INTEGRATING ALGEBRA AND PROOF IN HIGH SCHOOL

STUDENTS' WORK WITH ALGEBRAIC EXPRESSIONS INVOLVING VARIABLES WHEN PROVING

A qualifying paper

submitted by

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In partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in

Mathematics Education

TUFTS UNIVERSITY

May 2008

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Abstract

Nowadays, the learning and teaching of algebra is a central issue in the mathematics education research agenda (Gutiérrez & Boero, 2006; Stacey, Chick, & Kendal, 2004). In addition, students' obstacles when learning algebra have been largely documented (Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984; Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 2001; Demana & Leitzel, 1988; Filloy & Rojano, 1989; Kieran, 1981, 1985, 1989; Kuchemann, 1981; MacGregor, 1996; Steinberg, Sleeman, & Ktorza, 1990). Complementary, previous research claims that students' performance producing proofs using algebra is poor (Healy and Hoyles, 2000), and that students' opportunities to produce their own conjectures and proofs in school algebra are scarce (Friendlander & Hershkowitz, 1997; Harel & Sowder, 1998). The above sets the stage for the need to conduct more research in these two areas: algebra and proof.

The overarching research question of the study described in this paper is: *what are the consequences of an integrated approach to algebra and proof on students' mathematical knowledge while they work through a didactical sequence (i.e., the "Calendar Sequence")?* In particular, the goal of this paper is to report on the challenges that students faced in their work with variables, and equivalent expressions while engaged in producing and proving conjectures, and how these challenges were overcome. Previous studies on algebra and proof (Barallobres, 2004; Bell, 1993) are scant nonetheless promising in regards to students' production of proofs using algebra in an integrated approach. I claim that the results presented in this paper provide promising evidence that an integrated approach towards algebra and proof, such as that implemented in the Calendar Sequence, has a positive impact on students' use of algebra as a tool to prove. The following were the challenges identified that students faced and that were overcome

(a) the use of algebra to prove in contrast with a finite non-exhaustive set of numeric examples;
(b) the use of a single variable that could capture all cases at the same time;
(c) the number of dependent and independent variables to include in their expressions;
(d) how to set up relations among variables;
(e) how to obtain a simpler expression from a more complex algebraic expression;
(f) the use of conventions of algebra such as the use of parenthesis;
(g) the use of properties such as the use of distributive property.

In this study, a group of 9 high school students (9th and 10th graders) participated in fifteen one-hour-long lessons carried out by the author of this paper at their school in the Boston area, Massachusetts, United States of America.

Integrating Algebra and Proof in High School:

The Case of the Calendar Sequence

Introduction

Nowadays the learning and teaching of algebra is a central issue in the mathematics education research agenda (Gutiérrez & Boero, 2006; Stacey, Chick, & Kendal, 2004). Many studies have shown students' difficulties learning algebra, and its relationship to social and economic consequences (Chazan, 1996; Moses, 2001). In addition, students' obstacles when learning algebra have been largely documented (Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984; Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 2001; Demana & Leitzel, 1988; Filloy & Rojano, 1989; Kieran, 1981, 1985, 1989; Kuchemann, 1981; MacGregor, 1996; Steinberg, Sleeman, & Ktorza, 1990).

Healy and Hoyles' (2000) study shows that English students' performance producing proofs through the use of algebra is poor, even though students are aware of their teachers' expectation regarding the use of algebra. In addition, it is well known that the opportunities for students to produce their own conjectures and proofs in school algebra are scarce (Friendlander & Hershkowitz, 1997; Harel & Sowder, 1998). This sets the stage for the need to conduct more research in these two areas: algebra and proof.

The overarching research question of the study reported in this paper is: *what are the consequences of an integrated approach to algebra and proof on students' mathematical knowledge while they work through a didactical sequence (i.e., the "Calendar Sequence")?* Specifically, the goal of this paper is to report on the challenges that students faced in their work with variables and equivalent expressions while they were engaged in producing and proving conjectures, and how these challenges were overcome. Given the evidence provided by

Barallobres' (2004) work regarding students' learning of proof and proof construction using algebra, my hypothesis is that an integrated approach can provide meaning to students' learning of both algebra and proof. Another underlying hypothesis is that the technical work inherent, for instance, in the use of the distributive property, grouping, and factoring, has an epistemic value and is not simply an obstacle in students' learning of concepts (Artigue, 2003; Chevallard, 1992; Kieran, 2004; Lagrange, 2000, 2003).

In this investigation, these questions and hypotheses were explored through students' work on the "Calendar Sequence," (see the Appendix A for the complete Calendar Sequence) a *didactical sequence* (Brousseau, 1997) of twenty problems that was *engineered* (Artigue, 1988, 1994; Artigue & Perrin-Glorian, 1991; Douady, 1997) by the author of this paper as a way to promote a specific algebraic functioning: the production of new knowledge about a *system*[1] (Chevallard, 1985, 1989), such as the calendar, through the use of algebra to create a *model* (Chevallard, 1985, 1989) of the system. One of the assumptions underlying this research is that the use of variables, the expression of relationships among variables, and algebraic transformations to obtain equivalent expressions can foster the production of new knowledge about the system – or modeling process in Chevallard's (1985, 1989) terms. The context for the Calendar Sequence is the standard calendar or a variation of the standard calendar (e.g., an 8-day week).

[1] I follow Chevallard's (1989, p. 53) definitions of system, model, and process of modeling:

> First we will introduce a simplified scheme that supposes the existence of two entities: a **system**, mathematical or non-mathematical, and a (mathematical) **model** of that system. The **modeling process** consists schematically of three stages: (1) We define the system that we want to study by identifying the **pertinent** aspects in relation to the study of the system that we want to carry out, in other words, the set of **variables** through which we decide to cut off from reality the domain to be studied. . We name these variables with the letters x, y, w, a, b, c, etc., we will come back to the question about the use of the variables later. (2) Now we build a model by establishing a certain number of relations R, R', R'', etc., among the variables chosen in the first stage, the model of the systems to study is the **set of these relations**. (3) We 'work' the model obtained through stages 1-2, with the goal of producing **knowledge** of the studied system, knowledge that manifests itself by new relations among the variables of the system. (p. 53. Bolded text in original)

Regarding students' activities, throughout the didactical sequence they were prompted to analyze a problem-situation (from the Calendar Sequence), to produce a conjecture, and to use algebra to model the situation, to prove the conjecture's truth-value, and, most importantly, to understand the truth-value of the conjecture (Arsac et al., 1992b; Balacheff, 1987; Barallobres, 2004). Previously, Bell (1993) has highlighted the potential of using the regular calendar as a fruitful context for the learning of algebra by briefly showing some examples of students' work in algebra involving variables, algebraic expressions, and the use of algebra for proving (Bell's work is discussed extensively in the section *Analysis of the didactical sequence* in this paper). In Bell's (1993) words " [problems related with the calendar] have proved to be good ways of getting pupils to use algebraic language in situations where it forms a natural means of communication. Note that opportunities for checking, and understanding the possibility of relations being true always, sometimes, or never, are built also" (p. 52).

Theoretical Framework

Didactics of Mathematics

The *Theory of Didactical Situations* (Brousseau, 1997), *Theory of Didactical Transposition* (Chevallard, 1985, 1989, 1989-1990, 1992) and *Didactical Engineering* (Artigue, 1988, 1994; Artigue & Perrin-Glorian, 1991) are the theoretical tools that were used to design the study described in this dissertation.

I adopt the framework of Didactics of Mathematics (corresponds to the theories mentioned in the paragraph above) for designing didactical sequences to teach mathematical concepts, and for analyzing events related to the teaching and learning of mathematics. One of the principles of this field, according to Chevallard, Bosch, & Gascón (1997), consists in assuming that the comprehension and explanation of a mathematical-didactical event cannot be reduced to learners' psychological, motivational, and attitudinal factors. In general terms, Chevallard invites us to look beyond these, and focus as well on the mathematical knowledge involved. In other words, from the perspective of Didactics of Mathematics the theorization of the didactical phenomena is carried out through a certain model of the mathematical knowledge taught. Chevallard's approach in the Theory of Didactical Transposition (1992; , 1997) focuses on a system that includes the institutions at the source of the knowledge one aims to teach and the institutions targeted by this teaching. By institution Chevallard means the educational organization (level, location, etc.) where the teaching and learning takes place. Artigue (1994) provides a succinct explanation regarding how the didactical transposition theory is implemented:

> This is done by questioning the constitution and life of this knowledge, while remaining particularly attentive to the economy and ecology of the knowledge to be taught. One questions the possible viability of the content one whishes to promote while considering the laws that govern the functioning of the teaching system. One tries to foresee the deformations it is likely to undergo; one tries to ensure that the object can live and therefore develop within the teaching system without too drastically changing its nature or becoming corrupted. (p. 28)

Within the field of Didactics of Mathematics, I focus on *The Theory of Didactical Situations* (TDS) (Brousseau, 1986, 1997, 1999) as an important source. In TDS,

> The student learns by adaptation to a milieu which is a factor of contradictions, difficulties, unbalances, similar to the way human society does. This knowledge, product of the student's adaptation, manifests itself by the new answers which are proof of the learning. (Brousseau, 1986, p. 11)

In Chevallard et al.'s (1997) words, "to learn mathematical knowledge means to adapt to a specific *adidactical*[2] situation of that knowledge" (p. 216).

Chevallard et al. (1997) consider that knowledge (knowing[3]) produced by adaptation to a *milieu* is insufficient in and of itself, since what learners have to appropriate and comprehend (apprehend) is cultural knowledge. So, teachers' work is of a historical and cultural nature that totally differs from the nature of learners' work. The work of giving mathematics a cultural status and of integrating it within a whole system of knowledge is within the realm of the teacher.

In the TDS there are three phases in any didactical situation, according to the function of knowledge and according to the different features of the subjects' activity (Bessot, 2003):

> *Action situation*: the student elaborates implicit knowledge as a mode of action on the milieu (through actions).
>
> *Formulation situation*: the student makes explicit the implicit model of action.
>
> *Validation situation:* the empirical validation from the milieu becomes insufficient; the student must elaborate intellectual proofs (see below for an explanation of different types of proofs; Balacheff, (1987) in order to convince an opponent and herself.

These three phases were included in the design of the present study in order to allow mathematical objects to function in different ways: according to the knowledge status (knowledge to solve a calculation, knowledge used to know how to do something, knowledge to

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[2] "Une situation adidactique est une situation à finalité didactique (c'est-à-dire organisée par l'enseignant) où le sujet agit *comme si* la situation était non didactique (le sujet répond indépendamment des attentes de l'enseignant): *il y a alors dans la situation didactique des elements qui forment un milieu adidactique antagoniste de l'eleve*" (Bessot, 2003, p. 8). My translation: "An adidactical situation is a situation that has a didactical goal (meaning that it is organized by the teacher) where the subject behaves as if the situation were non-didactical (the subject responds independently of the teacher's expectations): now we have, within the didactical situation, the elements that conform an adidactical milieu antagonist to the student."

show why something is true or not). In the Calendar Sequence, I intended to promote these three ways of knowledge functioning. Considering Problem 1 from the Calendar Sequence (see Figure 1 below and Appendix A), for instance, the *action situation* is the first stage of the problem, where students have to find where to place the 2x2-calendar-square in order to obtain the biggest outcome. In this stage of the problem, students will "always" obtain the same outcome, which can be interpreted as contradictory with the explicit goal of the problem (i.e., to find the largest outcome). Student knowledge at this point indicates that there is no outcome variation regarding where the 2x2-calendar-square is placed. After this first stage, students have to communicate their findings to their classmates; this is the *formulation situation*. After this, the teacher asks, "why does the outcome always seem to be the same?" When the student engages in this task, it becomes necessary to produce an intellectual proof. This process of producing an intellectual proof involves the *validation situation.*

3 In French, there is a distinction between 'savoir' (knowledge) and 'connaissance' (knowing). 'Savoir' is mathematical knowledge as the set of mathematical objects that appears in the scientific domain of mathematics. 'Connaissance' is a version of this 'savoir' in the student, which is personalized, contextualized, and temporalized.

Problem 1

Part 1

Consider squares of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be $\begin{matrix}1 & 2\\ 8 & 9\end{matrix}$. These squares will be called 2x2 calendar squares. Calculate the difference between the products of the numbers in the extremes of the diagonals. Find the 2x2 calendar square that gives the biggest outcome. You may use any month of any year that you want.

Example:

JANUARY 2007						
S	M	T	W	T	F	S
	$\boxed{1}$	$\boxed{2}$	3	4	5	6
7	$\boxed{8}$	$\boxed{9}$	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

(1) 1 x 9 = 9

(2) 8 x 2 = 16

(3) 9 - 16 = -7

Part 2

Show and explain why the outcome is going to be -7 always.

Figure 1. Problem 1 from the Calendar Sequence.

At the heart of the validation phase is the production of proofs[4]. Balacheff (1987) differentiates between two types of proofs, the *pragmatic* and the *intellectual*. The *pragmatic proofs* are linked to the singularity of the event that constitutes them and are dependent on

[4] A *proof* is an *explanation* accepted by a given community at any given moment. Note that this usually differs from a *demonstration*. A demonstration is accepted by the mathematical community and is made up of a set of mathematical statements organized following established rules.

contingent material conditions. For example, a "proof" of a universal proposition with one numeric example or, in geometry, a "proof" using the particularities of the drawing. In Problem 1 (see Figure 1 and Appendix A), one way of proving the conjecture through the use of a pragmatic proof could be by using a set of non-exhaustive numeric examples to show that the outcome is always -7[5]. The *intellectual proofs* are detached from action; they are inscribed in relations and objects expressed through language and on calculations carried out with them. In Problem 1 (see Figure 1 and Appendix A), one way of proving the conjecture through the use of an intellectual proof could be to express the relations using algebraic notation and transforming the initial expression. From Balacheff's (1987) perspective, validation is the dialectic between proofs and refutations. Balacheff (1987) has studied the treatment of refutations within a geometric context. He has shown how the effect of a counterexample exceeds the mere refutation or change of the conjectures – in cases where this is possible. A counterexample can also lead someone to change definitions, to revise hypotheses, and to change the background knowledge or his/her rational base, or even to reject the counterexample itself.

While Balacheff (1987) developed a characterization of student proofs according to their use of knowledge, language, and personalization[6], Arsac et al.'s (1992b) goal was introducing students to deductive reasoning. In order to do this, Arsac et al. (1992b) designed a set of five didactical sequences proposing tasks where students have to deal with conjectures, proofs, quantifiers, and counterexamples. Arsac and his colleagues used the categorization of proofs proposed by Balacheff (1987) – pragmatic and intellectual — to analyze students' work. In their work, they showed that sometimes, when a student denies a counterexample, it might be because

[5] This issue is particularly important in the case of the calendar sequence since there are only a finite number of different 30 day months, 31-day months, 28 day months and 29-day months [7 in each instance]. Thus there are only 28 possible configurations of a one-month calendar sheet and it is perfectly possible and legitimate to prove the -7 conjecture by exhaustive demonstration (Judah Schwartz's personal communication).

it is taken by the student as a particular or exceptional case or as an exception to the rule. Arsac et al. (1992b) found that, for students, the presence of a counterexample was not enough to invalidate a universal statement; just one counterexample did not have enough strength to invalidate a statement. Piaget (1975) refers to this type of behavior as *alpha-behavior*:

> when a perturbation occurs that alters goal-directed behavior, or an attempt to predict or to explain, this perturbation gives rise to an attempt at compensation or even of suppression. […] We shall call this attempt to cancel perturbations *alpha-behavior*. (Piaget, 1975, p. 807)

Since students learn by adaptation to a situation, in their learning process they can produce answers that may seem wrong from an "expert" point of view. However, their answers will make sense, to some degree, within their own framework. From this perspective, learners' errors do not correspond to a lack of knowledge, but can relate to a way of producing knowledge that may have proved fruitful in past situations. Errors should be considered as sources of information about the way learners think about a particular mathematical object or within a certain mathematical area (Charnay, 1994). Vergnaud (1996), in his Theory of Conceptual Fields, proposes the concept of *theorem-in-action.* This turns out to be a very useful concept because it allows for the explicit differentiation and connection between student knowledge and knowledge from the expert's viewpoint. Vergnaud (1996) provides the following definition for theorems-in-action:

> A *theorem-in-action* [italics in original] is a proposition that is held to be true by the individual subject for a certain range of the situation variables. It follows from this definition that *the scope of validity of a theorem-in-action can be different from the real*

[6] Personalization is the degree to which a subject constructs a proof grounding it on her particular actions.

theorem, as science would see it [italics added]. It also follows that a theorem-in-action can be false. (p. 225)

Knowing mathematics is more than knowing definitions and theorems; it also involves recognizing the conditions in which to use the definitions and theorems as an instrument, to solve problems as well as to pose proper questions. Moreover, *knowing mathematics* involves the production of conjectures and their proof, the creation of models, symbolizations, languages, concepts, and theories (Chevallard, Bosch, & Gascón, 1997). Chevallard et al.'s (1997) position regarding knowing mathematics is aligned with Brousseau's (1997) three stages in the process of the teaching and learning of a mathematical concept (*action, formulation, and validation* situations). Since students' activity in the *action situation* involves the construction of implicit models, in the *formulation situation* students make explicit their models, and in the *validation situation* students produced proofs to justify their answers or solutions. Regarding the "Calendar Sequence," the problems in the sequence (see Appendix A) are an opportunity to create an algebraic model of the system, to recognize the use of equivalent expressions, to apply the distributive property, to produce conjectures about the behavior of the outcome, and to search for reasons that could explain this behavior.

As part of my design of the Calendar Sequence, I used the principles of *Didactical Engineering* (M. Artigue, 1988, 1994; M. Artigue & Perrin-Glorian, 1991; Douady, 1997), which is an operationalization of the TDS (Brousseau, 1997) and of the *Theory of Didactical Transposition* (Chevallard, 1985, 1989). The expression "Didactical Engineering" emerged within the field of didactics of mathematics in France in the early 1980s in order to label a type of work that is comparable to the work of an engineer. While engineers base their work on the scientific knowledge of their field and accept the control of theory (it allows them to create a model of the situation and work on it), they are forced to work with more complex objects than the refined objects of science and therefore to manage problems that science is unwilling or not yet able to tackle. Didactical Engineering approaches two issues: (1) the relationship between research and action within the system of teaching, and (2) the place assigned within research methodologies to "didactical performances" in class. Artigue (1994) points out that "the expression [Didactical Engineering] has become polysemous, designating both productions for teaching derived from or based on research and a specific research methodology based on classroom experimentations" (p. 30). Schematically, the methodology of Didactical Engineering includes the following stages (Douady, 1999): (1) Choose a teaching object in the current program; (2) Place the mathematical context in relation to the teaching tradition; (3) Develop hypotheses about the difficulties of the students and set the basis for a didactical engineering; (4) Develop an engineering, proceed to the a-priori analysis; (5) Implement the sequence and make an a-posteriori analysis of the collected data; (6) Reproduce the implementation; (7) Test the knowledge students were supposed to acquire in questions for which students' knowledge are adapted tools; and (8) Compare the output of the students and their skills with your a-priori expectations, and conclude with the relevance of the didactical hypotheses.

Mathematical activity from a modeling perspective

Since my goal in this study was to study the processes of learning and teaching within the field of mathematics, I will make explicit what I understand by mathematical activity within an educational institution, and what features allow us to define an activity as mathematical.

I agree with Chevallard, Bosch, and Gascón's position (1997) that an essential characteristic of a mathematical activity consists in building a (mathematical) model about systems (intra-mathematical or extra-mathematical contexts) to be studied, to use it, and to produce an interpretation of the obtained results. In others words, the mathematical activity can be characterized as that of making (mathematical) models of (intra or extra- mathematical) systems. But what does it mean to build a mathematical model? Chevallard and his colleagues underline three aspects concerning this activity: the routine utilization of preexisting mathematical models; the learning of models as well as the way of using them; and the creation of mathematical knowledge. For example, consider the system formed by two right triangles (ignoring their position in the plane) as shown in Figure 2. This system is well known in the theory of geometry. We can build a metric model from the system where: a and b are the measures of the sides; d is the measure of the diagonal; the measure of the area is S; u and v are the measures of the angles formed by the sides and diagonal. Building this mathematical model[7] we have the following relationships among the variables: $S=ab$, $d^2=a^2+b^2$, $u=\mathrm{arctg}(b/a)$, $v=\mathrm{arctg}(a/b)$.

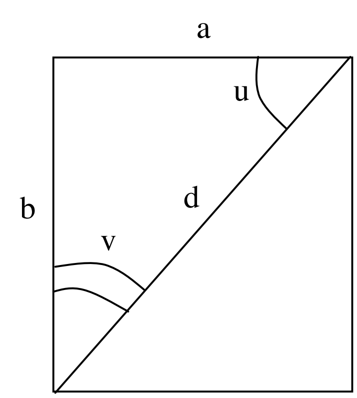

Figure 2. System formed by two right triangles

Very simple mathematical work on the model provides us with new information: the system that initially was parameterized by the measures of the sides a and b can be parameterized also in terms of d and u, where $a=d \cos(u)$ and $b=d \sin(u)$. In the example just shown, the

[7] The relations invoked in this model constitute a set of possible relations among the mentioned variables given the fact that the construction of a model is goal oriented.

model is constructed on a mathematical system; in this case, we say that the context is intra-mathematical. In cases where the system is formed by non-mathematical objects we will refer to the context as extra-mathematical.

Inherent to the discussion of what is mathematical activity is the discussion of what is mathematics. Following Nickson's study (1994), two different conceptions can be distinguished in terms of the nature of mathematics as a discipline: on the one hand the "formalist" tradition, and on the other hand, the "growth and change" view of mathematics. Within the "formalist" view, the foundations of mathematical knowledge are in no way social, but lie outside human action; mathematics is waiting "out there" to be discovered. In addition, from this tradition, "mathematics is considered to consist of immutable truths and unquestionable certainty" (Nickson, (1994), p. 11). Within the "growth and change" view, Nickson follows Kuhn and Popper's work, and proposes that mathematical knowledge as a field has a subjective aspect involved in the selection of theories. The role that judgment and choice (within a particular value system) play in knowledge construction is emphasized by this point of view. In the author's words, "This value dimension of how knowledge comes into being, and also how it comes to be superseded and changed, means that it is not only a social phenomenon but a cultural one" (Nickson, 1994, p. 11). My research is inscribed within the second point of view, since it includes the cultural dimension where the agreements and discussions among scientists can be explained as a variable that shapes knowledge at a certain period of time. In other words, this perspective includes the construction of arguments as one dimension in the making of mathematics. This position can be translated into classroom practice and how mathematics may appear to students: the discussion of ideas among students is very important so that they have authorship of the mathematical products of the class, just as mathematicians have authorship of the mathematical objects they create.

If we assume both this cultural perspective of mathematics and the requirement of creating new knowledge, which seems to be intrinsic to a mathematical activity, we have to ask ourselves: how can learners *do* mathematics, since they have to learn what has already been created? Following Chevallard, Bosch, and Gascón's (1997) work, I will consider the production of new mathematical knowledge by relating the term *new* to the subject who is learning. As Chevallard et al. (1997) state, "the learners may not create new knowledge for other human beings but they may create new mathematics for themselves in terms of the class group" (p. 56). It is my interpretation that this last statement does not inhibit the production of new mathematical knowledge by students in a broader sense. I believe that Chevallard et al. are emphasizing the fact that in schools we need as educators to create an artifact-situation where students (re-) create knowledge that already has a cultural status.

Charnay (1994) points to another essential characteristic of a mathematical activity: the notion of anticipation. Often, the mathematical activity requires elaborating a strategy that allows one to *see* the kind of expected result and designing a possible way of approaching the problem *without necessarily doing*. For example, in Problem 1 from the Calendar Sequence (see Figure 1 and Appendix A), I expect that students will anticipate a dependency between the outcome and the place within the month where the 2x2-calendar-square is placed. Before starting to solve the problem, some students will think that the outcome will be bigger when the 2x2-calendar-square is placed at the end of the month because numbers are larger as the month progresses. Other students will think that because there is a subtraction involved in the operator, the smaller the numbers involved (i.e., placing the 2x2-calendar-square towards the beginning of the month), the bigger the outcome since you are subtracting a smaller number.

In my design of the Calendar Sequence, I have addressed both the creation of new knowledge by the class group and the role that anticipation plays in mathematical learning. In terms of *the creation of new knowledge*, the sequence encourages students to produce new knowledge about the behavior of a given system (i.e., the Calendar Sequence) through the use of algebraic tools. For instance, in Problem 1 of the Calendar Sequence, after doing some initial calculations, students arrive at the conclusion that the outcome for the problem is always -7. However, students still need to show that this is *always* going to be the case. Specific empirical results may only inform students about a tendency or pattern in the problem, but not about all possible cases. At this point, algebra becomes the optimal tool to solve this problem by allowing students to create an algebraic model of the Calendar Sequence system, by choosing variables and relationships among those variables, as well as writing the expressions and transforming them into equivalent expressions. Through this modeling process, students may produce new knowledge: no matter what day, month, year, or location on the calendar of the 2x2-calendar-square, the outcome is always an invariant.

Regarding *anticipation*, it is addressed in two ways in the proposed sequence. On one hand, I took into account students' anticipations when designing the sequence of problems. For instance, as I mentioned above, in Problem 1 (see Appendix A) I considered that students would expect different outcomes as a function of the different days, months, and/or years where the 2x2-calendar-square is placed. On the other hand, this sequence enhances students' anticipations through the use of algebraic transformations. For instance, when they obtain the outcome for Problem 1, expressed in a general form, it looks like this: $a(a+8)-(a+1)(a+7)$. At first sight, this expression does not say anything about the behavior of the outcome. Students need to anticipate what type of algebraic transformation, in Boero's (2000, p. 99) sense "anticipation which allows us the process of transformation to be directed towards simplifying and resolving the task", they want to apply in order to obtain a more readable expression, with the goal of explaining the behavior of the outcome.

The Notion of Mathematical Problem

It is broadly accepted that dealing with problems is at the core of mathematics. But what is meant by "mathematical problem" within this study? Charnay cites Bachelard's conception of problem: "For a scientific spirit any knowledge is an answer to a question. If there were no question then there cannot be scientific knowledge. Nothing comes alone, nothing is given. Everything is constructed" (1994, p. 51). Following both authors, the principal function of "problems" is to provide an opportunity for students to ask questions about mathematics, motivating them to construct answers to their own questions. Another function of problems (Charnay, 1994) is to provide a system that can be modeled using a mathematical model, turning mathematical objects into tools to solve problems.

Mason (1996b) also analyses *the status of being a problem* in terms of the match between "the problem" and the "subject." In his own words: "a particular collection of words is not in itself a problem, merely a collection of words. If those words engender a state of *problématique*, then a problem has emerged, for the person" (p. 189).

Charnay (1994) specifies some essential characteristics for a task to become a real problem for learners. First, the task has to be approachable by students through the use of their knowledge system. Second, it should allow the students to use their knowledge as a tool. Third, it has to offer enough resistance to make old knowledge appear as insufficient and promote the construction of new relations; that is, it is desirable to promote an experience of “intellectual challenge” for learners. When designing the Calendar Sequence, I had these principles in mind, and applied them to the best of my knowledge drawn from my own experience as an educator, but also from data collected in past studies (Bell, 1995; Mason, 1996a, 1996b; Mason & Johnston-Wilder, 2005). In terms of how the first and second conditions above were addressed in the Calendar Sequence, students can approach all the problems by trying particular cases using their prior knowledge as a tool (use of operations, properties of numbers, etc.). However, their prior knowledge would likely be insufficient to solve the problem. I anticipated that students' prior knowledge would seem insufficient whenever a general question was posed, for instance, in Problem 1 of the Calendar Sequence, students are asked to show that for any location of the 2x2-calendar-square in the calendar of any month and year, the outcome will be always -7.

