ON THE AUTOMORPHISM GROUPS OF UNIVERSAL RIGHT-ANGLED COXETER GROUPS

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Abstract

We investigate the combinatorial and geometric properties of automorphism groups of universal right-angled Coxeter groups. McCullough-Miller space is virtually a geometric model for the outer automorphism group of a universal right-angled Coxeter group, $\mathrm{Out}(W_n)$. As it is currently an open question as to whether or not $\mathrm{Out}(W_n)$ is CAT(0) or not, it would be helpful to know whether McCullough-Miller space can always be equipped with an $\mathrm{Out}(W_n)$-equivariant CAT(0) metric. We show that the answer is in the negative. This is particularly interesting as there are very few non-trivial examples of proving that a space of independent interest is *not* CAT(0). We also show that an otherwise promising finite index subgroup of $\mathrm{Out}(W_n)$ is not a right-angled Coxeter group.

To Allison and my parents, without whom I would never had made it here.

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Chapter 1

Introduction

Geometric group theory studies the large scale geometry of groups, i.e., the geometric properties of an infinite, finitely generated group that do not depend on (at minimum) a choice of finite generating set. Mikhail Gromov [9] popularized the field by studying *hyperbolic* and CAT(0) *groups*, which generalize geometric properties of the classical theory of the fundamental groups of negatively and non-positively curved Riemanninan manifolds.

A CAT(0) *metric space* is a geodesic metric space such that geodesic triangles are no fatter than corresponding Euclidean triangles with the same side lengths. This condition generalizes many results from the classical theory of non-positively curved Riemannian manifolds. Groups that act properly discontinuously and cocompactly by isometries on a CAT(0) space are called CAT(0) *groups*, and through this action, inherit many of the metric properties of the space. CAT(0) spaces are contractible, have quadratic isoperimetric inequalities, admit a natural boundary at infinity, have a well-defined notion of angle, and have orthogonal projections onto convex subspaces. The standard reference on CAT(0) groups and spaces is [3].

Definition. Let X be a geodesic metric space, and let $a, b, c \in X$. Consider any geodesic triangle Δabc (that is, the union of any three geodesic segments: $[a, b]$, $[b, c]$, and $[a, c]$), and consider the comparison triangle $\Delta\overline{abc}$ in $\mathbb{E}^2$. See Figure 1.1. If p and q are points on $[a, b] \cup [b, c] \cup [a, c]$, then there exists comparison points $\overline{p}$ and $\overline{q}$ on the boundary of $\Delta\overline{abc}$ such that distances between corresponding points measured along the boundaries of the triangles are identical in both spaces.

X is then called *a CAT(0) space* if for all Δabc and all such p, q, it is the case that:

$$d_X(p, q) \leqslant d_{\mathbb{E}^2}(\overline{p}, \overline{q}).$$

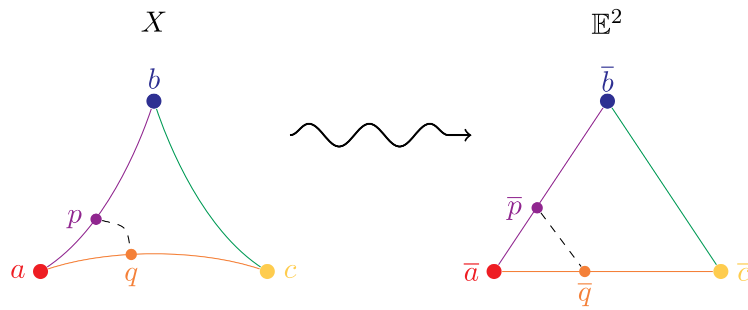

Figure 1.1: Comparison triangles for the CAT(0) condition in a geodesic metric space.

Definition. If a finitely generated group G acts on a CAT(0) space X properly discontinuously, co-compactly, and by isometries, then G is called a *CAT(0) group*.

CAT(0) groups are a generalized notion of non-positive curvature for groups. Unlike Gromov's δ-hyperbolic groups, the property of being a CAT(0) group is not a quasi-isometric invariant [19]. Furthermore, even if a group has a natural geometric model, the failure of that model to be CAT(0) doesn't preclude the possibility of the group acting geometrically on a different metric space which is CAT(0). Thus, it can be a more subtle question to determine when a group is CAT(0) or not.

In the 1930s, H.S.M. Coxeter introduced abstract Coxeter groups as a generalization of groups generated by geometric reflections. Their subsequent study has connected many areas of algebra, geometry, and combinatorics.

Definition. Given a finite simple graph Γ, the *right-angled Coxeter group defined by* Γ is the group $W = W_\Gamma$ generated by the vertices of Γ. The relations of W_Γ declare that the generators all have order 2, and adjacent vertices in Γ commute with each other.

Right-angled Coxeter groups (commonly abbreviated RACGs) have a rich combinatorial and geometric history. They each act properly discontinuously and co-compactly by isometries on a metric space, called a Davis complex [6]. Gromov [9] showed this space to be CAT(0) for RACGs, and Moussong showed [16] that all Coxeter groups are in fact CAT(0) groups.

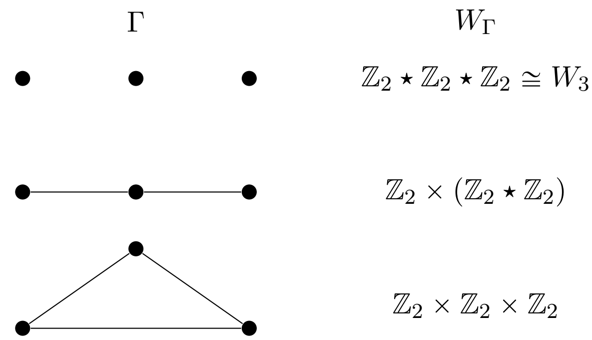

Figure 1.2: Some examples of defining graphs Γ and their RACGs W_Γ.

The combinatorial nature of RACGs makes them useful in studying their CAT(0) geometry as they admit a biautomatic structure as well as a geodesic normal form. Thus, they have effective solutions to the word and conjugacy problems. They are also *rigid*, which means a given RACG cannot arise from two different defining graphs [8, 7, 13, 20]. Thus, all of the combinatorial information of the group is contained in the graph Γ.

Example. One particularly interesting class of examples is the universal right-angled Coxeter groups, W_n, whose defining graph is the empty graph on n vertices. For instance, the group $W_4 = \left\langle a_1, a_2, a_3, a_4 \mid a_1^2 = a_2^2 = a_3^2 = a_4^2 = 1 \right\rangle$ is a right-angled Coxeter group and so is CAT(0).

The automorphisms of right-angled Coxeter groups are generated by automorphisms that come in three varieties [4, 8, 12]:

1. Graph symmetries, which are automorphisms of W_Γ induced by graph automorphisms of Γ. For instance, if two vertices of Γ are adjacent to the same set of vertices, then W_Γ has an automorphism which exchanges those two generators and leaves all other generators fixed.

2. Partial Conjugations, which conjugate a certain set of generators, D, by a particular generator a_i while leaving all other generators fixed. The combinatorics of Γ constrain which subsets D of the generators result in automorphisms of W_Γ for each a_i.

3. Transvections, which send a_i to $a_i a_j$ for a particular pair of generators and leave all other generators fixed.

Definition 1.1. Following [18], we denote by $x_{i,D}$ the partial conjugation of W_Γ defined by:

$$a_j \mapsto \begin{cases} a_i a_j a_i & \text{if } j \in D \\ a_j & \text{if } j \in [n] \backslash D \end{cases}$$

We call $x_{i,D}$ the *partial conjugation with acting letter a_i and domain D.*

If $\operatorname{St}(a_i)$ is the star of the vertex a_i in Γ, then $x_{i,D}$ is an automorphism of W_n if and only if D is a union of connected components of $\Gamma \backslash \operatorname{St}(a_i)$.

When D is a single connected component of $\Gamma \backslash \operatorname{St}(a_i)$, we follow [5] and call $x_{i,D}$ an *elementary partial conjugation.*

Any automorphism of a group must send involutions to involutions, and the only involutions of W_Γ are conjugates of commuting products of its generators [1]. Furthermore, no commuting products of generators are conjugate to one another in W_Γ [6], and so any automorphism of W_Γ must permute the conjugacy classes of commuting products of the generators. Thus, $\operatorname{Aut}(W_\Gamma)$ acts on the set of conjugacy classes of commuting products of the generators, whose kernel is denoted $\operatorname{Aut}^0(W_\Gamma)$.

Definition. $\operatorname{Aut}^0(W_\Gamma)$ consists of all automorphisms of W_Γ that map each vertex to a conjugate of itself.

$\operatorname{Aut}^0(W_\Gamma) \lhd \operatorname{Aut}(W_\Gamma)$ is generated by the set of all partial conjugations or the set of all elementary partial conjugations [17, 12].

The quotient of $\operatorname{Aut}^0(W_\Gamma)$ by the inner automorphisms gives a subgroup $\operatorname{Out}^0(W_\Gamma)$ of the full outer automorphism group. This quotient splits, and $\operatorname{Out}^0(W_\Gamma)$ is isomorphic to a subgroup of the full automorphism group. In fact, a full decomposition of the automorphism group was given in [10]:

Theorem (Gutierrez-Piggott-Ruane)**.**

$$\operatorname{Aut}(W_\Gamma) = \underbrace{\left(\operatorname{Inn}(W_\Gamma) \rtimes \operatorname{Out}^0(W_\Gamma)\right)}_{\operatorname{Aut}^0(W_\Gamma)} \rtimes \operatorname{Aut}^1(W_\Gamma)$$

Now $\mathrm{Inn}(W_\Gamma) \cong W_\Gamma/Z(W_\Gamma)$, and the center of a RACG is the subgroup generated by the vertices of Γ connected to all other vertices [10]. W_Γ then splits as $W_{\Gamma'} \times Z(W_\Gamma)$, where Γ' is the induced graph in Γ of the non-central vertices. Thus, $\mathrm{Inn}(W_\Gamma) \cong W_{\Gamma'}$ is a RACG itself.

Additionally, for a RACG W_Γ, $\mathrm{Aut}^1(W_\Gamma)$ is a subgroup of $\mathrm{GL}(n, 2)$, and so is a finite group [10]. So, both $\mathrm{Aut}^1(W_\Gamma)$ and $\mathrm{Inn}(W_\Gamma)$ have well-understood large scale geometry. Therefore, studying the geometry of $\mathrm{Aut}^0(W_\Gamma)$, or even $\mathrm{Aut}(W_\Gamma)$, relies on understanding the geometry of $\mathrm{Out}^0(W_\Gamma)$.

Since $\mathrm{Aut}^0(W_\Gamma)$ and $\mathrm{Out}^0(W_\Gamma)$ are generated by involutions (the partial conjugations), it is a natural question to ask:

Question. For a given RACG W_Γ, are $\mathrm{Aut}^0(W_\Gamma)$ or $\mathrm{Out}^0(W_\Gamma)$ themselves RACGs or even just CAT(0) groups?

To answer this, we need not just a generating set but a full finite presentation for $\mathrm{Aut}^0(W_\Gamma)$ and $\mathrm{Out}^0(W_\Gamma)$ and preferably a geometric model for each to act upon. A full presentation for $\mathrm{Aut}^0(W_\Gamma)$ is given in both [12, 17], and McCullough-Miller space will give one such potential geometric model for the simpler case of $\mathrm{Out}(W_n)$ [18].

For W_n, there are no transvections and $\mathrm{Aut}^1(W_n)$ consists of only the graph symmetries and so is isomorphic to Σ_n, the symmetric group on n letters. Since W_n has trivial center, $\mathrm{Inn}(W_n) \cong W_n$. Thus in the case of W_n, we have the decomposition:

Corollary.

$$\mathrm{Aut}(W_n) = \underbrace{\left(W_n \rtimes \mathrm{Out}^0\left(W_n\right)\right)}_{\mathrm{Aut}^0(W_n)} \rtimes \Sigma_n$$

$$\mathrm{Out}(W_n) = \mathrm{Out}^0\left(W_n\right) \rtimes \Sigma_n$$

Remark. When we write $x_{i,D} \in \mathrm{Out}^0(W_n)$, we can think of $\mathrm{Out}^0(W_n)$ as either a subgroup of $\mathrm{Aut}(W_n)$, in which case $x_{i,D}$ is a single automorphism, or else as a subgroup of $\mathrm{Out}(W_n)$, in which case $x_{i,D}$ is an equivalence class of automorphisms that differ by inner automorphisms. In the former case, both the acting letter i and the domain D are uniquely determined by the group element $x_{i,D}$. In the latter case, this is almost true. The acting letter i is determined, but there are exactly two domains that result in the same outer automorphism class, namely $x_{i,D} = x_{i,D^c}$, where $D^c = [n] \backslash \{D \cup \{i\}\}$. If we need to pick a unique representative for $x_{i,D}$, we follow [18] and choose the D that does *not* contain the smallest possible index (which is usually 1, unless 1 is the acting letter, in which case it is 2).

What about the geometry of $\mathrm{Out}^0(W_n)$? While $\mathrm{Aut}(W_3)$ is known to be CAT(0) [19] and $\mathrm{Out}^0(W_3) \cong W_3$, for $n \geqslant 4$, it was open as to whether or not $\mathrm{Aut}^0(W_n)$ or $\mathrm{Out}^0(W_n)$ is a right-angled Coxeter group or even a CAT(0) group.

For each of the groups $G = \mathrm{Out}^0(W_n)$ or $\mathrm{Out}(W_n)$, we might ask the following questions:

1. Is G a right-angled Coxeter group?
2. Is G a CAT(0) group?
3. Is there an accurate geometric model for G, i.e., a geodesic metric space X such that $\mathrm{Isom}(X) \cong G$?

Adam Piggott [18] proved that McCullough-Miller space is an accurate combinatorial and topological model for $\mathrm{Out}(W_n)$, although we show in Chapter 5 that it cannot be promoted to a true geometric model for either $\mathrm{Out}(W_n)$ or $\mathrm{Out}^0(W_n)$. We also prove in Chapter 4 that $\mathrm{Out}^0(W_n)$ is not a right-angled Coxeter group.

In particular, we prove the following main theorems:

Theorem 4.2. $\mathrm{Out}^0(W_n)$ *is not a right-angled Coxeter group.*

Theorem 5.12. *There does not exist an* $\mathrm{Out}^0(W_n)$*-equivariant (or* $\mathrm{Out}(W_n)$*-equivariant) piecewise Euclidean (or piecewise hyperbolic)* CAT(0) (CAT(−1)) *metric on* K_n *for* $n \geqslant 4$.

Chapter 2

Hypertrees

The following Chapter is inspired by the exposition in [18].

An accurate geometric model for $\mathrm{Out}^0(W_n)$ is given by McCullough-Miller space, which was originally defined using a simplicial complex associated to labeled bipartite trees [15]. However, an equivalent definition of the space is derived through a complex of labeled hypertrees [14].

The connection between hypertrees and $\mathrm{Out}^0(W_n)$ is encapsulated in the following main theorem of this section.

Theorem 2.9. *Let $x_{i_1,D_1}, x_{i_2,D_2}, \ldots, x_{i_p,D_p}$ be partial conjugations in $\mathrm{Out}^0(W_n) \leqslant \mathrm{Aut}^0(W_n)$. Then there exists a hypertree $\Theta \in \mathcal{HT}_n$ that carries all of the*

$$x_{i_1,D_1}, x_{i_2,D_2}, \ldots, x_{i_p,D_p}$$

if and only if they pairwise commute.

First, we must define the relevant concepts.

Definition 2.1. A *hypergraph* Γ is an ordered pair (V_Γ, E_Γ) consisting of a set of *vertices* V_Γ and a set of *hyperedges* E_Γ, where for each $e \in E_\Gamma$, $e \subseteq V_\Gamma$ and $|e| \geqslant 2$. Often we will label the vertices which leads to a *labeled hypergraph*, and we say that Γ is a *(labeled) hypergraph on* V_Γ. A hypergraph in which every edge contains exactly two vertices is a *(simple) graph.*

We consider two equivalences on the class of hypergraphs. First, two hypergraphs Γ and Γ' are *isomorphic as unlabeled hypergraphs* if there exists a bijection $f : V_\Gamma \to V_{\Gamma'}$ such that for each subset $S \subseteq V_\Gamma$, $f(S) \in E_{\Gamma'}$ if and only if $S \in E_\Gamma$. f is then called a *hypergraph isomorphism.* Second, two hypergraphs Γ and Γ' are *isomorphic as labeled hypergraphs* if $V_\Gamma = V_{\Gamma'}$ and the identity map $V_\Gamma \to V_\Gamma$ is a hypergraph isomorphism. Unless stated otherwise, labeled hypergraphs will be considered up to

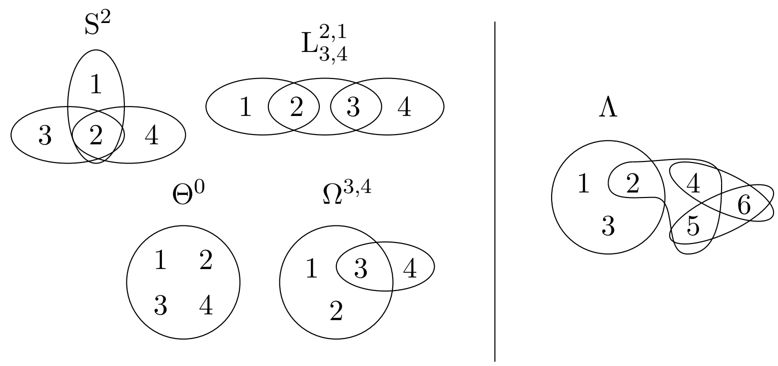

Figure 2.1: Examples of hypergraphs: Θ^0, S^2, $L^{2,1}_{3,4}$, and $\Omega^{3,4}$ are hypertrees. S^2 and $L^{2,1}_{3,4}$ are trees. Λ is a hypergraph but not a hypertree, since both $4 \to 6 \to 5$ and $4 \to 5$ are simple walks in Λ.

labeled hypergraph isomorphism.

