RIGID MEASURES ON THE TORUS

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Abstract

In this dissertation, we study a problems in smooth ergodic theory. Given a measure μ on a manifold M, we wish to characterize all smooth dynamics preserving μ. We consider measures μ supported on the two-torus and study the group of μ-preserving diffeomorphisms. For μ invariant under an Anosov diffeomorphism, we find conditions for which the group of μ-preserving diffeomorphisms is 'essentially' cyclic.

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CHAPTER 1

INTRODUCTION

1.1 SMOOTH DYNAMICS

In general, a smooth dynamical system (of class C^r) is defined by specifying the following data:

— a manifold M equipped with a C^r differential structure;

— a Lie group (or Lie monoid) G;

— a group action $\phi\colon G \times M \to M$ such that the map ϕ is C^r.

Classically, the study of smooth dynamics has primarily focused on one-parameter actions:

The discrete case: G is the group $\mathbb{Z}$ or the monoid $\mathbb{N}$. Here the action is generated by a single C^r map $f := \phi(1, \cdot)\colon M \to M$.

The continuous case: G is the group $\mathbb{R}$ or the monoid $\mathbb{R}_{\geq 0}$. In the setting where $G = \mathbb{R}$, the group action ϕ is a *flow*—the solution to a system of ordinary differential equations specified by the vector field X which in local coordinates is defined by

$$X_p = \frac{d\phi(t,p)}{dt}\Big|_{t=0}.$$

We note that we lose no generality in assuming the manifold M is C^∞. Indeed any C^k manifold, $k \geq 1$, is C^k diffeomorphic to C^∞ manifold M'.[1] A C^k group action on M then induces a C^k action on M'. Properties we wish to deduce about the action on M can be

[1] See for example [Hir94, Theorem 2.10].

deduced from the action on M'. We thus assume throughout the thesis that the underlying manifolds are all C^∞.

In this thesis, the primary interest is on actions of discrete one-parameter groups. However, many of the problems presented, particularly those in Chapter 4, are motivated by results from the theory of actions of higher-rank abelian groups.

1.2 INVARIANT STRUCTURES IN SMOOTH DYNAMICS

Given a group acting on a manifold M, one typically finds many auxiliary structures on M preserved by the action. The following question provides a paradigm through which one might hope to classify and study specific dynamical systems.

Question 1.1. *Given a group action on a space, identify, classify, and study the properties of various 'structures' left invariant by the action.*

Examples of auxiliary structures that are of common interest in the dynamics literature include:

topological structures: including fixed points, periodic orbits, closed subsets, splittings of the tangent space, and foliations;

geometric structures: including (conformalilty classes of) Riemannian metrics, connections, and horizontal subbundles;

measure theoretic structures: including σ-algebras, Borel measures, and measure classes.

For many families of dynamical systems, there are well developed theories regarding the existence and properties of various invariant structures. We present a few well-studied examples from the literature.

One-parameter hyperbolic actions: For an Axiom A diffeomorphism, Anosov diffeomorphism, or *expanding map* (e.g. the map $x \mapsto 2x \mod 1$ on the circle $\mathbb{R}/\mathbb{Z}$), on a compact manifold (see Chapter 2 for definitions) there exist periodic orbits of arbitrarily large period and many distinct closed invariant subsets. Furthermore, there exist an uncountable number of mutually singular ergodic measures with positive

dimension; in particular the equilibrium states for Hölder continuous functions, presented in Chapter 3, provide a such a family of measures.

Higher-rank abelian actions: In contrast to one-parameter hyperbolic actions is the theory of higher-rank algebraic actions. Unlike the one-parameter setting, for higher-rank hyperbolic actions there exist relatively few invariant measures and closed invariant subsets. For instance, Rudolph showed that any measure on $\mathbb{R}/\mathbb{Z}$, ergodic under the abelian semi-group action generated by

$$x \mapsto 2x \mod 1 \quad \text{and} \quad x \mapsto 3x \mod 1,$$

is either Lebesgue or has zero dimension [Rud90]. Similar dichotomies hold for certain algebraic $\mathbb{Z}^k$-actions, $k \geq 2$, on $\mathbb{T}^d$ with Anosov elements [KS96] and for diagonal actions on the homogeneous space $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$, $k \geq 3$, [EKL06]. Extensions of these results to non-algebraic and nonuniformly hyperbolic settings have also appeared in [KK01], [KK07], and [KRH10].

One-parameter unipotent actions: For a Lie group G and a one-parameter unipotent subgroup $U \subset G$, consider the action of U on the homogeneous space G/Γ for some lattice $\Gamma \subset G$. Such an action is called a *unipotent flow* ϕ_U^t on G/Γ. In contrast to one-parameter hyperbolic actions, Ratner's *measure classification theorem* guarantees the existence of relatively few invariant probability measures for the flow ϕ_U^t. In particular, the only ergodic ϕ^t-invariant probability measures are *homogeneous*; that is, such measures are the image of a Haar measure on a coset of a closed subgroup. See for instance [Mor05] for a precise statement.

In contrast to Question 1.1, we also consider the following natural, but far less studied, problem:

Question 1.2. *Given an auxiliary structure on a manifold, classify—or find nontrivial constraints on—the set of dynamics preserving the structure.*

In considering Question 1.2, we often impose additional regularity or dynamical hypotheses to make the problem more tractable. For instance, when the auxiliary structure is a

measure, we might focus on classifying all measure-preserving dynamics that act with positive entropy. For closed subsets, orbits, or splittings of the tangent bundle we might focus only on dynamics that preserve those structures and act with some degree of hyperbolicity. For invariant foliations we might impose some uniform or asymptotic volume expansion of the foliation under the dynamics.

We present some pertinent results from the literature that address Question 1.2.

Codimension-1 basic sets. In [Ply71, Theorem 3] Plykin showed that any basic set $\Lambda \subset M$ for an Axiom A diffeomorphism with $\dim(\Lambda) = \dim M - 1$ is either an attractor or a repeller for the ambient dynamics.

Codimension-1 Anosov maps. The Franks-Newhouse Theorem [Fra70, New70] shows that any codimension-1 Anosov diffeomorphism is topologically conjugate to a hyperbolic toral automorphism. That is, preservation of a continuous invariant splitting combined with the dynamical hypothesis of uniform exponential growth forces the system to be, up to a continuous change of variables, algebraic.

Invariant connections. In [BL93] Benoist and Labourie show (using the primary result from [BFL92]) that any Anosov diffeomorphism with smooth stable and unstable distributions that preserves a smooth connection is smoothly conjugate to an infranil-automorphism.

1.3 OUTLINE OF THE MAIN RESULTS

Consider a continuous self-map of a compact metric space $f\colon X \to X$. The existence of at least one f-invariant Borel probability measure is guaranteed by the Krylov-Bogolyubov theorem. When X is a manifold and f is a diffeomorphism exhibiting some degree of hyperbolicity, there are typically many mutually singular invariant probability measures. In Chapter 4 we are interested in understanding to what degree such a measure μ uniquely determines the ambient dynamics. In particular, we are interested in questions of the following nature.

Problem 1.3. *Given a Borel probability measure μ supported on a compact manifold M, describe* $\mathrm{Diff}^r(M;\mu)$*, the group of μ-preserving C^r diffeomorphisms.*

For instance, if μ is preserved by some diffeomorphism f, for $g \in \mathrm{Diff}^r(M;\mu)$ we are interested in understanding any nontrivial relationships between f and g.