Previous Studies

In this section, I will focus on different approaches to algebra employed by researchers in the study of the teaching and learning of algebra at the middle and high school levels in the last thirty years (1977-2006). The goals of this section are to review the researchers perspectives on algebra, to group their studies taking into account their foci, and to synthesize their epistemological foundations. With the notion of didactical transposition, Chevallard (1997) enlightened the process that objects of knowledge go through in order to become objects of teaching. My interest resides on the perspectives that the research community holds regarding algebra. Among other things, a researcher perspective is shown through their choice of problems used to collect data. The selection of problems used to do research is an important step in the process of shaping *the* mathematical object subject to research, in this case algebra. In the last ten years, there have been a variety of works reviewing where we come from as a research community, where we are in the domain, and what are the next steps for the future. Along these lines, the following works can be identified as seminal: *Approaches to Algebra* (Bednarz, Kieran, & Lee, 1996), *The Future of the Teaching and Learning of Algebra* or ICMI Study, (Stacey, Chick, & Kendal, 2004), and the chapter *Research on the Teaching and Learning of Algebra*, in the *Handbook of Research on the Psychology of Mathematics Education* (Kieran, 2006). These scholastic works sum up, from different perspectives and along different axes, the evolution of the research questions the community has asked itself along the last three decades. All of these reports agree on the importance of developing further research in the learning and teaching of algebra. The ICMI Study (Stacey, Chick, & Kendal, 2004) and *Approaches to Algebra* (Bednarz, Kieran, & Lee, 1996) are volumes entirely dedicated to algebra that show not only that copious research has been done in the domain, but also reveal the emergence of numerous new research questions.

Adopting different perspectives, researchers and educators have underlined the importance of algebra at the K-12 level. Moses (2001) draws attention to the role that algebra plays in the United States' (US) school system and society. He claims that, "the idea of citizenship now requires not only literacy in reading and writing but literacy in math and science" (Moses, 2001, p. 12). Especially because we are living in a technological era where "the visible manifestation of the technological shift is the computer, the hidden culture of computers is math" (Moses, 2001, p. 13). Algebra has been assigned the role of being the place where, young people learn the necessary symbolism that eventually becomes the tool to control technology. In addition to that, Moses (2001) states that algebra is the school subject that works as a filter in US society (while in France, for instance, geometry plays that role); this filter separates the youth that will go to college from those who will not. At risk groups identified by

Moses are Latino, Black-American, and poor White American students. Stacey and Chick (2004) agree with Moses (2001) when they state that:

> Algebra is often described as a gateway to higher mathematics, not least because it provides the language in which mathematics is taught. Consequently, it is important that all students be given a genuine opportunity to learn algebra. Without this, they are cut off from many occupations, either because algebra is really used there or because it is specified as a preliminary qualification. (Stacey & Chick, 2004, p. 2)

Following Moses (2001), in order to give youth access to a full citizenship, society needs to make their students learn and succeed in algebra in order to be included in economic life in an active way.

Chazan (1996) agrees with Moses in the identification of the groups of students who are at risk in algebra, as well as in the concern about equality in society and education and the role that algebra plays in this respect. Chazan and the ICMI report (Stacey, Chick, & Kendal, 2004) identifies mass schooling as a triggering factor in the emergence and identification of failure in the learning of algebra. Chazan (1996) even questions the pertinence of teaching algebra for all in the current conditions:

> School algebra policies are contested because they have implications for access to college. Although we need to act to address inequalities in our society that limits access to college, I believe it is wrong headed to force students to take a class that almost half of the students fail (Chazan, 1996, p. 475).

The target solution for Chazan seems to be a curricular reform which should take into account all students, “this curriculum must be intended for a broad range of people –those who are not college-intending, as well as those who are" (Chazan, 1996, p. 475).[8] Stacey and Chick (2004) considering that we are now faced with teaching algebra in high school to a broader spectrum of the population state that: "the challenge has been to reconceptualize algebra as a subject that does have a relevance to students and to do this in a way that the students themselves can perceive the relevance" (Stacey & Chick, 2004, p. 2). To the same goal, Kaput (1995; , 1998) proposed to "transform algebra from an engine of inequity to an engine of mathematical power" (Kaput, 1995, p. 2) by *algebrafying* the K-12 curriculum, thus starting the teaching of algebra in elementary school: "The key to algebra reform is integrating reasoning across all grades and all topics –to *algebrafy* school mathematics" (Kaput, 1995, p. 2). In Kaput's view algebra should include the following: (1) generalizing and formalizing patterns and constraints, (2) syntactically-guided manipulation of formalisms, (3) study of structures and systems abstracted from computations and relations, (4) study of functions, relations, and joint variation, and (5) cluster of (a) modeling and (b) phenomena controlling languages. As it will be discussed in the sixth section, a group of researchers in Spain and France have been working towards an *algebraization* of the mathematics curriculum (e.g., Bolea, Bosch, & Gascon, 1999, 2003; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascon, 1993-1994). A difference between these two perspectives is when to start the *algebraization* of the curriculum. For Kaput (Kaput, Carraher, & Blanton, 2007) algebra should start in the early grades of primary school in order to avoid the arithmetic-algebra transition. This seems not to be the case for the curriculum *algebraization* proposed by Bolea, Bosch & Gascón (1999, 2003) which focuses on algebra as a modeling tool, building on the notion of mathematical model

[8] In his research studies, Chazan (1996) has developed an algebra curriculum for high school bearing on the notions of variable, functions, and the use of real-life contexts.

developed ealier by Chevallard (1985, 1989, 1989-1990). Although Bolea et al (1999, 2003) addressed the teaching of *elementary* algebra they do not make a case for starting algebra early.

Complementary, many studies have pointed out students' difficulties when learning algebra. Many of these studies seem to attribute these difficulties to developmental reasons. Research of this nature has highlighted the difficulties children in middle and high school have. Among the difficulties named are that children in middle and high school (see Carraher & Schliemann, 2007 and Schliemann, Carraher, & Brizuela, 2006 for thorough reviews):

(a) believe that the equals sign only represents a unidirectional operator that produces an output on the right side from the input on the left (e.g., Booth, 1984; Kieran, 1981, 1985; Vergnaud, 1985, 1988a);

(b) focus on finding particular answers (e.g., Booth, 1984);

(c) do not recognize the commutative and distributive properties (e.g., Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 2001; Demana & Leitzel, 1988; MacGregor, 1996);

(d) do not use mathematical symbols to express relationships among quantities (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Vergnaud, 1985; Wagner, 1981);

(e) do not comprehend the use of letters as generalized numbers or as variables (e.g., Booth, 1984; Kuchemann, 1981; Vergnaud, 1985);

(f) have great difficulty operating on unknowns (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg, Sleeman, & Ktorza, 1990); and

(g) fail to understand that equivalent transformations on both sides of an equation do not alter its truth value (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg, Sleeman, & Ktorza, 1990).

One of the goals of this paper is to present evidence regarding high school students not only being able to tackle the above-mentioned obstacles but also going further. I believe that as a research community we still need to develop and implement mathematical activities and document students' learning in order to identify elements that foster algebraic learning.

Goals, Structure, and Organization of Previous Studies

I will review and synthesize a variety of studies on the teaching and learning of algebra at the middle and high school levels. For each of the studies, the goals of this review are to provide (whenever present in the source):

(1) A brief overview;

(2) An identification of the aspects of algebra that are central to the study and the underlying epistemology, and sample mathematical problems used by the researchers;

(3) An identification of students' learning outcomes and/or theoretical concepts that are central to the study.

The selection includes papers from 1976 to 2006 directly related with algebra. As selection criteria the word "algebra" had to appear either in the title, abstract, or keywords of the article. A diversity of articles from Europe, North America, and South America are presented in the corpus material. The material includes books, articles in journals, and conference proceedings. I am not including important areas of recent development such as Early Algebra (e.g., Carraher, Schliemann, & Brizuela, 2004; Davydov, 1962; Kaput & Educational Resources Information, 2000; Schliemann, Carraher, & Brizuela, 2006), and teachers' knowledge (e.g., Arcavi & Bruckheimer, 1983; Arcavi & Bruckheimer, 1984; Ball, 1988; Even, 1993; Richardson, 2001). They are beyond the purpose of this literature review that, as mentioned above, is the teaching and learning of algebra at the middle and high school levels. It might seem paradoxical that I do not include research perspectives on teachers' knowledge given that the theme of this literature review includes 'teaching'; however, I am focusing on the teaching aspects regarding the features of the activities and problems used to collect data, as well as the conditions in which the students produced knowledge. This is not directly related to teachers' knowledge as explored in the cited literature (e.g., Arcavi & Bruckheimer, 1983; Arcavi & Bruckheimer, 1984; Ball, 1988; Even, 1993; Richardson, 2001).

First, I will present a brief history of the evolution of research on the teaching and learning of algebra. Second, I will discuss researchers' perspectives addressing the relation between arithmetic and algebra. Many research studies have pointed to the conceptual change that students have to achieve in order to understand algebra, e.g.; the different meanings of the equal sign, the different ways of solving problems in arithmetic and algebra. Usually in arithmetic we apply operations to numbers and obtain results after each operation; but in algebra, we usually do not start solving a problem using the given numbers, doing calculations with them, and obtaining a numeric result. In algebra, students have to identify the unknowns, variables and relations among them, and express them symbolically in order to solve the problem (e.g., Behr, Erlwanger, & Nichols, 1976; Booth, 1984; Herscovics & Kieran, 1980; Kieran, 1979, 1981; Kuchemann, 1981; Vergnaud, 1984, 1988; Vlassis, 2004) . Third, I will cover research in the area of generalization and patterns. Fourth, I will discuss research on algebra and proof. Fifth, I will concentrate on the approach to algebra from a functional perspective. Also in this section, I will address the use of computational environments in the teaching and learning of algebra, since many of these environments use a functional perspective. Sixth, I will focus on the modeling approach to algebra.

I partially grouped the studies according to the classification developed by Bednarz, Kieran, and Lee (1996), inspired by the contributions at the colloquium on *Research Perspectives on the Emergence and Development of Algebraic Thought* held in Montreal in May 1993 and organized jointly by the *Centre Interdisciplinaire de Recherche sur l'Aprentissage te le Developpement en Education* (CIRADE) and the Mathematics Department of the Universite du Quebec a Montreal.[9] The approaches to algebra that this group identifies are the following: Historical, Generalization, Problem Solving, Modeling, and Functional. I took from this classification the following perspectives: Generalization, Problem Solving, Modeling, and Functional. I will be using this classification only in part because, first, I want to adapt it to my own interests and, second, because, as reflected in the ICMI Study (2004), this categorization seems to represent approaches taken in English speaking countries but not non-English speaking countries such as Spain and France. Adding new categories allows considering the diversity of approaches worldwide, including the section on *Algebra and Proof* . A third reason for the chosen categories is to group the studies according to their underlying epistemology and conceptions of algebra. Research on the relation between algebra and proof is scarce. This causes us to neglect one of the main uses

[9] Lee (1997) has developed another classification of studies in algebra. This classification emerged through raising the question "What is Algebra?" to researchers in the field, mathematicians, teachers, and students. The categories that were developed from this study are Algebra as a school subject, Algebra as generalized arithmetic, Algebra as a tool, Algebra is a language, Algebra is a culture, Algebra is a way of thinking, and Algebra is an activity. Given that I want to analyse the relation between the approach to algebra and the research findings, these categories are not fertile for the type of review I want to develop. Kaput (1995, 1998) defined algebra as including the following: (1) generalizing and formalizing patterns and constraints, (2) syntactically-guided manipulation of formalisms, (3) study of structures and systems abstracted from computations and relations, (4) study of functions, relations, and joint variation, and (5) cluster of (a) modeling and (b) phenomena controlling languages. Since these categories were created in order to define what algebra should include but not to describe approaches to algebra or a classification of

of algebra as a tool, which is the power entailed in transforming expressions and allowing to read information that was "hidden" in the expression before the set of transformations was applied. Algebra becomes a powerful tool when we try to find f'(2) with f(x)=x^2 using

$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{2} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$. If we try to calculate this limit using the current quotient expression, we arrive to an indetermination of the type $\frac{\to 0}{\to 0}$ when $h \to 0$.

Thanks to algebraic transformations we can re-write the expression $\lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$ as

$\lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$ using the definition of "squaring" and distributing. However, if we now try to calculate the limit, we will again obtain that $\frac{\to 0}{\to 0}$ when $h \to 0$. Therefore we need to try something else and keep re-writing. Using arithmetic we can transform the last expression as follows: $\lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h}$. Again, we are not there yet since we have that

$\frac{\to 0}{\to 0}$ when $h \to 0$. If we factor *h* in the numerator we obtain $\lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} \frac{h(4+h)}{h}$. If we try to calculate the limit in this format we again obtain that $\frac{\to 0}{\to 0}$ when $h \to 0$. Canceling *h* in the numerator and denominator we finally obtain $\lim_{h \to 0} \frac{h(4+h)}{h} = \lim_{h \to 0} 4 + h = 4$.

studies to algebra therefore I will not be using this classification. However, since many researchers focus on (1), (4) or (5), Kaput's components are included in sections two, five and six correspondingly.

Brief History of Algebra in Mathematics Education

For this brief history, I used mainly two sources: *The future of the teaching and learning of algebra* (Stacey, Chick, & Kendal, 2004), and the *Handbook of Research on the Psychology of Mathematics Education* (Gutiérrez & Boero, 2006). Kieran (Gutiérrez & Boero, 2006) breaks the 1977-2006 period into three sub-periods and identifies the main themes approached by researchers in algebra. The first sub-period corresponds to the years 1977-2006, where the main themes of study were transition from arithmetic to algebra, variables and unknowns, equations and equation solving, and algebra word problems. The second sub-period corresponds to the years from the mid-1980s to 2006 with use of technological tools[10] and a focus on multiple representations and generalization as main topics. The third sub-period of time matches up with the years from the mid-1990s to 2006 where the main subjects of study have been algebraic thinking among elementary school students, algebra for teacher/teaching, and dynamic modeling of physical situations and other dynamic algebra environments.

From Kieran's perspective, up until the mid-1960s, "algebra was a paper-and-pencil activity, focusing primarily on transformational work" (Kieran, 2004, p. 25). At that moment, algebraic meaning-making was addressed in the introduction of algebra books and was approached by translating arithmetical sentences into algebraic expressions. In the 1970s there was a change in the way algebra was taught, given the modern math movement and the importance given to problem solving. Kieran (2004) identifies the development and introduction of technology in education as crucial factors for changes that took place in the mid-1980s. Research conducted in the 1970s and 1980s provided evidence that students were struggling with learning algebra, and the main issues identified related to the arithmetic-algebra transition, and the re-conceptualization required to become proficient in algebra (e.g., the meaning of the equal sign). As a result of this, members of the research community started conducting teaching experiments in order to try alternative ways of teaching algebra, emphasizing meaning making. Following Kieran (2004), the evolution of the field seems to result in learning of algebra through meaning making activities, but with very limited attention to transformational activities.

Indeed, in the UK, for example, the search for meaning and the consequent suppression of symbolism led to a situation in the early 1990s where students were doing hardly any symbol manipulation (Sutherland, 1990). In various countries, problem solving, by whatever means, had all but replaced traditional algebra. "*The hope was that, in focusing on algebraic understanding (however this might be defined), the techniques would take care of themselves*" (Kieran, 2004, p. 27. Emphasis added).

However, the techniques did not take care of themselves. Artigue (Artigue, Defouad, Duperier, Juge, & Lagrange, 1998; M. Artigue, 2003) and her research group found compelling evidence for this dichotomy between meaning-making and technique or transformational activities in the teaching and learning of algebra. This is one of the topics in the research agenda in the area of learning and teaching of algebra: the study of activities that achieve two goals simultaneously: understanding of algebra and construction of meaning on one hand, and proficiency of technique on the other (Artigue, Defouad, Duperier, Juge, & Lagrange, 1998; M. Artigue, 2003; Kieran, 2004).

Analyzing the different themed-groups in the International Group for Psychology of Mathematics Education (PME) research community, one of the aspects that Kieran (2006) identifies is the broadening of sources of meaning in the teaching and learning of algebra.

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[10] Many researchers that developed technological tools to address the teaching and learning of school algebra chose

Kieran then proposes (see Figure 3) a modification of Radford's (2004) sources of meaning in mathematics education. For Radford (2004), meaning in school algebra is produced from three primary sources: (i) the algebraic structure itself, (ii) the problem context, and (iii) the exterior of the problem context (e.g., linguistic activity, gestures and bodily language, metaphors, lived experience, image building, etc.).

Kieran (2006) modified Radford's point (i) on algebraic structure to consider meaning from within mathematics, where the author includes algebraic structure, and meaning from multiple representations (see Figure 3).

1. Meaning from *within mathematics*:
   1.a. Meaning from *the algebraic structure itself*, involving the letter symbolic form.
   1.b. Meaning from *multiple representations*.
2. Meaning from *the problem context*.
3. Meaning derived from that which is *exterior to the mathematics/context problem* (e.g.: linguistic activity, gestures and bodily language, metaphors, lived experience, image building, etc.).

Figure 3. Sources of meaning in algebra (adapted from Radford [2004] by Kieran [2006, p. 32]).

Summarizing, from Kieran's (2006) viewpoint one of the main accomplishments of the PME research community is a broadening of sources of meaning for the teaching and learning of algebra. In addition to that, the research community now has, as a next goal to be achieved, the task of bringing together technique and meaning.

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to focus on the notion of function (see Schwartz & Yerushalmi, 1992; Yerushalmi & Schwartz, 1993; Yerushalmi & Shternberg, 2001; Yerushalmi, 2000)

School Algebra as a mathematical domain

Nowadays there are complementary characterizations of algebra that focus on different aspects of this mathematical domain.

From a cultural point of view, Lee (1996) points out that it is helpful to think about algebra as a mini-culture within the culture of mathematics. In this way, the author conceives of algebra as formed by the following: sets of activities, specific language, interactions between knowledge and language, and the interaction between algebra and other mathematical cultures. Algebra as a culture has its own way of communicating, and its own selection of topics, structures, and codes. Lee (1996) considers the introduction of algebra in school as a process of enculturation. The author recognizes generalization activities and the use of symbolic language as two central features of algebraic culture, coinciding with Mason (1996a).

In addition, Mason (1996a), from a history of mathematics perspective, identifies four important sources of algebra: expressing generalities, possibilities and constraints, rearranging and manipulating, and generalized arithmetic. Moreover, the author considers the articulation between generality and particularity an essential feature of algebraic work; the author points out that an expert manages to go spontaneously from the particular to the general, and from the general to the particular. This spontaneous flexibility between the particular and the general seems like a desirable learning objective since it is one of the main features of algebra (i.e., how the particular informs the general, and how the general informs the particular). One of the objectives of the "Calendar Sequence" is to address the general-particular relation when using particular outcomes to make hypotheses about the general behavior of the outcome.

Related to the general-particular articulation, Mason (1996a) reflects upon a typical circumstance in the mathematics class: the teacher's use of examples as a didactical tool. The author questions the effectiveness of such a practice. When a teacher presents students with an example, the meaning of the example is very different for the teacher and for the students. For the teacher, it is *part of a whole system* and that is why it is shown as an example *of* something else. For the students, it is a *totality* in itself and they have to decide which of the features of the example are particular to that example and which are general. In the "Calendar Sequence," the general is built through the analysis of the behavior of the outcome at different levels of generality; the particular cases can be obtained through replacing the parameters with a particular value.

The meaning of algebra has developed and broadened throughout history, going from process to object. Mason (1996a), referring to Gattegno, states that "algebra as a disciplined form of thought emerged when people became aware of the fact that they could operate on objects (numbers, shapes, expressions), and could operate on those operations" (p. 74). Another characteristic feature of algebraic thinking requires the possibility of seeing writing symbols both as expressions and values, as well as objects and processes. Mason (1996a) also makes the distinction between the act of writing expressions using symbols and the act of experiencing the meaning of those expressions.

Adopting an epistemological viewpoint and assuming that generalization is an essential feature of algebraic activity, Radford (1996) distinguishes in his work two dimensions in the functioning of generalization: as a primary need for knowledge and as norm-driven (norm in the sense of an epistemic norm[11]). The author stresses, "a superficial look at the history of mathematics leads us to the impression that all mathematics is about generalizing. A closer look suggests that, if we accept generalization as an epistemic norm, it could not function alone but

[11] Norm that regulates the production of mathematical knowledge.

may be related to another probable epistemic norm, namely the problem solving epistemic norm" (p. 108). Generalization, in combination with problem solving, motivates the search for knowledge. If we compare a case of simple generalization (e.g., write an expression for the n^{th} term of a given sequence) with the type of generalization that I am proposing in the Calendar Sequence, we can identify two basic differences. The first has to do with the goal of generalization: in the case of the Calendar Sequence, I anticipate that students will be surprised in terms of how the outcome behaves and how the generality is discovered. The second difference has to do with having to use algebraic expressions to explain *why* an outcome behaves in such a way. Another interesting issue that Radford (1996) brings up regarding generalization is the fact that generalization depends on the nature of the mathematical objects; therefore, a variety of approaches involving different mathematical objects will contribute to constructing diverse senses of generalizing among students.

Going back to the question "*what is algebra?*," Wheeler (1996), from a multidimensional perspective, claims that three dimensions constitute this field: algebra is a symbolic system, algebra is a calculus, and algebra is a representational system; however, each one of these dimensions alone is not algebra. In other words, algebra is a symbolic system but it is also more than a symbolic system, and so on regarding the other dimensions. Arcavi (1994) addresses the question of *"what is algebra?"* by concentrating his attention on describing and discussing behaviors that illustrate what he claims are examples of *symbol sense*. Arcavi's (1994) symbol sense would be given, among other issues, by:

- students' decision making about the pertinence of an algebraic representation when facing a certain problem (e.g., in the Calendar Sequence, the awareness of the use of variables to represent the problem, and the expression of all variables as a function of just one variable);

- students' acknowledgement and awareness of the fact that when transforming an algebraic expression into an equivalent expression, it might be possible to "read" information that was hidden or implicit in the original expression (e.g., this is one of the goals in the Calendar Sequence: to offer students problems where algebraic transformations allow one to read information that couldn't be read from the initial expression. In Problem 1 [see Appendix A], the initial expression is a(a+8)-(a+1)(a+7) while an equivalent expression is -7);
- -students' ability to read information in an algebraic expression (i.e., n.(n-1).(n+1) is a multiple of 6, for any n whole value. For instance, in the Calendar Sequence, in Problem 1 [see Appendix A], when students encounter -7 after transforming the initial expression, my hope was that students would learn how to read the expression in terms of the context: "The outcome does not depend on where the 2x2-calendar-square is placed, it is an invariant of the system under the current constraints");
- students' sensibility in analyze an expression and making decisions before rushing into algorithmic procedures (i.e., to analyze that the solution set of the equation $\frac{2x+3}{4x+6}=2$ is empty because for any $x \in \Re - \left\{-\frac{3}{2}\right\}$ the left-hand side of the equation is $\frac{1}{2}$);
- students' flexibility to turn to different forms of representation in order to solve a problem. In particular, their ability to appeal to the Cartesian representation for making decisions regarding the conditions on an algebraic expression (e.g., if one wants to determine the values of $x \in \Re / x^2 + 3x - 4 > 2$, it can be solved by determining the

values that are solutions of the associated equation and imagining that it is a concave parabola);

- students' integration of different algebraic knowledge for making a decision (e.g., if one wants to determine if two polynomial algebraic expressions of a second degree are equivalents, it is enough to verify that both expressions give the same numeric values when evaluating for any three numbers);
- students' ability to discern whether a solution set of an equation, system of equations, and inequalities is formed by zero, one, finite, or infinite elements.

Arithmetic and Algebra

This section addresses studies on the relationship between arithmetic and algebra. I will discuss the work by Kieran (1981), Balacheff (2000), and Filloy and Rojano (1989).

Currently, given the organization of the curriculum in the majority of Western countries, students' work with arithmetic precedes their work with algebra. Students may have up to eight years of schooling before they start working on algebra. Research addressing the arithmetic-algebra relation (e.g., N Balacheff, 2000; Bednardz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1981; Wheeler, 1996) has focused mainly on two issues. The first issue can be described as the attempt to characterize both mathematical domains –arithmetic and algebra-, to identify similarities and differences, and to trace the evolution of their relationship in History. The second issue addressed by research on the transition from arithmetic to algebra, is the design and/or analysis of instructional activities. In this section, I will be exploring these two issues.

Many studies have shown that students maintain, as long as they can, their arithmetical interpretation of different mathematical objects and tools when solving problems (e.g., Filloy & Rojano, 1989; Filloy, Rojano, & Rubio, 2001; Kieran, 1981).

Within these studies, Kieran's (1981) analysis of the transition from arithmetic to algebra focuses on equations and students' interpretation of the equal sign. Kieran (1981) opens the discussion regarding the role of the equal sign in arithmetic and algebra from Kindergarten to college. The researcher emphasizes kindergarteners' interpretation of the equal sign as the "do something sign". In addition, she provides evidence that students do not conceive of the equal sign as an equivalence when they are presented with a number sentence such as "6 + 7 = 5+ 8". They do not see both sides of the equal sign as two different names for the same number or two different representations of the same number. Students have the idea that after the equal sign, the result of the calculation needs to be stated. Other students perceive the number sentence as two different problems, the problem " 6 + 7 = ", and the problem " 5 + 8 =."

Kieran (1981), who considers that the interpretation of the equal sign should be that of equivalence, points out that "the symbol which is used to show equivalence, the equal sign, is not always interpreted in terms of equivalence by the learner" (p. 317). In the analysis of the meaning of the equal sign within the context of equation solving, the researcher claims that,

> … the ability to consider an algebraic equation as an expression of equivalence because both sides have the same value does not seem to be sufficient for an adequate conceptualization of the equation solving. For not only does equation solving involve a grasp of the notion that right and left sides of the equation are equivalent expressions, but also that each equation can be replaced by an equivalent equation (p. 323).

One has to be aware, however, that the equal sign can function in different ways. Consider, for instance, the case of an equation where the equal sign is used to establish a condition on a set. Taking an example from Kieran (1981, p. 323), the problem is presented as "Solve for x: 2x+3=5+3". From a mathematical point of view, this way of asking someone to solve an equation, usually presented at school, is an incomplete request that does not specify what type of number is x. Does x belong to the set of the integers, to the natural numbers, to the complex numbers? We can think of an equation as an object that defines a subset within a set (usually referred to as solution set). We can re-write Kieran's example in the following ways:

1. Determine the set of values of x in Z such that x+1 = x+2.
2. Determine the set of values of x in Z that makes the equality (x+1 = x+2) true.
3. $\{x \in \mathrm{Z} / x+1 = x+2\}$ = S.

In this particular case, for all values of x the equality is false. This fact contradicts the notion of equivalence or equivalent expressions. In a sense, two expressions that are equivalent in a set can be interchanged for one another. In the case of the equation presented by Kieran (1981), 2x+3=5+x, the solution set is x=2. That means that the equality is true only for x=2 and not for any other value. From this it follows, for instance, that if we plug in x=3, we are going to obtain $2 \times 3 + 3 \neq 5 + 3$.

The solution set can be empty, finite, or infinite. When the solution set is strictly included in the set –the solution set is a proper set- where the unknowns are defined, we cannot talk about equivalence. It is true that the equal sign denotes an equivalence relation in the set of the real numbers, since it can be proved that the equality on the set of real numbers is reflexive, symmetric, and transitive. In my view, Kieran (1981) is overseeing the different uses and functions of the equal sign; while in the real numbers it is true that the equal sign verifies the properties of an equivalence relation, when using the equal sign in an equation-solving context it represents a condition within a number set. The meaning of the equal sign depends on, at least, the context and the task for which it is used.

Filloy and Rojano (1989) provide a different perspective from Kieran's (1981), identifying conceptual and symbolic changes which mark differences between arithmetical and algebraic thought in the individual. Some of these hallmarks are the interpretation of letters, the notion of equality, and conventions for coding operations and transformations in the solution of equations. These authors postulate the existence of a cut, "a break in the development concerning operations on the unknown" (Filloy & Rojano, 1989, p. 19). Filloy and Rojano developed this idea based on an analysis of the strategies and methods for solving equations found in the pre-symbolic algebra textbooks of the 13th, 14th, and 15th centuries. The solution strategies for equations such as $x^2 + c = 2bx$ and $x^2 = 2bx + c$ are absolutely different from each other.

> This difference would not exist if the authors had had recourse to the rule of transposing terms from one side of an equation to the other for, at the syntactical level, the two equations would then be similar. But this facility would already imply an advanced ability to operate on the unknowns in the equations. (Filloy & Rojano, 1989, p. 19)

Considering different types of equations, Filloy and Rojano (1989) proposed a categorization for types of equations. They divide the realm of equations –linear and on one unknown- into arithmetical and non-arithmetical equations. The authors associate the "arithmetical equation" with the "arithmetical" notion of the equal sign. Filloy and Rojano (1989) agree with Kieran (1981) regarding the conception of the equal sign in arithmetic in the following way, "the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown); the right side represents the consequence of having performed such operations" (Filloy & Rojano, 1989, p. 19). The authors show evidence that this type of equation, i.e., Ax+B=C, can be solved "undoing" the operations. In this sense, if we present students only with this type of equations they can solve them using arithmetic tools. Work with such equations fails to introduce them to the algebraic world. Students need to face non-arithmetical equations, i.e., Ax+B=Cx+D, in order to promote their need to use algebraic tools. The solution of non-arithmetical equations involves operations drawn from outside of the domain of arithmetic (i.e., operations on the unknown). The authors hypothesize that the introduction of the non-arithmetical equations should be framed by problems involving different contexts. They propose the use of two contexts: the balance model and the geometrical model. Figures 4 and 5 show a geometrical representation of the equation Ax+B=Cx where A, B, and C are given positive integers and C > A.