A *simple walk* from v to v' in Γ is a sequence of alternating hypervertices and hyperedges $v = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} v_p = v'$ where $\{v_i, v_{i+1}\} \subseteq e_{i+1}$ for all $0 \leqslant i \leqslant p-1$, $v_i \neq v_j$ for all $0 \leqslant i \neq j \leqslant p$, and $e_i \neq e_j$ for all $1 \leqslant i \neq j \leqslant p$.

A *hypertree* is a hypergraph Γ where for all $v, w \in V_\Gamma$, there exists a unique simple walk from v to w in Γ. A hypertree which is also a graph is a *tree*.

Remark ([18])**.** The set of hypertrees on a set S is in one-to-one correspondence with the set of bipartite labeled trees whose labeled vertices are in bijection with S.

Definition 2.2. For each positive integer n, let $[n] := \{1, 2, \ldots, n\}$. Consider $\mathcal{HT}_n$, defined to be the set of hypertrees on $[n]$ up to labeled hypergraph isomorphism.

Given hypertrees $\Theta, \Theta' \in \mathcal{HT}_n$, we say that Θ' is obtained from Θ by *a single fold* if there exists distinct hyperedges $e, e' \in E_\Theta$ such that $e \cap e' \neq \varnothing$ and

$$E_{\Theta'} = \big(E_\Theta \backslash \{e, e'\}\big) \cup \{e \cup e'\},$$

i.e., $E_{\Theta'}$ is the result of replacing e and e' in E_Θ by their union (which is still a hyperedge). Since e and e' are required to intersect, folding a hypertree results in a hypertree. For each pair $\Theta, \Lambda \in \mathcal{HT}_n$, we write $\Theta \leqslant \Lambda$ and say that Θ is a result of *folding* Λ if Θ may be obtained from Λ by a (possibly empty) sequence of folds.

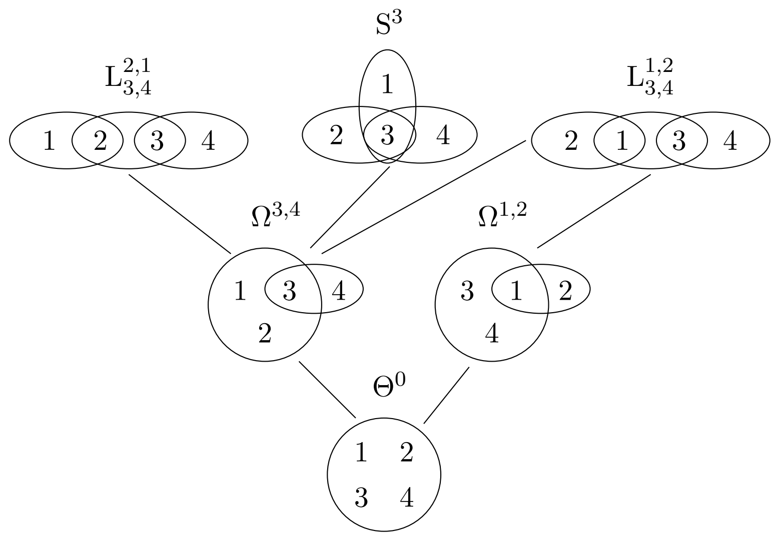

Figure 2.2: The lines represent folding relations on hypertrees in $\mathcal{HT}_4$. So $\Theta^0 \leqslant \Omega^{3,4} \leqslant \mathrm{L}^{2,1}_{3,4}$, S^3, $\mathrm{L}^{1,2}_{3,4}$, while $\Theta^0 \leqslant \Omega^{1,2} \leqslant \mathrm{L}^{1,2}_{3,4}$.

Then $(\mathcal{HT}_n, \leqslant)$ is a partially ordered set called the *hypertree poset of rank n*. We will often abuse notation and refer to this partially ordered set by $\mathcal{HT}_n$.

Definition 2.3. The simplicial realization of $(\mathcal{HT}_n, \leqslant)$ is the *hypertree complex of rank n*, HT_n. This means that HT_n is a simplicial complex whose vertices are in bijective correspondence with the set of hypertrees in $\mathcal{HT}_n$ and where $\Theta_1, \Theta_2, \ldots, \Theta_k$ span a k-simplex in HT_n if and only if (up to reordering) $\Theta_1 \leqslant \Theta_2 \leqslant \cdots \leqslant \Theta_k$ in $\mathcal{HT}_n$. Since maximal chains in $\mathcal{HT}_n$ involve folding trees a single fold at a time, the dimension of HT_n is $n-2$.

Remark. For $n = 4$, $|\,\mathcal{HT}_4\,| = 29$ and the height of $\mathcal{HT}_4$ is 3. Thus, HT_4 is a simplicial 2-complex.

Now Σ_n acts on $\mathcal{HT}_n$ in an obvious way: Each permutation of $[n]$ just permutes the labels of the hypertrees, which preserves the partial order, and so is an order automorphism of $\mathcal{HT}_n$. This action by order automorphisms of $(\mathcal{HT}_n, \leqslant)$ naturally extends to an action by simplicial automorphisms on HT_n [18]. One might wonder: Are there any other hidden automorphisms of either $\mathcal{HT}_n$ or HT_n? It turns out the answer is “no”.

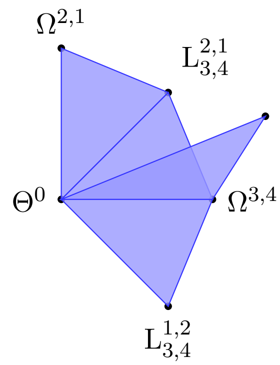

Figure 2.3: A portion of the hypertree complex, HT_4.

Theorem 2.4 (Piggott [18])**.** *For all integers $n \geqslant 3$,*

$$\mathrm{Aut}(\mathrm{HT}_n) \cong \mathrm{Aut}(\mathcal{HT}_n) \cong \Sigma_n,$$

where $\mathrm{Aut}(\mathrm{HT}_n)$ *is the set of simplicial automorphisms of* HT_n*,* $\mathrm{Aut}(\mathcal{HT}_n)$ *is the set of order isomorphisms of $\mathcal{HT}_n$, and Σ_n is the symmetric group on n letters.*

Thus, HT_n provides an accurate (topological) model and $\mathcal{HT}_n$ provides an accurate (combinatorial) model for Σ_n. If we endowed HT_n with any Σ_n-equivariant metric, for instance a piecewise Euclidean one with equilateral triangles, then Σ_n would act by isometries and so HT_n would be an accurate geometric model for Σ_n as well.

Definition 2.5. A hypertree $\Theta \in \mathcal{HT}_n$ has between one and $n-1$ hyperedges, and the *height* of Θ is defined to be one less than its number of hyperedges. Notice that hypertrees of height $n-2$ are actually trees.

We note a few special classes of hypertrees:

1. There is a unique hypertree of height zero, denoted Θ_n^0.

2. $\mathcal{S}_n = \{\mathrm{S}_n^j \mid j \in [n]\}$, the set of *star trees*, where S_n^j is the hypertree of height $n-2$ (tree) whose hyperedges (edges) are exactly $\{i, j\}$ for $i \neq j$.

3. $\mathcal{L}_n$, the set of *line trees*, which are the trees (hypertrees of height $n-2$) in which exactly two vertices are leaves.

4. $\mathcal{M}_n^1 = \{\Omega_n^{i,j} \mid i \neq j \in [n]\}$, the set of *omega hypertrees*, are those hypertrees of height 1 that contain the hyperedges $\{i, j\}$ and $[n]\backslash\{j\}$.

Two elements in one of these classes are isomorphic as unlabeled hypertrees, and so the action of Σ_n on $\mathcal{HT}_n$ acts transitively on each of these classes. Additionally, in W_4, this list actually exhausts all possible hypertrees.

Question. What does HT_n have to do with $\mathrm{Out}^0(W_n)$?

It turns out that hypertrees encode commuting relations in $\mathrm{Out}^0(W_n)$.

Definition 2.6. A hypertree $\Theta \in \mathcal{HT}_n$ *carries* a partial conjugation $x_{i,D}$ if and only if for all $d \in D$, $j \in [n]\backslash D$, the simple walk from d to j visits i.

A general automorphism $\alpha \in \mathrm{Out}^0(W_n)$ is *carried* by Θ if and only if there exists partial conjugations $x_{i_1,D_1}, x_{i_2,D_2}, \ldots, x_{i_p,D_p} \in \mathrm{Out}^0(W_n)$ such that $\alpha = x_{i_1,D_1}x_{i_2,D_2}\cdots x_{i_p,D_p}$ and x_{i_j,D_j} is carried by Θ for each $1 \leqslant j \leqslant p$.

For this definition, we may think of $\mathrm{Out}^0(W_n)$ as either a subgroup or a quotient of $\mathrm{Aut}^0(W_n)$. Inner automorphisms are trivially carried by all hypertrees, since the only element of $[n]\backslash D$ is i. Thus, the notion of a hypertree carrying an automorphism is actually well-defined up to outer automorphism class. In particular, we can use this fact to freely switch between representatives $x_{i,D} = x_{i,D^c}$ in $\mathrm{Out}^0(W_n)$, where $D^c = [n]\backslash\{D \cup \{i\}\}$. For notation, also let $\widetilde{D} = D \cup \{i\}$ and $\widetilde{D^c} = D^c \cup \{i\}$.

Remark 2.7. Hypertrees of height h carry 2^h automorphisms in $\mathrm{Out}^0(W_n)$, including the identity automorphism, and if $\Theta \leqslant \Lambda$, then Λ carries all the automorphisms that Θ does [18]. In fact, the 2^h automorphisms carried by Θ all commute and generate a $\mathbb{Z}_2^h$, which follows from Theorem 2.9 below.

Lemma 2.8 (Gutierrez-Piggott-Ruane)**.** *Let x_{i_1,D_1} and x_{i_2,D_2} be partial conjugations in $\mathrm{Out}^0(W_n) \leqslant \mathrm{Aut}^0(W_n)$. Then they commute if and only if one of the following four cases hold:*

1. $i_1 = i_2$
2. $i_1 \neq i_2$*,* $i_1 \in D_2$*,* $i_2 \notin D_1$*,* $D_1 \subseteq D_2$*, i.e.,* $i_1 \neq i_2$*,* $\widetilde{D_1} \cap \widetilde{D_2^c} = \varnothing$*.*

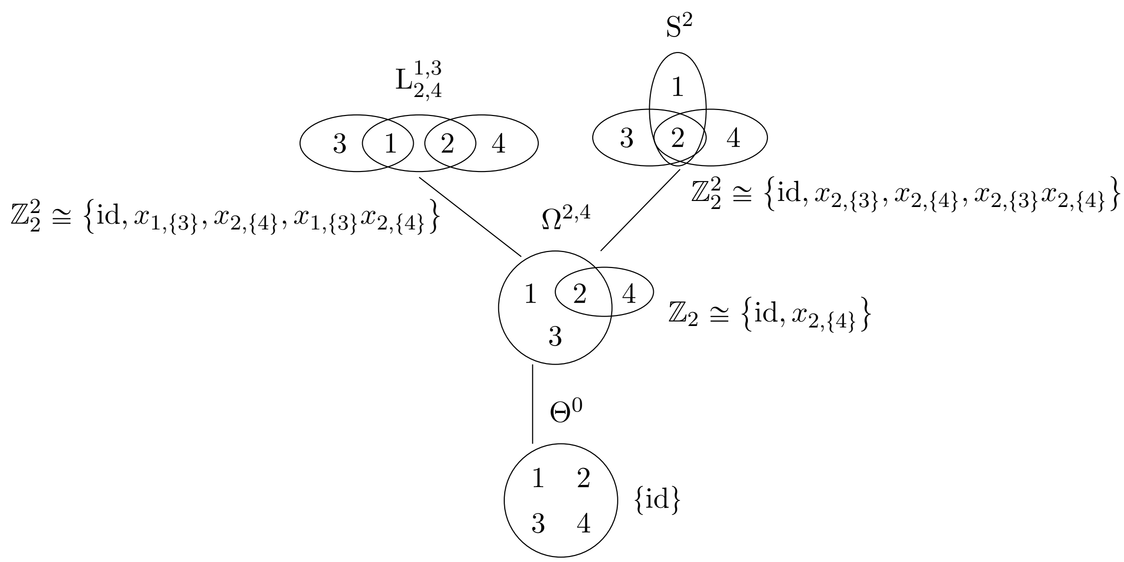

Figure 2.4: A portion of $\mathcal{HT}_4$ and the automorphisms in $\mathrm{Out}^0(W_4)$ carried by each hypertree.

3. $i_1 \neq i_2$, $i_1 \notin D_2$, $i_2 \in D_1$, $D_2 \subseteq D_1$, *i.e.*, $i_1 \neq i_2$, $\widetilde{D_1^c} \cap \widetilde{D_2} = \varnothing$.

4. $i_1 \neq i_2$, $i_1 \notin D_2$, $i_2 \notin D_1$, $D_1 \cap D_2 = \varnothing$, *i.e.*, $i_1 \neq i_2$, $\widetilde{D_1} \cap \widetilde{D_2} = \varnothing$.

Proof. This follows from Lemma 4.3 in [10]. ☐

Theorem 2.9. *Let $x_{i_1,D_1}, x_{i_2,D_2}, \ldots, x_{i_p,D_p}$ be partial conjugations in $\mathrm{Out}^0(W_n) \leqslant \mathrm{Aut}^0(W_n)$. Then there exists a hypertree $\Theta \in \mathcal{HT}_n$ that carries all of the*

$$x_{i_1,D_1}, x_{i_2,D_2}, \ldots, x_{i_p,D_p}$$

if and only if they pairwise commute.

Proof. One direction is Lemma 4.4 in [18], and is reproduced here for convenience:

Suppose that Θ carries each of the x_{i_j,D_j} for $j \in \{1, \ldots, p\}$. If $i_k \neq i_l$, then x_{i_k,D_k} and x_{i_l,D_l} commute by Lemma 1.1 in [15]. Because the $\mathbb{Z}_2$ factors in W_n are abelian (or just directly from the definition of partial conjugation), whenever $i_k = i_l$, then x_{i_k,D_k} and x_{i_l,D_l} commute.

Conversely, suppose that $x_{i_1,D_1}, x_{i_2,D_2}, \ldots, x_{i_p,D_p}$ pairwise commute.

We will build the hypertree Θ inductively.

Let Θ_1 be the hypertree on $[n]$ that has two hyperedges: one containing $\widetilde{D_1} = D_1 \cup \{i_1\}$ and the other containing $\widetilde{D_1^c} = D_1^c \cup \{i_1\} = [n] \backslash D_1$. Any simple walk from D_1 to D_1^c must pass through i, so Θ_1 carries x_{i_1,D_1}. In fact, the only automorphisms carried by Θ_1 are the identity and x_{i_1,D_1}.

Now inductively assume that there is a hypertree Θ_{k-1} on $[n]$ that carries $x_{i_1,D_1}, x_{i_2,D_2}, \ldots, x_{i_{k-1},D_{k-1}}$ for $1 \leqslant k-1 \leqslant p-1$ and that x_{i_k,D_k} commutes with all automorphisms carried by Θ_{k-1}. Since Θ_{k-1} is a hypertree, every hypervertex is in at least one hyperedge, and any two hyperedges are either disjoint or else intersect in exactly one hypervertex. Consider x_{i_k,D_k}, and denote the hyperedges of Θ_{k-1} by E_{k-1}. Now define E_k to be the set of non-empty intersections between the hyperedges of Θ_{k-1} and either D_k or D_k^c, i.e.,

$$E_k := \left(\left\{E \cap \widetilde{D_k} \mid E \in E_{k-1}\right\} \cup \left\{E \cap \widetilde{D_k^c} \mid E \in E_{k-1}\right\}\right) \backslash \varnothing,$$

and let Θ_k be the hypergraph defined on $[n]$ with E_k as its hyperedges. Suppose that both $E^1 = E \cap \widetilde{D_k}$ and $E^2 = E \cap \widetilde{D_k^c}$ are non-empty. We claim that $i_k \in E$ and so $E^1 \cap E^2 = \{i_k\}$:

Θ_{k-1} is a hypertree that carries at least one non-identity automorphism, so it has at least 2 hyperedges, and thus there is a neighboring hyperedge to E, E', such that $E \cap E' = \{m\}$ for some $m \in [n]$. If $m = i_k$, then $i_k \in E$. Otherwise, suppose that $m \neq i_k$. Since Θ_{k-1} is a hypertree, $\Theta_{k-1} \backslash \{m\}$ is disconnected. Let D_m be the connected component of $\Theta_{k-1} \backslash \{m\}$ that contains $E \backslash \{m\}$ and D_m^c be the union of the rest of the components. Then Θ_{k-1} must carry x_{m,D_m}. Thus, by assumption, x_{i_k,D_k} commutes with x_{m,D_m}. If $i_k \notin E$, then $i_k \notin D_m$ since $E \subseteq D_m$. Also, the non-empty element of E^1 can't be i_k, so it must be an element of $E \cap D_k$, i.e., $E \cap D_k \neq \varnothing$. But the same is true for E^2, $E \cap D_k^c \neq \varnothing$. By Lemma 2.8, this leaves only the option that $m \in D_k$ and $D_m \subseteq D_k$, which contradicts that $E \cap D_k^c \neq \varnothing$. Thus, i_k must be in E.

Suppose that $v = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} v_p = v'$ is the unique simple walk in Θ_{k-1} from v to v'. In Θ_k, each e_j is partitioned into (at most) two hyperedges,

$e_j^1 = e_j \cap \widetilde{D_k}$ and $e_j^2 = e_j \cap \widetilde{D_k^c}$. If v_{j-1} and v_j are both in the same hyperedge, say e_j^1, then just replace e_j with e_j^1 in the walk above. Otherwise, without loss of generality, $v_{j-1} \in e_j^1$ and $v_j \in e_j^2$. Since both e_j^1 and e_j^2 are non-empty, from above we know that $e_j^1 \cap e_j^2 = \{i_k\}$. Now replace $v_{j-1} \xrightarrow{e_j} v_j$ in the walk with $v_{j-1} \xrightarrow{e_j^1} i_k \xrightarrow{e_j^2} v_j$. This can only happen once in the walk since otherwise i_k would be in two non-consecutive hyperedges, and a different, shorter simple walk would have been possible in Θ_{k-1}. So this construction shows that Θ_k is a hypertree. It carries x_{i_k,D_k} since if $v_0 \in D_k$ and $v_p \in D_k^c$, then there must be some point in the walk where the hyperedges go from the e_j^1 to the e_j^2, at which point either i_k was already in the walk or else it gets inserted in the construction.