In Chapter 4 we consider Problem 1.3 for measures on the 2-torus invariant under an Anosov diffeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$. For an f-ergodic measure μ we define *Lyapunov exponents*

$$\lambda^s_\mu(f) := \lim_{n\to\pm\infty} \frac{1}{n}\log(\|Df^n_x v\|) \qquad\qquad \lambda^u_\mu(f) := \lim_{n\to\pm\infty} \frac{1}{n}\log(\|Df^n_x u\|)$$

where $v \in E^s_x \smallsetminus \{0\}$ and $u \in E^u_x \smallsetminus \{0\}$. In the case that f is *anisotropic* (for μ), in the sense that $\lambda^u_\mu(f) \neq -\lambda^s_\mu(f)$, we will show that the group $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ is 'essentially' cyclic. We note that the anisotropy condition implies that μ is singular with respect to volume. See Theorems 4.1 and 4.2 for precise statements.

CHAPTER 2

HYPERBOLIC DYNAMICS

The main mechanism for a smooth dynamical system to exhibit 'chaos' is through the presence of some form of hyperbolicity. Intuitively, hyperbolicity reflects an asymptotic exponential separation of nearby orbits under the dynamics. We present an introduction to the theory of uniform hyperbolicity followed by a brief presentation of the main results from the theory of nonuniform hyperbolicity needed in Chapter 4.

2.1 THE THEORY OF UNIFORM HYPERBOLICITY

The most studied notion of hyperbolicity in the literature is that of uniform hyperbolicity. Let M be a smooth manifold. For $U \subset M$, and a C^r embedding $f\colon U \to M$, $r \geq 1$, we say a subset $\Lambda \subset U$ is *invariant* if $f(\Lambda) = \Lambda$. A compact invariant set Λ is said to be *hyperbolic* if, for any Riemannian metric on M, there are constants C and $\mu < 1$, and a continuous Df-invariant splitting of the tangent bundle $T_xM = E^s(x) \oplus E^u(x)$ over Λ such that for every $x \in \Lambda$ and $n \in \mathbb{N}$

$$\|Df_x^n v\| \leq C\mu^n \|v\|, \quad \text{for } v \in E^s(x), \text{ and}$$
$$\|Df_x^{-n} v\| \leq C\mu^n \|v\|, \quad \text{for } v \in E^u(x).$$

The compactness of Λ allows us to find a metric on M, called the *adapted metric*, such that we may take $C = 1$ above. For the remainder, when working with a hyperbolic set we always fix the adapted metric and let d denote the induced distance on M.

We set

$$V^{\pm} = \bigcap_{n\in\mathbb{N}} f^{\pm n}(U).$$

When Λ is hyperbolic, there exists an $\epsilon > 0$ such that the sets

$$W^s_\epsilon(x) := \{y \in V^- \mid d(f^n(x), f^n(y)) < \epsilon, \text{ for all } n \geq 0\}, \text{ and}$$

$$W^u_\epsilon(x) := \{y \in V^+ \mid d(f^{-n}(x), f^{-n}(y)) < \epsilon, \text{ for all } n \geq 0\}$$

are C^r embedded open disks, called the *local stable* and *unstable* manifolds. Furthermore, there are $\lambda < 1 < \kappa$ such that for $x \in \Lambda, y \in W^s_\epsilon(x), z \in W^u_\epsilon(x)$ and $n \geq 0$ we have

$$d(f^n(x), f^n(y)) \leq \lambda^n d(x, y), \text{ and} \tag{2.1}$$

$$d(f^{-n}(x), f^{-n}(z)) \leq \kappa^{-n} d(x, z). \tag{2.2}$$

Note that (2.1) and (2.2) imply that $f(W^s_\epsilon(f^{-1}(x)) \subset W^s_\epsilon(x)$ and $W^u_\epsilon(x) \subset f(W^u_\epsilon(f^{-1}(x))$. For $x \in \Lambda$ we also have the sets

$$W^s(x) := \{y \in V^- \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}, \text{ and}$$

$$W^u(x) := \{y \in V^+ \mid d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}$$

called the *global* stable and unstable manifolds. Both $W^u(x)$ and $W^s(x)$ are C^r injectively immersed submanifolds. Note that in the case that f is invertible (i.e. when $f(U) = U$), we have

$$W^u(x) = \bigcup_{n\in\mathbb{N}} f^n\left(W^u_\epsilon(f^{-n}(x))\right) \cong \mathbb{R}^{\dim E^u(x)}, \text{ and}$$

$$W^s(x) = \bigcup_{n\in\mathbb{N}} f^{-n}\left(W^s_\epsilon(f^n(x))\right) \cong \mathbb{R}^{\dim E^s(x)}.$$

For proofs and more background in the theory of invariant manifolds for uniformly hyperbolic dynamics, we refer to [HP70].

2.1.1 **Anosov diffeomorphisms.** The principal examples of uniformly hyperbolic dynamics the are Anosov diffeomorphisms of compact manifolds. For a compact manifold M, we say that a diffeomorphism $f\colon M \to M$ is *Anosov* if the entire manifold M is a hyperbolic set. The standard examples of Anosov diffeomorphisms are perturbations of algebraic actions on tori and infranil-manifolds. Furthermore, the Franks-Manning Theorem shows

that *any* Anosov diffeomorphism of an infranil-manifold is, up to a continuous change of variables, algebraic. We make this precise via the following definition.

Definition 2.1. Given topological spaces X and Y and continuous maps $f\colon X \to X$ and $g\colon Y \to Y$, we say f and g are *topologically conjugate* if there exists a homeomorphism $h\colon X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes.

The Franks-Manning Theorem thus states that an Anosov diffeomorphism of an infranil-manifold is topologically conjugate to a hyperbolic infranil-automorphism [Man74, Theorem C]. Among of the oldest problems in modern dynamics is the conjecture that every Anosov diffeomorphism is conjugate to a hyperbolic infranil-automorphism; in particular it is believed that only infranil-manifolds support Anosov diffeomorphisms.

2.1.2 **Dynamical foliations.** By a *d-dimensional $C^{r,k}$ foliation $\mathscr{F}$* of an n-dimensional manifold M we mean a partition of M by immersed submanifolds $\{\mathscr{F}(x)\}_{x\in M}$, and a cover of M by open sets $\{U_\beta\}$ such that

1. the connected component of $\mathscr{F}(x)\cap U_\beta$ containing x, which we denote by $\mathscr{F}_{U_\beta}(x)$, is a C^r injectively immersed copy of $\mathbb{R}^d$ for all β and every $x \in U_\beta$;

2. there are coordinate maps

   $$\phi_\beta\colon \mathbb{R}^d \times \mathbb{R}^{n-d} \to U_\beta$$

   such that

   $$\phi_\beta(\mathbb{R}^d \times \{y\}) = \mathscr{F}_{U_\beta}(\phi_\beta(0, y));$$

3. on the intersection $U_\beta \cap U_\alpha$ the transition maps

   $$\phi_\beta^{-1}\circ\phi_\alpha\colon \phi_\alpha^{-1}(U_\alpha) \subset \mathbb{R}^n \to \mathbb{R}^n$$

   are C^k.

Here $\mathscr{F}(x)$ is called the *leaf through* x, $\mathscr{F}_{U_\beta}(x)$ is called the *local leaf through* x, and U_β is called a *foliation chart*. In general, given a foliation $\mathscr{F}$ of M and an open set $V \subset M$ we denote by $\mathscr{F}_V$ the *local foliation* of V whose leaf through x is the connected component of $\mathscr{F}(x) \cap V$ containing x.

For f an Anosov diffeomorphism of a manifold M, the partitions of M into stable and unstable manifolds induce foliations $\mathscr{F}^s$ and $\mathscr{F}^u$. When working with the foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ and an open set $V \subset M$ we write $W^\sigma_V(x)$ for the local leaf of $\mathscr{F}^\sigma$ through x. By a C^r *bifoliation chart* for the foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ we mean an open set $V \subset M$ and a C^r diffeomorphism

$$\phi\colon \mathbb{R}^u \times \mathbb{R}^s \to V$$

with

$$\phi\colon \{x\} \times \mathbb{R}^s \mapsto W^s_V(\phi(x,0)) \quad \text{and} \quad \phi\colon \mathbb{R}^u \times \{y\} \mapsto W^u_V(\phi(0,y)).$$

Here $u = \dim E^u$ and $s = \dim E^s$.

