sufficient accuracy, by their <u>averages</u> (over thickness)

$$\left.\begin{aligned}\bar{\sigma}_{xx} &= \frac{1}{h}\int_{-h/2}^{h/2} \sigma_{xx}\, dz \\ \bar{\sigma}_{yy} &= \text{----} \\ \bar{\sigma}_{xy} &= \text{---}\end{aligned}\right\}$$

Also: $\sigma_{zz} = 0$ on boundaries $\Rightarrow$ assume $\bar{\sigma}_{zz} \approx 0$

This treatment is <u>approximate</u>

The thinner the plate the better the accuracy

Plane σ : equilibrium : in terms of averages (over thickness)

First eq-n:

$$\frac{1}{h}\int_{-h/2}^{h/2}\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}\right)dz+\underbrace{\frac{1}{h}\int_{-h/2}^{h/2}\frac{\partial\sigma_{xz}}{\partial z}\,dz}_{\underbrace{\sigma_{xz}\big|_{h/2}}_{0}-\underbrace{\sigma_{xz}\big|_{-h/2}}_{0}}=0$$

$$\Rightarrow\ \boxed{\frac{\partial\bar{\sigma}_{xx}}{\partial x}+\frac{\partial\bar{\sigma}_{xy}}{\partial y}=0}$$

Analogously, 2-nd eq-n

$$\boxed{\frac{\partial\bar{\sigma}_{xy}}{\partial x}+\frac{\partial\bar{\sigma}_{yy}}{\partial y}=0}$$

Plane σ : Hooke's law: averaging over thickness

$$\begin{cases} \bar{\epsilon}_{xx} = \frac{1}{E}\left(\bar{\sigma}_{xx} - \nu\left(\bar{\sigma}_{yy} + \underbrace{\bar{\sigma}_{zz}}_{\approx 0}\right)\right) \\ \bar{\epsilon}_{yy} = \text{- - -} \\ \bar{\epsilon}_{xy} = \text{- - -} \end{cases}$$

Comparing Plane ϵ $\leftrightarrow$ Plane σ:

Plane-σ problem differs from plane-ϵ in replacing

$$\boxed{\nu \to \frac{\nu}{1+\nu}}$$

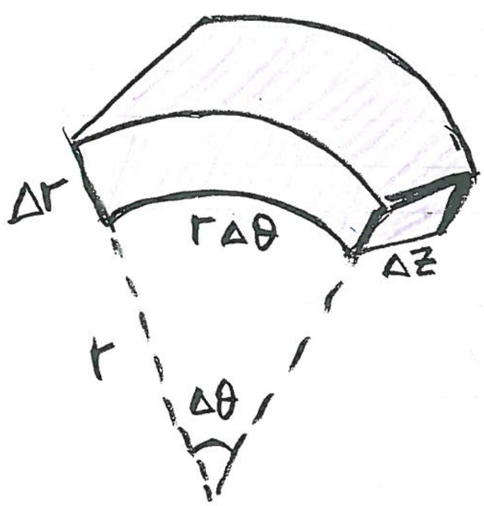
With this replacement, the problems are mathematically identical
(provided stresses in plane σ are understood as averages over thickness)

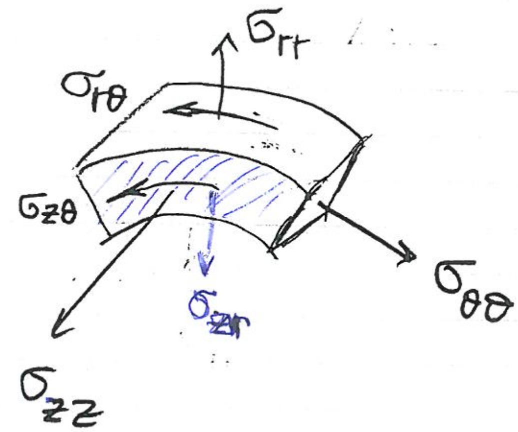
although they describe two physically different situations

Cylindrical coord system : used in 2D problems
(plane ϵ or plane σ)

pipes rotating disks

Material element bounded by coord lines





$(\sigma_{rr},\ \sigma_{\theta\theta},\ \sigma_{zz})$ – normal stresses

$\sigma_{\theta\theta}$ ← hoop stress

$(\sigma_{r\theta},\ \sigma_{\theta z},\ \sigma_{z\theta})$ – shear str.