It is immediate that $\Theta_{k-1} \leqslant \Theta_k$ since folding $E \cap \widetilde{D_k}$ and $E \cap \widetilde{D_k^c}$ into E is necessary only when both are non-empty. In particular, that means that Θ_k carries all of the automorphisms that Θ_{k-1} carried. Recall that x_{i_j,D_j} is carried by a hypertree Λ if and only if D_j is a union of connected components (other than the one containing the lowest index of $[n]\backslash\{i_j\}$) of $\Lambda\backslash\{i_j\}$ ([18]).

Let $D_{m_1}, D_{m_2}, \ldots, D_{m_l}$ be the connected components of $\Theta_k\backslash\{i_k\}$ other than the one with minimal index. These exactly correspond with the analogous connected components in Θ_{k-1} except that one is added each time a hyperedge (which had to contain i_k) was split. Since the number of outer automorphisms carried by a hypertree is 2^h (where $h =$ height $=$ number of hyperedges minus one), this unfolding increases the height by exactly the number of edges with $e_j^1 \cap e_j^2 = \{i_k\}$. All of the x_{i_k,D_j} carried by Θ_{k-1} are products of the $x_{i_k,D_{m_s}}$. So by a counting argument, all of the automorphisms carried by Θ_k are given by products of the x_{i_j,D_j} (with $i_j \neq i_k$ and $1 \leqslant j \leqslant k$) and the $x_{i_k,D_{m_s}}$ (with $1 \leqslant s \leqslant l$). It suffices to prove that all of these commute with the remaining automorphisms on our list.

Now, let $1 \leqslant j \leqslant p$, $1 \leqslant s \leqslant l$ and consider x_{i_j,D_j} and $x_{i_k,D_{m_s}}$. By construction, $D_{m_s} \subseteq D_k$ or $D_{m_s} \subseteq D_k^c$. It suffices to show that x_{i_j,D_j} or x_{i_j,D_j^c} commutes with $x_{i_k,D_{m_s}}$ or $x_{i_k,D_{m_s}^c}$ since $\mathrm{Out}^0(W_n)$ as a quotient is isomorphic to $\mathrm{Out}^0(W_n)$ as a subgroup of $\mathrm{Aut}^0(W_n)$. So without loss of generality, suppose that $D_{m_s} \subseteq D_k$, and so $\widetilde{D_{m_s}} \subseteq \widetilde{D_k}$. If $i_k = i_j$, this is trivial, so suppose not. By the definition of $x_{i_k,D_{m_s}}$, there is some component D of $\Theta_{k-1}\backslash\{i_k\}$ such that $D_{m_s} = D_k \cap D$, and thus $\widetilde{D_{m_s}} = \widetilde{D_k} \cap \widetilde{D}$. $x_{i_k,D}$ is then carried by Θ_{k-1} and so commutes with x_{i_j,D_j}. Now $\widetilde{D_{m_s}} \cap \widetilde{D_j} = \left(\widetilde{D_k} \cap \widetilde{D_j}\right) \cap \left(\widetilde{D} \cap \widetilde{D_j}\right)$, and if this is empty, we are done. Similarly, $\widetilde{D_{m_s}} \cap \widetilde{D_j^c} = \left(\widetilde{D_k} \cap \widetilde{D_j^c}\right) \cap \left(\widetilde{D} \cap \widetilde{D_j^c}\right)$, and if this is empty, we are done. Otherwise, all four intersections must be non-empty. But by Lemma 2.8, this forces $\widetilde{D_k^c} \cap \widetilde{D_j} = \varnothing$ and $\widetilde{D^c} \cap \widetilde{D_j} = \varnothing$, i.e., $i_k \notin D_j,\ i_j \in D_k,\ D_j \subseteq D_k$ and $i_k \notin D_j,\ i_j \in D,\ D_j \subseteq D$. Thus, $i_k \notin D_j,\ i_j \in D_{m_s},\ D_j \subseteq D_{m_s}$, and so we are done again. Thus, every automorphism on our list commutes with every automorphism carried by Θ_k. This completes the induction.

$\square$

In fact, examining the proof of Theorem 2.9, we actually proved a stronger corollary.

Corollary 2.10. *Given a hypertree $\Theta \in \mathcal{HT}_n$ and a partial conjugation $x_{i,D}$, then there exists an unfolding of Θ to a hypertree $\Lambda \geqslant \Theta$ that carries $x_{i,D}$ if and only if $x_{i,D}$ commutes with every automorphism carried by Θ.*

Now that we know how the hypertree complex encodes the commuting information of $\mathrm{Out}^0(W_n)$, we can use this to build a complex that $\mathrm{Out}^0(W_n)$ can act on.

Chapter 3

McCullough-Miller Space

McCullough and Miller originally [15] defined their complex using labeled bipartite trees, but McCammond and Meier [14] showed an equivalent way to define the space using $\mathcal{HT}_n$. Adam Piggott [18] then characterized the automorphism groups of these spaces.

McCullough-Miller space is constructed by taking a copy of $\mathcal{HT}_n$ for each element of $\mathrm{Out}^0(W_n)$ and then gluing these copies together according to the hypertree carrying relation.

Definition 3.1. First, define an equivalence relation $\sim$ on $\mathrm{Out}^0(W_n) \times \mathcal{HT}_n$ as follows: $(\alpha, \Theta) \sim (\beta, \Lambda)$ if and only if $\Theta = \Lambda$ and $\alpha^{-1}\beta$ is carried by Θ. Write $[\alpha, \Theta]$ for the $\sim$-equivalence class of (α, Θ) and let $\mathcal{K}_n$ be the set of $\sim$-equivalence classes.

Now, define a partial order $\leqslant$ on $\mathcal{K}_n$: $[\alpha, \Theta] \leqslant [\beta, \Lambda]$ if and only if Λ folds to Θ and $\alpha^{-1}\beta$ is carried by Λ, i.e., $\Theta \leqslant \Lambda$ in $\mathcal{HT}_n$ and $[\alpha, \Lambda] = [\beta, \Lambda]$.

McCullough-Miller space K_n is the simplicial realization of $(\mathcal{K}_n, \leqslant)$. We will often abuse notation and have $[\alpha, \Theta]$ refer to both its equivalence class in $\mathcal{K}_n$ as well as its corresponding vertex in K_n.

Remark 3.2. For a hypertree Θ of height h in $\mathcal{HT}_n$, Θ carries 2^h automorphisms, and so $[\alpha, \Theta]$ will be glued to $2^h - 1$ other copies of Θ. In particular, $\left[\alpha, \Theta_n^0\right]$ is a singleton, is not glued to any other element, and $\left[\alpha, \Theta_n^0\right] \leqslant [\beta, \Lambda]$ if and only if $[\alpha, \Lambda] = [\beta, \Lambda]$. These are called *nuclear vertices* of K_n. So K_n consists of partially glued copies of HT_n indexed by $\mathrm{Out}^0(W_n)$.

Recall that Σ_n acts on $\mathcal{HT}_n$ by permuting labels, and that $\mathrm{Out}(W_n) \cong \mathrm{Out}^0(W_n) \rtimes \Sigma_n$. So any $\alpha \in \mathrm{Out}(W_n)$ has a unique representative $\phi\sigma$, where $\phi \in \mathrm{Out}^0(W_n)$, $\sigma \in \Sigma_n$, and $\alpha = \phi\sigma$ in $\mathrm{Out}(W_n)$.

Definition 3.3. $\mathrm{Out}(W_n)$ acts on $\mathrm{Out}^0(W_n) \times \mathcal{HT}_n$ by:

$$\phi\sigma \cdot (\alpha, \Theta) = \big(\phi(\sigma\alpha\sigma^{-1}), \sigma\Theta\big)$$

Since $\mathrm{Out}^0(W_n) \trianglelefteq \mathrm{Out}(W_n)$, $(\sigma\alpha\sigma^{-1}) \in \mathrm{Out}^0(W_n)$, and so $\phi\sigma\alpha\sigma^{-1} \in \mathrm{Out}^0(W_n)$. The action of σ on Θ is by permuting the labels.

This action of $\mathrm{Out}(W_n)$ preserves $\sim$ as well as the partial order $\leqslant$. Thus, this descends to an action of $\mathrm{Out}(W_n)$ on $\mathcal{K}_n$ by order automorphisms as well as K_n by simplicial automorphisms [18].

Example. Let $(1\ 2) \in \Sigma_4$ be the transposition that exchanges 1 and 2, and let $(1-2-3-4)$ be the line (hyper)tree that contains the edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$.

$$\begin{aligned}
&\big(x_{1,\{3\}}, (1\ 2)\big) \cdot \big[x_{2,\{4\}}, (1-2-3-4)\big] \\
&= \big[x_{1,\{3\}}\left((1\ 2)x_{2,\{4\}}(1\ 2)^{-1}\right), (1\ 2)\cdot(1-2-3-4)\big] \\
&= \big[x_{1,\{3\}}x_{1,\{4\}}, (2-1-3-4)\big] \\
&= \big[x_{1,\{3,4\}}, (2-1-3-4)\big]
\end{aligned}$$

As with HT_n, this action induces an injective map from $\mathrm{Out}(W_n)$ into both $\mathrm{Aut}(\mathcal{K}_n, \leqslant)$ and $\mathrm{Aut}(K_n)$, and one might wonder whether or not there any other other hidden symmetries in these spaces. The answer is once again in the negative, and so these spaces serve as accurate combinatorial and topological models for $\mathrm{Out}(W_n)$.

Theorem 3.4 (Piggott [18], Thm 1.1)**.** *For $n \geqslant 4$,*

$$\mathrm{Aut}(\mathcal{K}_n, \leqslant) \cong \mathrm{Aut}(K_n) \cong \mathrm{Out}(W_n).$$

Remark 3.5. As in the case of HT_n and Σ_n, this shows that $\mathcal{K}_n$ is an accurate combinatorial model and K_n is an accurate topological or simplicial model for $\mathrm{Out}(W_n)$.

In fact, $\mathrm{Out}(W_n)$ acts on K_n properly discontinuously and co-compactly by simplicial automorphisms ([18]), but K_n has no *a priori* metric on it. To be an accurate *geometric* model, we will need to endow K_n with a metric to turn it into a geodesic metric space such that the action of $\mathrm{Out}(W_n)$ is by isometries. Then K_n will be quasi-isometric to $\mathrm{Out}(W_n)$ (and also its finite index subgroup, $\mathrm{Out}^0(W_n)$), and they will have the same large-scale geometry. There are many ways to do this, such as assigning the piecewise Euclidean metric with equilateral triangles to K_n, and we shall return to this idea in Chapter 6.

However, this metric does not turn K_n into a CAT(0) space. If we wish to use this space to show that $\mathrm{Out}(W_n)$ is a CAT(0) group, then we will need to pick a different metric. The metric will need to be CAT(0) as well as equivariant with respect to the $\mathrm{Out}(W_n)$ or $\mathrm{Out}^0(W_n)$ action on K_n. As we show in Chapter 5, no such (piecewise M_κ) metric turns out to exist.

Now, let $[\alpha, \Theta] \in K_n$ and suppose that $[\beta, \Theta']$ is another point where Θ and Θ' are isomorphic as unlabeled hypertrees. Since Θ' differs from Θ only in its labeling, there is a permutation $\sigma \in \Sigma_n$ such that $\sigma \cdot \Theta = \Theta'$ [18]. Since $\mathrm{Out}^0(W_n) \trianglelefteq \mathrm{Out}(W_n)$, $\sigma\alpha^{-1}\sigma^{-1} \in \mathrm{Out}^0(W_n)$, and thus $\phi = \beta\sigma\alpha^{-1}\sigma^{-1} \in \mathrm{Out}^0(W_n)$. Then we have that

$$\begin{aligned} \phi\sigma \cdot [\alpha, \Theta] &= \left[\phi(\sigma\alpha\sigma^{-1}), \sigma\Theta\right] \\ &= \left[\beta\sigma\alpha^{-1}\sigma^{-1}(\sigma\alpha\sigma^{-1}), \Theta'\right] \\ &= \left[\beta, \Theta'\right]. \end{aligned}$$

Thus, $\mathrm{Out}(W_n)$ acts transitively on the subsets of $\mathcal{K}_n$ where the $\mathrm{Out}^0(W_n)$ labels can be anything and the unlabeled hypertree isomorphism classes are preserved. Since the action of Σ_n on $\mathcal{HT}_n$ only permutes labels, it preserves unlabeled isomorphisms classes, and so the full action of $\mathrm{Out}(W_n)$ on $\mathcal{K}_n$ must as well. Thus, the quotient of $\mathcal{K}_n$ by $\mathrm{Out}(W_n)$ consists of one simplex for each unlabeled isomorphism class in $\mathcal{HT}_n$, glued along common edges.

$\mathrm{Out}^0(W_n)$ acts transitively on the labels of $\mathcal{K}_n$ but doesn't change the hypertree.

Thus, the quotient of $\mathcal{K}_n$ by $\mathrm{Out}^0(W_n)$ is the full hypertree complex $\mathcal{HT}_n$.

As noted in Definition 2.5, the unlabeled isomorphism classes in $\mathcal{HT}_4$ are precisely $\{\Theta_4^0\}, \mathcal{S}_4,\ \mathcal{L}_4,\ \mathcal{M}_4^1$ [18]. When we are only concerned with $n = 4$, we will drop the subscripts and use a more descriptive notation.

Notation 3.6. We will denote the hypertrees in $\mathcal{HT}_4$ as follows:

1. The hypertree with one hyperedge will be denoted Θ^0.
2. The star tree in $\mathcal{S}_4$ with central vertex i (generally called S_4^i) will be denoted S^i.
3. The line tree in $\mathcal{L}_4$ with hyperedges $\{j,i\},\ \{i,k\},\ \{k,l\}$ will be denoted $L_{k,l}^{i,j}$.
4. The hypertree in $\mathcal{M}_4^1$ which contains the hyperedges $\{i,j\},\ [n]\backslash\{j\}$ will be denoted $\Omega^{i,j}$.

Remark 3.7. The following describes the poset structure on the 29 elements of $\mathcal{HT}_4$ as well as the carrying relation. See also Figure 2.2. (Note that each listed partial conjugation might need to replace its domain with its complement to pick the representative not containing the minimal index.)

1. Θ^0 is a $\leqslant$-minimal element that only carries the identity.
2. $\Omega^{i,j}$ carries only the identity and $x_{i,\{j\}}$. It folds into Θ^0.
3. S^i carries the Klein 4-group of $\{\mathrm{id}, x_{i,\{j\}},\ x_{i,\{k\}},\ x_{i,\{j,k\}}\}$, where j and k are the non-minimal elements of $[4]\backslash\{i\}$ (and l is the minimal one). It folds into $\Omega^{i,j}$, $\Omega^{i,k}$, $\Omega^{i,l}$, and Θ^0.
4. $L_{k,l}^{i,j}$ carries the Klein 4-group of $\{\mathrm{id}, x_{i,\{j\}},\ x_{k,\{l\}},\ x_{i,\{j\}}x_{k,\{l\}}\}$. It folds to $\Omega^{i,j}$, $\Omega^{k,l}$, and Θ^0.

Examining the maximal chains in $\mathcal{HT}_4$, we see that every simplex in HT_4 has a vertex Θ^0, a vertex of the form $\Omega^{i,j}$, and a vertex of the form either $L_{k,l}^{i,j}$ or S^i. See Figure 2.3. Thus, every simplex in K_4 has a vertex $[\alpha, \Theta^0]$, a vertex of the form $[\alpha, \Omega^{i,j}]$, and a vertex of the form either $[\alpha, L_{k,l}^{i,j}]$ or $[\alpha, S^i]$ for

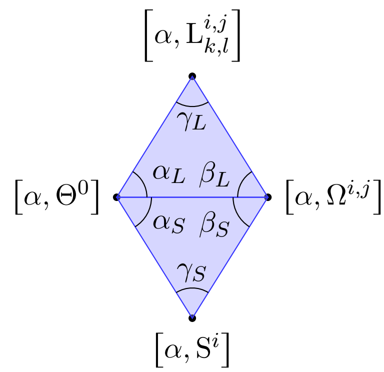

Figure 3.1: The fundamental domain for the $\mathrm{Out}(W_4)$ action on K_4, with associated angles after metrizing.

some $\alpha \in \mathrm{Out}^0(W_4)$. Since the action of $\mathrm{Out}(W_4)$ is transitive on these classes, a fundamental domain for the $\mathrm{Out}(W_4)$ action on K_4 is given by the union of the simplices spanned by $\left\{\left[\mathrm{id}, \Theta^0\right], \left[\mathrm{id}, \Omega^{1,3}\right], \left[\mathrm{id}, L^{1,3}_{2,4}\right]\right\}$ (called an L-simplex) and $\left\{\left[\mathrm{id}, \Theta^0\right], \left[\mathrm{id}, \Omega^{1,3}\right], \left[\mathrm{id}, S^1\right]\right\}$ (called an S-simplex). See Figure 3.1.

This description of K_4 and the action of $\mathrm{Out}(W_n)$ will be useful in Chapter 5.

Chapter 4

$\mathrm{Out}^0(W_n)$ is not a Right-Angled Coxeter Group

Another approach to determine whether or not $\mathrm{Out}^0(W_n)$ is CAT(0) would be to prove that it was a right-angled Coxeter group itself, since all RACGs are CAT(0). A presentation for $\mathrm{Aut}^0(W_\Gamma)$ (what Mühlherr calls $\mathrm{Spe}(W)$) is given in [17] as a semidirect product $\mathrm{Inn}(W_\Gamma) \rtimes \mathrm{Out}^0(W_\Gamma)$ and so a finite presentation can be extracted for $\mathrm{Out}^0(W_\Gamma)$ (after a few elementary Tietze transformations).