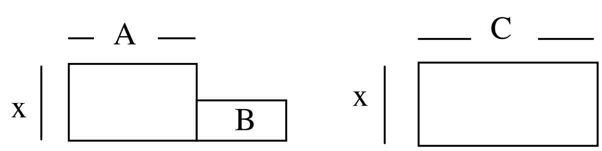


Figure 4: Representation of the equation Ax+B=Cx, using the geometrical model (from Filloy & Rojano, 1989, p. 19).

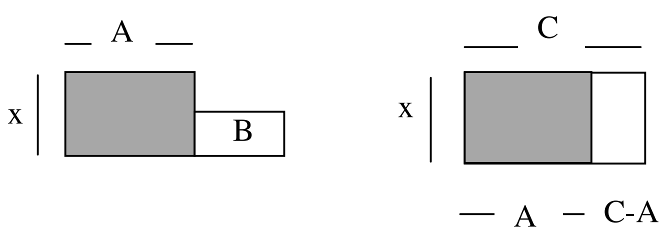


Figure 5: Representation of the equation comparison of areas as a way of giving meaning to the syntax (Filloy & Rojano, 1989, p. 19).

Figures 6 and 7 show a representation the same Ax+B=Cx equation in the context of the balance model (where A, B, and C are given positive integers and C > A).

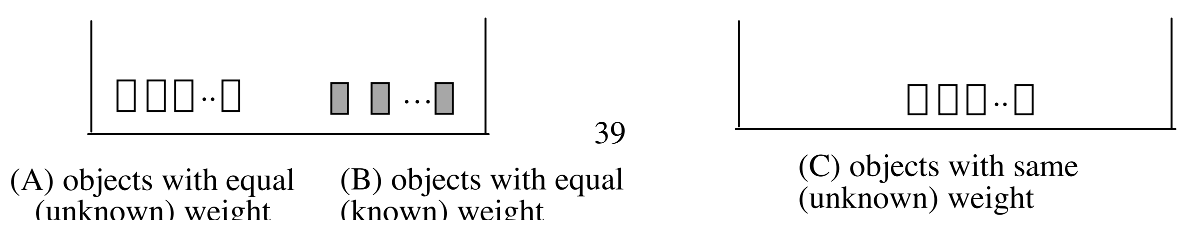


Figure 6. Representation of the set up of the equation AX+B=C in terms of weights (Filloy & Rojano, 1989, p. 19).

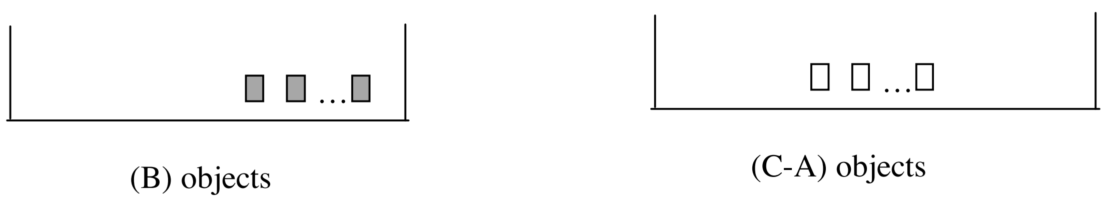


Figure 7. Representation of the solution of the equation AX+B=C in terms of weights (Filloy & Rojano, 1989, p. 19).

Filloy and Rojano (1989) identify different phenomena within their study results. In the first place, they conclude that "the correction of syntactic algebraic errors, and of the operational difficulties that occur in resolving complex problems or equations, cannot be left to be spontaneously resolved by children on the basis of their initial grasp of operational algebraic behaviour" (p. 24). Filloy and Rojano (1989) propose that "modeling" has two fundamental dimensions: *translation* and *separation*. Their work provides evidence that students that master the first component –translation- may be at a disadvantage at the *separation* phase. Students with mastery in translation "developed a tendency to stay and progress within the concrete context" (p. 25). In addition, students with a more syntactic tendency may develop obstacles when trying to abbreviate actions and produce intermediate codes (between the concrete and purely syntactic level). Filloy and Rojano (1989) point out the importance of the role of the teacher in "the development of the processes of detachment from, and negation of, the model, in order to lead towards the construction of the new notions" (p. 25).

Bednarz and Janvier (1996) present yet another perspective. Regarding the arithmetic-algebra transition, Bednarz and Janvier (1996) based their study on equations. They chose equations as the algebraic object to investigate differences between arithmetic and algebra, and to study students' obstacles in the equation-solving process. In their study, Bednarz and Janvier (1996) also proposed a categorization of the problems that are generally presented in algebra. Their analysis identified three types of problems: (1) problems of unequal sharing, (2) problems involving a magnitude transformation, and (3) problems involving non-homogenous magnitudes and a rate. In addition, Bednarz and Janvier (1996) analyzed features that are different in algebra and arithmetic. For instance, "arithmetic procedures are generally organized through the processing of known quantities, by attempts to create links between them in order to be able to operate on them. The unknown quantity appears at the end of the process" (Bednardz & Janvier, 1996, p. 119). Bednarz and Janvier (1996) label arithmetic problems as "connected" since a relationship can be easily established between two known data, leading to the possibility of arithmetic reasoning. On the contrary, in algebra the problems are "disconnected", meaning that no direct bridging can be established between the known data in order to operate on them to obtain the unknown. However, in algebra, "in operating on an unknown quantity, which we directly perceive as the one which will allow us to generate all the others, and doing it as if this quantity was known, the algebraic reasoning, just as with the arithmetic reasoning, *connects* the problem in some way" (Bednardz & Janvier, 1996, p. 128. Emphasis in original). In their study, where the students (12-13 year olds) solved three problems that involved equations, the main obstacles were:

- direct generation of an equation using a single unknown;
- substitution, which requires the passage to a single unknown;
- refusal to operate on the unknown;
- the students' symbolism used to present the relationships in the equation.

From a more theoretical perspective, Balacheff (2000) claims that arithmetic and algebra at school might be in competition for the same corpus of problems. If a student recognizes a problem as an arithmetic problem, then there is no need to use algebra when what he/she already knows and is familiar with (arithmetic) is enough to solve the problem. Therefore, this raises a need for thinking of a corpus of problems for the teaching and learning of algebra that requires algebra and can't be solved with the resources that arithmetic offers. Regarding this point, Balacheff (2000) agrees with Bednarz and Janvier (1996) regarding what they called "connected problems", and what Filloy and Rojano (1989) have called "arithmetical equation."
Differing from Bednarz and Janvier (1996) and Filloy and Rojano (1989), Balacheff (2000) does not focus on equations. Balacheff (2000) proposes to focus on another essential difference between algebra and arithmetic: the system of control. Although the corpus of problems and some tools are shared by arithmetical problems and introductory algebraic problems, arithmetic and algebra differ substantially in their system of control. Balacheff (2000) conceptualizes the transition from arithmetic to algebra as a shift from emphasis on a *pragmatic control* to a *theoretical control* in the solution of problems. In this sense, the anticipatory value of algebra could be seen as one of the possible keys to open the way to algebra (as long as it is not used for trivial problems). Furthermore, Balacheff (2000) highlights another difference between

arithmetic and algebra. On the one hand, in arithmetic the student can establish a parallel relationship between the computation and the referent world at all steps (Balacheff, 2000). On the other hand, algebra seems to require “the need for students to be able to associate meanings with the symbols being used, and to manipulate symbols independently of their meaning” (Balacheff, 2000, p. 253).

Avoid Summing up the different positions, research shows that students’ arithmetical interpretations of concepts are very stable. In this line, Kieran’s (Herscovics & Kieran, 1980; Kieran, 1979, 1981) analysis of the equal sign sheds light on the different roles that the equal sign plays in arithmetic and in algebra. Balacheff (2000) warns us about the choice of problems used for this purpose. The author advises us against using problems in the learning and teaching of algebra that could be solved using arithmetical tools; problems should challenge the student by requiring resources beyond arithmetic (e.g., use of letter to represent a generality, use of algebra to produce a proof, etc.). Ideally, students’ knowledge should be enough to approach the problem but insufficient to solve it, in order to promote disequilibrium, so that it is more likely that the students’ knowledge will develop to adapt to the problem. As discussed earlier in this section, Filloy and Rojano (1989), as a result of studying historical works on equation solving, proposed to divide the realm of linear equations on one unknown in two categories: arithmetical and non-arithmetical. Filloy and Rojano postulate the existence of a cognitive cut, “a break in the development concerning operations on the unknown” (1989, p. 19). In order to promote students’ meaningful learning of equation solving, Filloy and Rojano (1989) developed a sequence of equations to be solved using a geometrical context (areas of squares and rectangles) and an extra-mathematical context (balancing weights). Filloy and Rojano (1989) seem to address Balacheff’s (2000) concern regarding the competition between arithmetic and algebra for the same corpus of problems. Within the realm of equation solving, Filloy and Rojano (1989) identified the set of equations that could be solved with arithmetical tools, and the set that requires an algebraic set of tools. Balacheff's (2000) and Filloy and Rojano's (1989) works highlight the importance and usefulness of carrying out epistemological analyses of the mathematical domain (e.g., identification of arithmetic and algebraic equations). As mentioned before, Filloy and Rojano's (1989) contribution is in the realm of equation solving even though their analysis didn't led to a didactical solution to help children use the models in order to better understand equation solving; an interesting direction for the field would be to analyze and study sets of problems that require the use of algebra as a tool so that arithmetic and algebra problems do not compete for the same set of problems. Some of this will be discussed in *Algebra and Proof.*

Algebra, generalization, and patterns

Work developed by Mason (1996), Lee (1996), and Radford (1996) are illustrative of researchers' approaches to the teaching of algebra through pattern and generalization activities. For Mason, "generalization is the heartbeat of mathematics and appears in many forms" (1996, p. 65). From his perspective, a mathematics classroom, in order to be called so, should be permeated by the students' continuous expression of generality. In his work, Mason (1996) has mainly studied geometric and numeric patterns but "only to provide experiences which highlight the process [of generalization]" (p. 65). Mason (1996) analyzes many of the different forms of generalization that we can find in mathematics. Some examples of his analysis are particularly insightful. For instance, the author examines the case of theorems or propositions, in particular statements and the way they are expressed. Mason presents the example of the sum of the interior angles of a triangle. Let's state the proposition in the usual way: "The sum of the angles in a triangle is 180 degrees." Following Mason (1996), in this proposition the most important word is “a.” The word “a” is describing the whole set of triangles, each and every triangle or, in other words, any triangle. The second most important word is the modifying article “The.” If the triangle is changed, the sum remains invariant. The search and identification of invariants is a typical feature of mathematical work. The fact that the invariant is 180 degrees is of relatively little importance compared to the fact that it is invariant. Mason states: “The essence of the angle-sum assertion, and indeed, I conjecture, of most mathematical assertions, lies in the generality which can be read in it. There is some attribute that is invariant, while something else roams around a specified or implied domain of generality” (1996, p. 68). Another expression of generality in mathematics can be found, according to Mason (1996), in what is usually called generalized arithmetic. The structure of arithmetic, when expressed, produces algebra as generalized arithmetic.

In addition to what Mason mentions, modular arithmetic could be a fertile place not only to express generality but also to use algebraic notation and manipulations to obtain new information. For example, if we want to show *why* when we multiply an even number by an odd number the result obtained will always be an odd number, we can use algebraic notation to explain why. In particular, we need to move from “expressing” through the use of the language of algebra towards problems where the use of algebra is necessary to answer a question, to solve a problem, or to explain why. Mason’s perspective regarding the use of algebra as a tool to solve problems when needed coincides with Balacheff’s (2000) perspective that we need to change the fact that arithmetic and algebra compete for the same corpus of problems in school; Balacheff (2000) agrees with Mason (1996) in proposing to teach algebra using problems where algebra is a necessary tool. Mason (1996), focusing on *how* and *what degree of generality* can be read in an algebraic expression, analyzes the expression “1+3n”, where n is an integer. The expression

"1+3n", where n is an integer, represents any number that divided by 3 gives a remainder of 1. At the same time, it could represent a particular number for a particular n; in a way it is a number, in another way it is the structure of a number; it could also represent the rule to calculate a number. We could generalize even more departing from this expression in the following way: "r+3n" represents the numbers with a remainder "r" when divided by 3 – in the case $3>r\geq 0$-, the expression "r+kn" denotes any number that has a remainder of r when divided by k where $k>r\geq 0$. It would be very interesting to work with generality around the notion of parameter, since in these generalizations not all letters necessarily play the same role. We could consider "n" as a variable, and "r" and "k" as parameters. These issues will be at play in the Calendar Sequence described in this paper.

As explained earlier, Mason also explores the role of examples in the mathematics classroom. An example is not an example in and of itself, but it is an example *of* something broader. When a student is presented with an example, what he/she understands is probably a totality in itself. Until a person can see an example as an example *of* something, it probably has little meaning: "the whole notion of example depends upon and draws out the notion of generality" (Mason, 1996, p. 73). The student needs to construct an example as an example *of* something, and this can be done by reflecting on the particular aspects of the general.

In his search and study of experiences that could promote students' generalization processes, Mason (1996) stresses the intrinsic nature, in a generalization process, of the relation between the particular and the general. In order to promote awareness of the general, the author proposes the distinction between *looking at* and *looking through* when students are working on a sequence of exercises or problems. It is important to promote students' awareness of seeing *the general in the particular* and seeing *the particular in the general*. One of his suggestions, in order to address the articulation between the general and the particular, is to work explicitly on a set of exercises or problems as a whole and not just working *through them.* Mason (1996) criticizes sets of exercises that promote students' mindless practice. To move beyond this practice, we must study the problem as a whole. Let's exemplify with one activity that is representative of Mason's approach to algebra. The picture below (see Figure 8) shows a rectangle made up of two rows of four columns and of squares outlined by matches. How many matches would be needed to make a rectangle with R rows and C columns? (Mason, 1996, p. 80).

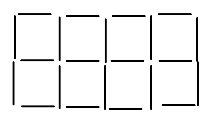


Figure 8. Rectangle made up of matches (from Mason, 1996, p. 80).

Mason (1996) claims that pattern generating and generality expressing might be more than appears at first sight. In a study with teachers where they had to solve the rectangle problem described above (see Figure 8), Mason found that there were many activities that students could learn from interacting with this problem. One of these activities was "multiple seeing," or finding several ways to see how to count the number of matchsticks and expressing these as general formulas. Another activity was "reversing seeing," or taking an equivalent arrangement of the expression and trying to arrange the counting process according to it. A different activity that the author mentions is "doing and undoing," taking doing as finding the number of matchsticks for a given number of rows and columns, and undoing as deciding if a given number of matchsticks was possible, and trying to characterize the form that such a number of matchsticks might take. For Mason (1996), the key in algebra is the interplay between the particular and the general towards generalization awareness in mathematics.

Lee (1996) has also assigned a central role to generalization activities in the initiation to algebra. However, Lee (1996) understands algebra as a mini-culture and the initiation into a culture is the first step in a long process. This acculturation process requires the learning of:

> What is sayable, what we talk about and how we talk about; what we do not talk about; what level of formality is used in various writing situations; what experiences and words are untranslatable; what are the gestures and symbols, the worlds of sense around objects, dates, rites; what are the sacred cows; what is the shared history of institutions, families, communities; what are the objects of thought; what is funny and what it isn't. (Lee, 1996, p. 89)

Lee (1996) proposes being initiated into this culture through generalization activities. She draws conclusions based mainly on two problems, the *consecutive numbers* problem, and the *dot rectangle* problem. In the consecutive numbers problem students are asked to: "show, using algebra, that the sum of two consecutive numbers (i.e., numbers that follow each other) is always an odd number" (Lee, 1996, p. 90). The dots rectangle problem is expressed as:

> The drawing on the left (see Figure 9) represents a set of overlapping rectangles. The first contains 2 dots. The second contains 6 dots. The third contains 12 dots. The fourth contains 20 dots. How many dots in the fifth rectangle? How many dots in the hundredth rectangle? How do you know? How many dots in the n^{th} rectangle? How do you know? (Lee, 1996, p. 90)

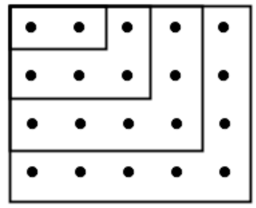


Figure 9. Overlapping Rectangles (from Lee, 1996, p. 90)

In the analysis of these two problems, Lee (1996) found that knowledge and practices that she had taken for granted were not available for a group of adult students taking a prerequisite algebra course at Concordia University in Canada. For example, when solving the dots problems, a student kept counting the dots, one by one, instead of counting by columns or rows using multiplication. For the researcher, the selection of the variables was "transparent," however, for that student it was not obvious. The researcher was also surprised by the fact that another student would not accept a single dot as a one by one rectangle: "she [the student] made us realize just how much our perceptions in algebra have already been trained and our potential for 'seeing' had essentially been whittled down over the past years" (Lee, 1996, p. 100).

An interesting finding in Lee's study (1996) was the fact that, after accepting that x and x+1 are any consecutive numbers, some students seemed to think of x+1 as always being an odd number, while x was definitively always even. Another finding was the fact that students were reluctant to use more effective strategies. For example, after one student showed that x-1, x, and x+1 were more efficient representations for three consecutive numbers for some problems, the other students did not spontaneously use this strategy. Lee (1996) shows that the students could "see" patterns perfectly; the problem was not "seeing" a pattern, but identifying a useful algebraic pattern. In addition, Lee (1996) claims that a generalization approach to algebra "immediately threw students into using letters as variables" (p. 105). Like Mason (1996), Lee (1996) considers that the interplay between "the specific" and "generalizing" is at the core of algebra. One of the meanings that can be given to equivalent expressions, for instance, is that of generating the same pattern. The difficulties that Lee (1996) encounters in this approach are obstacles at the perceptual level (seeing the intended pattern), at the verbalizing level (expressing the pattern clearly), and at the symbolization level (using n to represent the n^{th} array or number and then representing the number of dots in terms of this). Lee (1996) agrees with Mason (1996) when he claims that flexibility must be developed; therefore, this flexibility should be part of the school curriculum, since not all pattern perceptions are equally useful and in arithmetic flexibility is almost unnecessary.

Radford (1996), in his analyses of the work by Mason (1996) and Lee (1996), reminds us of carefully thinking about the role that we attribute to generalization in the teaching and learning of algebra. Radford (1996) states that a superficial look at the history of mathematics may lead us to conceptualize mathematical activity exclusively under the process of generalization. Radford's (1996) message is that there is more to mathematics than generalization: generalization depends on the mathematical objects that we are generalizing on. The author discusses two particular issues in geometric-numerical pattern generalizations. The first is related to the generalization of results, and the second has to do with the role that representations play in these kinds of patterns. Regarding the first issue, Radford (1996) points out that one of the goals of generalization is to produce a new result[12]; it is in this sense that generalization is not a concept but "a procedure allowing for the generation, within a theory and beginning with certain results, of new results" (p.108). In this generalization process, we cannot avoid the problem of validating a new result, as we usually do in mathematics. If we are working with generalization as a didactical device we should also be prepared to address the validity of the result.

[12] When considering the sequence of square numbers represented using points and squares, the new result is that the number of points arranged is a perfect square (square number).

In terms of the external representational system, representations are not independent of the goal of the activity or problem; they require an anticipation of the goal: “The problem that now arises is that of knowing which facets of the object should be kept in its representation” (Radford, 1996, p. 110). This last observation relates to what Mason (1996) and Lee (1996) refer to as something to be developed or learned by the students: pattern flexibility, moving among patterns, and seeing a useful algebraic pattern. In the majority of patterns presented by Mason (1996) and Lee (1996), the anticipation of the goal is related to the recognition of the variables involved in the particular pattern; i.e., in the matchsticks problem, the recognition of the rows and columns as variables is key in the development of a useful algebraic formula.

Generalization activities have proven fertile to promote students’ work with variables and algebraic expressions (Lee, 1996; Mason, 1996). Radford (1996), however, warns us that generalization is an activity, and that the product of that activity needs to be validated in mathematical terms.

Algebra and Proof

Unlike other domains such as algebra and functions, and algebra and modeling, the area of algebra and proof has been under-researched. In the United States, this might be a consequence of the fact that proof is almost exclusively studied in geometry courses in high school. This is not the case in England (Healy & Hoyles, 2000; Healy, Hoyles, Sowder, & Schappelle, 2002) and France (Barallobres, 2004). Many researchers (e.g., Balacheff, 2000; Barallobres, 2004; Brousseau, 1997; Mason, 1996a) have argued for the need to teach algebra using problems where algebra is a necessary tool to solve the problem, and one way of doing this is using algebra to prove.

For Chevallard (1985; , 1989; , 1989-1990), algebra is a modeling tool for: (a) setting up expressions representing the relations among variables; and (b) producing equivalent expressions that allow to read and infer properties that could not be read in the initial expression. These are two distinct features that define part of what algebra is and that separate algebra from arithmetic and other sub-domains within mathematics. If we could incorporate these distinctive features into the design of problems, we could explore the benefits of such an approach to algebra.

Barallobres (2004) has invested in this perspective towards algebra as a tool to model and to obtain new information through algebraic transformations. From an epistemological perspective, the algebra-proof approach seems promising since the approach to problems encourages students to use algebra as a tool, at least emphasizing aspects that are central to algebra in order to solve problems. The central issue in Barallobres' (2004) perspective is to profit from two distinctive features of algebra: algebra as a modeling tool and the use of algebraic transformations to obtain information that could not be read in the initial expression. This process of obtaining new information by transforming and interpreting algebraic expressions allows students to understand the reasons that make a proposition true. Since algebra becomes a tool to access the truth-value of a proposition and the reasons that make it true, algebra becomes a tool to prove.

Hoyles and Healy (Healy & Hoyles, 2000; Healy, Hoyles, Sowder, & Schappelle, 2002) carried out a large scale study on proof conceptions in algebra in the United Kingdom (UK). The study included a 70-minute survey administered to 2459 students from 94 classes in 90 schools. Students were high-attaining 14- and 15-year-olds. The study also included teacher and school surveys. The student survey included multiple-choice questions where the students had to choose the proof that would obtain the best mark, and the proof that would be closest to what they would do. This part of the survey was intended to provide information on students' views of what constituted a proof, its role, and its generality. In the second part of the survey, students were asked to construct their own proofs to offer insight into their competence in constructing proofs. Regarding the results of the multiple-choice questions, they show a big difference between students' choice for their own approach and their choice for the proof that would get the best mark. In fact, it turned out that the arguments that were the most popular for the students' own approaches were the least popular when it came to choosing for best mark, and vice versa. Along the same lines, students judged that their teachers would reward any argument, provided it contained some "algebra." From the survey administered to teachers, they appeared to overestimate the extent to which their students would make judgments that were based on mathematical content rather than simply on form.

The authors also found that students were much better at choosing correct mathematical proofs than at constructing them. The results also show that when students were asked to construct a proof for an unfamiliar statement, only 3% of them managed to produce a complete proof. The most popular form of argument was to provide empirical examples; if students tried to go beyond this pragmatic approach, they were more likely to give arguments expressed in an informal narrative style than to use algebra formally. These findings highlight the need to research the area of proof in algebra in order to understand why students don't use algebra when constructing their own proofs, and to develop new ways of integrating algebra and proof in the curriculum in such a way that students feel confident using algebra when proving. Hoyles and Healy (2000) state that,

> Although arguments that included algebra were the most popular among students for best mark, our results show that students knew that they would be highly unlikely to base their

> own arguments on similar algebraic constructions. In both multiple-choice questions, the algebraic arguments were the least frequently selected as the closest to the approach students would use, and algebra was used rarely as the language through which students attempted to write their own proofs. (p. 413)

A very interesting finding is that arguments that incorporated algebra were most likely to be viewed by the students neither as showing that the given statement was true nor as representing an easy way to explain to someone who was unsure about the truth value of a proposition. This is another aspect of the algebra-proof relation that needs to be investigated. Following the results of this study and regarding the explanatory power of algebra, it seems that students were put off from using algebra because it offered them little in the way of explanation; they were uncomfortable with algebra arguments and found them hard to follow. More than a quarter of the students in the sample had little or no idea of the meaning of proof and what it was for. The results show that the mathematics community needs to address the learning and teaching of algebra in relation to proof. It is surprising that we are missing one of the most important aspects of algebra: its explanatory power. We need to develop new ways of linking algebra and proof for students to experience the power of algebra as a tool.

Barallobres (2004), inspired by the French tradition in mathematics education, relies heavily on a careful design of the problems used in a teaching experiment. Barallobres (2004) designed a sequence of problems in a way that when students first explore the problem, the feedback from the teaching situation contradicts students' anticipations. This type of situation is not easy to design. Barallobres proposes an introduction to algebra through a proof perspective. To develop his work, Barallobres adopts Balacheff's (1982; , 1987; , 1988) categorization of proofs as *intellectual* and *pragmatic*. Regarding the roles of proofs, Barallobres adopts Arsac's

(Arsac et al., 1992a, 1992b; Arsac & Mante, 1997) position that proofs in a school setting can work in three different ways: (1) to decide, (2) to convince, and (3) to comprehend and know. Barallobres points to the fact that while a mathematician uses proof to convince himself about the truth value of a proposition, students can use other means to be *sure*, but not necessarily to know *why*. Within the process of using proof to introduce algebra at school, Barallobres (2004) views two dimensions that should be taken into account when designing the intervention: (1) construction of a proposition, and (2) the construction of the proof itself. The first dimension helps in promoting students' ownership of the task. Barallobres' (2004) goal in his teaching experiment was for students to move from the production of *pragmatic proofs* towards the production of *intellectual proofs*. The author designed the tasks in such a way that the search for reasons was linked to resolving a contradiction. In his work with a group of eight 12 year-old students, the one class was organized into fours parts. During the first part of the class, students were arranged in two groups of four students. Each group had to choose two natural numbers, the second number smaller than 3000. The goal of the game was to obtain the biggest number by carrying out the following set of calculations:

1. Multiply the two chosen numbers;
2. Add seven to the first number, and multiply the result by the second chosen number;
3. To the result obtained in step 2 take away the result obtained in step 1.

The group that obtained the biggest number would win. During this first part of the class, students played many different rounds until they got a sense of how the game worked. During the second part of the class, students were asked to produce a strategy that would allow them to win all the time. Students were also asked to explain why this happens. A goal of this phase in the teaching experiment was to make students realize that there are infinite solutions to the problem. During the third part of the class, students were asked to explain why there are infinite solutions. During the fourth part of the class, students were asked to explore how they could be sure that if we assigned 2999 for the second number, no matter what number we chose for the first one, there would be no other winning solution. Regarding the general findings of Barallobres' (2004) work, the mathematical task proved to be highly effective at prompting students to experience a contradiction between their expectations (finite solution vs. infinite solutions, dependence on two variables vs. dependence on one variable) and what happened when trying with specific numbers. The contradiction proved to motivate and encourage students to search for why and how this happens. Students' experience with contradiction led them to reflect on their actions. Barallobres found that the preferred strategy both in private and public student work was the use of a general example (see Balacheff, 1982; 1987;1988). The public interactions among students helped them to better understand the mathematical relations through their explanations. Another issue that this mathematical task helped accomplish was that students' work was more oriented towards understanding *why* and not only centered on determining the truth-value of the proposition. Students were easily convinced about the truth-value of the proposition but they were not convinced about why that was the case. The mathematical task proved effective in promoting students' experience of proof as a tool for explaining and answering.

The scarce research in the joint domain of proof and algebra described above and the poor performance of high-achievers in the UK described by Hoyles and Healy (Healy & Hoyles, 2000; Healy, Hoyles, Sowder, & Schappelle, 2002) point to the need to investigate what is happening in this domain. In addition, Barallobres' (2004) findings have proved that mathematical tasks involving the use of algebra and proof as tools are a fruitful context where students can learn to prove using algebra meaningfully. Approaching algebra through proof seems to offer a solution to the concern of many researchers (e.g., Balacheff, 2000; Barallobres, 2004; Brousseau, 1997; Mason, 1996a) regarding the lack of sets of problems where algebra becomes a necessary tool to solve a problem. Through an integrated approach to algebra and proof, we have been able to highlight at least two distinctive features of algebra: as a tool for modeling, and as a tool to read information after applying algebraic transformations. In this way, we see how Barallobres (2004) embraced Chevallard's (1985; , 1989; , 1989-1990) epistemological analysis of algebra, where he identifies two central aspects of algebra: as a modeling tool by setting up expressions representing the relations among variables, and as a tool to represent the production of equivalent expressions allowing to read and infer properties that couldn't be read in the initial expression.