In general, one must be careful with the regularity of the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$: typically each foliation is at best $C^{1+\alpha,\text{Hölder}}$ and one can only obtain $C^{\text{Hölder}}$ bifoliation charts for the foliations $\mathscr{F}^s$ and $\mathscr{F}^u$. However for codimension-1 Anosov diffeomorphisms and, in particular, for Anosov diffeomorphisms of $\mathbb{T}^2$, we obtain stronger transverse regularity of the foliations. The following proposition is well known and can be recovered, for example, from [PR02] and [PSW97, Theorem 6.1]. We note that Proposition 2.2 requires the hypothesis that the dynamics is at least $C^{1+\alpha}$. We recall that a diffeomorphism is said to be of class $C^{k+\alpha}$ for $k \in \mathbb{N}$ and $\alpha \in (0,1)$ if its derivatives of order k exist and are Hölder continuous with exponent at least α.

Proposition 2.2. *Let $f\colon M \to M$ be a $C^{1+\alpha}$ Anosov diffeomorphism such that $\dim E^u = \dim M - 1$. Then the unstable foliation $\mathscr{F}^u$ is $C^{1+\alpha,1+\alpha'}$ for some α'.*

In particular, if $M = \mathbb{T}^2$ then both the stable and unstable foliations are $C^{1+\alpha,1+\alpha'}$.

For U a foliation chart for $\mathscr{F}^u$ and $D, D' \subset U$ with D and D' transverse to each of the local leaves $\{\mathscr{F}_U(x)\}_{x\in U}$, we define the *unstable holonomy maps*

$$h_{D,D'}\colon G \subset D \to D'$$

by

$$h_{D,D'}\colon z \mapsto D' \cap \mathscr{F}_U(z)$$

when defined. As a consequence of Proposition 2.2 we obtain that the unstable holonomy maps $h_{D,D'}$ are $C^{1+\alpha'}$ for a $C^{1+\alpha}$ Anosov diffeomorphism of $\mathbb{T}^2$; in particular they are bi-Lipschitz. *Stable holonomy maps* are defined similarly.

2.1.3 **Local product structure.** Given a compact hyperbolic set $\Lambda \subset M$, it is always possible to find $0 < \delta < \eta$ such that for $x, y \in \Lambda$, $d(x, y) < \delta$ implies the intersection $W^u_\eta(x) \cap W^s_\eta(y)$ is a singleton.

Definition 2.3. We say that a hyperbolic set Λ has *local product structure* if, for η, δ as above, $d(x, y) < \delta$ implies $W^u_\eta(x) \cap W^s_\eta(y) \subset \Lambda$. In particular, this implies that the map

$$\phi\colon (W^u_\delta(x) \cap \Lambda) \times (W^s_\delta(x) \cap \Lambda) \to \Lambda$$

given by

$$\phi\colon (y, z) \mapsto W^s_\eta(y) \cap W^u_\eta(z)$$

is well defined and maps its domain homeomorphically onto its image.

A compact hyperbolic set Λ is called *locally maximal* if there exists an open set V containing Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$. For compact hyperbolic sets, local maximality is equivalent to the existence of a local product structure (see for example [KH95]). In particular, for an Anosov diffeomorphisms, the entire manifold has local product structure.

We make the following definitions.

Definition 2.4. Given a set Λ with local product structure and δ and η as above, we say a closed set $R \subset \Lambda$ is a *rectangle* or a *local product chart* if

1. $\sup\{d(x, y) \mid x, y \in R\} < \delta$,
2. R is proper, that is, R is equal to the closure of its interior (in Λ),
3. $x, y \in R$ implies $W^u_\eta(x) \cap W^s_\eta(y) \subset R$.

If R is a rectangle, we write $W^\sigma_R(x) := W^\sigma_\eta(x) \cap R$.

2.1.4 **Recurrence and spectral decomposition.** Consider a metric space X and a continuous map $f\colon X \to X$. A point $x \in X$ is said to be *nonwandering* for f if for any open set U containing x, there is some $n > 0$ such that $f^n(U) \cap U \neq \varnothing$; otherwise it is called *wandering*. We denote by $\mathrm{NW}(f)$ the set of all nonwandering points for f. We call an invariant set Λ *nonwandering* if $\Lambda \subset \mathrm{NW}(f)$.

An invariant set Λ is said to be *topologically transitive* under f if it contains a dense orbit. Alternatively, an invariant subset $\Lambda \subset X$ is topologically transitive if for all pairs of nonempty open sets $U, V \subset \Lambda$, there is some n such that $f^n(U) \cap V \neq \varnothing$. An invariant set Λ is said to be *topologically mixing* if, for all pairs of nonempty open sets $U, V \subset \Lambda$, there is some N such that $f^n(U) \cap V \neq \varnothing$ for all $n \geq N$. We note that it follows from [Man74, Theorem C] that Anosov diffeomorphism on tori (or more generally infranil-manifolds) are topologically transitive.

We say a diffeomorphism $f\colon M \to M$ is an *Axiom A* diffeomorphism if 1) the set $\mathrm{NW}(f)$ is hyperbolic and 2) $\mathrm{Per}(f)$ is dense in $\mathrm{NW}(f)$. Given an Axiom A diffeomorphism, (resp. a locally maximal hyperbolic set $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$) we have a partition, called the *spectral decomposition*, of the nonwandering points $\mathrm{NW}(f) = \Omega_1 \cup \cdots \cup \Omega_k$ (resp. $\mathrm{NW}(f|_V) = \Omega_1 \cup \cdots \cup \Omega_k$) where each Ω_j is a transitive hyperbolic set for f (see [KH95], [Sma67]). Given the spectral decomposition, we call the partition elements Ω_j above *basic sets*. In general, by a *basic set* we mean a locally maximal, topologically transitive, compact hyperbolic set $\Omega \subset \mathrm{NW}(f)$. In particular, for a transitive Anosov diffeomorphism, the entire manifold is a basic set.

2.2 FACTS FROM NONUNIFORM HYPERBOLICITY

Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a Riemannian manifold. We recall that there exists a Borel subset $\Lambda \subset M$, called the set of *regular points*, Borel functions $r\colon \Lambda \to \mathbb{N}$ and

$$\lambda_0(x) < \lambda_1(x) < \cdots < \lambda_{r(x)}(x)$$

on Λ, and a decomposition of the tangent space

$$T_x M = \bigoplus_{0 \le j \le r(x)} E^j(x)$$

over Λ such that (among other properties) for $x \in \Lambda$ and $v \in E^j(x) \smallsetminus \{0\}$

$$\lambda_j(x) := \lim_{n \to \pm\infty} \frac{1}{n} \log\left(\|Df_x^n(v)\|\right).$$

For $x \in \Lambda$, the numbers $\lambda_j(x)$ are called the *Lyapunov exponents* at x and the subspaces $E^j(x)$ are called the *Lyapunov subspaces* at x. By Oseledec's Theorem [Ose68] the set of regular points Λ has *full probability* in the sense that for any f-invariant Borel probability measure μ we have that $\mu(\Lambda) = 1$. Furthermore, we have that the splitting $T_x M = \bigoplus_{0 \le j \le r(x)} E^j(x)$ depends μ-measurably on the point x.