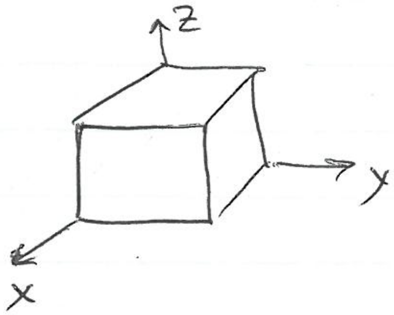
Case of axisymm stress distrib:

$$\sigma_{\theta r} = \sigma_{\theta z} = \sigma_{rz} = 0$$

(no shear stresses in this coord syst)

$\Rightarrow$ r, θ, z – princ. directions of $\underset{\sim}{\sigma}$

Equilibrium in cylindrical coord



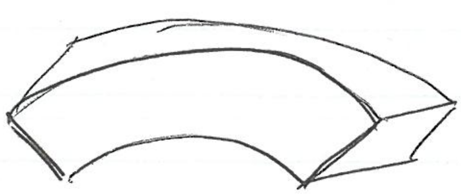
In cartesian: recall the meaning of eq-m eq-ns

$\dfrac{\partial \sigma_{xx}}{\partial x}$ characterizes imbalance of forces in X-dir. from front-rear pair

Should be equilibrated by imbalances of forces in X-dir. from two other pairs

$\dfrac{\partial \sigma_{xy}}{\partial y}$, $\dfrac{\partial \sigma_{xz}}{\partial z}$

In cylindrical:



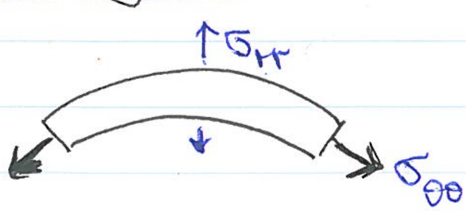
In r-direction:

$\dfrac{\partial \sigma_{rr}}{\partial r}$ – imbalance from top-bottom

should be equilibrated by

- $\dfrac{\partial \sigma_{r\theta}}{\partial \theta}$, $\dfrac{\partial \sigma_{rz}}{\partial z}$ but these shear stresses $= 0$ in cases of axial symmetry

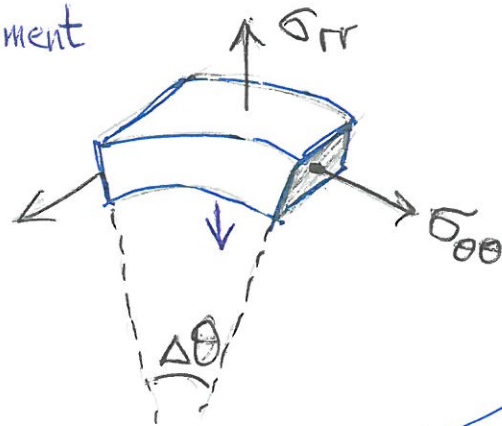
- Misalignment of $\sigma_{\theta\theta}$: due to curvilinearity of the element



$\Rightarrow \dfrac{\partial \sigma_{rr}}{\partial r}$ equilibrated by the misaligned $\sigma_{\theta\theta}$

Total force in r-direction

with account of misalignment



upper face

lower face

$$(\sigma_{rr} + \underbrace{\Delta\sigma_{rr}}_{\frac{\partial \sigma_{rr}}{\partial r}\Delta r})\underbrace{(r+\Delta r)\Delta\theta}_{\text{length of arc}} \cdot \Delta z \; - \; \sigma_{rr}\, r\, \Delta\theta\, \Delta z$$

misalignment

$$-2\cdot\sigma_{\theta\theta}\,\Delta r\,\Delta z \cdot \sin\frac{\Delta\theta}{2} \qquad = 0$$

Dividing over $\Delta V = r\Delta\theta \cdot \Delta r \cdot \Delta z$

$$\boxed{\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0}$$

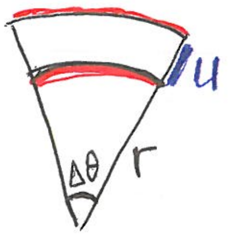
– the only eq-m eq-n in axisymm case

can be written as $\dfrac{d\sigma_{rr}}{dr}$

(no θ-dependence in axisymm case)

Axial symmetry case : $u_r \equiv u$ is the only displac. comp.

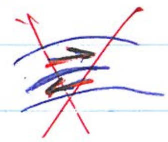
Strains in terms of u :



$$\epsilon_{rr} = \frac{du}{dr} \quad \text{obviously}$$

$$\epsilon_{\theta\theta} = \frac{(r+u)\,\Delta\theta - r\,\Delta\theta}{r\,\Delta\theta} = \frac{u}{r}$$

$\epsilon_{r\theta} = 0$ in this case (from symmetry)



Hooke's law

$u \rightarrow$ Strains $\longrightarrow$ stresses $\rightarrow$ substitute into equilibrium eq.