Functional perspective

Here, I will explore studies that take a functional perspective to algebra. A vast majority of these studies based their proposal on the use of technology and software, sometimes specially designed for those purposes (Heid, 1996; Kieran, Boileau, & Garançon, 1996; Moschkovich, Schoenfeld, & Arcavi, 1993; Rojano & Sutherland; Yerushalmi, 2000; Yerushalmi & Schwartz, 1992; Yerushalmi & Schwartz, 1993; Yerushalmi & Shternberg, 2001). These studies emphasize the power of technology to work simultaneously with different representations of functions (tabular, algebraic expressions, and graphs). However, the functional approach is not exclusive of projects that use computational environments (the work by Chazan [2000] and Doaudy [1999] are examples of the latter). The common aspect among all these projects that focus on functions is that students' work on multiple representations is highly emphasized.

In the next sections, I will discuss Heid's (1996), Schwartz and Yerushalmy's (1992), and Kieran, Bolieau, and Garançon's works (1996), in that order. Later, I will discuss the work by Chazan (2000) and the work by Doaudy (1999).

Heid's project (1996) addresses the introduction of algebra at school using a functional approach, considering the concept of variable as central. Heid (1996) states, "what makes the study of variables interesting is the study of functions on those variables" (p. 239). In this approach, the variables are used to describe real world quantities and functions to describe the relation among those quantities. Students study families of functions, their properties, and their relation to the real world, analyzing the meaning of various rates of change, roots, maximum and minimum values, and asymptotic behavior in contextual settings. The design of the activities requires that students solve the problems using multiple representations: graphical, numerical, and tabular. For Heid, *real world* context is a main pillar in her proposal.

Heid's group designed a beginning algebra curriculum for seventh, eighth, and ninth graders. In terms of the role of technology, in this project the computing tools (function graphers, curve fitters, table generators, and symbolic manipulators) were supposed to facilitate "explorations of algebra by providing students with continual access to numerical, graphical, and symbolic representations of functions, as well as to technology-intensive procedures for reasoning about algebraic expressions" (Heid, 1996, p. 240).

The curriculum is structured in the following way: (1) variables and functions; (2) calculators, computers, and functions; (3) properties and applications of linear functions; (4) quadratic functions; (5) exponential functions; (6) rational functions; (7) algebraic systems; (8) symbolic reasoning: equivalent expressions; (9) symbolic reasoning: equations and inequalities.

The authors present as one of the positive aspects of this curriculum the fact that Computer-Intensive algebra (CIA) does not focus (or include) by-hand symbolic manipulation as a formal part of the curriculum. However, the authors should be cautioned against replacing "manipulation" with their curriculum. The ideal situation would be an integration of both manipulation and a CIA. Heid (1996) stresses the fact that the CIA curriculum is designed to help students develop a solid understanding of why such rules are needed and of graphical and numerical meanings of equivalence of expressions. In terms of the required features of the software, following Heid, what is missing from today's algebra computer systems is the ability to translate from graphs to symbolic rules.

Heid's project was evaluated in terms of the *written*, *taught*, and *learned curriculum*. Regarding the analysis of the *written curriculum*, the CIA curriculum was compared with a popular 1960s textbook and a 1980s text used in a pilot CIA school. It was found that the current CIA curriculum asked for more complex questions than the other two texts. Considering the *taught curriculum*, the implementation of the CIA curriculum seems to lead not only to different classroom activities but also to a different set of roles, responsibilities, and challenges for teachers (e.g., facilitator, technical assistant, catalyst) and students (e.g., new responsibilities, new goals). The CIA classes seemed to spend more time than traditional classes on conceptualizing problems, on planning solutions, and interpreting answers. In the CIA classes, different content was managed in the whole group discussion, with more talk about applied problems, more comparison of different representations, and less time spent discussing step-by-step procedures. Taking into account the *learned curriculum*, students in the CIA curriculum performed significantly better than students in traditional classes. Whereas the traditional class students developed a concept of the letter as unknown, a majority of the CIA students developed a concept of the letter as variable as well as of unknown. The study found that students in this curriculum chose and used computers with significantly greater frequency than the scientific calculator and paper-and-pencil. Students chose and used symbolic representations three times as often as they chose and used tables and graphs. Students also showed that they were able to use different strategies with a single tool and representation.

Other researchers like Schwartz and Yerushalmy (1992) also developed and used computational environments for the introduction of algebra at school. Schwartz and Yerushalmy (1992) developed three different computational environments to address the teaching and learning of algebra: the Function Analyzer, the Algebraic Supposer, and the Function Comparator. Different from Heid's work (1996), Schwartz and Yerushalmy (1992) propose many tasks within a *mathematical* context. For Heid (1996), these would be considered tasks without context and with no relation with the real world; *real world* context is a main pillar in her proposal. Schwartz and Yerushalmy's (1992) perspective is to work with functions both as process and objects. The authors link function as a process with the symbolic expression, while they link the graph of a function to the object perspective. The authors propose to accomplish this process-object duality encouraging students to carry out unary and binary operations on functions. Schwartz and Yerushalmy (1992) consider functions as the central object of algebra, and the concept of variable. Consequently, the computational environments designed by Schwartz and Yerushlamy (1992) allow students to study the effects of various unary and binary operations on functions using different representations as well as to see the consequences of their activities both symbolically and graphically. One of the limitations of other software currently available is that it is not possible to directly manipulate any representation other than the symbolic representation. Schwartz and Yerushalmy (1992), similar to Heid (1996), claim that a fully symmetrical environment would allow users to both manipulate the symbolic representation (symbolically) and see the graphical consequences of their actions, and to manipulate the graphical representation (graphically), and see the symbolic consequences of their actions.

As mentioned above, it seems that Schwartz and Yerushalmy (1992) think that the symbolic representation of a function reveals its process nature, while the graphical representation helps to make the function more entity-like. While Heid (1996) uses extra-mathematical contexts[13] in problems, Schwartz and Yerushalmy (1992) focus on unary and binary operations on functions. Within the unary operations, the authors take into account: (1) translation, from f(x) to f(x+a), and from f(x) to f(x)+a; (2) dilation and contraction, from f(x) to f(ax), and, from f(x) to af(x); and (3) reflection, from f(x) to f(-x), and, from f(x) to –f(x). The computational environment designed to work on this kind of problem is the Function Analyzer. The student can appreciate the result of the operations through looking at the graph of the function together with the symbolic expression that appears below the graph.

In addition, Schwartz and Yerushalmy (1992) propose to approach binary symbolic operations on functions; the goal is to provide students with the experience of transforming functions so that they are not constrained to remain in the same family of functions. For this purpose, Schwartz and Yerushalmy (1992) develop the Function Supposer. The Function Supposer includes the four typical arithmetic operations (addition, subtraction, multiplication, and division) as well as the composition between functions. An interesting feature of this software is that all the manipulations can be done without having to enter any symbolic expression. In order to promote students' work on real world problems, Schwartz and Yerushalmy (1992) designed the Algebraic

[13] Extra-mathematical context is used in Chevallard's (1985, 1989) sense. If the mathematical model is constructed on a mathematical system we say that the context is intra-mathematical. In the case that the system is formed by non-mathematical objects we will refer to the context as a extra-mathematical.

Supposer. Using this software, students are required to write the information of the problem in the following categories: (1) how many, (2) what, and (3) notes. Students are in charge of defining the relations among the given data, and, as a result, students produce a graph. In addition, Schwartz and Yerushalmy (1992) developed the Function Comparator. Within this software, the work on equations and inequalities is interpreted as a comparison of functions; for instance an equation is seen as f(x)=g(x), and, an inequality as f(x)>g(x), with all possible variations. Students specify the two functions to compare and the software provides the solution set in the x-axis. Regretfully, in their paper, Schwartz and Yerushalmy (1992) emphasize the design aspects and the pedagogical approach more than the presentation and analysis of students' work. Nonetheless, some descriptive data is presented to exemplify students' productions. The mathematics education community could enormously benefit from more research on the use of this pool of software in schools.

Like Schwartz and Yerushalmy (1992), Kieran, Bolieau, and Garançon (1996) analyzed the introduction of algebra using a computational environment they designed. Kieran, Bolieau, and Garançon's work (1996) encompasses a deep and careful analysis of the mathematical content (algebra, problem solving, functions, and variables), the CARAPACE computational environment (Contexte d'Aide a la Résolution Algorithmique de Problemes Algébriques dans un Cadre Évolutif), and students' learning (including pedagogical and psychological issues). This project (Kieran, Boileau, & Garançon, 1996) includes six studies carried out during a six year period. Kieran, Boileau, and Garançon (1996) have opened up a new avenue whereby students have been shown to be able to develop a deeper understanding for the process of translating problem situations into notational representations without the acquisition of equation solving skills. This approach is innovative since in schools, algebra is synonymous with equation solving.

Regarding their perspective on using a functional approach to algebra, Kieran, Boileau, and Garançon (1996) claim that their approach does not necessarily entail the study of functions. However, it does entail the use of letters as variables in opposition to the use of letters as unknowns. Nonetheless, the authors acknowledge that a functional approach comprises more than that. It includes viewing a function from the perspective of the relationship among the x- and the y-values. This seems very subtle and could be read as a contradiction, but the authors are referring to the dual perspective of function as an object and as a process. Kieran, Boileau, and Garançon (1996) explicitly state that while the study by Schwartz and Yerushalmy (1992) focuses on functions as object in addition to functions as process, in Kieran, Boileau, and Garançon's (1996) curriculum the main perspective is to deal with functions as process. The authors' emphasis is on the process, as the set of mathematical operations to be applied to the independent variable in order to produce the dependent variable. Kieran, Boileau, and Garançon (1996) give a clear example of this approach, intended to emphasize the process. When students confront problems of the type: "The price of an object after a tax of 15% is $23; what was its price before tax?", their first attempt is to perform an arithmetic operation. In contrast, when they are faced with a slightly different version of the same problem, for instance: "If we know the price of an object before tax, describe how to calculate its price after a tax of 15%", a definite improvement in the subsequent solution attempts by the student is observed. Students shift from an arithmetical-unknown version of the problem to an algebraic-functional process-oriented perspective.

Regarding their conception of algebra, Kieran, Boileau, and Garançon (1996) were inspired in their work by the history of algebraic writing, taking into account different periods and stages, going from a rhetorical language in the solution of problems to a modern symbolic representation. The software design is inspired to support the students' development of this wide range of algebraic writing. Within the functional approach, problems involving the letter as an unknown can be re-conceptualized constructing equations with the functions at play. Another aspect that the authors consider intrinsic to algebra is the role of generalization. Consequently, setting up the general functional relation in the form of an algorithm involves a certain amount of generalizing for beginning algebra students.

In this section, I will describe the CARAPACE environment and provide a sample problem since one of the goals of this literature review is to provide the aspects of algebra that each work highlights; in addition, it seems fruitful to illustrate the approach through sample mathematical problems. The software's goal in terms of students' learning is to strength the algorithmic aspects of their algebraic language. In this context, the student must represent a functional situation in the form of a program that tells the computer how to perform certain arithmetic calculations. This environment (see Kieran, Boileau, and Garançon, 1996) has been used by 12 to 16 years old students during a period of six years. For example, students are presented with the following problem:

> "Carine works part time in the neighbourhood. She sells subscriptions to a magazine. She earns \$20 a week, plus a bonus of \$4 for each subscription sold."

Within CARAPACE, students find a first screen where they are asked for the input values, operations and nature of output values (see Figure 10). In Figure 11, a potential student answer for the subscriptions magazine problem stated in Figure 8 is shown.

Request values for:

Carry out these calculations:

Show values of:

Figure 10. First screen where students are asked for the input values, operations, and nature of the output values.

Request values for:

Number of subscriptions

Carry out these calculations:

Number of subscriptions x 4 gives total bonus
20+total bonus gives total salary

Show values of:

Total salary

Figure 11. Potential student answer for the subscriptions magazine problem.

Regarding other features of the software, the environment does not allow the use of the equal sign. After completing the first screen, the student can try with numerical examples that are kept in a table and a graph. The highest syntactical level that CARAPACE accepts is the usual set of expressions used in elementary algebra, such as: "ax^2+bx+c gives y", without incorporating the use of the equal sign. The absence of the use of the equal sign might present an obstacle in students' understanding of algebra and later learning. The equal sign is a foundational symbol in mathematics that conveys different meanings as discussed before in the second section of this paper. Avoiding the use of the equal sign evades the arousal of the difficulties about the different meanings of the equal sign; it doesn't solve the obstacles students face when learning about the equal sign.

Regarding the graphical interphase, Kieran, Boileau, and Garançon (1996) chose to show the graphs in a discrete domain, not in a continuous way like most other software. The CARPACE environment allows the user to plot points (it doesn't plot the function) belonging to the graph of the functions; that is, the user must specify the points to be plotted, and must assist the computer in plotting the points in question. One of the main goals of the use of Cartesian graphs in CARAPACE is not only to help students in the solving of problems but also to allow for the discussion of particular questions, such as the number of solutions to a problem. In this environment, the cursor is an "adapted" cursor, it looks like another pair of axis and it is where students choose the components of the points to plot, see the scale of the graph, and modify it. A feature that this environment shares with others is the use of several representations as an integral part of a functional approach to algebra problem solving. In CARAPACE, the input is always the algorithm entered by the student; based upon the input the environment makes the graph and the tabular representation available. The user cannot change the graph or the tabular representation and cannot observe the changes in the algorithmic representation. This is a big disadvantage of this computational environment. I believe that one of the main advantages of using software in a functional perspective is that software facilitates working with multiple representations at once; it is very powerful to change one of the representations and see how this change impacts the other representations (we have seen that this is not the case for the suite of software developed by Schwartz & Yerushlamy [1989])

Kieran, Boileau, and Garançon (1996) found that the substitutions of numerical test values into algebraic representations of problems allows beginning algebra students to construct meaning for problem representations that may be different from those they have experienced in the past, while at the same time using solution methods based on familiar arithmetic techniques. The results showed that when students moved to the tabular display, they tended to forget the contextual information they had previously been relying upon. In addition, students exhibited an immediate ease in taking a functional approach and writing the algorithmic representation using forward operations.[14] The learning of algebra with CARAPACE seems to improve the transition from a more "natural" language to more standard algebraic representations. Carrying out initial numerical trials would appear to be extremely helpful for students to make sense of the word problem, to represent it, and to solve it. The use of significant names for variables can assist students in retaining the sense of a problem and in performing operations such as substitutions. However, this study uncovered the lack of any real motivation to simplify expressions in a computer-supported environment like CARAPACE. Students do not feel the desire to shorten their procedural representations because of the speed of the computer. The computer tool CARAPACE has supported the potential of algebra as a problem-solving tool. Not all technology-supported roads that intended to be algebraic lead to developing meaning for traditional algebraic representations and transformations. Introducing algebra with this kind of environment provides students with only part of the picture of algebra. The students' heuristic search for multiple solutions was much more effective when they used only a table of values representing the functions. Students gradually understood that there can be more than one solution to a problem. Students acquired ease in changing and thinking of different scales in the axis. Students rapidly improved in the development of successive approximation strategies in the graphical context. In cognitive psychology, thinking about variables is usually considered more complex than a single-value conception, as if it were a matter of development (Kieran, 1996). Kieran, Boileau, and Garançon's (1996) study clearly shows that beginning algebra instruction with a variable interpretation of the letter and later including single-valued situations seems to avoid the cognitive obstacles that can be encountered when one begins with a single-valued conception of letters and attempts to move on to a multi-valued interpretation. Consequently, Kieran, Boileau, and Garançon's (1996) work encouraged seeing the particular in the general and the general in the particular; in this aspect the authors agree with Mason (1996).

So far, in this section I have discussed a variety of works (i.e., Kieran, Boileau, & Garançon [1996], Schwartz & Yerushlamy [1989], and Heid, [1996]) within the functional approach to algebra that share the use of a computational environment to teach algebra at school. Next, I will discuss Chazan's (2000) work, and Douady's (1999) study. Both researchers used a functional approach to algebra, but without centering on the use of software.

In this section, I will discuss the work that Chazan (2000) developed in the Holt school[15] using a functional approach to algebra. Regarding the notion of function, Chazan (2000) chooses to

[14] Forward as opposed to inverse arithmetical operations that are usually used when solving an equation of this type (i.e., ax+b=c). Students first subtract b and then divide by a. In forward operations, required by the algebraic way of writing, we have to write multiplying by a and adding b.

[15] Holt was described as a suburban setting and Chazan taught a lower-track Algebra One course.

emphasize, "that functions are relationships between quantities where output variables depend unambiguously on input variables" (p. 84). Moreover, following Comte's definition,

> By his definition, functions are the mathematization of our theories about the relationships of dependence, causation, interaction, and correlation between quantities. Auguste Comte, in his early study of the nature of different sorts of knowledge, sees such theories as methods for determining the values of quantities inaccessible to measurement. (Chazan, 2000, p. 84)

The structure of the curriculum was as follows: (1) reading sketches of relationships between quantities, rather than the traditional introduction to graphing, which emphasizes that each point has exact coordinates; (2) working on number recipes; (3) making traditional coordinate graphs from tables of values and reading computer displays from *The Function Supposer* (J. Schwartz, Yerushalmy, & EDC, 1989); (4) working on families of functions; (5) linear functions; and (6) standard algebraic manipulations. Regarding the results of his work, Chazan (2000) found that even though students may have come to recognize relationships between quantities as important mathematical objects, understanding their relevance, that does not mean students saw why facility with one representation of these objects such as algebraic symbols was considered so important. Chazan (2000) presents results concerning problem about a linear function (Figure 12) given to students at Holt as part of their final exam. Students found this task quite challenging and they did not perform as strong as it was expected. After the exam some students were interviewed about this problem. The results were mixed. Students could do thoughtful work about the problem beyond what they have written in their exams. However, students couldn't produce the right answer. Students made progress understanding the slope-y-intercept form and treat mathematical tasks meaningful. However, we still have much work to do in tasks involving the writing of linear expressions or equations. Students didn't appreciate the purpose of mastering the writing of linear expressions. "Thus, while a relationships-between-quantities approach provides a way of telling students what Algebra One might be about, we have not completed the task of psychologizing the subject matter. Because there are other canonical representations of functions, the issue of justifying to students (in the present tense) the importance of work with symbolic representation must continue to be addressed" (Chazan, 2000, p. 107).

In other countries, like Canada, temperature is measured in degrees Celsius. We are familiar with the meaning of temperatures in Fahrenheit. So when traveling, it is useful to have a rule for changing temperature from Celsius to Fahrenheit.

The temperature at which water freezes is 0 degrees Celsius and 32 degrees Fahrenheit. The temperature at which water boils is 100 degrees Celsius and 212 degrees Fahrenheit. The relationship between temperature Celsius and temperature Fahrenheit is a linear rule.

Using the information given above:

a. Figure out the slope for your rule. Show your work.

b. Write a rule that will do the conversion.

c. Using your rule, is 40 degrees Celsius the temperature of a hot day?

Figure 12. Sample problem to address the function-algebra relation (Chazan, 2000, p. 99).

Douady (1999), also adopting a functional approach to algebra without relying on computational environments, developed a didactical sequence on polynomial functions based on the method of Didactic Engineering (M. Artigue, 1988, 1994; M. Artigue & Perrin-Glorian, 1991; Douady, 1997). Schematically, the basic principles of Didactic Engineering are: (1) chose a teaching object in the current program, (2) place the mathematical context in relation with the teaching tradition, (3) bring out hypotheses about students' difficulties and set the basis for a didactical engineering, (4) develop such an engineering, proceed to the a-priori analysis, (5) implement it and make an a-posteriori analysis of the collected data, (6) reproduce the implementation, under experimental control, after possible modification in view of the previous analysis, (7) test the supposedly acquired knowledge of the students in questions for which they adapted tools, and (8) compare the output of the students and their skill with expectations, and conclude about the relevance of the didactical hypotheses. The goal in Douady's sequence was to integrate the graphical representation in the study of polynomial functions and its roots, multiplicity, and signs. Participants were 15-16 years old French students. In the traditional curriculum, this topic is taught as a mere manipulation of algebraic expressions (or chains of signs) through factoring techniques. No allusion is made to Cartesian graphs of the polynomial as a function, nor to issues related to differentiability and continuity, at least implicitly. In Douady's (1999) proposal, students deal with problems where functions, graphs, and, algebraic expressions "are put in stage dialectically" (p. 113). Sample problems given to the students are shown in Figures 13 and 14 below. The problems seem to be interesting; regretfully, the author doesn't provide information on students' work.

1. *Calculators forbidden*

Giving x numerical values, you will get numerical values for the following expression:

$$f(x) = (x - 2)(2x - 3)(x + 5)(4x + 1)(1 - x)$$

Are they always positive ? Are they always negative question

Are they sometime positive, sometime negative, sometime zero ? Compute.

When you have an answer, call your teacher.

Figure 13. Sample problem to address the function-algebra relation (Douady, 1999, p. 113).

2. *Calculators forbidden*

$$f(x) = (x - 2)(2x - 3)(x + 5)(4x + 1)(1 - x)$$

Find a way which enables you to tell, very fast and reliably, when your teacher gives you a numerical value for x, whether the expression is > 0, < 0 or $= 0$.
Orders : *Only one answer accepted. Computing the expression is not allowed : it is too long.*
When you think you have a method, call your teacher.

Figure 14. Sample problem to address the function-algebra relation (Douady, 1999, p. 113).

The above set of studies within the functional approach shows use of computational environments to introduce algebra through a functional perspective, focus on the algebra and functional polynomials relation, and use of multiple representations in order to promote a better understanding of the concepts. The works under the functional approach have advanced a deeper understanding of the core of algebra (functions and variables) and the symbolic representations (reinterpretation of the letter as a variable, instead of the letter as an unknown). The researchers' innovation is grounded in the reinterpretation of the traditional algebraic objects (unknown, equations, and inequalities) from the perspective of functions. I believe that this innovation pushes for a re-conceptualization of our understanding of algebra within the mathematics education community. We can re-conceptualize: unknowns as specific values of a variable under particular conditions, and equations/inequalities as comparisons between functions. There are two other advantages to this approach. One of them is that the functional perspective calls for a multiple representation approach, since functions traditionally are represented in tabular, graphical, and formulaic forms whenever possible. The other advantage is that this approach calls for an integration of algebra and functions, that are two of the most important topics in the

K-12 mathematics curriculum. However, as Chazan (2000) found, it is still a challenge to design activities that will successfully communicate to students the need for using algebraic tools.

Algebra in the modeling perspective

Regarding the modeling perspective in the research of teaching and learning of algebra, it is not a surprise that there are different perspectives and understandings of the word "modeling[16]." On the one hand, some authors (e.g., Chazan, 1993; Janvier, 1996; Nemirovsky, 1996) employ the word modeling with an implicitly agreed upon meaning that could be understood as creating a mathematical model of a (mostly) non-mathematical reality (extra-mathematical context[17]), better known as a "real world" problem. These authors, however, emphasize different aspect/s of modeling. For instance, Nemirovsky (1996) emphasizes the work on modeling through mathematical narratives; Chazan (1993; , 2000) emphasizes modeling through a functional approach to algebra; and Janvier (1996), besides emphasizing a functional approach, takes the nature of the variables involved as a main factor ("pure" numbers and extensive and intensive magnitudes). On the other hand, other researchers (e.g., Bolea, Bosch, & Gascon, 1999, 2003; Chevallard, 1985; Chevallard, 1989, 1989-1990; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascon, 1993-1994) take the study of the concept of modeling in and of itself, as well as the meaning and implications of conceiving algebra as a modeling tool in the school curriculum, as central matters. That

[16] The literature on modeling in mathematics generally does not distinguish between defining mathematical measures and conjecturing what might be valid relationships among those measures and then verifying or contradicting the validity of those relationships. The word model in science means the latter. In contrast, in mathematics the word model is used to define measures that capture some salient attribute of a situation (Judah Schwartz's personal communication).

[17] I will use Chevallard's (1985, 1989, 1989-1990) classification of mathematical problem contexts: intra-mathematical (mathematical reality) and extra-mathematical (non-mathematical reality) since it is more fruitful than the "real world" vs. "non-real world" distinction, since, "real-world" is more subjective to the specifics features of a person and living conditions than extra-mathematical.

is, they focus on intra-mathematical contexts. The distinction between the two groups of researchers is not clear-cut since the authors in the first group would agree with intra-mathematical modeling activities; however, their emphasis is not on the "algebrafication" of mathematics as presented in the work of the second group of authors.

Regarding Chazan's (1993) perspective, which is a functional perspective to modeling with algebra, he mentions four benefits to such a perspective: (1) it helps students approach the solution of equations in diverse ways and suggests that students should not be limited to the traditional symbolic manipulations for solving equations; (2) the conceptualization of an equation as an equality of functions can be used to clarify the basic terminology of the algebra curriculum: "equations, identities, inequalities, and what are now called relations can all be treated as comparisons of two functions and solved with the same three solution methods" (1993, p. 23); (3) this conceptualization of equations helps clarify the utility of algebra; and (4) the dynamic dependence relationships captured by functions are accessible aspects of real world situations: "thus an approach which views equations as built up from functions which vary, instead of expressions which represent an unknown, may make algebraic modeling more natural and coherent for students" (1993, p. 23). Chazan proposes problems that are "real world problems," for instance, presenting the "bike-manufacturing company" problem illustrated in Figure 15, and adapted from materials provided by the Michigan Department of Commerce in training sessions for citizens contemplating starting a small business.

A bike-manufacturing example

The people who invested in starting this company did not know how to calculate break-even points. They began their business manufacturing bikes to order. They knew that with the shop that they had set up the average cost to make each bike was $60, including labour, materials, and everything else (except for rent and salary for the boss). The rent for the shop was $500 a month. The head of the company was paid $1000 a month salary. They

Figure 15. Sample problem with a real world context (Chazan, 1993, p. 23).

The example in Figure 15 illustrates that a central aspect in Chazan's modeling conception is real-world relevance. In Chazan's (2000) work, a strong Dewey influence can be found and thus understood as part of his epistemology, since one of Dewey's concerns was to provide instruction using objects and resources from the world surrounding students to promote learning with meaning. Nemirovsky coincides in this focus on real world phenomena:

> A shared goal among mathematics educators is having students come to be able to fluently use graphs and equations in the description and interpretation of events in the world. However, as currently taught, often these representations arise out of nothing—and so have to be imposed on students as notations devoid of personal meaning. To change this situation it is essential to identify and nurture the students' domains of everyday experience that may offer a fertile background for the growth of mathematical ideas. (1996, p. 197)

Nemirovsky (1996) differs from Chazan (1993, 2000) in his approach in that for Nemirovsky mathematical narratives are a central aspect to his approach; as mentioned above, for Chazan a functional approach is the central perspective. A mathematical narrative is a narrative articulated with mathematical symbols. An example of a mathematical narrative is the following:

(a) First it rained more and more and it started to become steady (pointing to a piece of the graph in Figure 16).

(b) then it rained steadily (marking piece b of the graph in Figure 16).

(c) then it rained more and more (pointing to piece c of the graph in Figure 16).

(Nemirovsky, 1996, p. 200)

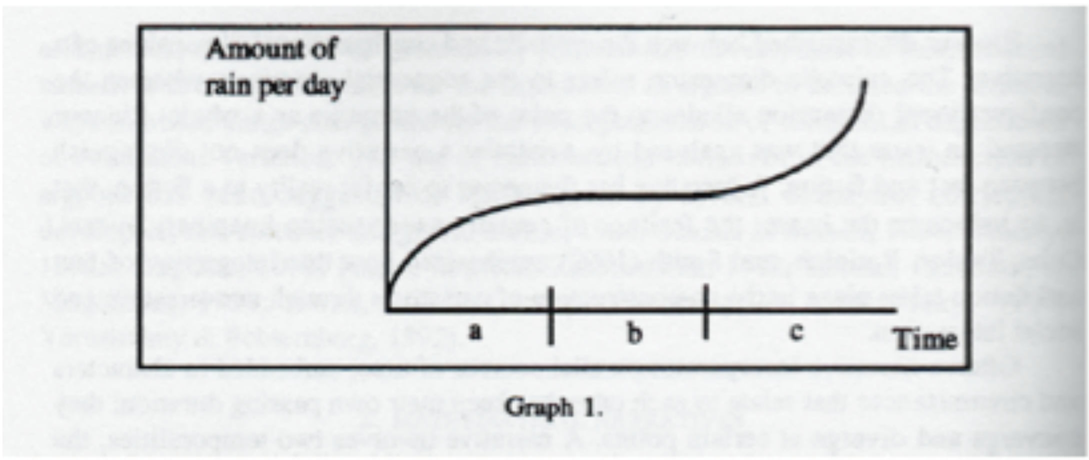


Figure 16. Graph used by Nemirovsky (1996, p. 200) to promote the work with narratives in mathematics.