For every $x \in \Lambda$ and $0 \le i \le r(x)$ with $\lambda_i(x) < 0$ there exists a $C^{1+\alpha}$ injectively immersed $\left(\sum_{\lambda_j(x) \le \lambda_i(x)} \dim E^j(x)\right)$-dimensional manifold $\widetilde{W}^i(x)$ defined by

$$\widetilde{W}^i(x) := \left\{ y \in M \mid \limsup_{n \to \infty} \frac{1}{n} \log(d(f^n(y), f^n(x))) \le \lambda_i(x) \right\}$$

with

$$T_x \widetilde{W}^i(x) = \bigoplus_{\lambda_j(x) \ge \lambda_i(x)} E^j(x)$$

called the i^{th} *stable Pesin manifold*. Similarly, *unstable Pesin manifolds* $\widetilde{W}^i(x)$ exist for $x \in \Lambda$ with $\lambda_i(x) > 0$ defined by

$$\widetilde{W}^i(x) := \left\{ y \in M \mid \limsup_{n \to \infty} \frac{1}{n} \log(d(f^{-n}(y), f^{-n}(x))) \le -\lambda_i(x) \right\}.$$

See, for example, [Rue79, Section 6] for statements regarding the existence and properties of local and global stable and unstable manifolds for nonuniformly hyperbolic diffeomorphisms. A standard reference on the theory of nonuniform hyperbolicity is [BP07].

We note that the $C^{1+\alpha}$ regularity of the dynamics is essential to obtain Pesin manifolds. (See, for example, [Pug84].) In contrast, a diffeomorphism or embedding need only be C^1 to obtain stable and unstable manifolds at every point of a uniformly hyperbolic set.

In Chapter 4 we will primarily be interested in applying Pesin's theory to Anosov diffeomorphisms of the 2-torus. We observe that for $M = \mathbb{T}^2$ and f Anosov, at any regular point x we have $r(x) = 1$ and $\lambda_0(x) < 0 < \lambda_1(x)$. In this context and write $\lambda^s = \lambda_0$ and

$\lambda^u = \lambda_1$ for the *stable* and *unstable Lyapunov exponents*. Clearly in this context, for any regular point $x \in \mathbb{T}^2$ we have $\widetilde{W}^1(x) \subset W^u(x)$ and $\widetilde{W}^0(x) \subset W^s(x)$.

CHAPTER 3

FACTS FROM MEASURE THEORY

In this chapter we present some of the basic results and constructions from the theory of measure-preserving transformations. All finite measure spaces $(X, \mathscr{A}, \mu)$ will be assumed to be *Lebesgue* or *standard* measure spaces, in that they are measurably isomorphic to the union of the interval $[0,1]$ equipped with the Lebesgue measure and a countable number of atoms. We refer to [Roh52] for background and proofs of elementary results. Our primary interest will be in measures spaces obtained by equipping a manifold with the completion of a Borel probability measure; these measure spaces are well known to be Lebesgue.

3.1 TRANSFORMATIONS OF MEASURE SPACES AND POINTWISE DIMENSION OF MEASURES

We begin with some elementary definitions. Consider measurable spaces $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ and a map $g\colon X \to Y$.

— We say the transformation g is *measurable* if $g^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}$.

— For a measure μ on $(X, \mathscr{A})$ and measurable $g\colon X \to Y$, we define the *push-forward* measure $g_*\mu$ on $(Y, \mathscr{B})$ by $(g_*\mu)(B) = \mu(g^{-1}(B))$.

— For $g\colon (X, \mathscr{A}) \to (X, \mathscr{A})$ measurable, we say g is *μ-preserving* if $g_*(\mu) = \mu$.

— For $g\colon (X, \mathscr{A}) \to (X, \mathscr{A})$ a μ-preserving transformation we say μ is g-ergodic (or, less commonly, that g is μ-ergodic) if the only g-invariant subsets of X are null or conull. Formally, we mean that $\mu(A \Delta g^{-1}(A)) = 0$ implies $\mu(A) = 0$ or $\mu(X \smallsetminus A) = 0$.

— For measures ν, μ defined on $(X, \mathscr{A})$ we say ν is *absolutely continuous with respect to* μ if $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \in \mathscr{A}$. We denote this by $\nu \ll \mu$.

— For $\nu \ll \mu$ we denote by $\frac{d\nu}{d\mu}(y)$ the unique μ-integrable function with the property that

$$\nu(A) = \int_A \frac{d\nu}{d\mu}(y)\, d\mu(y)$$

called the *Radon-Nikodym derivative*.

Pointwise dimension of measures

For a metric space X and a locally finite Borel measure μ we define the extended-real-valued *upper* and *lower pointwise dimension* functions

$$\overline{\dim}(\mu, x) := \limsup_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$$
$$\underline{\dim}(\mu, x) := \liminf_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$$

where $B(x,\epsilon)$ denotes the metric ball of radius ϵ at x and the *pointwise dimension*

$$\dim(\mu, x) := \lim_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$$

wherever the limit is defined.

The above quantities are related to the more familiar Hausdorff dimension of a subset of $\mathbb{R}^n$ via the following well known proposition.

Proposition 3.1 ([You82, Proposition 2.1])**.** *Let μ be a non-atomic, finite Borel measure on $\mathbb{R}^n$ and let $\mu(\Lambda) > 0$. Suppose there are uniform estimates*

$$\underline{\delta} \le \liminf_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} \le \limsup_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} \le \overline{\delta}$$

for every $x \in \Lambda$. Then $\underline{\delta} \le \dim_H(\Lambda) \le \overline{\delta}$ where $\dim_H$ denotes the Hausdorff dimension of the set Λ.

Behavior of pointwise dimension under bi-Lipschitz maps

In Chapter 4 we will be interested in the behavior of the pointwise dimension functions under bi-Lipschitz (and hence Borel measurable) transformations. We state the following definitions and Proposition which will be of use in Chapter 4.

Let ν and μ two locally finite Borel measures on $\mathbb{R}^m$ with $\nu \ll \mu$. Recall that for a measurable set $A \subset \mathbb{R}^m$, a point y is said to be a *μ-density point of A* if

$$\lim_{r\to 0} \frac{\mu(B(y,r)\cap A)}{\mu(B(y,r))} = 1.$$

For a locally finite Borel measure on $\mathbb{R}^n$, it is well known that any measurable set is equivalent, modulo a null set, to its set of density points.[1] We say that y is a *bounded (ν,μ)-density point* if there is some $N \in (0,\infty)$ such that y is both a μ- and ν-density point of the set

$$\left\{ x \in \mathbb{R}^m \mid \frac{1}{N} \le \frac{d\nu}{d\mu}(x) \le N \right\}.$$

We note that $\nu \ll \mu$ implies ν-a.e. point is a bounded (ν,μ)-density point.

Proposition 3.2. *Let μ and ν be locally finite Borel measures on $\mathbb{R}^m$. Let $g\colon \mathbb{R}^m \to \mathbb{R}^m$ be a bi-Lipschitz homeomorphism with $g_*(\mu) \ll \nu$. Then for each bounded $(g_*(\mu),\nu)$-density point y we have*

1. $\overline{\dim}(\nu, y) = \overline{\dim}(\mu, g^{-1}(y))$;

2. $\underline{\dim}(\nu, y) = \underline{\dim}(\mu, g^{-1}(y))$.

In particular, 1 and 2 hold for $(g_*\mu)$-a.e. point y.

Proof. Write $J(y)$ for the Radon-Nikodym derivative $J(y) := \dfrac{dg_*\mu}{d\nu}(y)$. For $N \in \mathbb{N}$, define $V_N := \{y \mid 1/N \le J(y) \le N\}$. Consider the inequality

$$\frac{g_*\mu(B(y,r))}{\nu(B(y,r))} = \frac{\int_{B(y,r)} J(z)\ d\nu(z)}{\nu(B(y,r))} \ge \frac{1}{N}\frac{\nu(B(y,r)\cap V_N)}{\nu(B(y,r))}$$

Since y is a ν-density point of V_N for some N, we have that $\dfrac{g_*\mu(B(y,r))}{\nu(B(y,r))}$ is bounded away from 0 as $r \to 0$.