Recall that a generating set for $\mathrm{Out}^0(W_n)$ is given by the set of partial conjugations $\mathcal{P}^0 = \{x_{i,D}\}$, where $i \in [n]$, j is the minimal index in $[n]\backslash\{i\}$, and D is a non-empty subset of $[n]\backslash\{i,j\}$ ([10]) . Also remember that $\widetilde{D} = D \cup \{i\}$ and $\widetilde{D^c} = D^c \cup \{i\}$. There are also some obvious classes of relations in $\mathrm{Out}^0(W_n)$:

1. (R1) $x_{i,D}x_{i,D} = \mathrm{id}$
2. (R2) $x_{i,D}x_{i,D'} = x_{i,(D \cup D')\backslash(D \cap D')}$
3. (R3) $[x_{i,D_i}, x_{j,D_j}] = 1$ if $\widetilde{D_i} \cap \widetilde{D_j} = \varnothing$, $\widetilde{D_i^c} \cap \widetilde{D_j} = \varnothing$, or $\widetilde{D_i} \cap \widetilde{D_j^c} = \varnothing$. (See Lemma 2.8).

Some elementary Tietze transformations on the main Theorem in [17] give the following.

Theorem 4.1 (Mühlherr [17])**.** *A finite presentation for* $\mathrm{Out}^0(W_n)$ *is given by the generators* $\mathcal{P}^0$ *and the set of relations given by the union of the classes (R1), (R2), and (R3).*

Now, we are in the situation where we have a finite presentation for a group and wish to know whether or not it is a RACG. It is generated by involutions, its abelianization is $\mathbb{Z}_2^{n(n-2)}$, and there are no obvious automorphisms of a finite order greater than 2. So the obvious invariants do not rule out the possibility yet.

In joint work with Andy Eisenberg, Kim Ruane, and Adam Piggott [5], we show that given a finite presentation of a group, there is a procedure that can determine whether or not that group is a right-angled Coxeter group and if so, construct the defining graph Γ. It uses a new invariant of a group, its *involution graph*, which is a graph that corresponds to all of the conjugacy classes of involutions of a group and the commuting relations between them. While our procedure is not in general a computable algorithm, in many particular instances of interest it is computable, and can prove that either the group is not a RACG or else construct its defining graph and often an explicit isomorphism.

Let us attempt to apply this theorem to our presentation for $\mathrm{Out}^0(W_n)$ and prove the following theorem.

Theorem 4.2. $\mathrm{Out}^0(W_n)$ *is not a right-angled Coxeter group.*

To do this, we will construct a portion of the involution graph and show that such a graph cannot appear as an induced subgraph of the involution graph of *any* right-angled Coxeter group. All of the following definitions are from [5].

Definition 4.3. Let G be a group. The *involution graph of* G, denoted Δ_G, is a graph defined as follows. The vertices are the conjugacy classes of involutions in G. Two vertices $[x]$ and $[y]$ are connected by an edge if there exist representatives gxg^{-1} and hyh^{-1} which commute with each other.

The involution graph of a right-angled Coxeter group is a special type of graph called a *clique graph*.

Definition 4.4. Let Γ be a finite simple graph. A *clique in* Γ is a set of pairwise adjacent vertices. The *clique graph of* Γ is the finite simple graph $\Gamma_K = (V_K, E_K)$ whose vertices correspond to nonempty cliques in Γ, and such that vertices are adjacent if the corresponding union of cliques is also a clique in Γ.

Definition 4.5. Let Γ be a graph with maximal cliques Γ_i, and write Γ_I for the intersections of maximal cliques.

1. We say that Γ *satisfies the maximal clique condition* if, for all I, there exists an integer k_I such that $|\Gamma_I| = 2^{k_I} - 1$.

2. If Γ satisfies the maximal clique condition, we will say that Γ *satisfies the inclusion-exclusion condition* if, for each J,

$$\sum_{I \supsetneq J} (-1)^{|I \setminus J|+1} k_I \leqslant k_J.$$

Theorem 4.6 (C.-Eisenberg-Piggott-Ruane [5]). *The involution graph of any right-angled Coxeter group is a clique graph, and any clique graph satisfies the maximal clique condition as well as the inclusion-exclusion condition.*

The following are some useful facts, which we will need later, abouts certain right-angled Coxeter quotients of $\mathrm{Out}^0(W_4)$.

Remark 4.7. Following the presentation of $\mathrm{Out}^0(W_3)$ from Theorem 4.1 (and using generators $y_{i,D}$ instead of $x_{i,D}$ to distinguish the $n = 3$ and $n = 4$ cases), we find that $\mathcal{P}^0 = \{y_{1,\{3\}}, y_{2,\{3\}}, y_{3,\{2\}}\}$. For each of these automorphisms, $\widetilde{D}$ and $\widetilde{D^c}$ contain at least two indices each, but since there are only three indices total in [3], these extended domains can never be disjoint. Thus, there are no relations of the form (R3). Also, there are no partial conjugations in $\mathcal{P}^0$ that have the same acting letter, and so there are no relations of the form (R2) either. Thus, the full presentation for $\mathrm{Out}^0(W_3)$ is given by:

$$\mathrm{Out}^0(W_3) = \left\langle y_{1,\{3\}}, y_{2,\{3\}}, y_{3,\{2\}} \mid y_{1,\{3\}}^2 = y_{2,\{3\}}^2 = y_{3,\{2\}}^2 = \mathrm{id} \right\rangle \cong W_3.$$

Thus, $\mathrm{Out}^0(W_3) \cong W_3$ and so is a right-angled Coxeter group.

We also remark that nothing was special about naming the vertices of W_3 as $\{1,2,3\}$. The same analysis holds if they are named $\{1,2,4\}$, $\{1,3,4\}$, or $\{2,3,4\}$.

Definition 4.8. For each $k \in [4]$, consider the copy of $\mathrm{Out}^0(W_3)$ that has vertex names $j \in [4]\backslash\{k\}$. Let m_k be the minimal index in $[4]\backslash\{k\}$.

Let $\varphi_k : \mathrm{Out}^0(W_4) \mapsto \mathrm{Out}^0(W_3)$ be defined as

$$\varphi_k(x_{i,D}) := \begin{cases} \mathrm{id} & \text{if } i = k,\ D = \{k\}, \text{ or } D^c = \{k\} \\ y_{i,D\backslash\{k\}} & \text{if otherwise and } m_k \notin D\backslash\{k\} \\ y_{i,D^c\backslash\{k\}} & \text{if otherwise and } m_k \in D\backslash\{k\} \end{cases}$$

By checking that each of the relation families (R1), (R2), and (R3) are preserved under the operations of either removing k from D or D^c or by sending certain generators to the identity, we can see that each map φ_k is a surjective homomorphism onto $\mathrm{Out}^0(W_3) \cong W_3$.

Remark 4.9. From the definition of φ_k, we collect the following facts. The proofs are elementary group theory exercises and left to the interested reader.

1. The kernel of φ_k, $\ker \varphi_k$, is the normal closure of the subgroup of $\mathrm{Out}^0(W_4)$ generated by the partial conjugations of the form $x_{k,D}$, $x_{i,\{k\}}$, and $x_{i,[4]\backslash\{k\}}$. There are six such generators (and one of them is redundant). For instance, $\ker \varphi_4$ is the normal closure of the subgroup generated by

$$\{x_{4,\{2\}}, x_{4,\{3\}}, x_{4,\{2,3\}}, x_{1,\{4\}}, x_{2,\{4\}}, x_{3,\{4\}}\}.$$

2. The images of $\ker \varphi_k$ under the abelianization map of $\mathrm{Out}^0(W_4) \to \mathbb{Z}_2^8$, denoted $\overline{\ker \varphi_k}$, is isomorphic to $\mathbb{Z}_2^5$.

3. The intersection of two of these abelianization images of kernels, say, $\overline{\ker \varphi_k}$ and $\overline{\ker \varphi_j}$, is isomporhic to $\mathbb{Z}_2^2$. For instance, $\overline{\ker \varphi_2} \cap \overline{\ker \varphi_4} = \langle \overline{x_{2,\{4\}}}, \overline{x_{4,\{2\}}} \rangle$. The intersection of three of them (and thus of all four of them) is trivial. Thus, the intersection of any three $\ker \varphi_k$ is contained in the commutator subgroup of $\mathrm{Out}^0(W_4)$.

4. Each $\overline{x_{i,D}} \in \mathbb{Z}_2^8$ is contained in exactly two $\overline{\ker \varphi_k}$.

5. For i, j, k, and l distinct in $[4]$, $x_{i,\{j\}}, x_{i,\{j,k\}} \notin \ker \varphi_k$.

We will also need the following lemmas.

Lemma 4.10. *Every involution in* $\mathrm{Out}^0(W_n)$ *is conjugate to a unique (up to reordering) product of commuting partial conjugations from* $\mathcal{P}^0$.

Proof. Suppose that $\alpha \in \mathrm{Out}^0(W_n)$ is an involution, and let $G = \langle \alpha \rangle \cong \mathbb{Z}_2$. Recall that McCullough-Miller space, K_n, is a contractable, finite dimensional simplicial complex that admits an action by $\mathrm{Out}^0(W_n)$ by simplicial automorphisms [15]. This restricts to an action of G on K_n by simplicial automorphisms.

Suppose that G acts freely on K_n. All of the claims below are from [11]. Then in fact this is a covering space action, and so $G \cong \pi_1(\mathrm{K}_n / G)$. But since K_n is contractible, K_n / G is a $\mathrm{K}(G, 1)$ space. K_n / G, like K_n, must also be a two dimensional Δ-complex, which thus has trivial i^{th} simplicial homology for $i > 2$. Since this is a $\mathrm{K}(G, 1)$ space, that implies that $G \cong \mathbb{Z}_2$ has trivial homology for $i > 2$. But an actual $\mathrm{K}(\mathbb{Z}_2, 1)$ space is the infinite-dimensional real projective space, which has non-trivial homology at arbitraily high orders. This is a contradiction, and so G cannot act freely on K_n, and so α must fix a point in K_n.

Since α acts as a simplicial automorphism, if it fixes a point, it must fix a simplex, i.e., either a vertex, an edge, or an entire face of K_n. But then G is a subgroup of the stabilizer of that simplex, and so G is conjugate to the stabilizer of a simplex in the fundamental domain for the action. However, the stabilizers of the fundamental domain given by the copy of the hypertree complex with vertices $[\mathrm{id}, \Theta]$ are exactly the automorphisms carried by the hypertree Θ. But the hypertrees at height h carry exactly 2^h automorphisms, and these are exactly given by the products of commuting partial conjugations from $\mathcal{P}^0$ (Theorem 2.9 and [18]). Thus, α must be conjugate to one of these products of commuting partial conjugations. Since they each project to distinct elements in the abelianization, this product is unique, up to reordering.

□

Lemma 4.11. *In* $\mathrm{Out}^0(W_n)$, *if* α *and* β *are distinct products of commuting partial conjugations from* $\mathcal{P}^0$, *then there exist conjugates of* α *and* β *that commute if and* *only if α and β commute.*

Proof. One direction is trivial. Conversely, assume that conjugates $x = \gamma\alpha\gamma^{-1}$ and $y = \delta\beta\delta^{-1}$ commute. Since these are involutions, that means that their product $z = xy$ is an involution as well. By Lemma 4.10, z is conjugate to a product of commuting partial conjugations from $\mathcal{P}^0$, namely the reduced word $c_1c_2\cdots c_k$. α and β are also products of commuting generators from $\mathcal{P}^0$, namely, $\alpha = a_1a_2\cdots a_m$ and $\beta = b_1b_2\cdots b_l$ with both words reduced.

Letting the generators that appear both among the a_i and b_j move to the end of α and the beginning of β, we see that $\alpha\beta = a_1a_2\cdots a_tb_sb_{s+1}\cdots b_l$ is a reduced product of distinct generators, and so it maps into the abelianization with a 1 in each component for a remaining a_i or b_j. But $z = c_1c_2\cdots c_k$ is a reduced word that maps to the same element in the abelianization as $\alpha\beta$, and so the c_p correspond exactly to the remaining a_i and b_j. But the c_p were all pairwise commuting, and so the same is true for the remaining a_i and b_j. But then α and β commute. □

Proof of Theorem 4.2. Suppose for the sake of contradiction that $\mathrm{Out}^0(W_n)$ were a right-angled Coxeter group. Then by Theorem 4.6, its involution graph would satisfy the inclusion-exclusion condition. Let us build part of the involution graph of $\mathrm{Out}^0(W_n)$.

First to see how this works, we'll do it for $n = 4$, and we'll build the entire involution graph.

Consider the product of commuting involutive generators in $\mathcal{P}^0$.

Lemma 2.8 tells us which pairs of involutions in this set commute with each other. We collect this information in Figure 4.1. Lemma 4.11 tells us that the missing edges are truly missing in the involution graph, i.e., Figure 4.1 is the full involution graph of $\mathrm{Out}^0(W_4)$. Lemma 4.10 tells us that the cliques in Figure 4.1 are maximal, since there are no other conjugacy classes of involutions in $\mathrm{Out}^0(W_n)$ other than commuting products of partial conjugations.

Now, let Γ_1 be the maximal clique in $\Delta_{\mathrm{Out}^0(W_4)}$ containing $\{x_{1,\{3\}}, x_{1,\{4\}}, x_{1,\{3,4\}}\}$, and consider all of the maximal cliques that intersect Γ_1 in at least one vertex. Γ_2

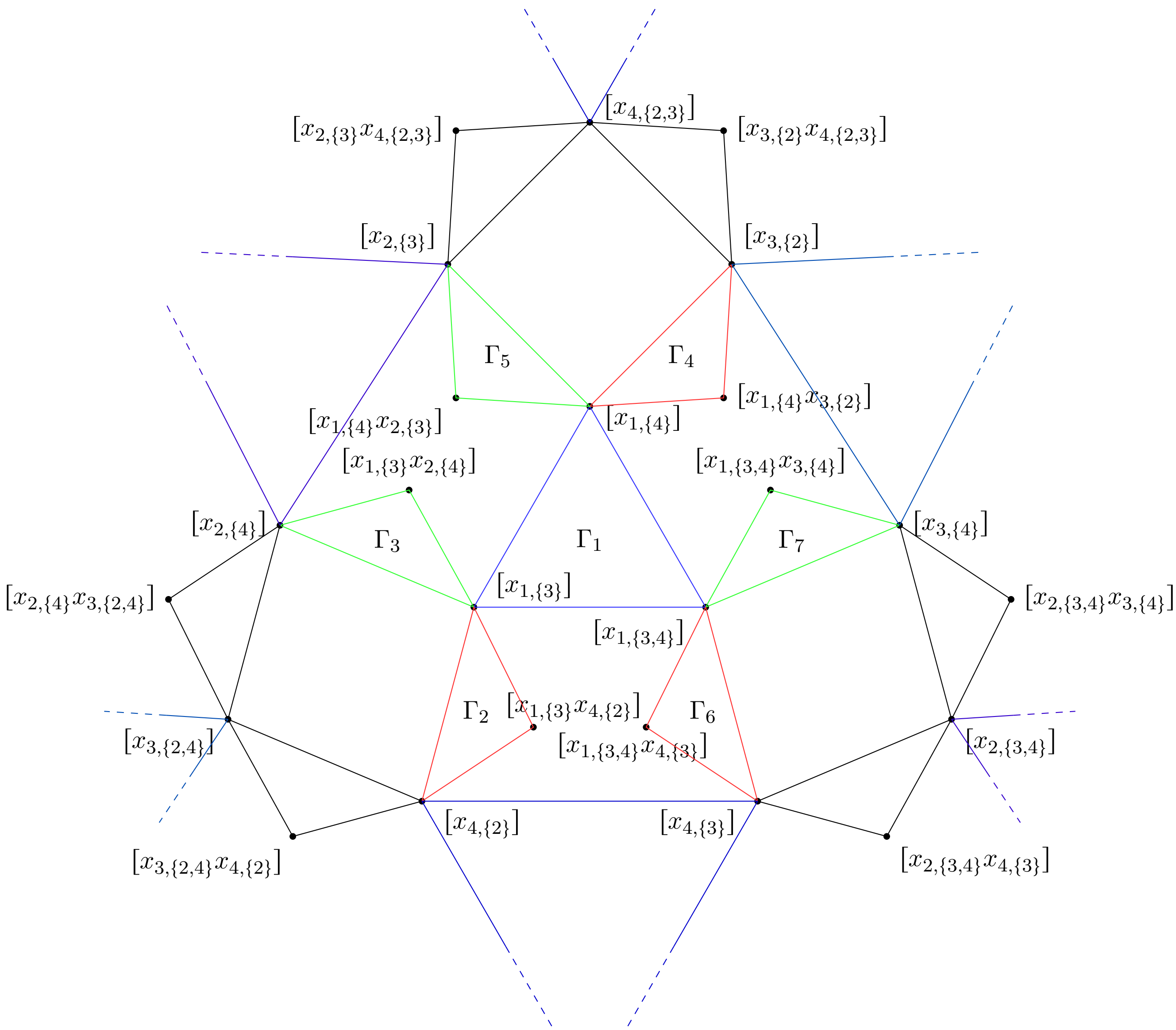

Figure 4.1: The full involution graph of $\mathrm{Out}^0(W_4)$, $\Delta_{\mathrm{Out}^0(W_4)}$. The triangles are the maximal cliques, and the seven of them intersecting the clique Γ_1 are named. Note that the dotted lines are connected to the solid lines of the same color on the other side of the graph.

will be the maximal clique in $\Delta_{\mathrm{Out}^0(W_4)}$ containing $\{x_{1,\{3\}}, x_{4,\{2\}}, x_{1,\{3\}}x_{4,\{2\}}\}$, Γ_3 will be the maximal clique in $\Delta_{\mathrm{Out}^0(W_4)}$ containing $\{x_{1,\{3\}}, x_{2,\{4\}}, x_{1,\{3\}}x_{2,\{4\}}\}$, Γ_4 will be the maximal clique in $\Delta_{\mathrm{Out}^0(W_4)}$ containing $\{x_{1,\{4\}}, x_{3,\{2\}}, x_{1,\{4\}}x_{3,\{2\}}\}$, Γ_5 will be the maximal clique in $\Delta_{\mathrm{Out}^0(W_4)}$ containing $\{x_{1,\{4\}}, x_{2,\{3\}}, x_{1,\{4\}}x_{2,\{3\}}\}$, Γ_6 will be the maximal clique in $\Delta_{\mathrm{Out}^0(W_4)}$ containing $\{x_{1,\{3,4\}}, x_{4,\{3\}}, x_{1,\{3,4\}}x_{4,\{3\}}\}$, and Γ_7 will be the maximal clique in $\Delta_{\mathrm{Out}^0(W_4)}$ containing $\{x_{1,\{3,4\}}, x_{3,\{4\}}, x_{1,\{3,4\}}x_{3,\{4\}}\}$. The intersections of these maximal cliques are denoted Γ_I where $I \subseteq \{1,2,3,4,5,6,7\}$. For instance, $\Gamma_1 \cap \Gamma_2 = \Gamma_{1,2}$. See Figure 4.1.