Another set of studies (Bolea, Bosch, & Gascón, 1999, 2003; Chevallard, 1985, 1989, 1989-1990; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascón, 1993-1994) takes a different perspective towards the modeling approach in algebra. As described previously in *Mathematical activity from a modeling perspective* as part of this paper, Chevallard, Bosch, and Gascón (1997) claim that an essential characteristic of a mathematical activity consists in building a (mathematical) model about systems (intra-mathematical or extra-mathematical contexts) to be studied, to use it, and to produce an interpretation of the obtained results. In others words, the mathematical activity can be characterized as making (mathematical) models of (intra or extra- mathematical) systems. The authors underline three aspects involved in building a mathematical model: the routine utilization of pre-existing mathematical models, the learning of models as well as the way of using them, and the creation of mathematical knowledge. Figure 17, shows an example of a metric model of a geometrical system where the relations involved are S=ab, $d^2=a^2+b^2$, u=arctg(b/a), and v=arctg(a/b).

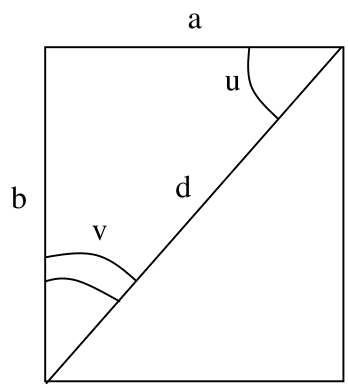


Figure 17. System formed by the two right triangles

This last perspective of mathematical activity coincides with Chevallard, Bosch, and Gascón's position (1997), characterizing mathematical activity as that of constructing (mathematical) models of (intra or extra-mathematical) systems. Regarding the activity of algebraic modeling, Gascón (1993-1994) bases his proposal on Viète and Descartes' works, where the principal feature of algebraic modeling is the systematic introduction of a literal representation to designate the unknowns and the given data, since it provides the advantage of studying a general case and the structure of the problems, and not only obtaining the unknown value. The French-Spanish research group formed by Chevallard, Bolea, Bosch, and Gascón consider that as a consequence of creating an algebraic model, the structure of mathematical problems can be studied as the most important feature of algebra. Gascón (1993-1994) provides an outline of what he calls a new conception of elementary algebra as having the following features: (1) elementary algebra consists of the study of a certain field of problems that contain not only arithmetic problems but also geometric construction problems[18], simple combinatorics to determine a finite set's cardinality, and level sets[19]; (2) the algebraic method provides a global symbolization of the relationships among the given data and the unknowns of the problem without distinguishing essentially among them; in other words, this method's main goal is to make explicit the formal structure of these relationships; (3) the language at play involves symbols that can be interpreted as unknowns, general numbers, variables, and parameters; (4) algebraic manipulation allows us to determine the "existence conditions" of the unknown object, as well as the form of dependence of each variable in relation to the other variables within the system; and (5) the algebraic modeling activity allows us to determine the "existence conditions" of objects other than the ones that we initially wanted to study; as any modeling activity, it allows us to invent new problems regarding the studied system. Along the same lines, Bolea, Bosch, and Gascón (1999; 2003) state that,

> Elementary algebra does not appear as a self-contained mathematical work comparable to other works studied in academic core courses (such as arithmetic, geometry, statistics, etc.), but rather as a modeling tool to be (potentially) used in all mathematical curricular works and which appears to be more or less used in them. (p. 138)

In addition to the features provided by Gascón (1993-1994), Bolea, Bosch, and Gascón (1999, 2003) offer more characteristics:

> In particular, the more a mathematical work is algebraized, the more it enables us to describe the different types of problems that can be solved, as well as the necessary

[18] For instance, "Construct with ruler and compass a triangle ABC given the side c=AB, the height h_C from vertex C, and the median m_A from vertex A."

[19] A detailed description of these problems can be found in Gascón (1989).

> conditions for solutions to exist, their possible uniqueness and their structure.... An indicator of the algebraization degree of a given mathematical work is linked to the possibility of considering, describing and handling the global structure of the above-mentioned problems [various types of problems that arise from a given question].... In an algebraized work, we use parameters and variables systematically. (p.142)

So far we have discussed two different perspectives towards the modeling approach to algebra; on the one hand, Nemirovsky (1996) and Chazan (1993), who consider real world (problem contexts) central to their work, and, on the other hand, the French-Spanish group (Bolea, Bosch, & Gascón, 1999, 2003; Chevallard, 1985, 1989, 1989-1990; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascón, 1993-1994) that emphasizes the modeling role of algebra to create models not only from real-world phenomena (extra-mathematical contexts) but also from within mathematics (intra-mathematical contexts). In comparing these two different approaches to modeling, the one proposed by the French-Spanish research group (Bolea, Bosch, & Gascon, 1999, 2003; Chevallard, 1985, 1989; Chevallard, Bosch, & Gascón, 1997; Gascon, 1993-1994) is broader and at the same time more specific; it doesn't take for granted what we understand to be the objects of algebra[20]; it incorporates parameters as a central component of algebra; and it captures its potential instrumental power linked to its historical roots (Descartes and Viète). The other approach, exemplified by Nemirovsky (1996) and Chazan (2000), deals with a more limited pool of mathematical concepts and objects, and are focused on making mathematical problems relevant (only) by using real world problems. As a closing remark, the French-Spanish group (e.g., Bolea, Bosch, & Gascon,

[20] Chevallard (1997) claims that one of the main tasks of the *didactician* is to care about the principle of "epistemological vigilance;" that is, to analyze the *distance* between the object of knowledge and the teaching object. In order to be taught, an object of knowledge goes through a process of transformations in order to become an object of teaching; this process is known as *didactical transposition*.

1999, 2003; Chevallard, 1985, 1989; Chevallard, Bosch, & Gascón, 1997; Gascon, 1993-1994) push the mathematics education community towards a re-conceptualization of what we understand as algebra. Similarly, the researchers who argue for a functional approach (as presented in earlier) to algebra (e.g., Heid, 1996a; Kieran, Boileau, & Garançon, 1996; Moschkovich, Schoenfeld, & Arcavi, 1993; Yerushalmi, 2000; Yerushalmi & Schwartz, 1992; Yerushalmi & Schwartz, 1993; Yerushalmi & Shternberg, 2001) also pushed the field to re-conceptualize what we understand by algebra (beyond the concept of equation), how we teach it (beyond teaching equation solving), and why we do it that way. The functional approach has highlighted that algebra is not mainly about equation solving but also abotechout variables and functions. In addition, we have now new technological tools that allow students to work with multiple representations of the same object simultaneously.

Concluding remarks

Within the research community on the learning and teaching of algebra, there has been a shift from the importance of considering and introducing the letter as an unknown towards considering the letter as a variable, as well as a complementary shift from considering equations as the main object of algebra towards considering functions as its main object. Regarding the modeling perspective, we find two different conceptions: the approach that emphasizes the use of real world contexts, and the approach that proposes the use of algebra as a modeling tool for any kind of context (intra- and extra-mathematical). Regarding a generalization perspective, research (e.g., Lee, 1996; Mason, 1996) has shown the central role that generalization activities play in the learning of algebra. Problems that foster generalization activities have proven fertile in promoting students' work with variables and algebraic expressions (e.g., Lee, 1996; Mason, 1996). Radford (1996), however, warns us about the fact that generalization is an activity, and that the product of that activity needs to be validated in mathematical terms. Underlying these different approaches to algebra we can identify different epistemologies guiding the research. The French-Spanish group proposes a re-conceptualization of algebra as a modeling tool for extra- or intra-mathematical systems. After describing the different approaches to algebra, one could ask how these different approaches relate to each other: Are they complementary? Are they ordered in a hierarchy? Are they compatible?

In order to answer these questions, I would like to recall Brousseau (1997) and Bell's (1995) work. From a general perspective, regarding mathematics knowledge, Brousseau (1997) states:

> The meaning of a piece of mathematical knowledge is defined, not only by the set of situations in which this knowledge is realized as a mathematical theory (semantic in Carnap's sense), not only by the set of situations in which the subject has come across it as a means of solving a problem, but also by the set of conceptions, of previous choices which it rejects, of errors which it avoids, the economies it procures, the formulations that it re-uses, etc. (p. 81)

From a local perspective, regarding algebra knowledge, Bell (1995) states:

> In general, the approach advocated is to learn the algebraic language in a way similar to that in which the mother tongue is learned; that is by using it in order to communicate, with oneself and with others, in the course of activities defined by the three main modes of activity already described: (a) generalizing, (b) forming and solving equations, and (c) working with functions and formulae. (p. 50)

The presented approaches should be treated as complementary since each approach highlights different dimensions of what we call algebra.

Lastly, this review has shown the scarce research on the algebra-proof relation. On the one hand, Healy and Hoyles (2000) show how poorly UK students performed in the construction of proofs that required the use of algebra. On the other hand, Barallobres' (2004) findings show the potential of teaching algebra in a proof production context. Consequently, this points to the need for further research to be conducted to understand better what learning problems involving both algebra and proof could entail. This paper seeks to respond to this need by addressing the following research question: *what are the consequences of an integrated approach to algebra and proof on students' mathematical knowledge while they work through a didactical sequence (i.e., the "Calendar Sequence")?*

Another issue discussed in this paper is the relation between meaning and technique in the teaching and learning of algebra. As argued in the first section in this literature review, there have been changes in the emphasis placed on technique and meaning within the field of research on the learning and teaching of algebra. After a first emphasis on the importance of technique and procedures to solve equations, there was a shift towards learning algebra meaningfully, accompanied by an underestimation of the role of technique (and formal writing of algebraic expressions). I believe that it is now time to reconcile both meaning and technique through investigations of how these two essential components of algebra can empower each other and promote a meaningful learning of algebra. That is one of the goals of this paper and will be addressed by adopting the calendar as the context for the problems assuming students' familiarity with the relations embedded in the calendar (i.e., two consecutive number in the calendar have a difference of 1), and by a heavy use of symbols and transformations.

Methods

Participants

In this study, a group of 9 high school students (9^{th} and 10^{th} graders, see Table 1) participated in fifteen one-hour-long lessons that took place in Francis W. Parker Charter Essential School in the Boston area, Massachusetts, United States of America. Francis W. Parker Charter Essential School[21] is a public charter school that serves students in grades 7 to 12. It was established in 1995 under the Massachusetts Education Reform Act of 1993, and currently serves 365 students from 40 surrounding towns in north central Massachusetts. Parker is known for its nontraditional educational philosophy; it is a member of the Coalition of Essential Schools, a leading organization for education reform. The school takes its name from Francis Wayland Parker, a 19th-century pioneer of the progressive school movement.

Parker draws about 365 students from the surrounding area in north central Massachusetts. Students come to Parker because they (or their parents) are looking for a better education than the one provided by the local public schools. Applicants often fall into one of two categories: academically successful students frustrated by the lack of opportunity and challenge in the local public schools or students whose personalities, attitudes, or learning styles have proven to be incompatible with the mainstream public schools and are looking for an alternative.

Because Parker consistently receives enrollment applications at a level several times the number of openings available (there were 287 applications for 65 spots for the 2006-2007 school year), admission is by random lottery; some applicants are placed on a wait-list. (The exception is for siblings of current Parker students, who are guaranteed a spot.) Application is open to residents of 70 Massachusetts towns in 46 school districts, but, in practice, the student body is somewhat self-selecting. The school's isolated location in Devens, a decommissioned army base in central Massachusetts, and the lack of school busing, mean that any student attending needs to have a ride to and from school. This makes it difficult for low-income students from nearby urban areas such as Lowell and Leominster to attend.

On its website, Parker claims "the socioeconomic, ethnic, and educational characteristics of the student body closely reflect the general population of the region." While this is true to the extent that the surrounding area is mostly white, there are significant minority and low-income populations in the area that are not really represented. The student body is almost exclusively middle- to upper middle-class white: in the 2005-2006 school year, the student body was 92.6% white, 0% African-American, 4.4% multi-racial, 2.2% Hispanic, and .8% Asian. Only .5% of students came from low-income families; 8.8% had Special Education needs (IEP), compared to 16.9% in the state, while an additional 15.1% had 504 Plans.

The faculty members of the mathematics and science department developed their own integrated science-mathematics curriculum. In addition, the school had mixed-aged groups instead of the traditional grade levels. Thus, all students were in the 9th/10th grade group. The group of students was diverse in terms of their mathematical performance and gender. There were four female and five male students. They worked in the same groups of three for the duration of the intervention. Students were recruited by their mathematics teacher and by myself. After the mathematics teacher provided a brief introduction to the project, I presented it to the students. One week later, 9 students volunteered to participate.

[21] Information obtained from http://profiles.doe.mass.edu/home.asp?mode=so&so=2036-13&ot=12&o=2035&view=all and http://www.parker.org/ .

	Student Name	Gender	Grade
Group 1	Abbie	Female	$9^{th}/10^{th}$
	Desiree	Female	$9^{th}/10^{th}$
	Grace	Female	$9^{th}/10^{th}$
Group 2	Audrey	Female	$9^{th}/10^{th}$
	Chris	Male	$9^{th}/10^{th}$
	Janusz	Male	$9^{th}/10^{th}$
Group 3	Brian	Male	$9^{th}/10^{th}$
	Cory	Male	$9^{th}/10^{th}$
	Tyler	Male	$9^{th}/10^{th}$

Table 1. Group of student participants in the study.

Procedure

<u>Lessons</u>: Fifteen one-hour lessons were held once a week. These lessons were part of the regular school schedule but not part of their regular mathematics classes. Each lesson was videotaped and the written work was collected and scanned for analysis. Since I was both the researcher and teacher in the classroom, and the lessons were videotaped with a camera standing on a tripod, each group within the class (there were three groups of three students each) was also audiotaped during class in order to document their work and conversations.

<u>Individual interviews</u>: Students were interviewed individually twice during the intervention: half-way through the intervention as well as at the end. In these interview sessions, students were asked to solve a problem (new to the students but similar to the problems discussed in class; see Appendix A) from the Calendar Sequence and the interviewer asked questions in order to get a deeper understanding of students' problem solving processes.

From the total of fifteen classes, in this paper I will report on data collected during lessons 1 and 2 (Problem 1), lesson 7 (Problem 16), and during the individual interview (Problem 17) conducted mid-way through the intervention. Problems 1, 16, and 17 were implemented during the first half of the study where students worked with variables, algebraic expressions, and derivations. During the second half of the study, students' work was centered on the notion of parameters, which will be the focus of another analysis. I chose to focus in this set of problems because the level of mathematical complexity increases, and also because a mathematical narrative can be build using their mathematical structure. Problem 1, deals with a 2x2-calendar-square with cross multiplication as the calculation producing the outcome, Problem 16 involves a 3x3-calendar square with a slightly different calculation from Problem 1; Problem 17 concerns a 3x3-calendar square with the cross-product. The problems (1, 16, and 17) involve variables.

A-priori Analysis of the Didactical Sequence

The following section outlines the rationale that supports the decisions for the design of the Calendar Sequence, including the mathematical concepts at stake, the didactical variables involved in the design, and their relation to the target knowledge I wanted students to develop by the end of the sequence.

As I mentioned previously, for the design of the Calendar Sequence I used the principles from *Didactical Engineering* (Artigue, 1988, 1994; Artigue & Perrin-Glorian, 1991; Douady, 1997), which incorporates theoretical principles of the *Theory of Didactical Situations* (TDS; (Brousseau, 1997) and the *Theory of Didactical Transposition* conceived by Chevallard (Chevallard, 1985; , 1989). Schematically, the methodology of Didactical Engineering includes the following stages (Douady, 1999): (1) Choose a teaching object in the current program; (2)

Place the mathematical context in relation with the teaching tradition; (3) Develop hypotheses about the difficulties the students will face or encounter and set the basis for a didactical engineering; (4) Develop an engineering and proceed to the a-priori analysis; (5) Implement the sequence and make an a-posteriori analysis of the collected data; (6) Reproduce the implementation; (7) Test the knowledge students were supposed to acquire in questions for which students' knowledge are adapted tools; and (8) Compare the output of the students and their skill with expectations, and conclude with the relevance of the didactical hypotheses. In this paper, Stages (1) and (2) have been developed in the *Introduction* and *Literature Review* sections, stages (3) and (4) are addressed in the *Literature Review* and *A-priori Analysis of the didactical Sequence* -this section-, stage (5) is addressed in the *Methodology* and *Results* sections.

Problem 1 (Figure 1) will provide an example of the type of problem that constitutes the Calendar Sequence. The rest of the problems can be found in Appendix A. Appendix B includes a table containing the variables considered in the design of the sequence. The *sequence at-a-glance* (Appendix C) summarizes in a graphical display the variables, calculation, and outcome involved in each of the problems in the sequence.

Analysis of Problem 1. In this problem, students are prompted to analyze the nature of the outcome of the described calculation (subtraction of the cross product). I expected that students would anticipate some kind of variation in the outcome in relation to the set of days where the operator is applied. I anticipated that students would find out, through exploration, that the same outcome is always obtained, no matter where they apply the operator. Here the problem for the students is to find out why this happens, and whether this is "always" going to be the case. At this stage in the problem, from a mathematical point of view, algebra becomes a tool to solve the problem; without algebra, students cannot provide a general explanation about the behavior of the outcome. Based on my previous teaching experiences, and as shown in different studies (Arsac et al., 1992a), students usually over-generalize a result that they observe to be true in a finite set of cases.

Thus, one of the challenges in Problem 1 (as well as in problem 16 and 17[22]) is to show the limitations of this strategy (e.g., using a non-exhaustive finite set of examples to prove that proposition is true) and to encourage students to use algebra as *the tool* that allows them to express all cases using a unique expression. Since I consider that variables, unknowns, parameters, algebraic expressions, and equivalent transformations are at the core of school algebra, a goal of this didactical sequence was to promote the following activities: choosing relevant variables, setting up relations among variables and/or parameters, expressing a set of variables as a function of one of the variables, making decisions about transformations (i.e., which transformations to apply), transforming into equivalent expressions, working with equivalent expressions, deciding when to stop transforming, and reading and interpreting the chosen final expression in terms of their hypotheses and the context.

In Problem 1, a model of the 2x2-calendar-square would look like the one shown below in Figure 18.

$$\boxed{\begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} \equiv \begin{matrix} a & a+1 \\ a+7 & a+8 \end{matrix}}$$

Figure 18. Representation of a 2x2-calendar-square.

[22] As mentioned previously, in this paper I will present data collected when students were working on problems 1, 16, 17.

Using the representation shown in Figure 18, the outcome and its equivalent expressions are shown in Figure 19 below. For problems 16 and 17 and the rest, a similar analysis and diagrammatic representation can be found in Appendix C.

When a calendar begins on the first day of the week, the value of Cell (r,c)=7(r-1) + c, where r means row and c column.

The days of the month that fall on the same day of the week are equivalent modulo 7. This structure allows row column positions in the table to be related to positions in the number sequence.

By identifying the table positions with respect to one of the cells, a <u>relative frame of reference</u> comes into play. For instance, a+8 serves to identify a day that is one week plus one day ahead of the reference day, a. This corresponds to a downward hop of 1 week followed by a day forward hop to the right of 1. If a is on the last day of the week, moving forward by one day requires advancing to the first cell of the following row.

The *a-notation* is relative to an arbitrary day *a*. Row-column notation is relative to the beginning of the month.

$$
\begin{aligned}
a(a+8)-(a+7)(a+1) &= a^2+8a-(a^2+a+7a+7)= \\
&= a^2+8a-a^2-a-7a-7= \\
&= a^2-a^2+8a-a-7a-7= \\
&= -7
\end{aligned}
$$

Figure 19. Transformation of the initial outcome into equivalent expressions.

Structure of the problems

An analysis of the structure of the problems and didactical sequence will allow us to be aware of the constraints that I chose in the design among all other possibilities, to examine a-priori in a careful way what kind of environment I was hoping to offer to the students to approach the concepts that I wanted them to learn. Students' production of knowledge not only depends on cognitive or psychological aspects, but also on the "artificial" environment: the sequence of problems that we, as educators, create in order to promote student's mathematical learning.

In terms of the context of the problem, the calendar[23] can be seen as formed by three components: (1) the mathematical template; (2) the mathematical operator that is formed by (2.a) the mathematical shape and (2.b) the mathematical calculation; and (3) the outcome (see Figure 20).

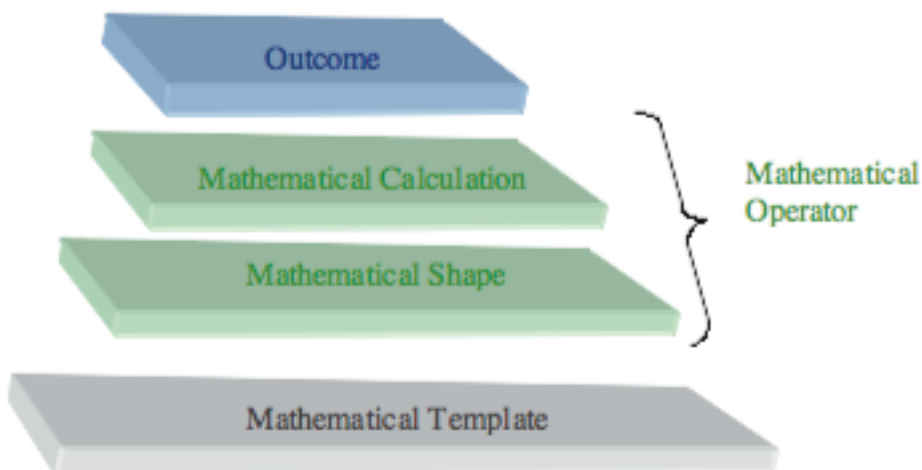


Figure 20. Structure and elements of the context.

[23] A calendar is a sequence. It is also a **table**. As a table it has a finite (generally small) number of rows and columns that intersect to form cells, each of which is a natural number in the sequence. Cell values start at 1 and increment by one from left to right within a row; when a row is filled; one proceeds to the beginning of the next row, extending the sequence. (David W. Carraher's personal communication)

Now, using Problem 1 (Figure 1), I will illustrate the structure of the context (Figure 20). The mathematical template in this case is a linear sequence that starts with 1 and ends in 28, 29, 30 or 31, depending on the month. The increment from one term to the next is 1. One peculiarity of the mathematical template in this particular case is that the usual arrangement of our calendar is in seven columns. Therefore, in order to move from any number to the one right below (in the same column), it is enough to add seven.

The mathematical operator is formed by both the mathematical shape and the mathematical calculation. The mathematical operator is applied to the mathematical template. The mathematical shape in this case is a 2x2 square and it is named a 2x2-calendar-square. The repertoire of possible shapes is square, triangle, rectangle, and cross. The mathematical calculation is a set of operations involving some of the numbers where the shape is applied. In the case of Problem 1, it involves multiplication and subtraction. Finally, the last component of the context is the outcome. In Problem 1, the initial outcome and its transformations are displayed in Figure 19.

Analysis of the didactical sequence

In this section, I will describe the issues that are at stake in the Calendar Sequence from a design point of view. First, I will discuss the rationale to support the idea that algebra is an optimal tool to solve problems in the Calendar Sequence. Second, I will discuss the increase in the degree of difficulty by the inclusion of variables, one parameter and later two parameters. Third, I will address the relation algebra and proof in this Sequence. Fourth, I will provide a detailed analysis as to “what of algebra” is included (e.g.; relation among variables, transformations, etc.) Last, I will address how generalization is played throughout the problems in the sequence, and the constraints among the different values for the variables and parameters established by the relations among them (e.g.; the dimension of the square at play in problem 1 cannot be larger than 8 when working with a 7-day week).

In the Calendar Sequence, one of the main goals is to promote the use of algebra as the optimal tool, from the mathematical point of view, for approaching the problem, given its constraints. In Problem 1, to show that the outcome is always -7 , the use of a letter allows us to capture and to represent all possibilities, as well as to show understand by looking at the structure why the outcome is always -7. The idea is to promote an experience of needing to use algebraic tools in order to solve the problem. For example, in the first problem of the sequence (see appendix A, Problem 1), students have to decide where to place the 2x2-calendar-square in order to obtain the biggest outcome after performing the indicated calculations. Since the outcome of the calculation is an invariant, regardless of the day, month, and year where the 2x2-calendar-square is placed, the question here is "why does this happen?," "why do we obtain the same outcome for any day, month, and year?." Here, in terms of the design of the sequence, I foresaw that students would expect a variation in the outcome in terms of day, month, and/or year. To counter this expectation, I wanted to present students with evidence that would contradict their anticipations. As described earlier, Barallobres (2004) used this strategy to generate intrigue among his students. This strategy consists in designing a problem in a way that when the students explore the problem, the feedback from the situation[24] contradicts students' anticipations. From the mathematical point of view, the optimal solution to the question posed by the problem is provided by the use of algebraic tools: using variables to denote the days, expressing the relations among the four variables as a function of one of the variables, expressing the calculation and/or outcome of the calculation in terms of the chosen variable, transforming the expressions into equivalent expressions where students are able to read new information that in the original or intermediate expressions could not be read, and interpreting the displayed information in the algebraic expression in terms of the calendar. After this, students have to articulate what they read from the expressions, what is happening in the calendar, and produce an explanation. I anticipated that students would probably not use algebra in order to solve the problem, and this is when students would face a new intellectual challenge: "how do I show that the outcome is an invariant?" In this situation, I articulated this question in different ways until students provided the idea of using letters as a way of capturing "all the possible numbers" at once, and through only one expression.

The sequence was designed to promote students' need to use algebra throughout the problems in the sequence. Throughout the sequence, the degree of difficulty and complexity of the problems also increases. By the end of the sequence, students were working not only with variables and relations among variables but also with parameters to analyze dependence of the outcome in relation to the constraints of the calendar (duration of the week and weeks per month), and features of the operator (formed by the shape and the calculation). Mid-way through the sequence (e.g., see Problem 6 in Appendix A), students were required to parameterize the length of the shape of the mathematical operator, while at the same time keeping track of what happens with the outcome in relation to the days where the shape is applied.

Analyzing the sequence from a complementary perspective, in this sequence the role of algebra is to produce a proof. In this regard, this study would join other studies linking algebra and proof (Barallobres, 2004; Dorier, Robert, & Rogalski, 2002; Healy, Hoyles, Sowder, & Schappelle, 2002; Panizza, 2001; Uhlig, 2002). Students were expected to produce conjectures regarding the way the system behaves. In order to validate and to show why a property under certain constraints is true (e.g., in Problem 1, the outcome is a constant), students needed to use algebra in order to represent a whole set of values for the variable. The same is true in the case of a property involving a parameter of the model (e.g., in Problem 1, the outcome is -7 because it is the number of days per week).

After explaining the main goals of the sequence, I will now move to the next level of analysis within the mathematical content. As I mentioned above, students worked in particular with: variables, choosing relevant variables, setting up relations among variables, expressing a set of variables as a function of one variable, making decisions about transformations (i.e., which transformations apply), working with equivalent expressions, deciding when to stop transforming, and reading and interpreting the chosen final expression in terms of their hypothesis and in terms of the context. In addition to that, students worked with parameters and variables at the same time, and analyzed the behavior of the outcome as a function of the variables and parameters. Students were also required to solve the opposite task: instead of being required to analyze the outcome, they had to define the operator in order to obtain an outcome given the constraints (e.g., see Problem 5, in Appendix A).

In this sequence, another algebraic aspect that is at stake is the constraints on the variable and parameters when creating a mathematical model of the system. For example, in the case of the analysis of l days per week, r minimum weeks per month, and a nxn-calendar-square, they need to establish that $n < \min \{l, r\}$.