Similarly,

$$\frac{g_*\mu(B(y,r))}{\nu(B(y,r))} = N\frac{g_*\mu(B(y,r))}{\int_{B(y,r)} N\ d\nu(z)} \le N\frac{g_*\mu(B(y,r))}{\int_{B(y,r)\cap V_n} J(z)\ d\nu(z)} = N\frac{g_*\mu(B(y,r))}{g_*\mu(B(y,r)\cap V_N)}$$

which implies that $\dfrac{g_*\mu(B(y,r))}{\nu(B(y,r))}$ is bounded away from ∞ as $r \to 0$ since y is a $(g_*\mu)$-density point of V_N for some N.

[1] See for instance the Lebesgue-Besicovitch Differentiation Theorem [Tay06, Theorem 11B.4].

In particular the expression

$$\log\left(\frac{g_*\mu(B(y,r))}{\nu(B(y,r))}\right)$$

is bounded above and below for sufficiently small $r>0$. Hence

$$\limsup_{r\to 0}\frac{\log\left(\frac{g_*\mu(B(y,r))}{\nu(B(y,r))}\right)}{\log r}=\liminf_{r\to 0}\frac{\log\left(\frac{g_*\mu(B(y,r))}{\nu(B(y,r))}\right)}{\log r}=0.$$

We have[2]

$$\begin{aligned}\overline{\dim}(\nu,y)&:=\limsup_{r\to 0}\frac{\log(\nu(B(y,r)))}{\log r}\\&=\limsup_{r\to 0}\frac{\log(\nu(B(y,r)))}{\log r}+\lim_{r\to 0}\frac{\log\left(\frac{g_*\mu(B(y,r))}{\nu(B(y,r))}\right)}{\log r}\\&=\limsup_{r\to 0}\frac{\log(g_*\mu(B(y,r)))}{\log r}\end{aligned}$$

and similarly

$$\underline{\dim}(\nu,y)=\liminf_{r\to 0}\frac{\log(g_*\mu(B(y,r)))}{\log r}.$$

Since g is assumed bi-Lipschitz, for each y we may find $L>0$ and $0<C<1$ such that $d(y,z)<L$ implies

$$\frac{1}{C}d(y,z)\le d(g^{-1}(y),g^{-1}(z))\le Cd(y,z).$$

Thus, for sufficiently small $r>0$ we have

$$B(g^{-1}(y),r/C)\subset g^{-1}(B(y,r))\subset B(g^{-1}(y),Cr)$$

and thus

$$\frac{\log(\mu(B(g^{-1}(y),\frac{r}{C})))}{\log\frac{r}{C}+\log C}\le\frac{\log(g_*\mu(B(y,r)))}{\log r}\le\frac{\log(\mu(B(g^{-1}(y),Cr)))}{\log(Cr)-\log C}.\tag{3.1}$$

Applying the $\limsup_{r\to 0}$ and $\liminf_{r\to 0}$ operators to both sides of (3.1) yields the desired results.

□

3.2 CONDITIONAL MEASURES AND ENTROPY

Given a Lebesgue space $(X,\mathscr{A},\mu)$, we consider a partition ξ of X by measurable sets. Given $x\in X$ we write $\xi(x)$ for the partition element of ξ containing x. We denote the space of equivalence classes by X/ξ, and the projection map by $p\colon X\to X/\xi$. The σ-algebra $\mathscr{A}$

[2]We use here the identity $\limsup_{n\to\infty}a_n+\liminf_{n\to\infty}b_n\le\limsup_{n\to\infty}(a_n+b_n)\le\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n$.

then induces a σ-algebra $\mathscr{A}/\xi$ on X/ξ. Defining the push-forward measure $p_*\mu = \mu \circ p^{-1}$ on X/ξ we have that $(X/\xi, \mathscr{A}/\xi, p_*\mu)$ is a measure space.

Our primary interest will be in partitions that are *measurable* in the sense that the σ-algebra $\mathscr{A}/\xi$ on X/ξ is countably generated. More precisely, ξ is measurable if there exists a countable collection $\{A_j\}$ of measurable subsets of X/ξ such that for any $x \in X$ there is a sequence j_k with $\xi(x) = \bigcap_{k\in\mathbb{N}} A_{j_k}$. Equivalently, ξ is a measurable partition of the Lebesgue space $(X, \mathscr{A}, \mu)$ if and only if the induced measure space $(X/\xi, \mathscr{A}/\xi, p_*\mu)$ is Lebesgue.[3]

Given a measurable partition ξ of Lebesgue space (X, μ), it is well known (see, for example, [Roh52]) that we may find a collection of measures $\{\tilde{\mu}_x^\xi\}_{x\in X}$, called a *family of conditional probability measures*, with the following properties:

1. $\tilde{\mu}_x^\xi = \tilde{\mu}_y^\xi$ for $y \in \xi(x)$;
2. $\tilde{\mu}_x^\xi(\xi(x)) = 1$ and $\tilde{\mu}_x^\xi(X \smallsetminus \xi(x)) = 0$ for μ-a.e. x;
3. for measurable subsets $A \subset X$, the function $x \mapsto \tilde{\mu}_x^\xi(A)$ is measurable and
$$\mu(A) = \int_X \tilde{\mu}_x^\xi(A)\ d\mu(x);$$
4. the family is unique in the sense that any other collection of measures satisfying (1)-(3) is equivalent to $\{\tilde{\mu}_x^\xi\}_{x\in X}$ on a set of full measure.

We will need the following straightforward observation.

Claim 3.3. *Let (X, μ) be a Lebesgue space, ξ a measurable partition, and $g\colon X \to X$ a measure-preserving transformation. Write $\eta = g^{-1}(\xi) := \{g^{-1}(C) \mid C \in \xi\}$. Let $\{\tilde{\mu}_x^\eta\}_{x\in X}$ and $\{\tilde{\mu}_y^\xi\}_{y\in X}$ be families of conditional measures for the partitions η and ξ, respectively. Then for μ-a.e. x we have*
$$g_*(\tilde{\mu}_x^\eta) = \tilde{\mu}_{g(x)}^\xi.$$

Proof. If the claim fails, then there is a set $Y \subset X$ of positive measure with the property that for each $x \in Y$ there is a set $A_x \subset \xi(x)$ with
$$g_*(\tilde{\mu}_{g^{-1}(x)}^\eta)(A_x) \neq \tilde{\mu}_x^\xi(A_x).$$

[3]Indeed if $(X/\xi, \mathscr{A}/\xi, p_*\mu)$ is Lebesgue (in particular, separable) then ξ is countably generated; the other direction follows from [Roh67, Section 1.5].

(We note that even in the case that g is not invertible, the notation $\tilde{\mu}^{\eta}_{g^{-1}(x)}$) is unambiguous since for $y, y' \in g^{-1}(x)$ we have $\eta(y) = \eta(y')$ whence $\tilde{\mu}^{\eta}_{y} = \tilde{\mu}^{\eta}_{y'}$.) We suppose without loss of generality that $g_*(\tilde{\mu}^{\eta}_{g^{-1}(x)})(A_x) > \tilde{\mu}^{\xi}_{x}(A_x)$ for all x in a subset $Y' \subset Y$ of positive measure. Letting $A = \bigcup_{y \in Y'} A_y$ we have

$$g_*(\tilde{\mu}^{\eta}_{x})(A) \geq \tilde{\mu}^{\xi}_{g(x)}(A)$$

for all x, where the inequality is strict on a set of positive measure. We then have

$$\int_X g_*(\tilde{\mu}^{\eta}_{x})(A)\ d\mu(x) > \int_X \tilde{\mu}^{\xi}_{g(x)}(A)\ d\mu(x) = \int_X \tilde{\mu}^{\xi}_{x}(A)\ d\mu(x) = \mu(A).$$