Since we are assuming that $\mathrm{Out}^0(W_4)$ is a RACG, Theorem 4.6 says that it satisfies the maximal clique condition, so let k_I be the integer such that $|\Gamma_I| = 2^{k_I} - 1$. From Figure 4.1, we see that $k_1 = 2$, $k_{1,i} = 1$ for $i \in \{2,3,4,5,6,7\}$, $k_{1,i,i+1} = 1$ for $j \in \{2,4,6\}$, and $k_I = 0$ for all other intersections.

So we see that

$$\sum_{I \supsetneq \{1\}} (-1)^{|I \setminus J|+1} k_I = \sum_{i=2}^{7} k_{1,i} - k_{1,2,3} - k_{1,4,5} - k_{1,6,7} = 6 - 3 = 3 \geqslant 2 = k_1,$$

contradicting the inclusion-exclusion condition of Theorem 4.6. Thus, $\mathrm{Out}^0(W_4)$ must not have been a RACG after all.

Now, we'll generalize the proof for $\mathrm{Out}^0(W_n)$. Suppose that $\mathrm{Out}^0(W_n)$ is a right-angled Coxeter group. By Lemmas 4.10 and 4.11, a full system of representatives (see [5]) for the involution graph of $\mathrm{Out}^0(W_n)$ is given by commuting products from $\mathcal{P}^0$. By Theorem 2.9 and Remark 2.7, intersections of maximal cliques in the involution graph correspond to commuting products of partial conjugations, which correspond to the automorphisms carried by a hypertree, except the identity. Since $\mathcal{HT}_n$ is a lattice ([14]), every intersection of maximal cliques can be associated to a hypertree, and the intersection relationships are exaclty encoded in the partial order in $\mathcal{HT}_n$. Thus, the maximal cliques are of size $2^{n-2} - 1$ and correspond to labeled trees, and the intersection of maximal cliques given by the automorphisms carried by a hypertree Θ at height h is of size $2^h - 1$.

Consider the intersection of maximal cliques given by the automorphisms carried by the hypertree Θ with hyperedges $\{\{1,3\},\{1,4\},\{1,2,F\}\}$, where $F=\{5,6,\dots n\}$. Then Θ carries $\langle x_{1,\{3\}}, x_{1,\{4\}}\rangle \cong \mathbb{Z}_2^2$. If this intersection of maximal cliques is Γ_J, then $k_J = 2$. Each of the involution graph vertices $x_{1,\{3\}}, x_{1,\{4\}}, x_{1,\{3\}}x_{1,\{4\}}$ generate a further intersection of maximal cliques of size 2, since there is a partial conjugation that commutes with each one of them but not the other two, e.g., $x_{4,\{2\}}$ commutes with $x_{1,\{3\}}$ but not $x_{1,\{4\}}$ nor $x_{1,\{3\}}x_{1,\{4\}}$. Each of these intersection of maximal cliques is associated to a hypertree, e.g., the hypertree $\Omega_n^{1,3}$ is associated to its carried automorphism $x_{1,\{3\}}$.

However, we have an indexing problem, since the same set of vertices might show up in different intersection of maximal cliques, and we need to count each one of these (with parity) in the inclusion-exclusion formula. To do so, we note that the maximal cliques containing a vertex correspond to the trees above the relevant hypertree. So in the indexing, the $I \supsetneq J$ correspond with all non-empty subsets of the collection $\mathcal{C}$ of trees Λ above $\Omega_n^{1,3}$ but not above Θ in $\mathcal{HT}_n$, with the odd subsets of $\mathcal{C}$ contributing a $+1$ and the even subsets of $\mathcal{C}$ contributing a -1 to the inclusion-exclusion sum. As already noted, $\mathcal{C}$ is non-empty and finite for each of the three vertices, and the set of odd subsets of a non-empty finite set is always bijective with the set of even subsets. Thus, since we exclude the empty subset, which is even, there is always a net of $+1$ in the sum for each vertex.

Since there are three such vertices, we have again the inclusion-exclusion formula

$$\sum_{I\supsetneq J}(-1)^{|I\setminus J|+1}k_I = 1+1+1 = 3 \geqslant 2 = k_J,$$

which contradicts Theorem 4.6. Thus, $\mathrm{Out}^0(W_n)$ cannot be a right-angled Coxeter group after all. $\square$

Chapter 5

Metrizing McCullough-Miller Space

In this chapter, we show that K_n admits no G-equivariant $\mathrm{CAT}(\kappa)$ M_κ-polyhedral structure for $G \cong \mathrm{Out}(W_n)$ or $G \cong \mathrm{Out}^0(W_n)$, $n \geqslant 4$, and $\kappa \leqslant 0$.

This is analogous to a result in Bridson's thesis [2] for $\mathrm{Out}(F_n)$ (for $n \geqslant 3$).

We shall need the following foundational theorem on curvature in polyhedral complexes. Gromov stated it without proof in [9], and Bridson proved it in full generality in [2].

Theorem 5.1 (Gromov's Link Condition [9, 2, 3])**.** *For $\kappa \leqslant 0$, a 2-dimensional M_κ-complex with finitely many isometry classes of polyhedrons is* $\mathrm{CAT}(\kappa)$ *if and only if it is simply connected and the link of each vertex is globally* $\mathrm{CAT}(1)$ *if and only if it is simply connected and for each vertex v, every injective loop in the link of v,* $\mathrm{Lk}(v)$*, has length at least 2π.*

For 2-dimensional complexes, this condition reduces to the following.

Theorem 5.2 (Gromov, Bridson [9, 2, 3])**.** *If X is a 2-dimensional* $\mathrm{CAT}(\kappa)$ *simplicial M_κ-complex for $\kappa \leqslant 0$, and $\alpha_i \in (0, \pi]$ are the angles at each corner of a simplex in the complex, then $\Sigma_T \alpha_i \leqslant \pi$, where T are the interior angles of a simplex, and $\Sigma_\gamma \alpha_i \geqslant 2\pi$, where γ are the angles around an injective loop in a link of a vertex.*

In particular, if the system of inequalities in the α_i given by Theorem 5.2 is unsatisfiable, then X admits no M_κ-polyhedral structure of non-positive curvature.

5.1 The $\mathrm{Out}(W_4)$ Case

We would now like to use Theorem 5.2 to show that no appropriate $\mathrm{CAT}(\kappa)$ metric can be assigned to K_4. So suppose that K_4 *has* been given an $\mathrm{Out}(W_4)$-equivariant metric that makes K_4 a $\mathrm{CAT}(\kappa)$ M_κ-simplicial complex.

Definition 5.3. Since the metric is $\mathrm{Out}(W_4)$-equivariant, it suffices to assign an angle to each corner of each simplex in the fundamental domain of the action in order to specify an angle in every corner of every simplex of K_4. So let the angles be defined as follows:

1. In any L-simplex, let α_L be the vertex angle of $\left[\alpha, \Theta^0\right]$, let β_L be the vertex angle of $\left[\alpha, \Omega^{i,j}\right]$, and let γ_L be the vertex angle of $\left[\alpha, L^{i,j}_{k,l}\right]$.
2. In any S-simplex, let α_S be the vertex angle of $\left[\alpha, \Theta^0\right]$, let β_S be the vertex angle of $\left[\alpha, \Omega^{i,j}\right]$, and let γ_S be the vertex angle of $\left[\alpha, S^i\right]$.

By Theorem 5.2, we know that the angles must satisfy the following inequalities:

$$\alpha_L + \beta_L + \gamma_L \leqslant \pi \tag{5.1}$$

$$\alpha_S + \beta_S + \gamma_S \leqslant \pi \tag{5.2}$$

To determine the other inequalities, we need to understand what the links of the vertices in K_4 look like. It suffices to consider the links of the vertices in a fundamental domain.

Example 5.4. We start with the link of $\left[\mathrm{id}, \Omega^{1,3}\right]$.

In K_4, $\left[\mathrm{id}, \Omega^{1,3}\right]$ is adjacent to $[\alpha, \Lambda]$ whenever either $\Omega^{1,3} \leqslant \Lambda$ and $\mathrm{id}^{-1}\alpha = \alpha$ is carried by Λ, or else $\Theta \leqslant \Omega^{1,3}$ and $\mathrm{id}^{-1}\alpha = \alpha$ is carried by $\Omega^{1,3}$. In the former case, since $[\mathrm{id}, \Lambda] = [\alpha, \Lambda]$ for any α carried by Λ, it suffices to consider the representatives $[\mathrm{id}, \Lambda]$. In the latter case, α might not be carried by Θ, so the different $[\alpha, \Theta]$ will result in different vertices.

Since $\Omega^{1,3}$ carries the identity and $x_{1,\{3\}}$, $\left[\mathrm{id}, \Omega^{1,3}\right] = \left[x_{1,\{3\}}, \Omega^{1,3}\right]$, and so $\left[\mathrm{id}, \Omega^{1,3}\right]$ is adjacent to $\left[\mathrm{id}, \Theta^0\right]$ and $\left[x_{1,\{3\}}, \Theta^0\right]$, which are different vertices.

On the other hand, the hypertrees greater than $\Omega^{1,3}$ in $\mathcal{HT}_4$ are the ones that it can unfold into, namely, $L^{1,3}_{2,4}$, $L^{1,3}_{4,2}$, and S^1. So $\left[\mathrm{id}, \Omega^{1,3}\right]$ is also adjacent to $\left[\mathrm{id}, L^{1,3}_{2,4}\right]$, $\left[\mathrm{id}, L^{1,3}_{4,2}\right]$, and $\left[\mathrm{id}, S^1\right]$. These 5 vertices are the only ones adjacent to $\Omega^{1,3}$ in K_4.

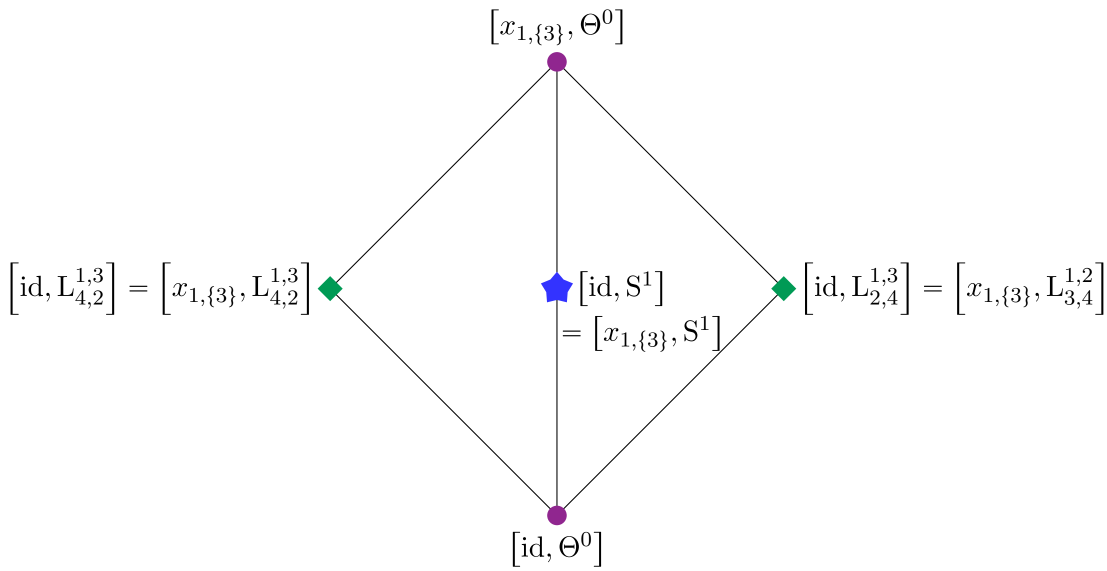

Figure 5.1: The link of $\left[\mathrm{id}, \Omega^{1,3}\right] = \left[x_{1,\{3\}}, \Omega^{1,3}\right]$ in K_4. The blue stars are star trees, the green diamonds are line trees, and the purple circles are nuclear vertices.

In the link, vertices are connected by an edge if they share a simplex in K_4 and the length of that edge is given by the angle with vertex $\left[\mathrm{id}, \Omega^{1,3}\right]$ in that simplex. So the line trees and star tree are never connected to each other, but the nuclear vertex is connected to each whenever the label matches up. The link is shown in Figure 5.1.

Reading off the injective loops that go around the large square as well as one of the smaller squares, we use Theorem 5.2 to get the inequalities:

$$\beta_L + \beta_L + \beta_L + \beta_L \geqslant 2\pi \tag{5.3}$$
$$\text{i.e., } \beta_L \geqslant \frac{\pi}{2}$$

$$\beta_L + \beta_L + \beta_S + \beta_S \geqslant 2\pi \tag{5.4}$$
$$\text{i.e., } \beta_L + \beta_S \geqslant \pi$$

Example 5.5. Next, we examine the link of $\left[\mathrm{id}, L_{2,4}^{1,3}\right]$.

Since $L_{2,4}^{1,3}$ carries $\{\mathrm{id},\ x_{1,\{3\}},\ x_{2,\{4\}},\ x_{1,\{3\}}x_{2,\{4\}}\}$, $\left[\mathrm{id}, L_{2,4}^{1,3}\right] = \left[x_{1,\{3\}}, L_{2,4}^{1,3}\right] = \left[x_{2,\{4\}}, L_{2,4}^{1,3}\right] = \left[x_{1,\{3\}}x_{2,\{4\}}, L_{2,4}^{1,3}\right]$, and so $\left[\mathrm{id}, L_{2,4}^{1,3}\right]$ is adjacent to $[\mathrm{id}, \Theta^0]$, $[x_{1,\{3\}}, \Theta^0]$, $[x_{2,\{4\}}, \Theta^0]$, and $[x_{1,\{3\}}x_{2,\{4\}}, \Theta^0]$ which are different vertices.

$\left[\mathrm{id}, L_{2,4}^{1,3}\right]$ is also adjacent to the vertices with these same four labels and with hypertree $\Omega^{1,3}$ or $\Omega^{2,4}$, but since each of these vertices has two representatives (e.g., $[\mathrm{id}, \Omega^{1,3}] = [x_{1,\{3\}}, \Omega^{1,3}]$), this results in only four new adjacent vertices in K_4.

In total, there are 8 adjacent vertices. In the L-simplices in K_4, the nuclear vertices are connected to both $\Omega^{i,j}$ vertices, and each of those vertices carry one non-identity automorphism, and so are connected to two nuclear vertices. Calculating all of these adjacencies, we see that the link graph is a single cycle of length 8, as shown in Figure 5.2.

Reading off the single injective loops in the cycle, we use Theorem 5.2 to get the inequality:

$$\begin{aligned} 8\gamma_L &\geqslant 2\pi \qquad (5.5) \\ \text{i.e., } \gamma_L &\geqslant \frac{\pi}{4} \end{aligned}$$

Example 5.6. Now, we construct the link of $[\mathrm{id}, S^1]$.

Since S^1 carries $\{\mathrm{id},\ x_{1,\{3\}},\ x_{1,\{4\}},\ x_{1,\{3\}}x_{1,\{4\}}\}$, $[\mathrm{id}, S^1] = [x_{1,\{3\}}, S^1] = [x_{1,\{4\}}, S^1] = [x_{1,\{3\}}x_{1,\{4\}}, S^1]$, and so $[\mathrm{id}, S^1]$ is adjacent to $[\mathrm{id}, \Theta^0]$, $[x_{1,\{3\}}, \Theta^0]$, $[x_{1,\{4\}}, \Theta^0]$, and $[x_{1,\{3\}}x_{1,\{4\}}, \Theta^0]$ which are different vertices.

$[\mathrm{id}, S^1]$ is also adjacent to the vertices with these same four labels and with hypertree $\Omega^{1,2}$, $\Omega^{1,3}$, or $\Omega^{1,4}$, but since each of these vertices has two representatives (e.g., $[\mathrm{id}, \Omega^{1,3}] = [x_{1,\{3\}}, \Omega^{1,3}]$), this results in only six new adjacent vertices in K_4.