In general, another central aspect of the algebraic work in particular but also of the mathematical work at stake in this sequence is the construction of generalizations (Lee, 1996; Mason, 1996a, 1996b). In the calendar sequence, there are many generalizations that are constructed inside each problem and throughout different problems within the sequence. Within each problem, students were expected to generalize the behavior of the outcome under particular constraints. Throughout the sequence of problems, students were asked to generalize the behavior of the outcome for different operators, for example different dimensions for the shape and different calculations, as well as for different calendars varying mostly in terms of days per week. Students were prompted to generalize using parameters for the number of days per week and for the dimensions of the shape. The different dimensions of generalization that I just mentioned are represented in Figure 21 below.

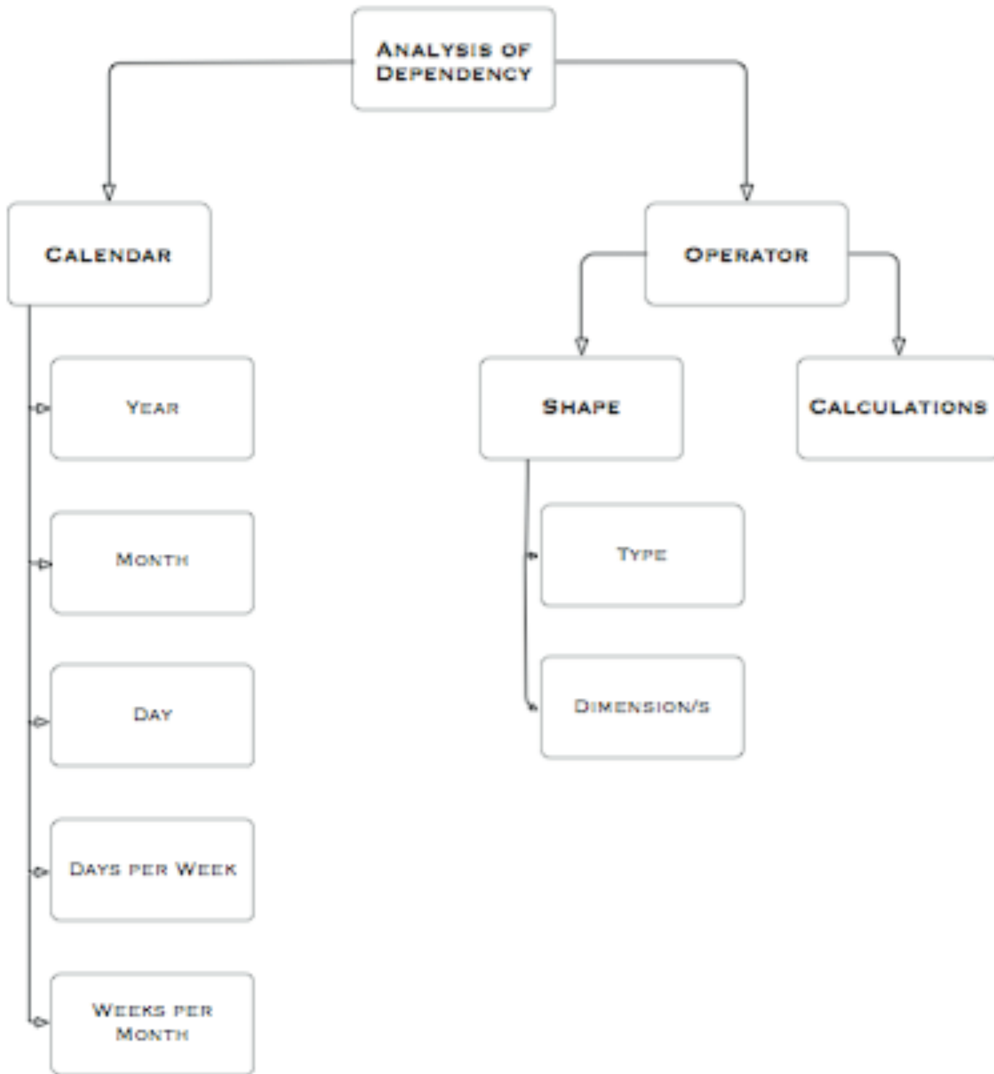


Figure 21. Dimensions of generalizations.

The following provides an example of how these different dimensions of generalizations were built along the sequence. If we consider the pair formed by Problem 1 (see Appendix A) and Problem 8 (see Appendix A), the second problem can be considered as a generalization of Problem 1 for the parameter dimension of the square shape. If we consider the Problems 1, 9, and 18, Problem 18 (see Appendix C) can be seen as a generalization of Problems 1 and 9. Now if we consider all of these problems (1, 8, 9, and 18) together, and Problem 15 (see Appendices A, B, and C), the sequence can be seen as a generalization on both parameters: days-per-week and dimension of the calendar-square. In Problem 1, the outcome is -7; in Problem 8 the outcome is $-7(n-1)^2$; in Problem 9 the outcome is -9; in Problem 18 the outcome is –d; and in Problem 15 the outcome is $-d\,(n-1)^2$ (see Table 2). A detailed analysis of each of the problems in the Calendar Sequence regarding the problem variables can be found in Appendices B and C.

	Outcome	Parameterization	Values of the Parameters	
Problem 1	-7	-	n = 2	d = 7
Problem 8	$-7(n-1)^2$	Dimension of calendar square	N	d = 7
Problem 9	-9	-	n=2	d = 9
Problem 18	-d	Days per week	n=2	d
Problem 15	$-d\,(n-1)^2$	Days per week Dimension of the calendar square	N	d

Table 2. Generalization through the parameterization on days per week and dimension of the calendar-square.

Another aspect of the algebraic work that is at stake in the Calendar Sequence is the constraints on the parameters and variables, for example in Problem 15, $n<\min\{d , r\}$ where r is the number of weeks per month.

As mentioned earlier, in his proposal for an algebra curriculum, Bell (1995) proposes that it should be learned as a language grounded in activities such as generalizing, forming and solving equations, and working with functions and formulae. He suggests the use of the Line

Patterns problem (see Figure 22) and also Middle and Corners problem (see Figure 23), where the generalization aspect of the algebraic work is at play according to him.

LINE PATTERNS

Take a copy of the Multiplication Square.

Find the box showing

14
21
28

Notice that 14 + 28 = 42
and 2 x 21 = 42

Try this other box in the same vertical line

28
35
42

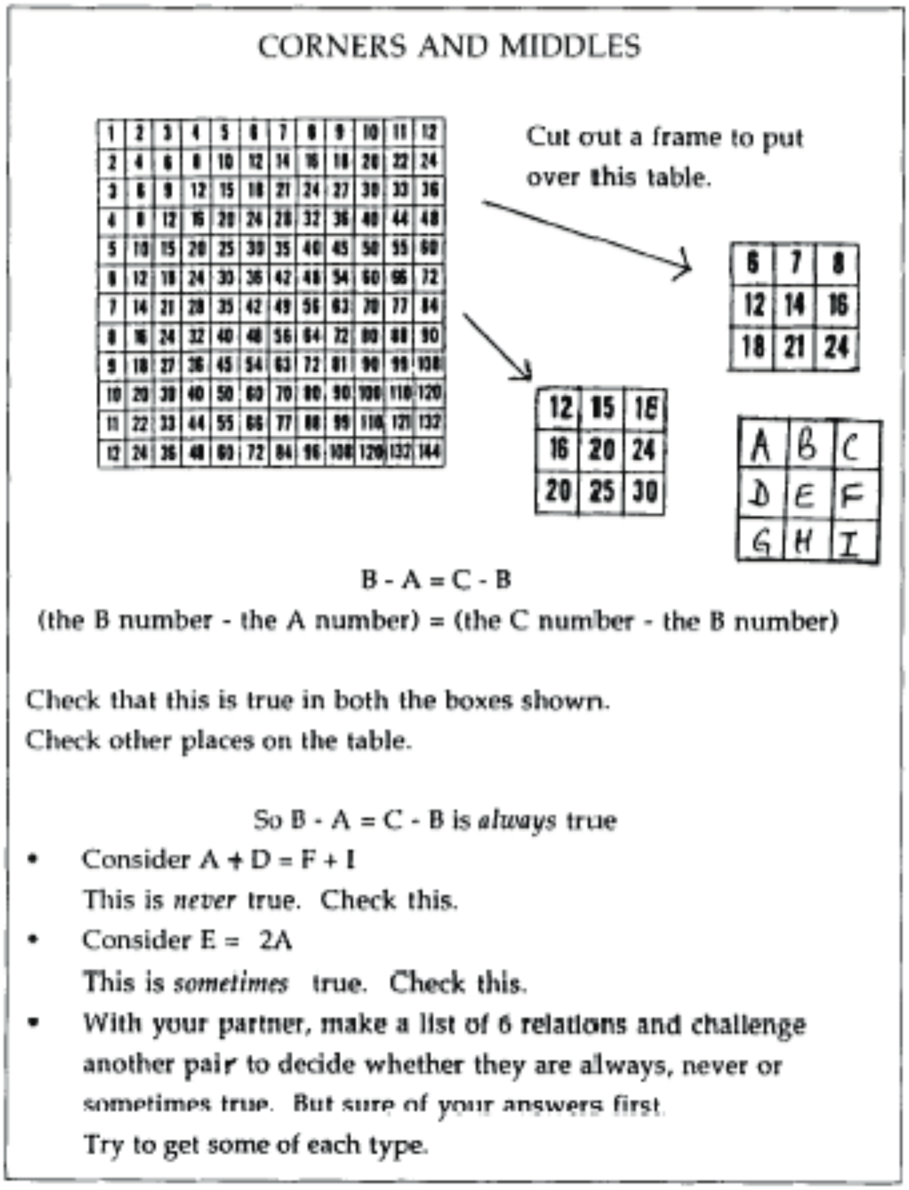
Is it still true that T + B = 2M?

Try a few more, in the same line.
Try some similar boxes in other lines.
Try horizontal boxes also: Does L + R = 2M?
left right middle

Try lines of 4 and 5 numbers. Write what you find.
Use letters if you wish.

Figure 2A.

Figure 22. Line Patterns Problem in Bell (1995, p. 52).



CORNERS AND MIDDLES

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	96	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	90
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Cut out a frame to put over this table.

6	7	8
12	14	16
18	21	24

12	15	18
16	20	24
20	25	30

A	B	C
D	E	F
G	H	I

B - A = C - B

(the B number - the A number) = (the C number - the B number)

Check that this is true in both the boxes shown.
Check other places on the table.

So B - A = C - B is *always* true

- Consider A + D = F + I
  This is *never* true. Check this.
- Consider E = 2A
  This is *sometimes* true. Check this.
- With your partner, make a list of 6 relations and challenge another pair to decide whether they are always, never or sometimes true. But sure of your answers first.
  Try to get some of each type.

Figure 2B.

Figure 23. Corners and Middles in Bell (1995, p. 53).

Bell (1995) shares the written work of a student named Julia (14 years old). Julia is working with the regular calendar and applying (what I call) the 3x3-calendar-square. Julia is able to successfully choose the variables, express the relations among the variables as a function of only one of them, represent the indicated calculation, transform the expression into equivalent expressions, and relate the final expression – in this case a number — to the system. In addition to that, Julia is able to produce a conjecture, and using algebra as a tool, she proves this conjecture. Bell (1995) presents three examples of students' written work constructing conjectures through generalizations and using algebra as a tool to prove in the calendar. Even though Bell does not provide any quantitative data or analysis about these examples, I think that they constitute a good indicator that this type of context might be fertile to promote the use of algebraic tools to prove conjectures produced by the students. In the Calendar Sequence that I implemented in my study, the process of parameterization makes it different from Bell's problem (1995). In addition to this, my goal was to generate evidence in order to show what type of knowledge students produce during the sequence.

Results

In general, students were engaged and intrigued when solving problems from the sequence. Problems seemed to be accessible for students and at the same time they offered students new challenges. In many opportunities, students verbally expressed these opinions. In addition, at the end of the intervention, they completed an opinion survey where they expressed that they enjoyed the lessons and the problems, that they learned algebra, and felt better prepared for the mathematics part of the Massachusetts Comprehensive Assessment System (MCAS), the mandated state test. In their schooling, students have *learned* to use letters as unknowns in the context of equation solving and letters as variables in the context of studying linear and quadratic functions. In the Calendar Sequence, students faced a series of new challenges involving the *same old*[25] objects (letters, algebraic expressions); however, the objects were also *new* in that these problems demanded new uses for these objects. Algebra in this case is not about "finding the value of x" or just about expressing the functional relations among variables. In the Calendar Sequence, new dimensions of algebra are called upon. In the first place, the overarching goal of algebra is to prove a conjecture produced by students themselves. This is something totally new for them. Second, students' task is not only to choose variables and express the relations among them; this is just an intermediate step in order to obtain an expression that they can work with afterwards, and, by transforming (making derivations on) the initial expression, hopefully arrive at something that will help them make a case to prove their conjecture.

A new dimension of algebra is opened up and the power of algebra is revealed for students: by transforming (using derivations) we obtain re-writings of the initial expressions (this

[25] I am using *old* to emphasize that these were objects that they had previously worked with in the context of equation solving and functional relations. However, when we use these *old* objects in new ways, the object changes, incorporating new aspects from the new way of functioning.

is not an aimless task like traditional polynomial-factoring); by analyzing the structure of the algebraic expression, students can obtain information that they did not have before and explain what is happening and why. For instance, in Problem 1, students learned that even though the initial expression was large, and had nothing to do with -7, transforming into equivalent expressions (i.e., deriving), they were able to obtain what they were looking for and prove their conjecture (see Table 5).

<table>
<tr><th colspan="2"></th><th colspan="2">Problem 1</th><th>Problem 16</th><th>Problem 17</th></tr>
<tr><td colspan="2" rowspan="2">Duration</td><td colspan="2">2 lessons (#1 and #2)</td><td rowspan="2">1 lesson (#7)</td><td rowspan="2">15-30 minutes (Individual Interview)</td></tr>
<tr><td>Day 1</td><td>Day 2</td></tr>
<tr><td colspan="2">Writing of initial expression</td><td>100% (8/8)</td><td>100% (8/8)</td><td>100% (8/8)</td><td>100% (9/9)</td></tr>
<tr><td colspan="2">Writing of Initial Expression using numbers</td><td>75% (6/8)</td><td>88% (7/8)</td><td>75% (6/8)</td><td>89% (8/9)</td></tr>
<tr><td colspan="2">Writing of Initial Expression using variables</td><td>100% (8/8)</td><td>100% (8/8)</td><td>100% (8/8)</td><td>100% (9/9)</td></tr>
<tr><td colspan="2">Initial and final use of exactly one independent variable</td><td>62% (5/8)</td><td>62% (5/8)</td><td>100% (8/8)</td><td>100% (9/9)</td></tr>
<tr><td>Initial use of more than one ind. Var.</td><td>Shift to more than one ind. Var. but less than initial</td><td>12.5% (1/8)</td><td>12.5% (1/8)</td><td>0%</td><td>0%</td></tr>
<tr><td>More than 1 independent variable</td><td>Shift to 1 independent variable</td><td>25% (2/8)</td><td>25% (2/8)</td><td>0%</td><td>0%</td></tr>
</table>

Table 5. Students' Initial Expression.

In what follows, I will present the results regarding the first half of the intervention where students solved problems involving variables (Problems 1, 16, and 17). The presentation of results is organized as follows:

1. Conjectures
2. Generic Model
3. Initial Expression

4. Treatment of Initial Expressions
5. Proof

During the first half of the intervention, students faced a series of challenges:

- Use of variables vs. use of numbers in order to prove the validity of the conjecture. (See sections *Conjectures* and *Proof*)
- Number of variables and dependence. (See sections *Generic Model* and *Initial Expression*)
- Use of the initial expression. (See sections *Initial Expression* and *Treatment of Initial Expression*)
- The possibility that a re-writing of the initial expression could be more manageable. (See sections *Initial Expression* and *Treatment of Initial Expression*)
- Transforming into equivalent expressions using *old*[26] properties (i.e., cancellation, associative, etc.). (See sections *Initial Expression* and *Treatment of Initial Expression*)
- New use of the initial algebraic expression: is not an equation to be solved but rather an expression from which we can derive other equivalent algebraic expressions. (See sections *Initial Expression* and *Treatment of Initial Expression*)

The points listed above had to be discussed and negotiated between the teacher researcher and the students during the intervention in small group and whole class discussions. I will present evidence that students' challenges tended to disappear towards the first half of the

[26] I am using *old* to emphasize that these were properties that the students had previously worked with in the context of equation solving and functional relations. However, when we use these *old* properties in new ways, the object changes, incorporating new aspects from the new way of functioning.

intervention. In addition, I will present students' written work to show how they attempted to overcome the challenges; in other words, how they constructed new uses for the *old* objects.

Conjectures

One of the goals of the didactical sequence was to promote students' construction of their own conjectures as opposed to the more traditional format "Prove that the following property P(x) is true" used in school geometry. The goal of having students producing their own conjectures was to give students a sense of ownership regarding their mathematical work. For example, in Problem 1, there were a variety of conjectures put forth by students regarding the possible outcome. After trying different numerical examples, students conjectured that:

- The square should be placed at the end of the month (where the biggest numbers are) in order to obtain the biggest outcome. Chris states: "A 2x2-square with larger numbers might have... might give... will give a larger answer."
- The location of the square matters but not sure about how maters. Abbie states: "it must matter" but they (Abbie, Desiree, and Grace) were not sure about where to locate it. Later, during group work, Abbie changed her mind and said, Abbie "I think it depends on what day of the week the month starts"
- It does not matter where the square is located since the outcome will be always -7. Tyler states: "We thought that not matter which one you are using it's always an outcome of -7."

During the first lesson (Problem 1) and after constructing their conjectures, all the students *proved* them by using a finite number of examples. This obstacle was anticipated when designing the sequence. There were two ways of managing the situation in order to get students to start moving away from this strategy. One of them was to ask students whether they were sure that this way of proving was enough to be sure that if we calculate the outcome for year 3067 or for 567 the outcome would always be -7. The other resource was to begin a conversation about why this phenomenon (outcome equals -7) happens. Once *established* that something general was necessary (that would represent all the possible cases), the use of letters was suggested by students. The exchange below illustrates the discussion among the members of the group – Brian, Cory, and Tyler- and me about what constitutes evidence to prove the hypothesis and how students came up with the idea of using letters to capture the general during class 1 when working on Problem 1:

Mara (from now on M): So, your hypothesis is that it doesn't matter where you place the square (2x2-calendar-square) you are always going to obtain -7?

Cory (from now on C), Tyler (from now on T): aha.

M: Ok, so, how do you gather evidence to prove that? That is your problem now. How do you do it? How did you come to that idea?

C: we...

M: How did you come to that idea? Aha

C: we tried examples

M: aha, yes

M: how many examples did you try?

C: 3 or 4...

M: in what month? ...using what months?

C: we used January, I think

T: yes

Brian (from now on B): yes

M: only January?

T: Oh, no, July

B: the days are always the same

T: where they are placed doesn't matter if you are making a full square [showing the case where one of the numbers in missing like in $\boxed{\begin{array}{cc} 23 & 24 \\ 30 & \end{array}}$]

T: that's a difference of 7, that's a difference of 7, that's a difference of 7 ...days that are...exactly ...that's exactly one week away, it's always 7 days away...

M: aha, yes

T: so, it is always 7 days away that ... they are all exactly like that...they might as well be the same thing

M: aha, because from one row to the next one...

B: yeah, so...

M: you can get the numbers always adding 7

T, B: aha

M: so, you tried with July 2006, what else did you try?

T: January 2007

C: 2007, yeah

M: That is the one that you have here. And, what else did you try?
T: that's pretty much it.
M: That's pretty much it ok. So, now...How can you be sure that with any month in any year, if somebody comes to you and they show you a calendar of the year 1920, let's say October 1920, how can you be sure that this is going to work?
C: Because it is always one week away still
M: aha
B: the days never change; the calendar is still the same...
T: except the leap year...is the only different
M: the one that has...
T: every four years, February the 29^{th}... but it doesn't matter because if you make a square is a week away
M: Ok, let's try something then. Can I write in this one?
T: Yes
M: How can you show that for any number this is going to be the case?
T: oh, so, you do like x
M: ok, you can try.
T: right, so like different variables... a, b, c, d...
M: work, gather information, why did you come up with the idea of using letters?
T: using variables?
M: yes, variables
T: **Because if you have a formula then it shows that it will always be the same...then**
M: aha
B: yeah
T: **mmm... then ... then...it will work for anything, right so actually...**
M: ok, try to work with that and see it that gives us something solid to prove.

In the group of Abbie, Desiree, and Grace, the idea of using letters appeared promoted by Abbie, as indicated in the exchange below. The exchange happened after they made a decision of working individually and then sharing their ideas. Abbie takes the lead,

Abbie (A from now on): all right so we know that it is always going to be -7...I'm pretty sure of that.
Desiree (D from now on): yeah...
A: all right, **so this is what I have x times x-8 minus, y times y-6, that's basically what we are doing.**
Grace (G from now on): say that again...
D: yes!
A: ha-ha, yes...ok. we are subtracting the product of this two...
D: aha
A: times ... wait... we are subtracting the product of these two from the product of these two...
D: aha

A: and the product of these two is this number times eight less than this number, so x times x-8, ...minus, y times y minus 6
D: wait wait wait... are we...yeah, ok, I think I'm following.
A: ok
G: but...
A: all I did is ... was
D: But shouldn't those be two different variables? Never mind...yeah...the whole week thing... so it doesn't make a different really
A: equals -7 no matter what. Let's try some numbers.

I approached the group after Abbie called me, and the following exchange took place:

M: yes, talk about it please!
A: we are trying to get everything in terms of this number
M: Oh, wow!!! What is that number? Can you draw the square for me?
D: like here Abbie...
A: 1, 9, 8,...
M: oh, ok...because it could be this or this other...sorry, go ahead...
A: so, we are trying to get everything in terms of this number, which we are calling x
M: ah, ok
A: alright so we are saying that it is going to be ...the product of this minus the product of this
M: aha, wonderful
A: and we know that this number is always going to be 8 less than this number, so x times x minus 8...
M: aha...
A: minus, an we know that this number is always going to be one less than this number so,
M: aha
A: it's x minus 1 times x minus 1 minus 6, cause we know that this number is going always to be 6 less than this number
M: ok, ok, let me see whether I understand. So you are saying...Oh, wait a minute... why did you start using x?
M: how come? You worked with numbers and suddenly I come and you are using x. I'm not saying it is wrong, I just want to know.
D: why did we jump to algebra?
M: yes
D: because it's a variable; so you can plug in any number in here so instead of using 1, 2, 8, 9 we could use 6,7, 13, 14 and it would still work for the problem as long as you plug in the right variables
A: you could use only 14 and figure out that these are 6, 7, 13
D: oh, yes.
M: Oh, can you repeat that idea?
A: if this, what we came up with, we do not need to know any of these numbers, we just need to know this one number

D: that number to get all the others...
A: all the others...
D: and, the answer
M: the answer?
M: Oh! And, did you get minus 7?
D: yes
Pause and short discussion among them
A: but we still have to prove that it is always equal minus 7, and why.
M: I agree. So, how did you name these?

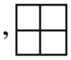
During lesson 7 and during the individual interview, students used numbers to produce a conjecture but not to prove it. Proving through the use of a finite set of examples did not appear to be an issue after Problem 1. In general, after having constructed a conjecture through their work with examples, all the students used letters to represent the calculation and prove their conjecture.

The Generic Model

As part of their work, all students used what Desiree labeled as *generic square* (see Figure 24). I took the name from her and changed it to a somewhat more general name: *generic model* is a diagram where students represented shape, dimension, variables (involved in the calculation), and the relations among them. All students (see Table 3) used this diagram throughout the sequence.

Brian, Cory, and Tyler started using the *generic model* as a result of analyzing whether the relation "+7" was an invariant. The following exchange illustrates that point:

Mara (M, from now on): That's pretty much it ok. So, now...How can you be sure that with any month in any year, if somebody comes to you and they show you a calendar of the year 1920, let's say October 1920, how can you be sure that this is going to work?
Cory (C, from now on): Because it is always one week away still
M: aha
Brian (B, from now on): the days never change; the calendar is still the same...
Tyler (T, from now on): except the leap year...is the only different
M: the one that has...
T: every four years, February the 29th... **but it doesn't matter because if you make a square is a week away**

M: Ok, let's try something then. Can I write in this one? [Here I probably draw a 2x2-square without the contents, i.e., ]
T: Yes
M: How can you **show that for any number** this is going to be the case?
T: oh, so, you do like x
M: ok, you can try.

As it can be seen in the exchange, I made profit of Tyler's suggestion about making a square, and a pushed the discussions introducing the challenge of doing that for any number.

In Abbie, Desiree, and Grace's group, after Abbie stated that they still had to prove that the outcome was always going to be -7 and why, I probably asked for the "names" in an empty square:

M: I agree. **So, how did you name these?**
A: x
M: Can you write it? That is x. What else now?
A: What else?
M: **Can you complete the other spots?**
A: x-8, x-1, x-6
M: are you sure?
A: Oh, wait... x minus 7
M: now we agree, wonderful.
A: x times x-8, minus, x-1 times x-7 equals -7.
M: it should
A: should
M: you can say...
A: alright
M: can you work this through?
A: alright
M: do you need my help or can you do it by yourself?
A: we can try...
M: yes, try please.

Students' *generic models* were varied in terms of the number of variables used, the relations expressed in the diagram, the position/s of the "independent variable/s," and the letters used to represent them (see Figures 24 and 25):

- Desiree and Chris (Figure 24) used only one independent variable, and each of them used different letters (*x* in the case of Desiree and *n* in the case of Chris) and different locations (lower right for Desiree and upper left for Chris).
- Brian and Cory (Figure 25) initially used more than one independent variable. In Brian's case, he used four independent variables (*a*, *b*, *c*, and *d*) and then he realized the relation between the numbers in the upper row and the numbers in the lower row, and as a consequence he added the +*7* to the lower row. In Cory's case (Figure 25, center), he initially used two independent variables (*a* and *b*) to represent the numbers in the upper row. Later, he became aware of the relation between the numbers in the first column and the numbers in the second column (Figure 24, right).

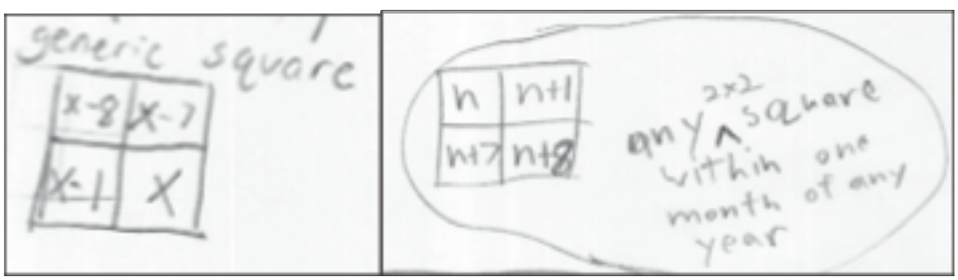


Figure 24. Generic Square for Problem 1. Desiree (left) and Chris (right).

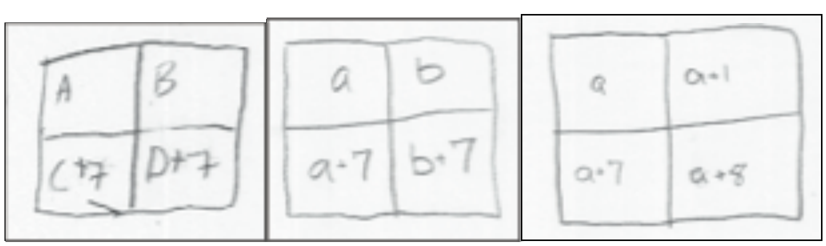


Figure 25. Generic Squares for Problem 1 Brian (left), Cory class 1 (center), and Cory class 2 (right).

As shown in Table 3, the use of a generic model (or generic diagram) was uniform throughout the duration of the study (100% of students used a diagram with variables for Problems 1, 16, and 17). Even though all students used a diagram with variables during the first lesson when they were solving Problem 1 (Table 2), this happened after some prompting and after discussing the limitations of proving with numbers. Thus, students did not use the *generic model* spontaneously when solving Problem 1. When solving Problem 1, 62.5% of the students (5/8) used only one independent variable. For Problems 16 and 17, the totality of the students used only one independent variable to solve the problem. I consider this important progress because students were able to identify the variables, identify relations among them, and express them using algebraic notation; these are the two first stages in the modeling process as described by Chevallard (1985; 1989).