We use here that $g_*\mu = \mu$, and hence $\int_X \phi(x)\ d\mu(x) = \int_X \phi(g(x))\ d\mu(x)$ for any measurable function ϕ. Finally, we have

$$\int_X g_*(\tilde{\mu}^{\eta}_{x})(A)\ d\mu(x) = \int_X \tilde{\mu}^{\eta}_{x}(g^{-1}(A))\ d\mu(x) = \mu(g^{-1}(A)).$$

Hence

$$\mu(g^{-1}(A)) > \mu(A),$$

a contradiction. □

3.2.1 **Dimension of measures along dynamical foliations.** Consider $f\colon M \to M$ a $C^{1+\alpha}$ diffeomorphism and denote by Λ its set of regular points (see Section 2.2). Let μ be an f-ergodic Borel probability measure on M, and note that the functions $r(x)$, $\lambda_i(x)$, and $\dim E^i(x)$ are constant on a set of full measure $\Lambda' \subset \Lambda$. For $x \in \Lambda'$ with $\lambda_i(x) > 0$ write $\widetilde{W}^i(x)$ for the ith unstable Pesin Manifold. The collection $\{\widetilde{W}^i(x)\}_{x\in\Lambda'}$ (and the measure-zero complement of its union) provides a partition of M. We say that a measurable partition ξ is *subordinate to* $\{\widetilde{W}^i(x)\}_{x\in\Lambda}$ if, for μ-a.e. x, we have $\xi(x) \subset \widetilde{W}^i(x)$ and $\xi(x)$ contains an open neighborhood of x in $\widetilde{W}^i(x)$.

Let ξ be a measurable partition subordinate to $\{\widetilde{W}^i(x)\}_{x\in\Lambda}$, and consider a family of conditional measures $\{\tilde{\mu}^{\xi}_{x}\}_{x\in M}$. We define measurable functions

$$\overline{\delta}^i(x) := \overline{\dim}(\tilde{\mu}^{\xi}_{x}, x) := \limsup_{\epsilon\to 0} \frac{\log \tilde{\mu}^{\xi}_{x}(B(x,\epsilon))}{\log \epsilon}$$

$$\underline{\delta}^i(x) := \underline{\dim}(\tilde{\mu}^{\xi}_{x}, x) := \liminf_{\epsilon\to 0} \frac{\log \tilde{\mu}^{\xi}_{x}(B(x,\epsilon))}{\log \epsilon}$$

Up to null sets, the functions $\overline{\delta}^i(x)$ and $\underline{\delta}^i(x)$ are independent of the choice of the partition ξ. Furthermore, by [LY85] we have equality

$$\overline{\delta}^i(x) = \underline{\delta}^i(x)$$

for μ-almost every x; we define $\delta^i(x)$ to be this common value.

Since the functions $\delta^i(x)$ are f-invariant, the assumption that μ is f-ergodic guarantees they are constant μ-almost everywhere. In the case that μ is non-ergodic, the functions $\overline{\delta}^i(x)$ and $\underline{\delta}^i(x)$ are defined by first passing to an ergodic decomposition (see [LY85] for details). For a regular point x, and i with $\lambda_i(x) < 0$, we may similarly construct pointwise dimension functions $\overline{\delta}^i(x)$ and $\underline{\delta}^i(x)$. Finally, we define the *stable* and *unstable* pointwise dimensions of the measure μ to be the measurable functions

$$\delta^u(x) = \max\{\delta^i(x) \mid \lambda_i(x) > 0\}, \text{ and}$$

$$\delta^s(x) = \max\{\delta^i(x) \mid \lambda_i(x) < 0\}.$$

A measure μ is said to be *hyperbolic* for a $C^{1+\alpha}$ diffeomorphism $f\colon M \to M$ (or simply *hyperbolic* when the ambient dynamics is understood) if $\lambda_i(x) \neq 0$ for μ-a.e. regular point x and every $0 \leq i \leq r(x)$. From [BPS99], for any ergodic hyperbolic measure μ we have

$$\overline{\dim}(\mu) = \underline{\dim}(\mu) = \delta^u + \delta^s, \tag{3.2}$$

where $\overline{\dim}(\mu)$, $\underline{\dim}(\mu)$, δ^u, and δ^s are the constant values attained μ-a.e. by the corresponding functions; in particular μ is *exact dimensional* in the sense that

$$\overline{\dim}(\mu, x) = \underline{\dim}(\mu, x) = \dim(\mu, x)$$

on a set of full measure.

For $x \in \Lambda$ we write $u(x) := \inf\{0 \leq i \leq r(x) \mid \lambda_i(x) > 0\}$. We say a measure μ is a *u-measure* if, for any $\{W^{u(x)}(x)\}$-subordinate measurable partition ξ and a corresponding family of conditional probability measures $\{\tilde{\mu}_x^\xi\}$, for μ-a.e. x the measure $\tilde{\mu}_x^\xi$ is absolutely continuous with respect to the induced Riemannian volume on $\widetilde{W}^{u(x)}(x)$. This is equivalent to the property that for μ-a.e. x

$$\delta^u(x) = \sum_{j \geq u(x)} \dim E^j(x)$$

which, in turn, is equivalent to the property that for μ-a.e. x and $j \geq u(x)$

$$\dim E^j(x) = \begin{cases} \delta^j(x) - \delta^{j+1}(x) & j < r(x), \\ \delta^j(x) & j = r(x). \end{cases}$$

We similarly define *s-measures*.

3.2.2 **Entropy, dimension, and Lyapunov exponents.** Consider a measure-preserving transformation T of a Lebesgue probability space (X, μ). Let ξ be a countable partition of X by measurable sets. We define the entropy of the partition ξ to be the quantity

$$H(\xi) := \sum_{C \in \xi} \mu(C) \log \mu(C) = \int_X \log(\mu(\xi(x)))\ d\mu(x)$$

(where by convention we define $0 \log 0 = 0$.) For $n \in \mathbb{N}$ define $T^{-n}(\xi)$ to be the partition of X consisting of the sets

$$\{T^{-n}(C) \mid C \in \xi\}.$$

For two partitions η and ξ we define the *joint partition*

$$\xi \vee \eta := \{C \cap D \mid C \in \xi, D \in \eta\}$$

and dynamical partitions

$$\xi_{-n} := \bigvee_{j=0}^{n-1} T^{-j}(\xi).$$

The *entropy of T relative to the partition ξ* is defined to be

$$h(T, \xi) = \lim_{n \to \infty} \frac{1}{n} H(\xi_{-n}).$$

Note that the assumption that ξ is finite or countable is needed in the above definition. However, alternative definitions of the quantity $h(T, \xi)$—coinciding with the above for countable partitions—allow one to extend the definition of $h(T, \xi)$ to uncountable partitions. See for example [Roh67].

We define the *measure-theoretic entropy* of the transformation T to be

$$h_\mu(T) = \sup\{h(T, \xi) \mid H(\xi) < \infty\}.$$

The measure theoretic entropy should be interpreted as a numerical measurement of the complexity of a measure-preserving transformation, and has become a fundamental tool in

the modern theory of dynamical systems. The measure theoretic entropy satisfies a number of natural properties:

1. $h_\mu(T^n) = nh_\mu(T)$ for $n \in \mathbb{N}$;

2. if T is invertible then $h_\mu(T) = h_\mu(T^{-1})$;

3. for two T-invariant measures μ and λ and $p \in [0,1]$ we have

$$ph_\mu(T) + (1-p)h_\lambda(T) = h_{p\mu+(1-p)\lambda}(T).$$

In general, entropy is difficult to calculate. However, in the context of a $C^{1+\alpha}$ diffeomorphism preserving a Borel probability measure, the following formula, first presented in [LY85], provides an elegant relationship between entropy, pointwise dimension, and Lyapunov exponents. For x a regular point of a $C^{1+\alpha}$ diffeomorphism f we define functions

$$\gamma_j(x) := \begin{cases} \delta^{r(x)}(x) & j = r(x), \\ \delta^j(x) - \delta^{j+1}(x) & u(x) \le j < r(x). \end{cases}$$

Note in the case that μ is ergodic, the functions $\gamma_j(\cdot)$ are a.e. constant. We then have the equality

$$h_\mu(f) = \begin{cases} \sum_{\lambda_j > 0} \gamma_j \lambda_j & \mu \text{ ergodic}, \\ \int \sum_{\lambda_j(x) > 0} \gamma_j(x)\lambda_j(x)\, d\mu(x) & \mu \text{ non-ergodic} \end{cases} \tag{3.3}$$

known as the *Ledrappier–Young entropy formula*. (3.3) was first established in [You82] for C^2 surface diffeomorphisms and in [LY85] for general C^2 diffeomorphisms. For a statement and proof in the $C^{1+\alpha}$ setting, we refer to [BP07]. By passing to f^{-1} a similar result to (3.3) holds for negative Lyapunov exponents and γ_k defined with respect to the corresponding stable pointwise dimension functions.