In total, there are 10 adjacent vertices. In the S-simplices in K_4, the nuclear vertices are connected to all three $\Omega^{i,j}$ vertices, and each of those vertices carry one non-identity automorphism, and so are connected to two nuclear vertices. Calculating all of these adjacencies, we see that the link graph is three cycles of length 6,

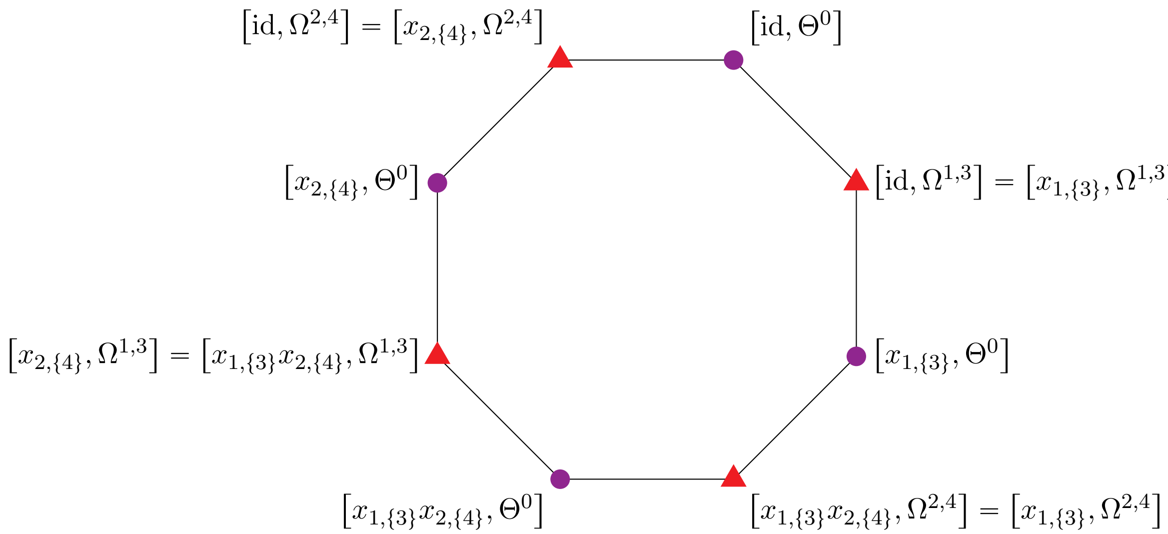

Figure 5.2: The link of $\left[\mathrm{id}, \mathrm{L}^{1,3}_{2,4}\right] = \left[x_{1,\{3\}}, \mathrm{L}^{1,3}_{2,4}\right] = \left[x_{2,\{4\}}, \mathrm{L}^{1,3}_{2,4}\right] = \left[x_{1,\{3\}}x_{2,\{4\}}, \mathrm{L}^{1,3}_{2,4}\right]$ in K_4. The purple circles are nuclear vertices, and the red triangles are elements of $\mathcal{M}^1_4$.

each glued to each other along paths of length 2, as shown in Figure 5.3.

Since all of the edges in the link have length γ_S, finding the smallest injective loop will give us an inequality that will imply all of the others. So, reading off the smallest injective loop in the link, which is a cycle of length 6, we use Theorem 5.2 to get the inequality:

$$\begin{aligned} 6\gamma_S &\geqslant 2\pi \\ \text{i.e., } \gamma_S &\geqslant \frac{\pi}{3} \end{aligned} \tag{5.6}$$

Example 5.7. Finally, we construct the link of $[\mathrm{id}, \Theta^0]$.

Since Θ^0 only carries the identity but is in every simplex in HT_4, $[\mathrm{id}, \Theta^0]$ is adjacent only to vertices with the same label but *any* hypertree, i.e., the vertices $\{[\mathrm{id}, \Lambda] \mid \Lambda \in \mathcal{HT}_4\}$ in K_4. So its link in K_4 is identical to its link in HT_4, which is given in Figure 5 in [18] and reproduced below in Figure 5.4.

It has 4 star vertices, 12 omega vertices, and 12 line vertices, for a total of 28. The star vertices are each connected to three omega vertices, the line vertices are

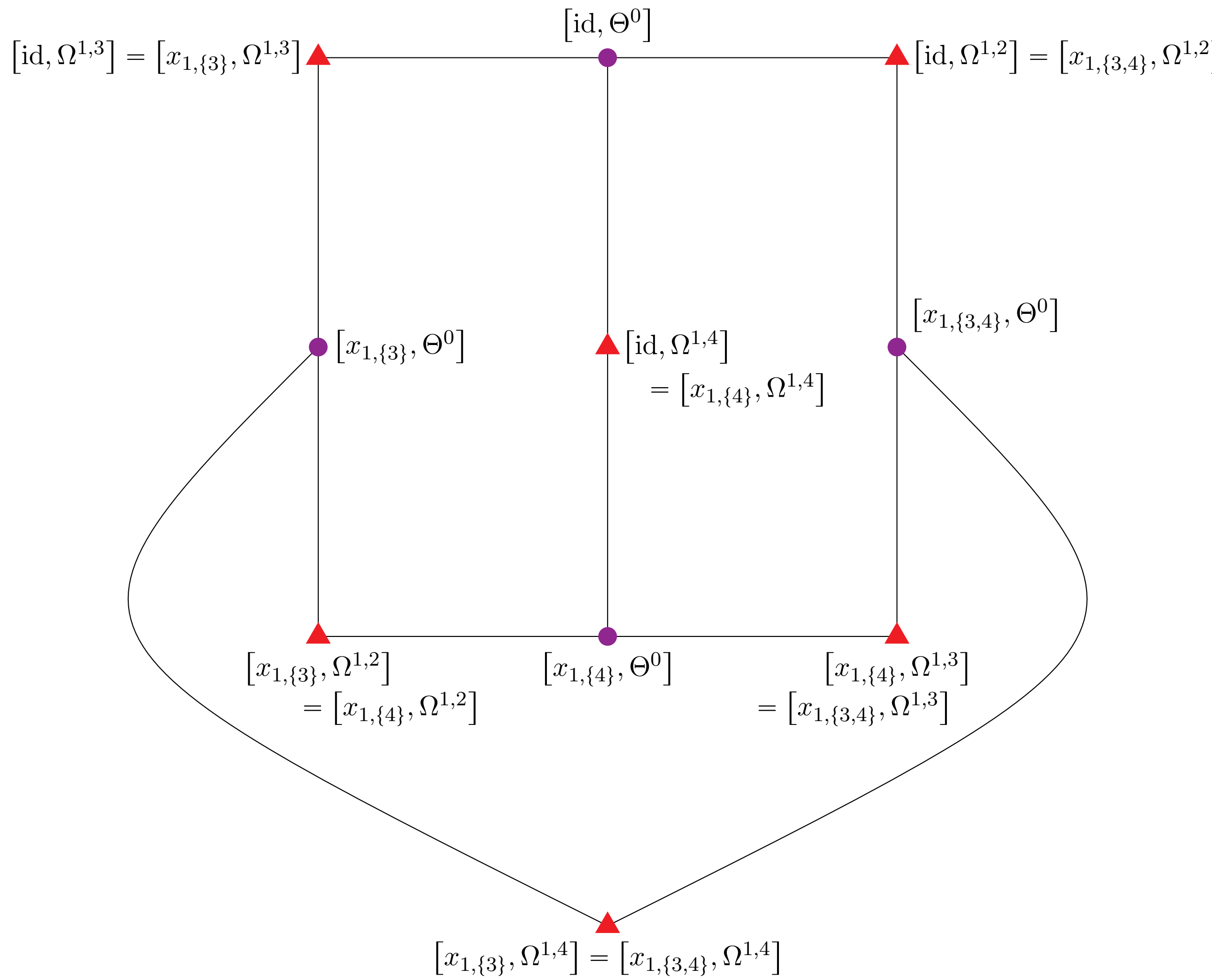

Figure 5.3: The link of $\left[\mathrm{id}, \mathrm{S}^1\right] = \left[x_{1,\{3\}}, \mathrm{S}^1\right] = \left[x_{1,\{4\}}, \mathrm{S}^1\right] = \left[x_{1,\{3,4\}}, \mathrm{S}^1\right]$ in K_4. It consists of three hexagons glued together. The purple circles are nuclear vertices, and the red triangles are elements of $\mathcal{M}_4^1$.

each connected to two omega vertices, and the omega vertices are each connected to one star and two line vertices. The link is made up of glued octagons, and we only need the smallest injective loops which wrap around each octagon. There are two types, so we once again use Theorem 5.2 to get the inequalities:

$$\begin{gathered} 8\alpha_L \geqslant 2\pi \\ \text{i.e., } \alpha_L \geqslant \frac{\pi}{4} \end{gathered} \tag{5.7}$$

$$\begin{gathered} 4\alpha_L + 4\alpha_S \geqslant 2\pi \\ \text{i.e., } \alpha_L + \alpha_S \geqslant \frac{\pi}{2} \end{gathered} \tag{5.8}$$

This is enough information to show that no angle solutions are possible.

Theorem 5.8. *There does not exist an* $\mathrm{Out}(W_4)$*-equivariant piecewise Euclidean (or piecewise hyperbolic)* $\mathrm{CAT}(0)$ $(\,\mathrm{CAT}(-1)\,)$ *metric on* K_4*.*

Proof. If there did exist such a metric, then by Theorem 5.2, there would exist angles $\alpha_L, \alpha_S, \beta_L, \beta_S, \gamma_L, \gamma_S \in (0, \pi]$ that satisfy Inequalities (5.1) - (5.8) above. Let us show that these are inconsistent.

$$\pi \geqslant \alpha_L + \beta_L + \gamma_L \geqslant \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = \pi \qquad \text{by (5.1), (5.7), (5.3), (5.5)}$$

$$\implies \alpha_L + \beta_L + \gamma_L = \pi \tag{5.9}$$

$$\implies \alpha_L = \pi - \beta_L - \gamma_L \leqslant \pi - \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \leqslant \alpha_L \qquad \text{by (5.3), (5.5), (5.7)}$$

$$\implies \alpha_L = \frac{\pi}{4} \tag{5.10}$$

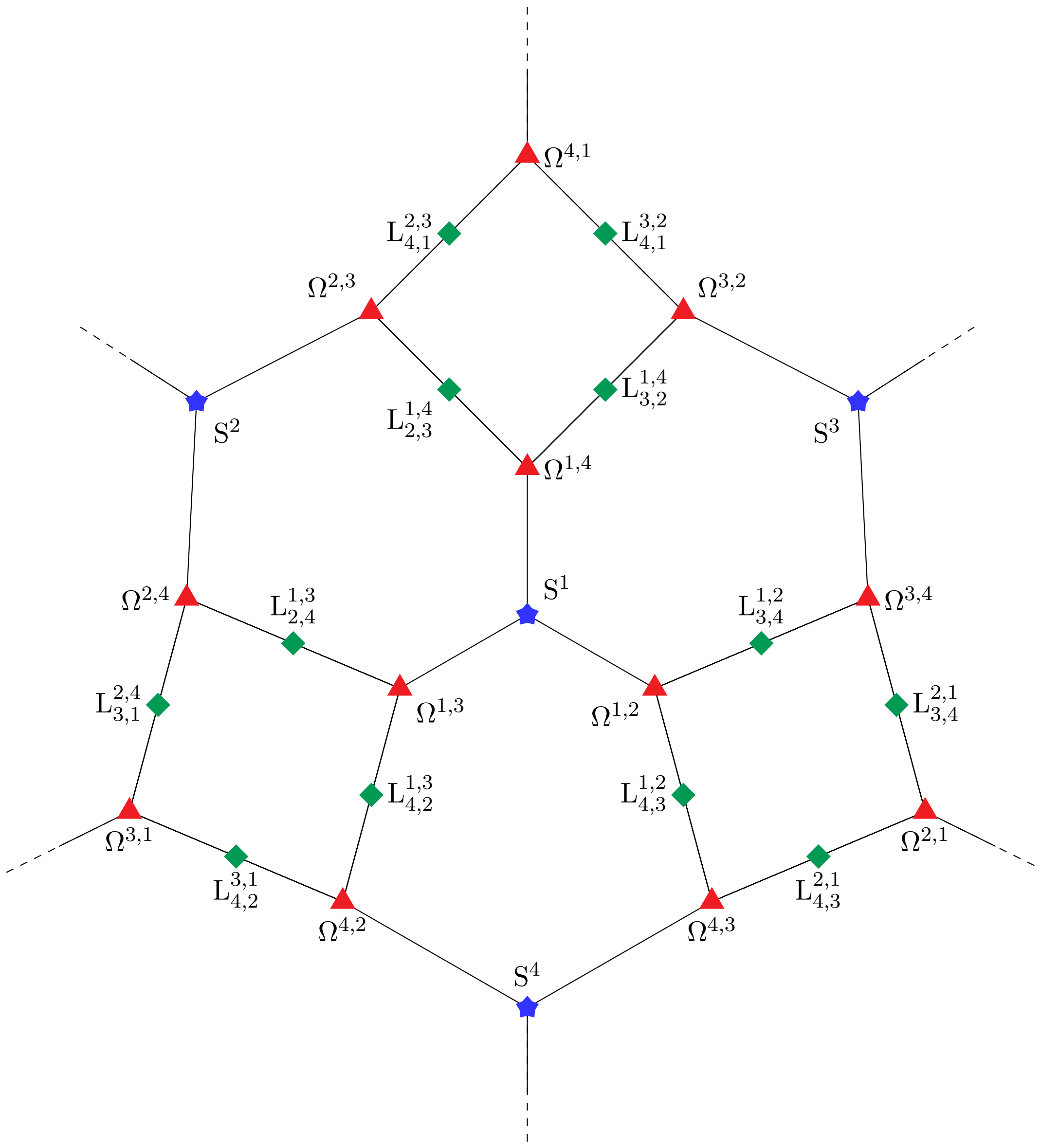

Figure 5.4: The link of $\left[\mathrm{id}, \Theta^0\right]$ in K_4. All of the adjacent vertices are labeled with $\mathrm{id} \in \mathrm{Out}^0(W_4)$, so this is the same as the link of Θ^0 in HT_4. The vertices are labeled with their corresponding hypertree. The blue stars are star trees, the green diamonds are line trees, and the red triangles are hypertrees in $\mathcal{M}^1_4$. Dashed lines connect to the other side of the link. (See Piggott [18] for another picture.)

$$\begin{aligned}\beta_L = \pi - \alpha_L - \gamma_L = \frac{3\pi}{4} - \gamma_L \leqslant \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2} \leqslant \beta_L \quad &\text{by (5.9), (5.10), (5.5), (5.3)} \\ \implies \beta_L = \frac{\pi}{2} \quad &(5.11)\end{aligned}$$

$$\begin{aligned}\gamma_L = \pi - \alpha_L - \beta_L = \pi - \frac{\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4} \quad &\text{by (5.9), (5.10), (5.11)} \\ \implies \gamma_L = \frac{\pi}{4} \quad &(5.12)\end{aligned}$$

$$\begin{aligned}\alpha_S \geqslant \frac{\pi}{2} - \alpha_L = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad &\text{by (5.8), (5.10)} \\ \implies \alpha_S \geqslant \frac{\pi}{4} \quad &(5.13)\end{aligned}$$

$$\begin{aligned}\beta_S \geqslant \pi - \beta_L = \pi - \frac{\pi}{2} = \frac{\pi}{2} \quad &\text{by (5.4), (5.11)} \\ \implies \beta_S \geqslant \frac{\pi}{2} \quad &(5.14)\end{aligned}$$

$$\begin{aligned}\gamma_S \leqslant \pi - \alpha_S - \beta_S \leqslant \pi - \frac{\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4} \quad &\text{by (5.2), (5.13), (5.14)} \\ \implies \gamma_S \leqslant \frac{\pi}{4} \quad &(5.15)\end{aligned}$$

$$\therefore \quad \frac{\pi}{3} \leqslant \gamma_S \leqslant \frac{\pi}{4} \qquad\qquad \text{by (5.6), (5.15)}$$

This is a contradiction, and so we are done.

□

5.2 The $\mathbf{Out}^0(W_4)$ Case

Since being a CAT(κ) group is not a property that is in general preserved under finite extension, it is possible that $\mathrm{Out}^0(W_4)$ is a CAT(κ) group, but $\mathrm{Out}(W_4)$ is not. So while K_4 could not be made into a CAT(κ) M_κ-simplicial complex that was equivariant with respect to the full $\mathrm{Out}(W_4)$ action, it is *a priori* possible that we could relax the requirement and obtain a metric only equivariant with respect to the induced $\mathrm{Out}^0(W_4)$ action. It turns out that this is still impossible.

In Section 5.1, the quotient of K_4 by $\mathrm{Out}(W_4)$ consisted of two simplices, and so only eight angle variables were necessary to consider. On the other hand, the quotient of K_4 by $\mathrm{Out}(W_4)$ is a full copy of HT_4, which consists of 24 L-simplices and 12 S-simplices, for a total of 36 simplices and so 108 angles. Our number of inequalities will rise as well. For instance, there will be 24 of type (5.1), 12 of type (5.2), and so on. There will even be additional forms of inequalities such as $\beta_{L^{i,j}_{k,l}} + \beta_{L^{i,j}_{k,l}} + \beta_{L^{i,j}_{l,k}} + \beta_{L^{i,j}_{l,k}} \geqslant 2\pi$, since in the link of $\left[\mathrm{id}, \Omega^{i,j}\right]$, the vertex angles connecting to the different line graphs could now be different. So our direct approach in Theorem 5.8 is too cumbersome to try again identically. Instead, we'll use the additional Σ_4 symmetry in the quotient HT_4 to simplify the calculations and prove the following theorem.

Theorem 5.9. *There does not exist an* $\mathrm{Out}^0(W_4)$*-equivariant piecewise Euclidean (or piecewise hyperbolic)* CAT(0) (CAT(-1)) *metric on* K_4.

First, we need to find a convenient way to name these 108 variables and describe their inequalities.

Definition 5.10. Suppose K_4 has been given an $\mathrm{Out}^0(W_4)$-equivariant metric to turn it into a M_κ-polyhedral complex. Since the metric is $\mathrm{Out}^0(W_4)$-equivariant, it suffices to assign an angle to each corner of each simplex in the fundamental domain of the action in order to specify an angle in every corner of every simplex of K_4. The fundamental domain is isometric to the quotient HT_4. So let the angles be defined as follows:

1. In any L-simplex, there are vertices of the form $\left[\alpha, \Theta^0\right]$, $\left[\alpha, L_{k,l}^{i,j}\right]$, and either $\left[\alpha, \Omega^{i,j}\right]$ or $\left[\alpha, \Omega^{k,l}\right]$. Since $L_{k,l}^{i,j}$ is the same labeled hypertree as $L_{i,j}^{k,l}$, we usually restrict the indexing to $i < k$, i.e., the smaller of the two is in the superscript. However, in the L-simplex, we also want to keep track of which Ω vertex is present. So we will subscript the angles in this simplex with $L_{k,l}^{i,j}$ where the $\{i, j\}$ superscript indicates which $\Omega^{i,j}$ is present. So for instance, $\alpha_{L_{2,4}^{1,3}}$ will be the vertex angle of $\left[\alpha, \Theta^0\right]$, $\beta_{L_{2,4}^{1,3}}$ will be the vertex angle of $\left[\alpha, \Omega^{1,3}\right]$, and $\gamma_{L_{2,4}^{1,3}}$ will be the vertex angle of $\left[\alpha, L_{2,4}^{1,3}\right]$. On the other hand, $\alpha_{L_{1,3}^{2,4}}$ will be the vertex angle of $\left[\alpha, \Theta^0\right]$, $\beta_{L_{1,3}^{2,4}}$ will be the vertex angle of $\left[\alpha, \Omega^{2,4}\right]$, and $\gamma_{L_{1,3}^{2,4}}$ will be the vertex angle of $\left[\alpha, L_{1,3}^{2,4}\right] = \left[\alpha, L_{2,4}^{1,3}\right]$.