$\Rightarrow$ equilibrium in terms of radial displac. u:

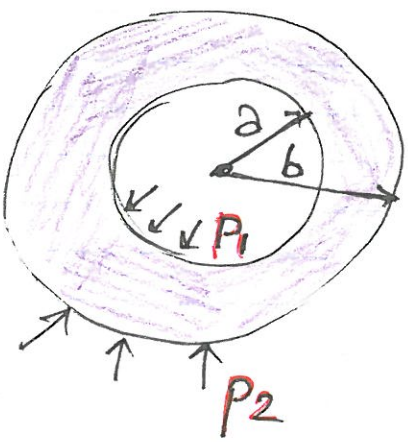
$$\frac{d}{dr}\left(\frac{du}{dr} + \frac{u}{r}\right) = 0$$

or, $u'' + \frac{1}{r}u' - \frac{1}{r^2}u = 0$ — hints at solution $u = r^{\alpha}$

$$\boxed{u = Ar + B\frac{1}{r}}$$

— displac. field in axisymm. problems

Example of working with cylindrical coord: pipe under internal pressure p_1
external p_2



Plane strain : stresses/strains independent of z (except : near ends)

Axial symmetry : displacement is purely radial $u_r \equiv u$

General sol'n: $u = Ar + B\frac{1}{r}$

$\Rightarrow$ derive strains ; stresses ; satisfy b.c.: – at $r=a$, $\sigma_{rr} = -p_1$
– $r=b$, $\sigma_{rr} = -p_2$

$$\Rightarrow \begin{cases} \sigma_{rr} = \dfrac{p_1 a^2 - p_2 b^2}{b^2 - a^2} - \dfrac{p_1 - p_2}{b^2 - a^2}\dfrac{a^2 b^2}{r^2} \\[2ex] \sigma_{\theta\theta} = \dfrac{p_1 a^2 - p_2 b^2}{b^2 - a^2} + \dfrac{p_1 - p_2}{b^2 - a^2}\dfrac{a^2 b^2}{r^2} \end{cases}$$

Case of internal pressure only :

$$\sigma_{rr} = -p_1 \frac{a^2}{b^2 - a^2}\left[\frac{b^2}{r^2} - 1\right]$$ compressive. Decreases from p_1 to 0 (inner → outer)

$$\sigma_{\theta\theta} = p_1 \frac{a^2}{b^2 - a^2}\left[\frac{b^2}{r^2} + 1\right]$$ tensile

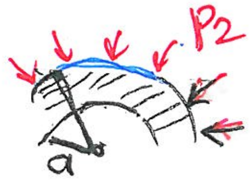
In the limit of <u>thin</u> pipe, internal pressure only:

$$\sigma_{\theta\theta} = p_1 \frac{a^2}{b^2 - a^2 \approx 2a \cdot h} \cdot \left(\frac{b^2}{r^2} + 1 \right)$$

h: thickness $b-a$

becomes <u>large</u> !

Strength ↘



Role of external pressure

reduces (tensile) hoop stress

$$\sigma_{\theta\theta} = \frac{1}{b^2 - a^2}\left[p_1 \cdot a^2\left(1 + b^2/r^2\right) - p_2 b^2\left(1 + a^2/r^2\right)\right]$$

Application to gun construction:

Cannon : several tubes, each heated, to slip over the interior one

- cools & contracts, compressing the interior one

$\Rightarrow$ reduces tensile $\sigma_{\theta\theta}$ due to firing

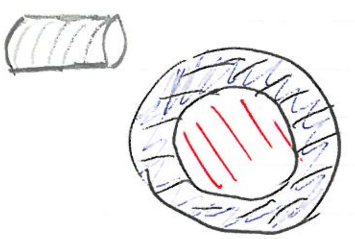
Stress σ_{zz} : depends on far-end conditions:

- Constrained cylinder : $\epsilon_{zz} = 0$

$$\epsilon_{zz} = \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta})$$

$$\Rightarrow \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

- Closed cylindrical vessel: eq-m for the bottom plate:



$$\sigma_{zz} \cdot \pi(b^2 - a^2) = p_1 \cdot \pi a^2 - p_2 \cdot \pi b^2 \quad \Rightarrow \text{find } \sigma_{zz}$$

Role of external pressure

reduces (tensile) hoop stress

$$\sigma_{\theta\theta} = \frac{1}{b^2 - a^2}\left[p_1 \cdot a^2\left(1 + b^2/r^2\right) - p_2 b^2\left(1 + a^2/r^2\right)\right]$$

Application to gun construction:

Cannon : several tubes, each heated, to slip over the interior one

- cools & contracts, compressing the interior one

$\Rightarrow$ reduces tensile $\sigma_{\theta\theta}$ due to firing