3.3 EQUILIBRIUM STATES

Consider $f\colon X \to X$ a homeomorphism of a compact metric space and $\phi\colon X \to \mathbb{R}$ a continuous function. We say an f-invariant measure μ is an *equilibrium state for ϕ with respect to f*—or simply an *equilibrium state for ϕ* when the dynamics is understood—if μ

maximizes the expression

$$h_\mu(f) + \int \phi \, d\mu$$

over all f-invariant probability measures.

We are primarily interested in the setting where $f\colon M \to M$ is a C^1 transitive Anosov diffeomorphism and $\phi\colon M \to \mathbb{R}$ is Hölder continuous or, more generally, when Λ is a basic set for a C^1 embedding f and $\phi\colon \Lambda \to \mathbb{R}$ is Hölder continuous. It is well known in this setting that there exists a unique equilibrium state, often denoted μ_ϕ, for ϕ. Furthermore, the equilibrium states μ_ϕ are f-ergodic, have full support in Λ, and have positive entropy $h_{\mu_\phi}(f)$. In addition, the equilibrium states exhibit a *local product structure*—defined formally in Theorem 3.4(e)—which mimics the topological local product structure of Λ. We refer to [Bow08] for more background in the theory of equilibrium states and [KH95, Chapter 20] for a more contemporary treatment.

For a transitive Anosov diffeomorphism f, there are three equilibrium states which are in some sense 'natural':

— the *forwards SRB* measure, the equilibrium state for

$$\phi^u := -\log\left(\det\left(Df\restriction_{E^u}\right)\right);$$

— the *backwards SRB* measure, the equilibrium state for

$$\phi^s := -\log\left(\det\left(Df^{-1}\restriction_{E^s}\right)\right);$$

— the *measure of maximal entropy*, the equilibrium state for $\phi \equiv 0$.

We note that when f is algebraic, these three measures coincide. When f is volume preserving, the forwards and backwards SRB measures coincide. In the non-algebraic, non-volume preserving setting, each of the three measures above generalizes certain properties of volume: the measure of maximal entropy is the unique measure whose canonical disintegrations are invariant under holonomies (see Remark 3.5 below), and the forwards (resp. backwards) SRB measure is the unique u- (resp. s-) measure for f.

3.3.1 **Product structure of equilibrium states.** We now investigate in more detail the structure of equilibrium states for hyperbolic dynamics. Let Λ be a basic set for a C^1

embedding f on a manifold (see Section 2.1.4 for definitions). Let $\phi\colon \Lambda \to \mathbb{R}$ be a Hölder continuous function and let μ be the associated equilibrium state. Recall that we may find $0 < \delta < \epsilon$ with the property that for all $x, y \in \Lambda$ with $d(x, y) \leq 2\delta$ the intersection

$$W^u_\epsilon(x) \cap W^s_\epsilon(y)$$

is contained in Λ and contains exactly one point. For such x, y we write

$$[x, y] := W^u_\epsilon(x) \cap W^s_\epsilon(y).$$

Given $x^s \in W^s_\delta(x)$ and $x^u \in W^u_\delta(x)$ we define the local holonomies

$$h^s_{x,x^s}\colon W^u_\delta(x) \to W^u_\epsilon(x^s) \qquad h^s_{x,x^s}\colon z \mapsto [x^s, z] \tag{3.4}$$

$$h^u_{x,x^u}\colon W^s_\delta(x) \to W^s_\epsilon(x^u) \qquad h^u_{x,x^u}\colon z \mapsto [z, x^u]. \tag{3.5}$$

The following theorem describes a local product structure for equilibrium states.

Theorem 3.4. *Let μ be the equilibrium state associated to a Hölder continuous function ϕ on Λ. Then for each $\sigma \in \{s, u\}$ there exists a family of measures $\{\mu^\sigma_x\}_{x\in\Lambda}$ such that*

a) the family $\{\mu^\sigma_x\}_{x\in M}$ is uniquely determined up to scalar multiplication, and $\mu^\sigma_x = \mu^\sigma_y$ for $x \in W^\sigma(y)$;

b) μ^σ_x is supported on $W^\sigma(x) \cap \Lambda$ and $\mu^\sigma_x(U) > 0$ for any nonempty open subset of $W^\sigma(x) \cap \Lambda$;

c) $f_*\mu^\sigma_x$ and $\mu^\sigma_{f(x)}$ are equivalent with

$$\frac{d(f_*\mu^u_x)}{d\mu^u_{f(x)}}(f(y)) = e^{\phi(y)-P(\phi)} \tag{3.6}$$

$$\frac{d(f_*\mu^s_x)}{d\mu^s_{f(x)}}(f(y)) = e^{-\phi(f(y))+P(\phi)} \tag{3.7}$$

for $y \in W^\sigma(x)$, where $P(\cdot)$ denotes the pressure functional

$$P(\phi) = \sup\left\{h_\nu + \int \phi \, d\nu\right\}$$

where the supremum is taken over all f-invariant measures;

d) for $x^s \in W^s_\delta(x)$ and $x^u \in W^u_\delta(x)$ we have

$$\frac{d\mu^u_{x^s}}{d((h^s_{x,x^s})_*\mu^u_x)}(\cdot) = e^{\omega^u_x(\cdot)} \tag{3.8}$$

$$\frac{d\mu^s_{x^u}}{d((h^u_{x,x^u})_*\mu^s_x)}(\cdot) = e^{\omega^s_x(\cdot)} \tag{3.9}$$

where

$$\omega^u_x(y) := \sum_{i=0}^{\infty} \phi(f^i(y)) - \phi(f^i([x,y])) \tag{3.10}$$

$$\omega^s_x(y) := \sum_{i=0}^{\infty} \phi(f^{-i}(y)) - \phi(f^{-i}([y,x])); \tag{3.11}$$

e) after suitable normalization, on local charts $[W^s_\delta(x)\cap\Lambda, W^u_\delta(x)\cap\Lambda]$ we have the product decomposition

$$d\mu(\cdot) = e^{\omega^u_x(\cdot)+\omega^s_x(\cdot)-\phi(\cdot)}\, d(\mu^u_x \times \mu^s_x)([x,\cdot],[\cdot,x]); \tag{3.12}$$

f) for any measurable partition ξ subordinate to $\mathscr{F}^u$, up to a normalizing constant, the family

$$\{e^{\omega^s_x-\phi}\mu^u_x\}$$

provides a family of conditional probability measures $\tilde{\mu}^\xi_x$;

Remark 3.5. For μ the measure of maximal entropy—the equilibrium state for $\phi \equiv 0$—Theorem 3.4(d) guarantees that the families of measures $\{\mu^u_x\}_{x\in\Lambda}$ and $\{\mu^s_x\}_{x\in\Lambda}$ are invariant under their respective holonomy maps. This well known property uniquely characterizes the measure of maximal entropy.