2. In any S-simplex, there are vertices of the form $\left[\alpha, \Theta^0\right]$, $\left[\alpha, \Omega^{i,j}\right]$, and $\left[\alpha, S^i\right]$. The indexing is much easier here, since adding the $\{i, j\}$ superscript uniquely specifies the star tree. So for instance, $\alpha_{S^{1,3}}$ will be the vertex angle of $\left[\alpha, \Theta^0\right]$, $\beta_{S^{1,3}}$ will be the vertex angle of $\left[\alpha, \Omega^{1,3}\right]$, and $\gamma_{S^{1,3}}$ will be the vertex angle of $\left[\alpha, S^1\right]$.

Notation. Throughout the rest of this section, we adopt the convention that when indexes i, j, k, and l appear in subscripts and superscripts of the hypertree or angle notation, it is assumed that the indexes are drawn from [4], are distinct, and that the listed inequalities hold for all such choices of the indices.

By Theorem 5.2, we get these inequalities for each simplex in HT_4:

$$\alpha_{L_{k,l}^{i,j}} + \beta_{L_{k,l}^{i,j}} + \gamma_{L_{k,l}^{i,j}} \leqslant \pi \tag{5.16}$$

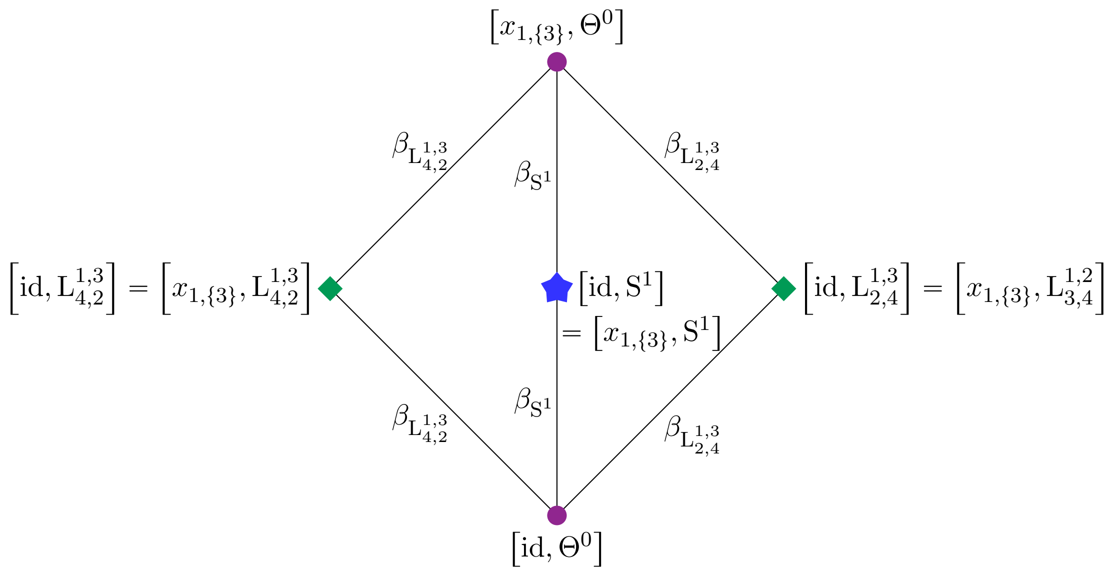

Figure 5.5: Another picture of the link of $\left[\mathrm{id}, \Omega^{1,3}\right]$ in K_4 with angle variables eqivariant with respect to the action of $\mathrm{Out}^0(W_4)$. For the $\mathrm{Out}(W_4)$ case, the picture is the same but we can ignore the indexing on the angles.

$$\alpha_{S^{i,j}} + \beta_{S^{i,j}} + \gamma_{S^{i,j}} \leqslant \pi \tag{5.17}$$

Now we need to re-examine injective loops in the links of vertices in K_4 to find appropriate inequalities. All of the links of vertices look identical to the links is Section 5.1 except that the angle labels now have (possibly different) indices. These indices are determined by the indices of the adjacent hypertrees but not the $\mathrm{Out}^0(W_4)$ label. See Figure 5.5.

$$\beta_{L^{i,j}_{k,l}} + \beta_{L^{i,j}_{k,l}} + \beta_{L^{i,j}_{l,k}} + \beta_{L^{i,j}_{l,k}} \geqslant 2\pi \tag{5.18}$$
$$\text{i.e., } \beta_{L^{i,j}_{k,l}} + \beta_{L^{i,j}_{l,k}} \geqslant \pi$$

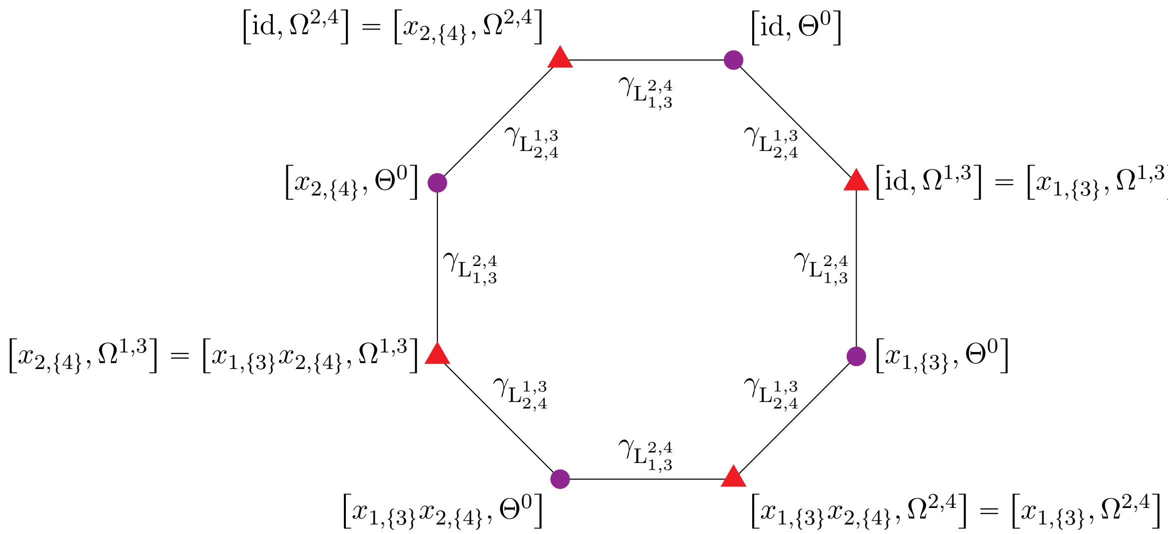

Figure 5.6: Another picture of the link of $\left[\mathrm{id}, \mathrm{L}^{1,3}_{2,4}\right]$ in K_4 with angle variables eqivariant with respect to the action of $\mathrm{Out}^0(W_4)$. For the $\mathrm{Out}(W_4)$ case, the picture is the same but we can ignore the indexing on the angles.

$$\beta_{L^{i,j}_{k,l}} + \beta_{L^{i,j}_{k,l}} + \beta_{S^{i,j}} + \beta_{S^{i,j}} \geqslant 2\pi \tag{5.19}$$

$$\text{i.e., } \beta_{L^{i,j}_{k,l}} + \beta_{S^{i,j}} \geqslant \pi$$

Notice that $\beta_{L^{i,j}_{l,k}} + \beta_{S^{i,j}} \geqslant \pi$, which is also an inequality derivable from that link, is included in Inequality (5.19) since our notation implicitly quantifies over the different possibilities for k and l.

We continue to examine injective loops in the links of vertices.

See Figure 5.6.

$$4\gamma_{L^{i,j}_{k,l}} + 4\gamma_{L^{k,l}_{i,j}} \geqslant 2\pi \tag{5.20}$$

$$\text{i.e., } \gamma_{L^{i,j}_{k,l}} + \gamma_{L^{k,l}_{i,j}} \geqslant \frac{\pi}{2}$$

See Figure 5.7.

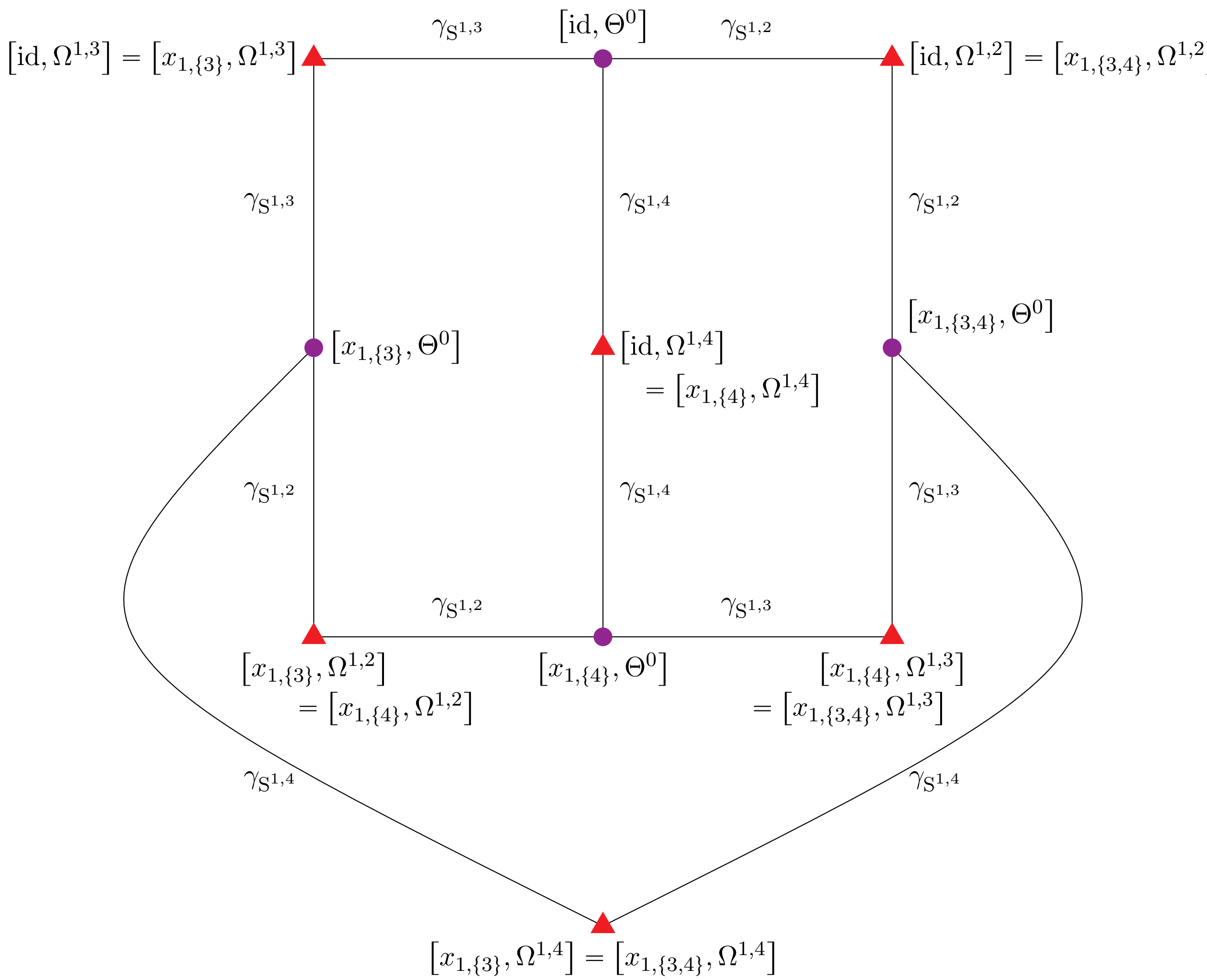

Figure 5.7: Another picture of the link of $[\mathrm{id}, \mathrm{S}^1]$ in K_4 with angle variables eqivariant with respect to the action of $\mathrm{Out}^0(W_4)$. For the $\mathrm{Out}(W_4)$ case, the picture is the same but we can ignore the indexing on the angles.

$$2\gamma_{S^{i,j}} + 2\gamma_{S^{i,k}} + 2\gamma_{S^{i,l}} \geqslant 2\pi \tag{5.21}$$

$$\text{i.e., } \gamma_{S^{i,j}} + \gamma_{S^{i,k}} + \gamma_{S^{i,l}} \geqslant \pi$$

See Figure 5.8.

$$\begin{aligned}\alpha_{L^{i,j}_{k,l}} &+ \alpha_{L^{k,l}_{i,j}} + \alpha_{L^{k,l}_{j,i}} + \alpha_{L^{j,i}_{k,l}} \\ &+ \alpha_{L^{j,i}_{l,k}} + \alpha_{L^{l,k}_{j,i}} + \alpha_{L^{l,k}_{i,j}} + \alpha_{L^{i,j}_{l,k}} \geqslant 2\pi\end{aligned} \tag{5.22}$$

$$\begin{aligned}\alpha_{L^{i,j}_{k,l}} &+ \alpha_{L^{k,l}_{i,j}} + \alpha_{S^{k,l}} + \alpha_{S^{k,j}} \\ &+ \alpha_{L^{k,j}_{i,l}} + \alpha_{L^{i,l}_{k,j}} + \alpha_{S^{i,l}} + \alpha_{S^{i,j}} \geqslant 2\pi\end{aligned} \tag{5.23}$$

On its own, the system of inequalities (5.16) - (5.23) is too complicated to try to solve by hand. However, we can exploit an additional unused symmetry, not of the metric space K_4, but of the inequalities themselves, namely that the system is invariant under the action of Σ_4 that permutes the labels in the subscripts. In fact, we have been implicitly using this symmetry to avoid the explicit quantification over $i, j, k, l \in [4]$ in the different classes of inequalities.

So now we explicitly note that Σ_4 acts on the set of 108 angles given in Definition 5.10 as follows. For any $\sigma \in \Sigma_4$:

$$\sigma \cdot \alpha_{L^{i,j}_{k,l}} = \alpha_{L^{\sigma(i),\sigma(j)}_{\sigma(k),\sigma(l)}} \quad \sigma \cdot \beta_{L^{i,j}_{k,l}} = \beta_{L^{\sigma(i),\sigma(j)}_{\sigma(k),\sigma(l)}} \quad \sigma \cdot \gamma_{L^{i,j}_{k,l}} = \gamma_{L^{\sigma(i),\sigma(j)}_{\sigma(k),\sigma(l)}}$$

$$\sigma \cdot \alpha_{S^{i,j}} = \alpha_{S^{\sigma(i),\sigma(j)}} \quad \sigma \cdot \beta_{S^{i,j}} = \beta_{S^{\sigma(i),\sigma(j)}} \quad \sigma \cdot \gamma_{S^{i,j}} = \gamma_{S^{\sigma(i),\sigma(j)}}$$

Definition 5.11. Given the angles defined in Definition 5.10, we define the following

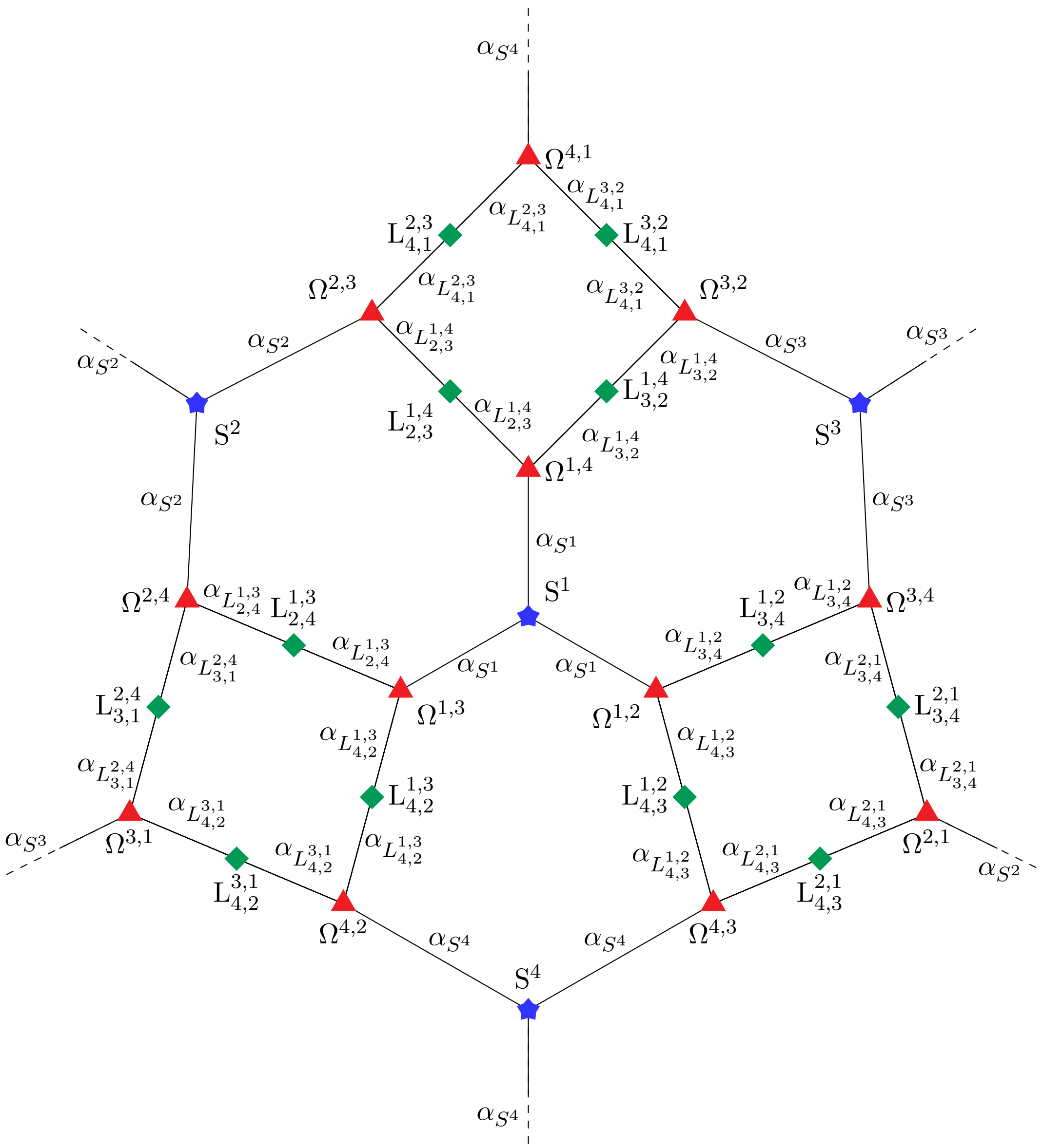

Figure 5.8: Another picture of the link of $\left[\mathrm{id}, \Theta^0\right]$ in K_4 with angle variables eqivariant with respect to the action of $\mathrm{Out}^0(W_4)$. For the $\mathrm{Out}(W_4)$ case, the picture is the same but we can ignore the indexing on the angles.