	Problem 1		Problem 16	Problem 17
Duration	2 lessons (#1 and #2)		1 lesson (#7)	15-30 min. (Individual Interview)
	Day 1	Day 2		
Use of diagram	100% (8/8)	100% (8/8)	100% (8/8)	100% (9/9)
Use of diagram with numbers	87.5% (7/8)	87.5% (7/8)	75% (6/8)	89% (8/9)
Use of diagram with variables	100% (8/8)	100% (8/8)	100% (8/8)	100% (9/9)
Initial and final use of one Ind. Var.	62.5% (5/8)	62.5% (5/8)	100% (8/8)	100% (9/9)

Table 3. Analysis of the way students used the generic diagram.

As shown in Table 4, students used different numbers of independent variables to represent the numbers involved in the calculation when solving Problem 1. Initially, only one student (1/8 or 12.5%) used four independent variables (*a, b, c*, and *d*) while two students (2/8 or 25%) used two independent variables (*a* and *b*). At the end of the second class (day 2), the same single student (1/8 or 12.5%) was still using four variables in the generic model, while in the case of the two students (2/8 or 25%) initially using two variables, there was a shift towards the use of one independent variable. Even though all the students used one independent variable in the generic model when solving Problems 16 and 17, there was a wide variation in terms of what specific letters were used to represent the independent variable (*a*, *x* and *y* in Problem 16, and *x*, *n*, and *a* in Problem 17) and its location within the generic model.

	Problem 1								**Problem 16**						**Problem 17**			
Duration	2 lessons (#1 and #2)								1 lesson (#7)						15-30 min. (Individual Interview)			
	Day 1				Day 2													
	Initial				**Final**						**Initial**		**Final**					
	Number of Ind. Var.	Students using that number of Ind. Var.	Letters used to represent variables	Location of Ind. Var.	Number of Ind. Var.	Students using that number of Ind. Var.	Letters used to represent variables	Location of Ind. Var.	Number of Ind. Var.	Students using that number of Ind. Var.	Letters used to represent variables	Location of Ind. Var.	Letters used to represent variables	Location of Ind. Var.	Number of Ind. Var.	Students using that number of Ind. Var.	Letters used to represent variables	Location of Ind. Var.
Number of Independent variables	4	12.5% (1/8)	abcd	All positions	4	12.5% (1/8)	abcd(100%, 1/1)	All positions	1	100%	x(62.5%, 5/8)	Lower left (25%, 2/8)	x(62.5%, 5/8)	Lower left (25%, 2/8)	1	100% (9/9)	x(56%, 5/9)	Lower middle (11%, 1/9)
	2	25% (2/8)	ab	Upper left and lower right (100%, 2/2)	2	0.0%	-	-			a(25%, 2/8)	Upper left (62.5%, 5/8)	a(12.5%, 1/8)	Upper left (75%, 6/8)			n(11%, 1/9)	Upper middle (33%, 3/9)
	1	62.5% (5/8)	x or n	Upper left (2/5) Lower right (3/5)	1	87.5% (7/8)	x(28.5%, 2/7) n(43%, 3/7) a(28.5%, 2/7)	Upper left (71%, 5/7) Lower right (29%, 2/7)			y(12.5%, 1/8)	Upper right (12.5 %, 1/8)	y(25%, 2/8)	Upper right (0 %)			a(33%, 3/9)	Lower right (11%, 1/9) Upper left (45%, 4/9)

Table 4. Number of independent variables that students used in the generic model.

When solving Problems 1 and 16, some students shifted from using more than one independent variable to a smaller number of independent variables (two or one); some students changed the letter representing the independent variable (i.e., initially using *a* and later using *y*); and some other students changed the location of the independent variable (i.e., from upper right to upper left). This all happened while they were working on the problems. Abbie, Desiree, and Grace started working with two variables x and y with no apparent initial relation:

Later they shift to one independent variable, becoming aware of this relation between x and y:

D: just plug in random numbers?
A: yeah
D: except no, like over 30, but I bet that if you go over 30 it wouldn't....
A [talking out loud the calculation that she is doing]: ...times 12 minus 8
A: oh, wait!
G: what?
A: y and x are also related because x is one less than y
A: ok, so we can go x minus 1 times x minus1 minus 6...
A: So, y and x are not like random numbers because if here it is 17 that would be...
A: So, we can say everything in terms of this number right here
D: ok
A: I'm so bad at explaining these things.
D: that's ok, I think I'm following.
D: don't erase it!!!!
A: I'm not...I'm not...I'm just re-writing it...
D: so I'm going to write down that that's what we are always doing and write down examples
A: ok, I'm going to try to talk to her [meaning Mara] about it

While solving Problem 17 at the end of the first half of the intervention, no students changed the number of independent variables, the letter used to represent it, and its location (Table 4). This could be read as an increased control over the process being modeled through the use of variables. This invariance could also be a consequence of working individually with Problem 17 during the interview. When students solved problems during class they worked in small groups, sometimes the members of a group held different opinions on some aspects of the problem. In general, this internal disagreement triggered a discussion followed by the change of opinion among some members of the group.

Setting up the Initial Expression

Once students represented *the generic situation,* they used it to represent the (*generic*) calculation involved in each particular problem. The students' first attempts to express the calculation involved in the problem, before starting to *work on* (applying properties to obtain an equivalent expression; for instance, the distributive property) the expression as an object is what I call the *initial expression*.

When working on Problem 1 during the first day, students didn't spontaneously write the *initial expression*. I had to negotiate with the students the potential use of writing the *initial expression*. A student (Desiree) came up with the idea that using the distributive property would allow them to later cancel things out. The idea that from a long expression, the -7 (very short expression!) could be obtained was something that had to be negotiated with them. After these interventions, students wrote down the initial expression (indicated in Table 5 by the 100% for Problem 1, day 1). However, as pointed out above, this did not happen spontaneously.

In Problem 1, after discussing what constitutes evidence to prove the conjecture, Brian, Cory, and Tyler, struggled with the number of variables, the relation among them when trying to write the calculation in general, as described in the exchange below:

> Cory (C, from now on): You have to add these first, give the thing numbers
> Tyler (T, from now on): no, no, you don't because look
> T: If you give it numbers this is actually b+7...oh, yeah, yeah, ...no, no, no
> Brian (B, from now on) it's the day of the week plus seven.
> T: right, right, so use...
> B: but how does it work? Day of the week plus seven?
> T: alright, alright, so it's plus 7
> B: why seven? Why can't it be plus zero? We are not adding seven.
> C: yeah, why are you adding seven?
> B: it should be multiplied a times d, and b times c
> T: No, no, don't use c and d, use a+7 and b+7. Because...
> C: Oh, yeah! It's a week away. Yes, he's right.
> T: because it's the same thing plus 7. It is whatever this number is plus 7.
> T:" since it is a week away, it will be + 7

C: cause they are a week away.
T: ok, so...we think we got this!
B: so, this is plus 6
T: no
C: no
T: well, this is in between this
C: that's because it is one day apart
T: wait, which one is bigger?
C: this one will be
T: this one is bigger? Oh, yeah , yeah, yeah, yeah.
T: this one is plus six cause is one day apart...
B: yeah. That's one day ahead, and that's one day behind
T: Wait. Maybe we can ....use a in all of them!
C: this is one day ahead...
T: no, it's more than that
B: it's 8 days ahead
C: yeah...of a week. And this is always one behind
T: wow, so this...wait , wait, wait...I'm gonna do another one... Couldn't it be a, a+1, a+7, a+8?... Then, it would be a, a+8, right? Minus...
C: a+7..
T: no, a+1
C: a+7
T: equals...
C: -7...
T: well, that's what we want the outcome to be, right?
C: watch, try it.
T: well, that is what it should be. Equals -7. All right so, fill it in with numbers.

They try with a=1. They decided that their expressions were correct after trying the numbers.

C: don't we want it to equal zero, though?
T: no, you don't. You want equal -7. Oh, Mara!
Mara: I will be there in a minute.

I approached Brian, Cory, and Tyler's group, and I asked them to explain what they had written as their *initial expression*:

M: Can you walk me through this? Can you explain to me what is this?
T: *a* is any top left number, *b* is any top right
M: aha:
T: and then, a, which is the top right, plus 7 , will be this
C: because they are one week away
T: and then, b plus 7 will be this

M: ok, yes.
T: now, if we do...write it out like that
M: ah, this is the calculation that I asked you to do, right?
T: right. In any case this will always be -7.
M: how do you know?
T: that's what we said.
M: ok , you made a lot of progress. What you want to show here is whether this is always going to be -7.
T: right.
M: is there anyway that we can work this out?
C: It is always going to be one week away from the top.
M: yes, I agree. This is fantastic, this is perfect. That is not in question. I totally agree with you.
T: I think that a and b, b is a+1
M: aha,
T: so, if we make that a+1 then we can cancel out both a's and that can be
M: did you hear his idea? Say it again please.
T: so, if we change b to a+1 because that is what it is equals to
C: aha
T: than we can cancel a , and then just have one here...
B: this might work.
M: try it. You are doing a wonderful job. Don't erase, you can write it down.

In Abbie, Desiree, and Grace's group, three different *initial expressions* were considered but just the last of them was written down by the three of them as part of their work. Those three attempts are illustrated by the following exchanges:

First attempt

Abbie: All right so we know that it is always going to be -7...I'm pretty sure of that
D: yeah...
A: all right, so this is what I have **x times x-8 minus y times y-6**, that's basically what we are doing.

Second attempt

A: Y and x are also related because x is one less than y
A: ok, so we can go x minus 1 times x minus1 minus 6...
A: So, y and x are not like random numbers because if here it is 17 that would be ...
A: So, we can say everything in terms of this number right here

Third attempt

M: I agree. So, how did you name these?

A: x
M: Can you write it? That is x. What else now?
A: What else?
M: Can you complete the other spots?
A: x-8, x-1, x-6
M: are you sure?
A: Oh, wait... x minus 7
M: now we agree, wonderful.
A: x times x-8, minus, x-1 times x-7 equals -7.
M: it should
A: should
M: you can say...
A: alright
M: can you work this through?
A: alright
M: do you need my help or can you do it by yourself?
A: we can try...
M: yes, try please.

On the other hand, when solving Problems 16 and 17, 100% of the students wrote the *initial expression* representing the calculation without the need for my intervention.

Regarding the number of independent variables used in the initial expression, we can see (Table 5) that it mirrors what happened with the number of independent variables used in the *generic model* (previous section). In Problem 1 (days 1 and 2), 37.50% of students (3/8) used more than one independent variable in the *initial expression*. Of this 37.5% of students, 12.5% (1/8) used more than two independent variables and then shifted to using two variables without shifting to one variable at the end in the *initial expression*. These students shifted to using one variable when they started applying properties in order to obtain equivalent expressions. This might indicate that the need to reduce to one variable is prompted by the need to obtain a more compact expression and that one way of doing this is by canceling inverse terms within an expression. This hypothesis is confirmed by the fact that in Problems 16 and 17, all students used one independent variable when writing the *initial expression*.

<table>
<tr><td></td><td colspan="6">Problem 1</td><td colspan="3">Problem 16</td><td colspan="3">Problem 17</td></tr>
<tr><td>Duration</td><td colspan="6">2 lessons (#1 and #2)</td><td colspan="3">1 lesson (#7)</td><td colspan="3">15-30 minutes (Individual Interview)</td></tr>
<tr><td rowspan="6">Use of a name/expression/variable to represent the outcome</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td></tr>
<tr><td rowspan="5">87.5% (7/8)</td><td rowspan="5">-7</td><td rowspan="5">100% (7/7)</td><td rowspan="5">87.5% (7/8)</td><td rowspan="5">-7</td><td rowspan="5">100% (7/7)</td><td rowspan="5">87.5% (7/8)</td><td>X+2</td><td>28.55% (2/7)</td><td rowspan="5">55.5% (5/9)</td><td rowspan="2">-48</td><td rowspan="2">60% (3/5)</td></tr>
<tr><td>y</td><td>28.55% (2/7)</td></tr>
<tr><td>n</td><td>14.3% (1/7)</td><td rowspan="3">x</td><td rowspan="3">40% (2/5)</td></tr>
<tr><td>b</td><td>14.3% (1/7)</td></tr>
<tr><td>y+16</td><td>14.3% (1/7)</td></tr>
</table>

Table 6. Students' use of an expression to represent the outcome.

<table>
<tr><td colspan="2"></td><td colspan="2">Problem 1</td><td>Problem 16</td><td>Problem 17</td></tr>
<tr><td colspan="2" rowspan="2">Duration</td><td colspan="2">2 lessons (#1 and #2)</td><td rowspan="2">1 lesson (#7)</td><td rowspan="2">15-30 minutes (Individual Interview)</td></tr>
<tr><td>Day 1</td><td>Day 2</td></tr>
<tr><td rowspan="2">Format of Initial Expression</td><td>Equation Format (Algebraic exp. = outcome)</td><td>87.5% (7/8)</td><td>87.5% (7/8)</td><td>100% (8/8)</td><td>55.5% (5/9)</td></tr>
<tr><td>Algebraic Exp. Format (without "= outcome")</td><td>12.5% (1/8)</td><td>12.5% (1/8)</td><td>0%</td><td>44.5% (4/9)</td></tr>
</table>

Table 7. Format of students' initial expression.

Something not anticipated by me was that in Problems 1 and 16, most of the students — 87.5% of students (7/8 in Table 6) — wrote an expression for the outcome after the equal sign. For Problem 1, 100% of the students (7/7) who wrote an expression representing the outcome after the equal sign used *-7* (see Figure 26). Since students are *using* what they want to prove, their procedure is logically wrong. This is not the way theorems are proved in mathematics. To prove a mathematical theorem, the consequent needs to be derived from the antecedent. Only 55.50% of the students (5/9) used an expression to represent the outcome in the *initial expression* when students solved the last problem (Problem 17) in the first half of the intervention. This improvement might indicate a movement towards a more formal way of writing derivations. For analysis purposes, whenever students used an expression to represent the outcome on the other side of the equal sign, I call the format of the *initial expression* an *equation* (Figures 26 and 27). Whenever students do not use another expression to represent the outcome, I call the format of the *initial expression* an *algebraic expression* (Figure 28). This distinction indicates the format of the initial expression. The *equation format* (Figures 26 and 27) of the *initial expression* resembles the structure of an equation ("initial expression = outcome"). Similarly, the *algebraic expression* (Figure 28) in an *initial expression* looks like a general algebraic expression in contrast to an *equation* ("algebraic expression [without equals to outcome]").

The fact that students use an *equation format* does not mean that they will apply a traditional equation-solving procedure in order to solve the problem. The bottom of Table 7 indicates that in Problem 1, most of the students — 87.50% of students (7/8) — used an *equation* format for the *initial expression* (before applying any properties for transforming); in Problem 16, all students (8/8) used an *equation format* for the *initial expression* (Figure 26); while in Problem 17, 55.56% of students (5/9) used an *equation format* for the *initial expression*.

The evidence provided in this section supports the claim that students adapted the algebraic symbolism they had been using in the context of equation solving, and used it in a new way to represent equivalent expressions with the goal of proving a conjecture as opposed to *finding the value/s of x*. Thus, I would argue, they are using *old objects* in *new* ways.

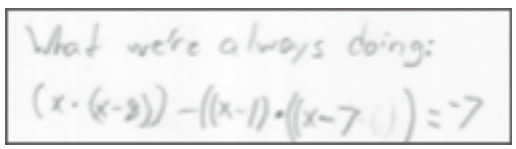


Figure 26. Grace's Initial Expression (Equation Format), Problem 1, Day 1.

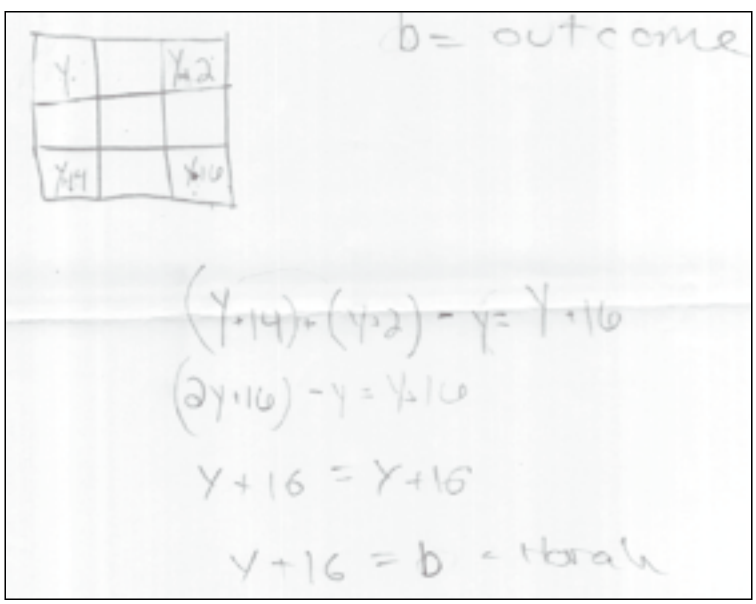


Figure 27. Tyler's Initial Expression (Equation Format), Problem 16.

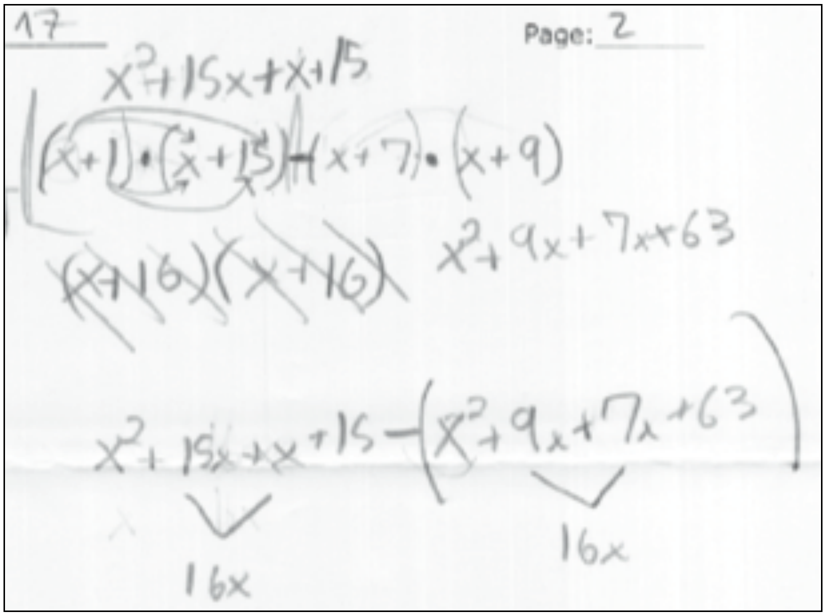


Figure 28. Janusz's Initial Expression (Algebraic expression format), Problem 17.

Treatment of Initial Expressions

Once students had in front of them the *initial expression*, they faced many challenges such as: what to do with that expression, how to obtain a -7 from it, and the use of properties (e.g., distributive). The following passage illustrates some of the challenges that Abbie, Desiree, and Grace faced in this moment of their work:

> A: **I don't know how to prove things like this.**
> D: I can help you with that but I have a question, the fact that this is x-7 has something to do with the fact that the answer is -7?
> A: ah...

After a while Abbie states:

> A: I think that we need to do it algebraically **instead of** thinking it through.

Then she continues,

> **A: this is backwards for me because usually I'm doing something and working towards making a formula and now I have a formula and I have to prove that it is true.**

D: This is part of why we are here because we wanna do what is...
M: ok, I can help you with that. Let me give you a hint. Oh, you put it on the other side... [Referring to the way she started "solving" like if it was an equation]
A: yeah, I was just messing around
M: ok. I understand because you are used to solving equations.
A: aha
M: so, an idea is the following, here, what you want to do is to work this out...
A: so only x
M: yes, to work this side in order to see whether you can get a -7. Because we don't know whether this part is true.
A: ok
M: ok? Because this is what we want to try to prove, right?
A: So we're trying to isolate x?
M: so, we want to try to see what is this...
A: so, we should just plug in numbers for x...
M: **no, let's say. Here we have, x times x-8 – x-1 times x-7, what if we apply distributive property here?**
[Applied distributive property]
D: Can we put the x's together?
M: yes.
M: Try to manipulate this to see what happens and call me.
D: I need a page 2.

In Brian, Cory, and Tyler's group, I also intervened to push for the idea of working on the *initial expression*:

M: ok, you made a lot of progress. What you want to show here is whether this is always going to be -7.
T: right.
M: is there anyway that we can work this out?
C: It is always going to be one week away from the top.
M: yes, I agree. This is fantastic, this is perfect. That is not in question. I totally agree with you.
T: I think that a and b, b is a+1
M: aha,
T: so, if we make that a+1 then we can cancel out both a's and that can be...
M: did you hear his idea? Say it again please.
T: so, if we change b to a+1 because that is what it is equals to
C: aha
T: than we can cancel a, and then just have one here...
B: this might work.
M: try it. You are doing a wonderful job. Don't erase, you can write it down.

After working for a while on their own, they called me to validate whether it was syntactically correct to cancel out the a^2's:

T: **Can we cancel the a^2's?**
M: kind of...
C: kind of?
M: I want you to do one more step first...
T: alright
M: what is the impact of this minus on the signs of the things here? Do you remember that?
B: do you mean get rid of the a's?
T : the minus affects the whole thing
M: aha, impacts the whole thing.
Confusion...
M: whenever you have this –(-3+x). If I want to make disappear the parenthesis, and this minus here... do you remember what do we do?
B: no
M: do you remember that we change the signs?
T, C: no
M: did you ever see it?
C, T, B: nooo!!!
M: are you in the same class as them (pointing to another group)?
C, T, B: yes.
M: they saw it, so you saw it.
Laughs...
T: but I don't know...
M: don't worry, I will remind you. If the signs are equal you get a +, if the signs are different you get a -.
M: do you follow? Or, do I repeat?
T: yes...
M: ok, so that is your challenge now and here. Try that challenge please and if you have any problems please call me.

This episode illustrates the type of interventions that were typical during class 1 and class 2, and that decreased without disappearing towards the end of the first part of the intervention, as well, as toward the end of the intervention in general. Many opportunities presented themselves as a consequence of the *need* to work the initial expression, subordinating the transformations towards an ulterior goal; the goal itself was not transforming an expression by the sake of transforming but towards proving that the initial expression would always end up being -7.

In order to analyze students' way of working from the initial expression, besides its format (*equation* or *algebraic expression*), I took into account the treatment that students gave to the *initial expression* and to its derivations into *equivalent expressions*. A posteriori, I defined three categories to classify students' use of the *initial expression* and its derivations:

(1) traditional *equation solving* strategy,

(2) *formal derivation* strategy, and

(3) the *hybrid* strategy.

Equation solving (traditional school equation-solving) strategies are those in which students maintained expressions on both sides of the equal sign on the same line, and applied the same operation on both sides of the equal sign. This last feature is important because it indicates that both sides of the equal sign are engaged in an equation solving strategy. *Formal derivation* strategies, such as Janusz's strategy (Figure 29), were those in which an expression representing the outcome did not appear on the other side of the equal sign after the expression indicating the calculation, and expressions equivalent to the initial one were derived through the application of properties (i.e., distributive, associative, cancellation, etc). *Hybrid strategies*, such as Cory's (Figure 30), looked like an equation solving strategy by maintaining the outcome after the equal sign, even though the subsequently derived algebraic expressions were obtained by applying properties, although not simultaneously, on both sides of the expression.

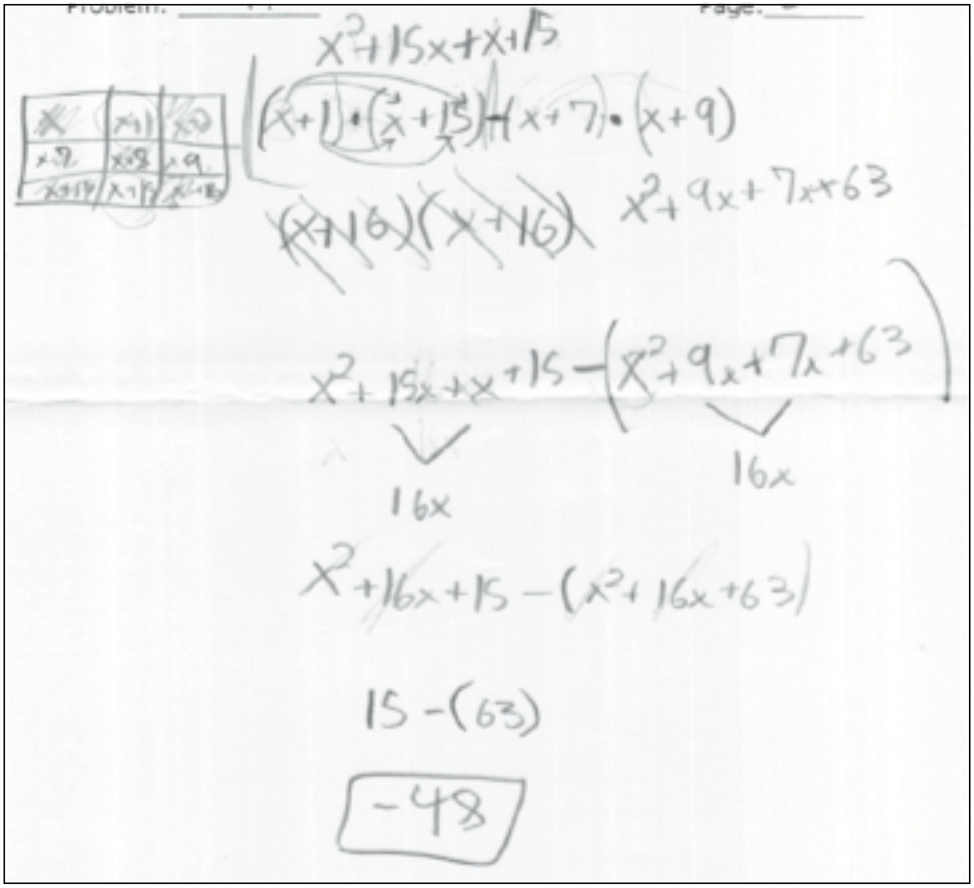


Figure 29. Janusz's hybrid strategy to prove his conjecture in Problem 16.

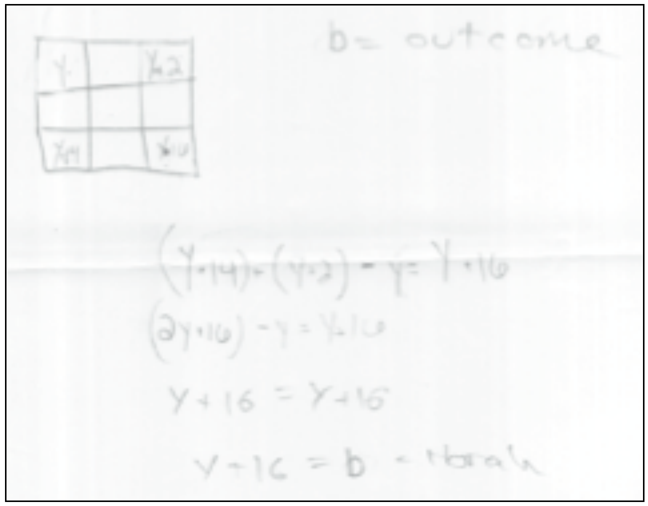


Figure 30. Cory's hybrid strategy to prove his conjecture in Problem 16.

Hybrid Strategy. Given that a majority of students used the *equation format* for the *initial expression* in Problems 1, 16, and 17, it seems contradictory to find no students used an *equation solving strategy*. However, the fact that no students used an *equation solving strategy* is not contradictory given the nature of the problems involved. At this point in the problem solving process, the nature of the students' task was to show that for any value of the independent variable, the outcome would always be -7 in the case of Problem 1; -48 in the case of Problem 17; and the number located in the lower right within the 3x3-calendar-square in Problem 16. *The students' task was not to "find the value of x."* In Problem 1[27], 75% of students (6/8, see Table 10) kept the outcome on the right hand side of the equal sign. However, none of these students used the expression representing the outcome or applied the same operation on both sides of the

[27] The percentages for Problem 1 should be considered in context because they reflect students' written work as a result of my frequent interventions questioning their decisions to try to push their thinking (e.g., my interventions directed to question students use of non-exhaustive finite set of numeric examples). This was not the case for Problems 16 and 17.

equal sign. This shows that the expression representing the outcome was not necessary for their derivations. In Problem 16, all students (100%, 8/8, see Table 9) continued to use an expression representing the outcome after the equal sign; however, none of them used an *equation solving strategy*. All students (100%, 8/8, Table 9) used a *hybrid strategy* to show that the outcome was the number located in the lower right within the 3x3-calendar-square. Students used properties (i.e., distributive, cancellation, associative) on one side of the equal sign to derive an equivalent expression without engaging both sides of the expression at any moment. This suggests that students were using *old tools* (algebraic notation, properties) in *new ways* (i.e., to show the invariance of the outcome). In Problem 17, 55.5% of students (5/9, see Table 9) applied a hybrid strategy. Since none of the students used an equation solving strategy, this indicates that there is a shift at the end of the first half of the intervention towards a formal derivation strategy.

