E. A Fundamental Result: The Mean Speed Theorem

1. The discussion from [198] to [205] has removed the conceptual obstacles facing the proposed definition of uniformly accelerated motion, but it has not provided the reader any way to conceptualize such motion
   a. As the preceding discussion of the alternative rule for uniform acceleration makes clear, not in a position to conceptualize it as we do today: $dv=a*dt$ and $v= ds/dt=a*t$, and so $s=1/2*at^2$
   b. For not even Galileo could visualize -- i.e. conceptualize -- the difference between the two rules from their mere statement
2. The "mean speed theorem" -- Proposition I -- bridges this conceptual gap by allowing one to think of uniformly accelerated motion in terms of a corresponding uniform motion
   a. I.e. space traversed in a given time is equal to the space traversed by an object moving uniformly throughout at half the speed the accelerating object has at the end of the time
   b. Licenses inferences about spaces traversed and times required that are parasitic on results for uniform motion, including inferences enabling measured values of varying velocities!
   c. I.e. mathematically reduces uniformly accelerated motion problems to uniform motion problems, thereby avoiding any need for reference to instantaneous speed
3. A trivial result once integrals are used as above, for simply substitute the value for v at the end of the time for $a*t$ in the formula for s: $s = [1/2*at]*t = [\text{mean } v]*t$
   a. Galileo had no way of "adding up" all the infinity of speeds to get the distance covered -- i.e. had no way of integrating
   b. {Nor did the Mertonians and others – in particular, Oresme, of whom Galileo must have been aware -- who had discovered the mean speed theorem before Galileo}
4. Galileo's "proof" invokes a one-to-one comparison of momenta of speed in the uniformly accelerated case with the momenta of speed in the uniform motion case [208f]
   a. Once one-to-one correspondence granted, the argument goes through
   b. The lacuna: what assures such a one-to-one correspondence? -- this is the non-geometric step
5. Huygens ultimately achieved a purely geometric proof of the mean speed theorem, though one that uses reductio ad absurdum twice
   a. Galileo was averse to using reductio ad absurdum proofs at all in mechanics
   b. So, he might have continued to prefer his proof

III. "The Third Day": The Evidential Difficulties

A. The Postulate: Empirical Motivation and Grounds

1. What immediately follows the definition of uniformly accelerated motion in the Latin text is a postulate: "the degrees of speed acquired by the same moveable over different inclinations of planes are equal whenever the heights of those planes are equal" [205]

a. Speed here clearly a scalar quantity -- direction irrelevant
b. Idea that inclined plane motion is an instance of naturally accelerated motion can be found in Leonardo da Vinci

2. The postulate used to establish Propositions 3, 4, and 5, which, in modern terms, amount to saying that the acceleration along an inclined plane is sin(angle) times the acceleration in vertical fall
   a. I.e. $v=g*\sin(\theta)*t$
   b. A result we get by resolving the acceleration of gravity g into components, whether of accelerations or of forces, something quite alien to Galileo, even allowing for Day Four
3. The obvious value of the postulate is that it provides a means for empirically investigating naturally accelerated motion generally, and hence free fall, that lessens the empirical difficulties
   a. Slows the process down so that it can more readily be observed: time of fall through height varies inversely with $\sin(\theta)$, but speed at every height along the plane the same as speed at same height if instead falling vertically
   b. Time extended without increasing velocity, so that resistance effects certainly not amplified in the process, and perhaps they are inherently lower in this situation too
4. First empirical defense of postulate turns on claim that interdicted pendulum always reaches the same height from which it started, so that same momenta always yield same height
   a. But momenta in descent = momenta in ascent, so that same height always yields same momenta regardless of pendulum arc
   b. Salviati cautions that such reasoning only analogous when applied to inclined plane, where process of acquiring momenta is different from that along a circular arc
5. Second defense of postulate, added in later editions, turns on the empirical fact (Stevin) that the vertical weight required to hold G in equilibrium on an inclined plane = weight of G $*\sin(\theta)$
   a. Lemma: impetus of descent varies as $\sin(\theta)$, for impetus of descent is proportional to minimum force or resistance needed to stop it, which by above varies with $\sin(\theta)$
      (1) Authorization for key step: "it is manifest that..." [[216]]
      (2) Key question: What exactly is "impetus of descent"? Whatever it is, momenta of speeds are proportional to impetuses of descent
      (3) Thus lemma turns on a static-to-dynamic inference!
   b. Postulate in later editions proved as a theorem, using corollary II of Proposition II, the lemma, and Proposition II from the equable motion part of the treatise [[218]]
6. As we will see, the postulate is strictly speaking false, though it is true so long as it is confined to separate types of motion
   a. Lacuna in proof: the momenta (accelerations) in rolling and in sliding not the same, even though impetuses of descent (forces) are the same along the same incline

b. In passing, also notice the point just before [[216]]: impossible that a heavy body should move naturally upward; variations of this principle will be seen again

7. Independently of the rolling vs. falling issue, Galileo's postulate introduces a proposal that will remain in the forefront for at least the next 150 years -- e.g. through Lagrange's equations of motion – for it voices a fundamental principle of modern mechanics

    a. Torricelli (1608-1647), Galileo's protegé, in his *De Motu Naturaliter Descendentium et Projectorum* of 1644 offers a different justification of the postulate, invoking a principle subsequently named after him and deriving the sine rule from it and the postulate from the sine rule (see Appendix)

    b. And Huygens, adopts a modified form of Torricelli's principle to justify pathwise-independence claims in general, not just for inclined planes, a decade or so later, and then uses it throughout his later work

    c. It then became a central element of modern mechanics from Huygens (and Leibniz)

B. Proposition II and the 1,3,5,... Pattern

1. Proposition II is not only the pivotal mathematical result for uniformly accelerated motion, but also the pivotal result from an empirical viewpoint

    a. Predicts a distinctive *pattern* in uniformly accelerated motion, akin to the pattern of retrograde loops that provided the basis for Ptolemy's bisection of eccentricity

    b. A pattern that should be observable as a phenomenon of "naturally accelerated motion," even if only to reasonable approximation as a consequence of air resistance etc.

2. Proposition II simply says that spaces traversed in uniformly accelerated motion are proportional to the squares of the times -- i.e. s is proportional to t^2

    a. First corollary then says that the increments of distance over a progression of equal times are as the 1,3,5,... progression

    b. Second corollary nothing more than times being proportional to square root of the distances

3. Proof of the theorem and the corollaries uses only Proposition IV from the section on equable motion and the mean speed theorem

    a. Postulate not needed for this proposition at all

    b. Calculus steps taken care of by the mean speed theorem: $s = (1/2*v_{max})*t$, but $v_{max} = a*t$

    c. Proof yields the further result that s varies as v_{max}^2 -- i.e. $s_1:s_2$ as $v_{1max}^2 : v_{2max}^2$

4. The Scholium [214] asserts that result holds equally for vertical fall and for motion along inclined plane, both of which are taken to be uniformly accelerated (in the absence of resistance)

    a. Scholium thus authorizing application of mathematical result to physical situations

    b. But, unlike Postulate, not making claim about how vertical fall and inclined plane accelerations are related to one another