six *average angles.*

$$\alpha_L := \tfrac{1}{24} \sum_{\sigma\in\Sigma_4} \sigma \cdot \alpha_{L^{1,2}_{3,4}} \quad \alpha_S := \tfrac{1}{24} \sum_{\sigma\in\Sigma_4} \sigma \cdot \alpha_{S^{1,2}}$$
$$\beta_L := \tfrac{1}{24} \sum_{\sigma\in\Sigma_4} \sigma \cdot \beta_{L^{1,2}_{3,4}} \quad \beta_S := \tfrac{1}{24} \sum_{\sigma\in\Sigma_4} \sigma \cdot \beta_{S^{1,2}}$$
$$\gamma_L := \tfrac{1}{24} \sum_{\sigma\in\Sigma_4} \sigma \cdot \gamma_{L^{1,2}_{3,4}} \quad \gamma_S := \tfrac{1}{24} \sum_{\sigma\in\Sigma_4} \sigma \cdot \gamma_{S^{1,2}}$$

Note that some angles might appear more than once in these sums as Σ_4 does not act freely on the set of angles. Also, notice that since Theorem 5.2 implies that each angle from Definition 5.10 is in $(0, \pi]$, then the new average angles in Definition 5.11 are also in $(0, \pi]$, since $|\Sigma_4| = 24$.

We can now prove Theorem 5.9:

Proof of Theorem 5.9. For each class of Inequalities (5.16) - (5.23), we take one instance of the inequality for each of the 24 possible assignments of distinct i, j, k, l from $[4] = \{1, 2, 3, 4\}$, and then add the instances together.

For instance, consider a particular instance of Inequality (5.16):

$$\alpha_{L^{1,2}_{3,4}} + \beta_{L^{1,2}_{3,4}} + \gamma_{L^{1,2}_{3,4}} \leqslant \pi$$

Each of the other 23 instances of this inequality is obtained by permuting the labels, i.e., by acting on each variable in the inequality by $\sigma \in \Sigma_4$. When we add the 24 instances of this inequality together, we get

$$\begin{aligned}
\sum_{\sigma\in\Sigma_4} \sigma \cdot \left(\alpha_{L^{1,2}_{3,4}} + \beta_{L^{1,2}_{3,4}} + \gamma_{L^{1,2}_{3,4}}\right) &\leqslant \sum_{\sigma\in\Sigma_4} \pi \\
\implies \sum_{\sigma\in\Sigma_4} \sigma \cdot \alpha_{L^{1,2}_{3,4}} + \sum_{\sigma\in\Sigma_4} \sigma \cdot \beta_{L^{1,2}_{3,4}} + \sum_{\sigma\in\Sigma_4} \sigma \cdot \gamma_{L^{1,2}_{3,4}} &\leqslant \sum_{\sigma\in\Sigma_4} \pi \\
\implies 24\alpha_L + 24\beta_L + 24\gamma_L &\leqslant 24\pi \\
\implies \alpha_L + \beta_L + \gamma_L &\leqslant \pi,
\end{aligned}$$

i.e., we recover Inequality (5.1).

In fact, this is general. For each class of inequalities (5.16) - (5.23), adding together all 24 instances of them indexed by the action of Σ_4 and then dividing by 24 implies the Inequalities (5.1) - (5.8) in the six average angles variables $\{\alpha_L, \beta_L, \gamma_L, \alpha_S, \beta_S, \gamma_S\}$.

So assuming the existence of an $\mathrm{Out}^0(W_4)$-equivariant metric on the M_κ-simplicial complex K_4 (for $\kappa \leqslant 0$) allowed us to derive six real numbers $\alpha_L, \beta_L, \gamma_L, \alpha_S, \beta_S, \gamma_S \in (0, \pi]$ that simultaneously satisfy Inequalities (5.1) - (5.8). But the proof of Theorem 5.8 shows that no such six numbers exist. This completes the proof.

$\square$

Note that these results do not immediately extend to K_n for $n \geqslant 5$ since there is no analogue of Theorem 5.2 for higher dimensional M_κ-polyhedral complexes. So the analogous theorem to Theorem 5.8 for $n \geqslant 5$ needs a different approach.

5.3 The $\mathrm{Out}^0(W_n)$ and $\mathrm{Out}(W_n)$ Case

To extend the results of this section to a general $n \geqslant 5$, we first notice that K_n has K_4 as a full subcomplex which is left invariant by $\mathrm{Out}^0(W_4)$ sitting as a subgroup in $\mathrm{Out}^0(W_n)$. We then wish to prove the following theorem.

Theorem 5.12. *There does not exist an* $\mathrm{Out}^0(W_n)$*-equivariant (or* $\mathrm{Out}(W_n)$*-equivariant) piecewise Euclidean (or piecewise hyperbolic)* $\mathrm{CAT}(0)$ $(\mathrm{CAT}(-1))$ *metric on* K_n *for* $n \geqslant 4$.

Note that Theorem 5.12 suffices for both the $\mathrm{Out}^0(W_n)$ as well as the $\mathrm{Out}(W_n)$ case, since if there were an $\mathrm{Out}(W_n)$-equivariant piecewise Euclidean (or piecewise hyperbolic) $\mathrm{CAT}(0)$ $(\mathrm{CAT}(-1))$ metric on K_n, then it would be $\mathrm{Out}^0(W_n)$-equivariant as well.

By Theorem 4.1, there are higher dimensional analogues to Definition 4.8.

Definition 5.13. Consider the subset $F = \{5, 6, \ldots, n\} \subset [n]$. Denote the partial conjugation generators of $\mathrm{Out}^0(W_n)$ by the usual $x_{i,D}$, and let the partial conjugation generators of $\mathrm{Out}^0(W_4)$ now be denoted as $y_{i,D}$.

Let $\varphi_{5^+} : \mathrm{Out}^0(W_n) \to \mathrm{Out}^0(W_4)$ be defined as

$$\varphi_{5^+}(x_{i,D}) := \begin{cases} \mathrm{id} & \text{if } i \geqslant 5,\ D \subset F, \text{ or } D^c \subset F \\ y_{i,D \setminus F} & \text{otherwise.} \end{cases}$$

Remark 5.14. By checking that each of the relation families (R1), (R2), and (R3) are preserved under the operations of either removing F from D or by sending certain generators to the identity, we can see that each map φ_{5^+} is a surjective homomorphism onto $\mathrm{Out}^0(W_4)$.

Furthermore, consider the map: $\psi_{5^+} : \mathrm{Out}^0(W_4) \to \mathrm{Out}^0(W_n)$ which is defined as $\psi_{5^+}(y_{i,D}) := x_{i,D}$. For each $y_{i,D}$, $x_{i,D} = \psi_{5^+}(y_{i,D})$ trivially satifies relation families (R1) and (R2), and since $F \subset D^c$ for all images of the map, the disjointness conditions in (R3) remain satisfied as well. (It's critically important here that none of the three disjointness conditions is $\widetilde{D_i^c} \cap \widetilde{D_j^c} = \varnothing$). Thus, ψ_{5^+} is a section of φ_{5^+}, and so $\mathrm{Out}^0(W_n)$ splits as a semidirect product. In particular, it contains $\mathrm{Out}^0(W_4)$ as a subgroup, which by abuse of notation we also denote by $\mathrm{Out}^0(W_4)$.

Now, we embed $\mathcal{HT}_4$ into $\mathcal{HT}_n$.

Definition 5.15. Let $\Theta \in \mathcal{HT}_4$ be a hypertree. Then to Θ, associate a hypertree $\widetilde{\Theta} \in \mathcal{HT}_n$, which is defined to be the hypertree on $[n]$ with the same hyperedges as Θ as well as the additional hyperedges $\{\{1, f\} \mid f \in F\} = \{\{1,5\}, \{1,6\}, \ldots, \{1,n\}\}\}$, i.e., put each remaining vertex in a hyperedge with the vertex 1. Denote the subset of $\mathcal{HT}_n$ given by all such $\widetilde{\Theta}$ as $\widetilde{\mathcal{HT}_4}$.

Remark 5.16. By adding or removing these hyperedges, we see that there is a bijection between $\mathcal{HT}_4$ and $\widetilde{\mathcal{HT}_4}$, this bijection respects folding, and so it is order-preserving from $(\mathcal{HT}_4, \leqslant)$ to $(\mathcal{HT}_n, \leqslant)$. Thus, it is also a simplicial automorphism from HT_4 into HT_n.

In order to see how this subcomplex sits in K_n, we need to see which partial conjugations are carried by each hypertree. For each $\widetilde{\Theta} \in \widetilde{\mathcal{HT}_4}$, if $y_{i,D}$ is carried by Θ, then $x_{i,D}$ is carried by $\widetilde{\Theta}$, since for $i \neq 1$, $1 \in D^c$, and for $i = 1$, F is its own union of connected components of $\widetilde{\Theta}\backslash\{i\}$. This also shows that $\widetilde{\Theta}$ carries $x_{1,F'}$ for any $F' \subseteq F$, which thus commutes with all the other carried partial conjugations by Theorem 2.9. If Θ is at height h and so has 2^h carried automorphisms, then $\widetilde{\Theta}$, with its $n-4$ additional hyperedges, is at height $h+n-4$, and so the 2^{h+n-4} automorphisms given by $\{x_{i,D}x_{1,F'} \mid x_{i,D} \in \mathrm{Out}^0(W_4) \text{ carried by } \Theta, F' \subseteq F\}$ exhaust all the automorphisms carried by $\widetilde{\Theta}$.

Next, we embed K_4 into K_n. Consider the subgroup $G = \langle x_{1,\{f\}} \mid f \in F\rangle \subset \mathrm{Out}^0(W_n)$, which is a product of $n-4$ commuting non-conjugate involutions, and so is isomorphic to $\mathbb{Z}_2^{n-4}$. G is thus a finite group acting on K_n by simplicial automorphisms.

Theorem 5.17. *The fixed point set of G in K_n is the set of simplices spanned by $[\alpha, \widetilde{\Theta}]$, where $\alpha \in \mathrm{Out}^0(W_4)$ and $\widetilde{\Theta} \in \widetilde{\mathcal{HT}_4}$. This set is simplicially isomorphic with K_4.*

Proof. If a simplicial automorphism fixes a simplex pointwise, then it fixes each vertex in that simplex. Conversely, since K_n is a flag complex, any simplicial automorphism that fixes each of the vertices in a simplex will fix the simplex they span.

So suppose that $[\alpha, \Lambda]$ is a vertex of K_n that is fixed by every element of G, i.e., for each subset $F' \subset F$,

$$x_{1,F'} \cdot [\alpha, \Lambda] = [x_{1,F'}\alpha, \Lambda].$$

By the definition of K_n, this happens precisely when $\alpha^{-1}x_{1,F'}\alpha$ is carried by Λ. But the automorphisms carried by a hypertree are products of pairwise commuting partial conjugations from $\mathcal{P}^0$ (by Theorem 2.9), and these commuting products all project injectively into the abelianization of $\mathrm{Out}^0(W_n)$. Thus, $\alpha^{-1}x_{1,F'}\alpha$ must be equal to $x_{1,F'}$, i.e., α commutes with every $x_{1,F'}$.

Additionally, this implies that Λ carries $x_{1,\{f\}}$ for each $f \in F$. Thus, $\{f\}$ must be a connected component of $\Lambda\backslash\{1\}$, i.e., $\{1, f\}$ is a hyperedge of Λ for each $f \in F$. Therefore, $\Lambda = \widetilde{\Theta}$ for some $\Theta \in \widetilde{\mathcal{HT}_4}$.

Now, since α commutes with every $x_{1,F'}$, we claim that $\alpha \in \mathrm{Out}^0(W_4) \times G$. We will induct on the word length of α.

If $\alpha = x_{i,D}$, then we know that $1 \notin D$ (by our naming convention for D). If $i \in F$, then $x_{i,D}$ will not commute with $x_{1,\{i\}}$ by Lemma 2.8, which contradicts our assumption. So $i \notin F$. If $i \neq 1$, then since $1 \notin D$, $i \notin F$, for $x_{i,D}$ to commute with $x_{1,F}$, Lemma 2.8 forces $D \cap F = \varnothing$, and so $x_{i,D} \in \mathrm{Out}^0(W_4)$. If $i = 1$, then $x_{i,D} = x_{1,D}x_{1,F'}$, where $D' \cap F = \varnothing$ and $F' \subset F$ (either might be empty). In that case, $x_{i,D}$ is again in $\mathrm{Out}^0(W_4) \times G$.

Now, inductively assume that $\alpha = \alpha' x_{i,D}$, where $\alpha' = \beta x_{1,F''} \in \mathrm{Out}^0(W_4) \times G$. Then $x_{i,D} = \alpha^{-1}\beta x_{1,F''}$ also commutes with every $x_{1,F'}$. But then by the base case, $x_{i,D} \in \mathrm{Out}^0(W_4) \times G$, and thus so is α.

Thus, we now have that every vertex in the fixed point set of G is of the form $[\beta x_{1,F'}, \widetilde{\Theta}]$ for $\beta \in \mathrm{Out}^0(W_4)$ and $F' \subset F$. But since $\beta^{-1}\beta x_{1,F'} = x_{1,F'}$ is carried by each $\widetilde{\Theta}$, we have that in K_n, $[\beta x_{1,F'}, \widetilde{\Theta}] = [\beta, \widetilde{\Theta}]$, and so the fixed point set of G is generated by $[\mathrm{Out}^0(W_4), \widetilde{\mathcal{HT}_4}]$. Since the carrying partial order of $\widetilde{\mathcal{HT}_4}$ is isomorphic to $\mathcal{HT}_4$, we have that the fixed point set of G is a simplicially isomorphic copy of K_4 which admits the same action of $\mathrm{Out}^0(W_4)$. By abuse of notation, we call this subcomplex K_4. $\square$

Now we can prove the main theorem of the section.

Proof of Theorem 5.12. Suppose that for $\kappa \leqslant 0$, there existed an $\mathrm{Out}^0(W_n)$-equivariant $\mathrm{CAT}(\kappa)$ M_κ-simplicial metric on K_n. Since there are only finitely many shapes, the metric is complete (Theorem 7.50 in [3]). Then the action by $\mathrm{Out}^0(W_n)$ would be by isometries, and so G is a finite group of isometries of the complete CAT(0) space K_n, and so the fixed point set of G, namely $\mathrm{K}_4 \subset \mathrm{K}_n$ by Theorem 5.17, is a convex subspace of K_n (by Corollary 2.8 in [3]), and so would inherit a CAT(0) M_κ-simplicial metric. Since the metric on K_n is $\mathrm{Out}^0(W_n)$-equivariant, and since $\mathrm{Out}^0(W_4)$ leaves K_4 invariant, the induced metric on K_4 is $\mathrm{Out}^0(W_4)$-equivariant as well. But this contradicts Theorem 5.9. $\square$

Chapter 6

Future Research

From Chapter 4, we know that $\mathrm{Out}^0(W_n)$ is not a right-angled Coxeter group, and by Chapter 5, we know that its natural combinatorial model K_n cannot show it to be CAT(0). So now we are left with two options.

1. If $\mathrm{Out}^0(W_n)$ is CAT(0), then we will need to investigate a different geometric model space in order to prove it.

2. If $\mathrm{Out}^0(W_n)$ is *not* CAT(0), then perhaps that can be detected with known invariants of CAT(0) geometry.

Both options are interesting areas for future research. In particular, all CAT(0) groups and CAT(0) metric spaces are known to satisfy an at most quadratic isoperimetric inequality [3]. Since isoperimetric inequality is a quasi-isometry invariant, we can study it either directly in the group $\mathrm{Out}^0(W_n)$ or in the model K_n by endowing it with any $\mathrm{Out}^0(W_n)$-equivariant metric, such as by declaring every edge to have length 1 and then taking the induced path metric. This turns all simplices into equilateral Euclidean simplices. This metric won't be CAT(0) as Theorem 5.12 promises, but it is still quasi-isometric to $\mathrm{Out}^0(W_n)$ via the action, and so will have the same optimal class of isoperimetric inequalities. Thus, we wish to in the future compute the isoperimetric inequality of either K_n or else $\mathrm{Out}^0(W_n)$ directly by more combinatorial and geometric methods. In particular, we will need to find a normal form for $\mathrm{Out}^0(W_n)$ and calculate its algorithmic and combinatorial group theoretic properties.

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