Student	Problem 1		Problem 16		Problem 17	
	Initial Expression Format	Strategy	Initial Expression	Strategy	Initial Expression	Strategy
Abbie	Equation	Hybrid	Equation	Hybrid	Equation	Hybrid
Desiree	Equation	Formal Derivation	-	-	Equation	Hybrid
Grace	Equation	Formal Derivation	Equation	Hybrid	Equation	Hybrid
Audrey	Equation	Hybrid	Equation	Hybrid	Algebraic	Formal Derivation
Chris	Equation	Hybrid	Equation	Hybrid	Algebraic	Formal Derivation
Janusz	Algebraic	Formal Derivation	Equation	Hybrid	Algebraic	Formal Derivation
Brian	Equation	Hybrid	Algebraic	Hybrid	Algebraic	Formal Derivation
Cory	Equation	Hybrid	Equation	Hybrid	Equation	Hybrid
Tyler	Equation	Hybrid	Equation	Hybrid	Equation	Hybrid

Table 8. Students' Initial Expression format and Strategy for Problems 1, 16 and 17.

		Problem 1		Problem 16	Problem 17
Duration		2 lessons (#1 and #2)		1 lesson (#7)	15-30 minutes (Individual Interview)
		Day 1	Day 2		
Use of the initial Expression	Equation solving strategy	-	0%	0%	0%
	Hybrid	-	75% (6/8)	100% (8/8)	55.5% (5/9)
	Formal derivation strategy	-	25% (2/8)	0%	44.5% (4/9)

Table 9. Students' Treatment of the Initial Expression in Problems 1, 16, and 17.

<table>
<tr><td colspan="1"></td><td colspan="6">Problem 1</td><td colspan="3">Problem 16</td><td colspan="3">Problem 17</td></tr>
<tr><td rowspan="2">Duration</td><td colspan="6">2 lessons (#1 and #2)</td><td colspan="3" rowspan="2">1 lesson (#7)</td><td colspan="3" rowspan="2">15-30 minutes (Individual Interview)</td></tr>
<tr><td colspan="3">Day 1</td><td colspan="3">Day 2</td></tr>
<tr><td rowspan="8">Use of a name/expression/variable to represent the outcome</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td><td>Students (%)</td><td>Used Expression</td><td>Students (%)</td></tr>
<tr><td rowspan="7">-</td><td rowspan="7">-</td><td rowspan="7">-</td><td rowspan="7">75% (6/8)</td><td rowspan="7">-7</td><td rowspan="7">100% (6/6)</td><td rowspan="7">100% (8/8)</td><td>X+2</td><td>12.5% (1/8)</td><td rowspan="7">55.56% (5/9)</td><td rowspan="4">-48</td><td rowspan="4">60% (3/5)</td></tr>
<tr><td>x+2 and y</td><td>12.5% (1/8)</td></tr>
<tr><td>n</td><td>12.5% (1/8)</td></tr>
<tr><td>y+16</td><td>12.5% (1/8)</td></tr>
<tr><td>"outcome"</td><td>25% (2/8)</td><td rowspan="3">x</td><td rowspan="3">40% (2/5)</td></tr>
<tr><td>y</td><td>12.5% (1/8)</td></tr>
<tr><td>y+16 and b</td><td>12.5% (1/8)</td></tr>
</table>

Table 10. Students' use of an expression to represent the outcome.

Formal Derivation Strategy. As shown in Table 9, in Problem 1, only two students (2/8 or 25%) used a *formal derivation* strategy. Surprisingly, in Problem 16 no students used this type of strategy, while in Problem 17 44.5% of students (4/9) used a *formal derivation* strategy.

This seems very promising given the fact that students were not encouraged to follow a particular strategy and my interventions were minimal in comparison with the interventions in class 1 and 2 (for Problem 1).

In both the *hybrid* and the *formal derivation strategies*, students applied properties on one side of the equal sign to derive an equivalent expression. This data is an indication that students learned how to use the *old algebra tools* in *new* ways, such as to prove a conjecture.

Proof

When solving Problems 16 and 17, all of the students used algebra as a tool to prove their conjecture. This was not the case when solving Problem 1, perhaps because it was the first lesson and this was a completely new way of working with algebra (see Table 11). A high percentage of students (87.50% or 7/8 in Problem 16 and 89% or 8/9 in Problem 17) used algebra "correctly" to produce a proof. I classified a proof as correct based on the use of either the *hybrid* or *formal derivation strategies*, even though it could be argued that the *hybrid strategy* is not logically correct since students write an expression representing the outcome after the equal sign. However, my point is that students wrote the expressions representing the outcome when using a *hybrid strategy* but they did not use the expression representing the outcome in their derivations. I see the use of the *hybrid strategy* as an adaptation of their algebraic tools being used in *new* ways. However, since I consider this adaptation temporary, I dedicated part of the second half of the intervention, which will be described below, to get students to shift completely towards a *formal derivation strategy*. Two of the strategies used by the teacher researcher to promote students' shift from a *hybrid* to a *formal derivation strategy* were:

(1) By incorporating parameters, it was very difficult for them to anticipate the outcome as a function of the variables and parameters, and consequently, it was very challenging for them to have an expression for the outcome.

(2) Incorporating more unpredictable variable outcomes in order for them to have an intuition regarding some aspect of the behavior of the outcome but not knowing exactly how it would behave in terms of the variables and parameters involved.

	Problem 1		Problem 16	Problem 17
Duration	2 lessons (#1 and #2)		1 lesson (#7)	15-30 minutes (Individual Interview)
	Day 1	Day 2		
Interpretation of result in writing	0%	87.5% (7/8)	50% (4/8)	0%
Complete and correct proof	0%	100% (8/8)	87.5% (7/8)	89% (8/9)

Table 11. Written interpretation of the result and students production of proofs.

Conclusions

First, as a general conclusion, I claim that the results presented in this paper provide promising evidence that an integrated approach towards algebra and proof, such as that implemented in the Calendar Sequence, has a positive impact on students' use of algebra as a tool to prove. Students used the algebraic notational system as a modeling tool and proved their conjectures.

Student Learning

The data presented in this paper provide evidence regarding the Calendar Sequence and its impact on student learning. Problems in the sequence fostered students' production of their own conjectures. Initially, in many of the lessons such as class 1 and class 2 (Problem 1) and class 13 (Problem 9), there were different conjectures at play. When students embarked on gathering evidence to prove their conjectures, the students gradually arrived at a shared conjecture. During the second half of the intervention, when we focused on parameters as well as variables, students' conjectures were more homogeneous than during the first part of the intervention. In part, this could be explained by students' familiarity with the system, i.e., the calendar, under study.

The data here presented also show that students were able to use variables meaningfully, and to establish relations among them to create a model of the situation (i.e., the *generic model*), and to use this model to express the calculations through their *initial expressions*. During the intervention, students developed strategies (*hybrid* and *formal derivation strategies*) to prove their conjecture through the use of algebra. These strategies consisted in the use of properties (i.e., associative, distributive, and cancellation) to *derive* an expression equivalent to the initial expression. During the first half of the intervention, when focused on problems that involved variables, the derivation strategies gravitated towards the hybrid strategy.

The following were the challenges identified that students faced and that were overcome while working on the Calendar Sequence:

(h) the use of algebra to prove in contrast with a finite non-exhaustive set of numeric examples;

(i) the use of a single variable that could capture all cases at the same time;

(j) the number of dependent and independent variables to include in their expressions;

(k) how to set up relations among variables;

(l) how to obtain a simpler expression from a more complex algebraic expression;

(m) the use of conventions of algebra such as the use of parenthesis;

(n) the use of properties such as the use of distributive property.

Students were able to overcome these obstacles; some, more easily than others. By the end of the intervention, students were able to use algebra as a tool to prove complex conjectures involving several variables.

In this study, students used algebra as a tool in a new way from the way they were typically used to using algebra. They began to use algebra to derive equivalent expressions in order to prove a conjecture. It is relevant to highlight that during the first part of the intervention (Problems 1, 16, and 17), the students did not use an *equation solving strategy*; this strategy might have been one of the expected strategies for students to use given the high percentage of students setting up their initial expressions as equations. This mismatch in the approach to the initial expression on one hand and the strategies for transforming the expressions on the other is very important; it provides support for the position that students adapted algebraic notation that was already familiar to them and began using it in a new way, with a new goal in mind: to *derive equivalent expressions to prove a conjecture* as opposed to *solving for x*.

As mentioned above, students encountered challenges when working on problems from the Calendar Sequence; however, students overcame the challenges and learned from them. The evidence presented in this paper counters claims regarding students capabilities to learn algebra and use it as a tool such as:

(a) believe that the equals sign only represents a unidirectional operator that produces an output on the right side from the input on the left (e.g., Booth, 1984; Kieran, 1981, 1985; Vergnaud, 1985, 1988a);

(b) focus on finding particular answers (e.g., Booth, 1984);

(c) do not recognize the commutative and distributive properties (e.g., Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 2001; Demana & Leitzel, 1988; MacGregor, 1996);

(d) do not use mathematical symbols to express relationships among quantities (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Vergnaud, 1985; Wagner, 1981);

(e) do not comprehend the use of letters as generalized numbers or as variables (e.g., Booth, 1984; Kuchemann, 1981; Vergnaud, 1985);

(f) have great difficulty operating on unknowns (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg, Sleeman, & Ktorza, 1990); and

(g) fail to understand that equivalent transformations on both sides of an equation do not alter its truth value (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg, Sleeman, & Ktorza, 1990).

The evidence presented in this paper along with Barallobres' (2001) and Bell's (1993) studies highlight that middle and high school students not only are able to tackle the above-mentioned obstacles but also going further. I believe that as a research community we still need to develop and implement mathematical activities and document students' learning in order to identify elements that foster algebraic learning.

Approaches to algebra

As mentioned in the *Previous Studies* section in this paper, there are different approaches to school algebra that have produced a wide variety of teaching experiments: arithmetic and algebra which focuses on the differences between arithmetic and algebra (e.g. Filoy & Rojano, 1989; Kieran, 1981); generalization approach which emphasizes the role of algebra to express general properties or statements, and relations (e.g. Mason, 1996; Lee, 1996); modeling approach that focus on algebra as a modeling tool (e.g. Chazan, 1993, 2000; Nemirovsky, 1996); functional approach which considers function, and its entailed concept, variables at the center of school algebra (e.g. Schwartz & Yerushlami, 1992).

The algebraic way of functioning at play in this sequence is different from algebra in an equation solving context where students apply operations on both sides of the equal sign simultaneously with the ultimate goal of determining the value/s of the unknown. Also, the approach taken in this paper *Integrating algebra and proof in high school* differs from a functional approach to algebra, where the focus is on the notion of function as process and object, and multiple representations are emphasized. This is not to say that the approach taken in this study should replace a focus on functions; to the contrary, this is a complementary perspective that highlights essential features (usually not present in the or functional approach) of algebra, focusing on its use as a modeling tool for a system in order to obtain new information. Proving this complementary character, as shown in the *Previous Studies* section, Chazan (1993, 2000) focused on a modeling approach centering on functions. The approach taken in this paper approach relies heavily on the notions of equivalence and transformations. The power of the algebraic notational system resides in that it allows us to obtain new information through the use of transformations, operating on the system (i.e., by applying transformations on expressions and obtaining equivalent expressions to the formers) and following its rules. Equation solving allows us to identify a set of values that satisfy a condition (or a set of conditions), and the letter plays the role of an unknown. On the other hand, functions are central to building the notion of the letter as a variable, capturing as many values of different nature as we want, as well as to understand a particular kind of correspondence between sets. Once these notions start to develop as constituents of algebra for the students, from an epistemological point of view it is our obligation, I believe, to offer students the experience of operating with the system of algebra in a new, perhaps more advanced, way. In this integrated approach to algebra and proof, the goal of objects such as variables, parameters and relations among them is to obtain new information about a given system. This feature is very important since it goes beyond expressing all cases at the same time. We want students to be able to operate with the algebraic notational system to operate on itself- to derive equivalent expressions that would provide new information about the studied phenomena that we did not have at the initial point. This feature is usually absent or downplayed in the school curriculum but also in research approaches to algebra. In addition, one of the most fruitful contexts to use this powerful feature is in the process of proving conjectures.

Theoretical tools

The Theory of Didactical Situations (Brousseau, 1997), the Modeling Perspective (Chevallard, 1985, 1989-1990), and Didactical Engineering (Artigue, 1988, 1991, 1994) have been essential tools in the conception and conceptualization of the work here presented. This paper constitutes an example of how the theories just mentioned can be used to design, develop, implement, and assess a sequence of problems with a thorough a-priori analysis of the mathematics involved along with a thorough a-priori analysis of previous research findings on students' learning. The Theory of Didactical Situations (Brousseau, 1997) proved essential in terms of structuring each problem (action, formulation, and validation situations) and analyzing its consistency with the goals in terms of the targeted mathematical notions. Chevallard's (1985, 1989-90) modeling perspective helped to conceptualize essential features of algebra, the role of algebra in the problems, and to structure the data analysis taking into account the three stages (i.e., identifying variables and parameters involved in the situation to be studied, setting up the relations among variables and parameters, and working the model –built by the two stages just mentioned- to gain information about the studied situation that was not available at the beginning of the modeling process) in the modeling process. Didactical Engineering was central in conceiving the study as a sequence of problems with pre-set goals as a whole. These theories provide us with a set of research tools that allow structuring the design, implementing the sequence, and conducting the data analysis.

Future Directions

According to the principles of Didactical Engineering, the testing of the didactical hypotheses requires the reproduction of the intervention in order to analyze the reproducibility of the Calendar Sequence, in this case. In addition, since the sample of students I worked with in this study was small and the sequence was only conducted once, future reproductions of the sequence should be carried out to analyze how general and stable the results here reported are. It is my goal and intention to carry out future interventions in a variety of schools and with teachers that work in the school system.

There are many analyses that remain to be done with the already collected data such as:

- Teacher Interventions and their role in students' overcoming of obstacles,
- Problems 6, 7, and 8 dealing with variables and parameters n (i.e., dimension of the shape),
- Problem 15 dealing with variables and two parameters (d and n),
- Students' generalizations across the whole sequence,
- Use of examples across the problems in the sequence,
- Interpretations of the final results within each group and in the whole group discussions,
- Students' spontaneous creation of new problems and analysis of the domain of validity of their conjectures.

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APPENDIX A

Abbie Nehring

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Problem 1

Part 1

Consider squares of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be $\begin{smallmatrix}1 & 2\\ 8 & 9\end{smallmatrix}$. These squares will be called 2x2-calendar-squares.

Calculate:

(1) The product between the number in the upper left corner and the number in the lower right corner.

(2) The product between the number in the upper right corner and the number in the lower left corner.

(3) To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

Find the 2x2-calendar-square that gives the biggest outcome. You may use any month of any year that you want.

Example:

JANUARY 2007						
S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

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(1) 1 x 9 = 9

(2) 8 x 2 = 16

(3) 9-16 = -7

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Problem 2

Part 1

Consider squares of three by three compound by the days of a certain month (See below).

For example, a square of three by three can be $\begin{matrix} 9 & 10 & 11 \\ 16 & 17 & 18 \\ 23 & 24 & 25 \end{matrix}$. These squares will be called 3x3-calendar-squares.

Calculate:

(1) The ~~product~~ sum between the number in the upper left corner and the number in the lower right corner.

(2) The ~~product~~ sum between the number in the upper right corner and the number in the lower left corner.

(3) To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

Find the 3x3-calendar-square that gives the smallest outcome. You may use any month of any year that you want.

Example

JANUARY 2007						
S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Figure 1

Integrating Algebra and Proof

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The calculation is:

1. 25+9=34
2. 23+11=34
3. 34-34=0

Problem 2

Part 2

Is there any other 3x3-calendar-square that could give a smaller number? Explain.

Is there any other 3x3-calendar-square that could give a bigger number? Explain.

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Problem 3

Analyze what happens in the case of a 4x4-calendar-square in the two following cases:

(1) Using the operation used in Problem 1.

(2) Using the operation used in Problem 2.

The Calendar Sequence Parker Charter School

Name: Desiree Jerome Date: 11/8/06

Problem: Chris' Problem Page: 1

Chris came up with a formula: $\boxed{-7(n-1)^2 = \text{the outcome}}$ n= the # of sides. The formula is for finding the solution with the number of days for a given square size. this could even with a month with an infinite number of days, only go to a 7x7 square, becau otherwise, the relationship between the days would change.

Problem 16

Part 1

Consider squares of three by three compound by the days of a certain month (See below).

For example, a square of three by three can be $\begin{matrix} 9 & 10 & 11 \\ 16 & 17 & 18 \\ 23 & 24 & 25 \end{matrix}$. These squares will be called

3x3-calendar-squares.

Calculate:

(1) The sum between the number in the lower left corner and the number in the upper right corner.

(2) To the number obtained in (1) subtract the number in the upper left corner. This result is your outcome.

Find the 3x3-calendar-square that gives the smallest outcome. You may use any month of any year that you want.

Example

JANUARY 2007						
S	**M**	**T**	**W**	**T**	**F**	**S**
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

The calculation is:

1. $23+11=34$
2. $34-9=25$

Problem 5

Part 1

Using 4x4-calendar-square in any month of any year, your task is to define a set of operations in such a way that:

(i) the outcome of the operation varies depending on the place of the month where the square is located, and,

(ii) the outcome obtained is always the first number in the square.

Example

JANUARY 2007						
S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Set of Operations:

(Operation 1)………………………………………………………

(Operation 2) ………………………………………………………

(Operation 3) ………………………………………………………

(Operation 4) Outcome for this square in this position in this month is 1.

You may use more than five operations. You may use addition, subtraction, multiplication, and/or division.

Problem 6

Part 1

You can work with any month of the calendar. Suppose we choose January 2007. In this case you are going to work with a rectangle shape with variable length and fixed height of one. These rectangles will be called 1xn-calendar-rectangles.

Your task is to choose the <u>length n</u> of a 1xn-calendar-rectangle in such a way that you get the biggest possible outcome when you do the following set of operations:

1. Add the numbers in the last and first places within the rectangle.
2. To the number obtained in step 1 subtract the number in the next to last place in the rectangle.

Example:

JANUARY 2007						
S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

In the example shown we are applying the set of operations in a 1x4-calendar-rectangle in the following way:

1. 8+11=19
2. 19-10=9

Write your conclusion

Problem 6

Part 2

Analyze how this outcome varies. Show why the outcome is always going to be the number on the second spot, indenpendently of the dimension of the rectangle.

Problem 7

Part 1

In this case, you will be working with a slightly different calendar. This calendar has a week of seven days but each month has an infinite number of weeks.
You will be working with a vertical rectangle whose width is one and height is n. These rectangles will be called nx1-calendar-rectangle.

Your task is:

-to decide **where** to place a nx1-calendar-rectangle and,
-to **determine** the value of n in the nx1-calendar-rectangle

in such a way that the outcome turns out to be the biggest possible. Do not us numeric examples.

The set of operations to carry out is:

(1) to the number in the next to last spot (at the bottom) within the rectangle take away the number in the second spot (at the top) within the rectangle.

Example:

JANUARY 2007

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	...	...

In this case we are using a 7x1-calendar-rectangle, and the operation to carry out is:

45-17=28

Problem 7

Part 2

Is there a bigger outcome? How do you know? Explain.

Identify what factors the outcome depends on.

Problem 8

Analyze the behavior of the outcome in the case of a nxn-calendar square and, when the month has infinite number of weeks but each week has seven days.

The set of operations is (same as Problem 1):

(1) The product between the number in the upper left corner and the number in the lower right corner.

(2) The product between the number in the upper right corner and the number in the lower left corner.

(3) To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

Example

JANUARY 2007						
S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69
70	71	72	73	74	...	...

In this case we are using a 5x5-calendar-square and the set of operations to perform is:

(1) 15 x 47=705

(2) 43 x 19=817

(3) 705-812=-107

Possible questions to explore:

Is the outcome a constant?

Does the outcome depend on any features of the month or dimensions of the square?

Problem 9

In <u>problem 1</u>, it was shown that the outcome resulting of the subtraction of the cross products when applying a square shape of dimension two by two is always -7.
Now, instead of working with a week of seven days, the week is going to be formed by 9 days.
Analyze the behavior of the outcome when using a 2x2-calendar-square with the following set of operations:

(1) The product between the number in the upper left corner and the number in the lower right corner.

(2) The product between the number in the upper right corner and the number in the lower left corner.

(3) To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

Aspects to be analyzed in the behavior of the outcome:

(1) Does the outcome vary depending on the position of the 2x2-calendar-square within the month?

(2) In comparison to the outcome of problem 1, does the length of the week affect the outcome?
If you answer 'yes', how does the change of number of days per week affect the outcome?
If you answer 'no', explain why.

Problem 18 ~~(Individual Interview #2)~~

In this case you will be working with a month that has **d days** in each week. The shape to use is a **2x2-calendar-square**.

The set of operations to carry out is the following:

(1) Multiply the numbers located in the upper left corner and lower right corner.

(2) Multiply the numbers located in the upper right corner and lower left corner.

(3) From the number obtained in (1) subtract the number obtained in (2). This is your outcome.

Your task is to analyze the behavior of the outcome in terms of its dependency on the dimensions of the square, the length of the week, and position of the square.

Problem 15

In this case you will be working with a month that has **d** days in each week. The shape to use is nxn-calendar-square.

The set of operations to carry out is the following:

(1) Multiply the numbers located in the upper left corner and lower right corner.

(2) Multiply the numbers located in the upper right corner and lower left corner.

(3) From the number obtained in (1), subtract the number obtained in (2). This is your outcome.

Your task is to analyze the behavior of the outcome in terms of its dependency on the dimensions of the square, the length of the week, and location of the square.

APPENDIX B

APPENDIX B

Problem Number	Line of development	Result for the problem	Dependency	It is requested to find/define/solve…	Is it possible?	Values of the variables						
						Mathematical template	Shape	Dimension	Calculation	Length of the week	Average weeks per month	Outcome
1	I	-7	Outcome doesn't depend on location on the month, month and year.	Biggest outcome	NO. The outcome is always -7.	$a_n=n$ with n in N<32	Square	2x2	The difference between the products of the numbers in the extreme of the diagonals	7	4	-7
2	I	0	Outcome doesn't depend on location on the month, month and year.	Smallest outcome	NO. The outcome is always 0.	$a_n=n$ with n in N<32	Square	3x3	(1) The sum between the number in the upper left corner and the number in the lower right corner. (2) The sum between the number in the upper right corner and the number in the lower left corner. (3) To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.	7	4	0
3.1	I	-63	Outcome doesn't depend on location on the month, month and year.	Analysis of situation	YES.	$a_n=n$ with n in N<32	Square	4x4	The difference between the products of the numbers in the extreme of the diagonals	7	4	-63
3.2	I	0	Outcome doesn't depend on location on the month, month and year.	Analysis of situation	YES.	$a_n=n$ with n in N<32	Square	4x4	(1) The sum between the number in the upper left corner and the number in the lower right corner. (2) The sum between the number in the upper right corner and the number in the lower left corner. (3) To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.	7	4	0
16	I	a+16	Outcome depends on the place where the square is located and the outcome is always the first number in the square plus 16	Smallest outcome	Yes.	$a_n=n$ with n in N<32	Square	3x3	(1) sum between the number in the lower left corner and the number in the upper right corner, (2) to the number obtained in (1) subtract the number in the upper left corner. This result is your outcome.	7	4	variable= a+16
5	I	Set of operations	Outcome doesn't depend on location on the month, month and year.	Define a set of operations	YES.	$a_n=n$ with n in N<32	Square	4x4	-	7	4	outcome 1
6	III	Rectangle length	Outcome doesn't depend on location on the month, month and year.	Find the length of the rectangle in a way of finding the biggest outcome	NO	$a_n=n$ with n in N<32	Rectangle	1xn	(1) add the numbers in the last and first places within the rectangle, (2) to the number obtained in step 1 subtract the number in the next to last place in the rectangle	7	4	outcome= a+1

Problem Number	Line of development	Result for the problem	Dependency	It is requested to find/define/solve…	Is it possible?	Values of the variables						
						Mathematical template	Shape	Dimension	Calculation	Length of the week	Average weeks per month	Outcome
7	III	Rectangle length	Outcome depends on the length of the rectangle	Find where to place the rectangle and the length of the rectangle.	NO, because the month is infinite and the length can be as big as you want. There isn't a	a_n=n with n in N	Rectangle	nx1	To the number in the next to last spot (at the bottom) within the rectangle take away the number in the second spot (at the top) within the rectangle.	7	Infinite	outcome = 7n-16
8	III	$-7(n-1)^2$	Outcome depends on the dimension of the square	Analyze where to place it and the length of the square.	Yes, students can obtain a formula	a_n=n with n in N	Square	nxn	The difference between the products of the numbers in the extreme of the diagonals.	7	Infinite	outcome= $7(n-1)^2$
9	II	-9	Outcome doesn't depend on location on the month, month and year.	Comparison with result obtained in problem 1	Yes	a_n=n with n in N	Square	2x2	the difference between the products of the numbers in the extreme of the diagonals	9	-	outcome= -9
18	II	-d	Outcome depends on the length of the weeks	Analyze the behavior of the outcome	Yes		Square	2x2	the difference between the products of the numbers in the extreme of the diagonals	d	-	outcome= -d
15	IV	It depends on the dimension of the square and the length of the week	It depends on dimension of the square and the length of the week	Analyze the behavior of the outcome	YES.	a_n=n with n in N	Square	nxn	(1) Multiply the numbers located in the upper left corner and lower right corner. (2) Multiply the numbers located in the upper right corner and lower left corner. (3) From the number obtained in (1), subtract the number obtained in (2).	d	-	outcome= $-(n-1)^2d$
17	I	-48	It doesn't depend on location, dimension, month, and year.	smallest outcome	NO	a_n=n with n in N<32	Square	3x3	the difference between the products of the numbers in the extreme of the diagonals	7	4	outcome= -48
12	II	The outcome depends on the length of the week but not on the position of the square.	It depends on d	Analyze the behavior of the outcome	YES.	a_n=n with n in N	Square	4x4	(1) Multiply the numbers located in the upper left corner and lower right corner. (2) Multiply the numbers located in the upper right corner and lower left corner. (3) From the number obtained in (1) subtract the number obtained in (2). This is your outcome.	d	-	-9d

APPENDIX C

Problem	First place in shape	Days per week	Shape	Dimension of shape	Picture	Calculation Diagram	Initial Equation	Outcome
1	a	7	Square	2x2			$a\cdot(a+8)-(a+7)(a+1)$	-7
2	a	7	Square	3x3			$a+(a+16)-[(a+14)+(a+2)]$	0
3.1	a	7	Square	4x4			$a\cdot(a+24)-(a+21)(a+3)$	-63
3.2	a	7	Square	4x4			$a+(a+24)-[(a+21)+(a+3)]$	0
16	a	7	Square	3x3			$(a+14)+(a+2)-a$	a+16
5	a	7	Square	4x4		to de defined by the student	to de defined by the student	a
6	a	7	Rectangle	1xn			$a+[a+(n-1)]-[a+(n-2)]$	a+1
7	a	7	Rectangle	nx1			$[a+7\cdot(n-2)]-(a+7)$	7n-21
8	a	7	Square	nxn			$a\cdot[a+(n-1)8]-[a+(n-1)\cdot 7]\cdot[a+(n-1)]$	$-7(n-1)^2$
9	a	9	Square	2x2			$[a\cdot(a+10)]-[(a+9)\cdot(a+1)]$	-9
18	a	d	Square	2x2			$[a\cdot(a+d+1)]-[(a+d)\cdot(a+1)]$	-d
15	a	d	Square	nxn			$a\cdot[a+(n-1)\cdot(d+1)]-[a+(n-1)\cdot d]\cdot[a+(n-1)]$	$-d(n-1)^2$
17	a	7	Square	3x3			$a\cdot(a+14)-(a+6)\cdot(a+8)$	-48
12	a	d	Square	4x4			$a\cdot(a+3d+3)-(a+3d)\cdot(a+3)$	-9d
Color key	1. variable a	1. variable 'a' 2. paramter in the shape dimension	1. Variable 'a' 2. Parameter days/week	1. Variable 'a', 2. Pameter days/week, 3. Parameter shape's dimension				

Figure 2