Complete proofs of Theorem 3.4 are missing from the literature, but partial proofs and sketches exist. We contribute another sketch here.

Proof sketch of Theorem 3.4. The existence of a family of measures satisfying (3.6) and (3.8) is derived in [Lep00, Theorem 2.3] and [PW01, Proposition 2.3]. We note however that both references contain minor errors in the statements and proofs of the results corresponding to (3.6) and (3.8).

(3.7) and (3.9) then follow from (3.6) in (3.8) by replacing f with f^{-1}. In particular, we check

$$\frac{df_*\mu_x^s}{d\mu_{f(x)}^s}(f(y)) = \frac{d\mu_x^s}{df_*^{-1}\mu_{f(x)}^s}(y) = \frac{d\mu_x^s}{df_*^{-1}\mu_{f(x)}^s}(f^{-1}(f(y))) = e^{-\phi(f(y))+P(\phi)}$$

where the last equality follows from (3.6) applied to f^{-1}.

Using (3.10) and the fact that $[x,[x',y]] = [x,y]$ we derive the identity

$$\exp(\omega_x^u(y)) = \exp(\omega_x^u([x',y]))\exp(\omega_{x'}^u(y)).$$

By (3.8) we have

$$d\mu_{x'}^u([x',\cdot]) = \exp(\omega_x^u([x',\cdot]))d\mu_x^u([x,\cdot])$$

and we verify that the expression on the right hand side of (3.12) is well defined; that is, the measure is defined independently of the choice of base point x. Furthermore, by (3.6) and (3.7) we verify that the measure defined by the right hand side of (3.12) is invariant under f. Indeed we have

$$\begin{aligned}
d(f_*\mu)(f(y)) &= e^{\omega_x^u(y)+\omega_x^s(y)-\phi(y)}\ d(\mu_x^u\times\mu_x^s)([x,y],[y,x]) \\
&= e^{\omega_x^u(y)+\omega_x^s(y)-\phi(y)}\ d(f_*(\mu_x^u\times\mu_x^s))([f(x),f(y)],[f(y),f(x)]) \\
&= e^{\omega_x^u(y)+\omega_x^s(y)-\phi(y)}e^{\phi([x,y])-\phi(f([x,y]))}\ d(\mu_{f(x)}^u\times\mu_{f(x)}^s)([f(x),f(y)],[f(y),f(x)]) \\
&= e^{\omega_{f(x)}^u(f(y))+\omega_{f(x)}^s(f(y))-\phi(f(y))}\ d(\mu_{f(x)}^u\times\mu_{f(x)}^s)([f(x),f(y)],[f(y),f(x)]) \\
&= d\mu(f(y)).
\end{aligned}$$

To verify Theorem 3.4(f) it is enough to check that the conclusion holds for a partition ξ *adapted* to some local chart $V = [W_\delta^s(x)\cap\Lambda, W_\delta^u(x)\cap\Lambda]$ in the sense that for all $x\in V$, we have $\xi(x)\cap V = W_V^u(x)\cap\Lambda$.[4] For $y\in W_\delta^u(x)\cap\Lambda$ and $z\in W_\delta^s(x)\cap\Lambda$ define functions

$$\begin{aligned}
g(y,z) &:= \exp\left(\omega_x^u([z,y]) + \omega_x^s([z,y]) - \phi([z,y])\right) \\
c(z) &:= \int_{W_\delta^u(x)} g(y,z)\ d\mu_x^u(y).
\end{aligned}$$

[4]Indeed, for any two measurable, $\mathscr{F}^u$-subordinate partitions, the families of conditional measures agree, up to renormalization, on their common refinement.

For $A \subset V$ measurable we have

$$\begin{aligned}
\mu(A) &= \int_{W^s_\delta(x)} \int_{W^u_\delta(x)} 1_A([z,y]) g(y,z)\, d\mu^u_x(y)\, d\mu^s_x(z) \\
&= \int_{W^s_\delta(x)} c(z) \left(\frac{1}{c(z)} \int_{W^u_\delta(x)} 1_A([z,y]) g(y,z)\, d\mu^u_x(y) \right) d\mu^s_x(z) \\
&= \int_{W^s_\delta(x)} c(z) \left(\frac{1}{c(z)} \int_{W^u_\delta(z)\cap A} g([x,\cdot],z)\, d\mu^u_x([x,\cdot]) \right) d\mu^s_x(z) \\
&= \int_{W^s_\delta(x)} \left(\int_{W^u_\delta(x)} g(y,z)\, d\mu^u_x(y) \right) \left(\frac{1}{c(z)} \int_{W^u_\delta(z)\cap A} g([x,\cdot],z)\, d\mu^u_x([x,\cdot]) \right) d\mu^s_x(z) \\
&= \int_{W^s_\delta(x)} \int_{W^u_\delta(x)} \left(\frac{1}{c(z)} \int_{W^u_\delta(z)\cap A} g([x,\cdot],z)\, d\mu^u_x([x,\cdot]) \right) g(y,z)\, d\mu^u_x(y)\, d\mu^s_x(z) \\
&= \int_V \left(\frac{1}{c(z)} \int_{\xi([z,y])\cap A} g([x,\cdot],z)\, d\mu^u_x([x,\cdot]) \right) d\mu([z,y]).
\end{aligned}$$

Hence the family of measures $\{\tilde{\mu}^\xi_q\}_{q\in V}$ defined by

$$\tilde{\mu}^\xi_q(A) = \frac{1}{c([q,x])} \int_{\xi(q)\cap A} g([x,\cdot],[q,x])\, d\mu^u_x([x,\cdot])$$

forms a family of conditional probability measures for the partition ξ. We have

$$\begin{aligned}
\tilde{\mu}^\xi_q(A) &= \frac{1}{c([q,x])} \int_{\xi(q)\cap A} g([x,\cdot],[q,x])\, d\mu^u_x([x,\cdot]) \\
&= \frac{1}{c([q,x])} \int_{\xi(q)\cap A} \exp(\omega^u_x(\cdot) + \omega^s_x(\cdot) - \phi(\cdot))\, d\mu^u_x([x,\cdot]) \\
&= \frac{1}{c([q,x])} \int_{\xi(q)\cap A} \exp(\omega^s_x(\cdot) - \phi(\cdot))\, d\mu^u_q(\cdot) \\
&= \frac{1}{c([q,x])} \int_{\xi(q)\cap A} \exp(\omega^s_q(\cdot) + \omega^s_x(q) - \phi(\cdot))\, d\mu^u_q(\cdot) \\
&= \frac{e^{\omega^s_x(q)}}{c([q,x])} \int_{\xi(q)\cap A} \exp(\omega^s_q(\cdot) - \phi(\cdot))\, d\mu^u_q(\cdot)
\end{aligned}$$

and the result follows. □

The uniform hyperbolicity of f and the Hölder continuity of ϕ ensure that the functions $\omega_x^\sigma(y)$ are well defined. By Claim 3.3 and Theorem 3.4(f), for a.e. $x \in \Lambda$ we expect

$$f_*(e^{\omega_x^s - \phi}\mu_x^u) = Ke^{\omega_{f(x)}^s - \phi}\mu_{f(x)}^u$$

for some constant K. We check that $K = e^{-P(\phi)+\phi(f(x))}$ works. Note that even for $x' \in W^u(x) \cap \Lambda$ the measures $e^{\omega_x^s - \phi}\mu_x^u$ and $e^{\omega_{x'}^s - \phi}\mu_{x'}^u$ differ by the constant factor $e^{\omega_x^s(x')}$. Hence it is expected that the rescaling K will depend on the point x.

CHAPTER 4

STATEMENT AND PROOF OF RESULTS

The problem considered in this thesis is of the following nature:

> *Given a Borel probability measure μ on a manifold M, classify—or find non-trivial constraints on—the group of μ-preserving diffeomorphisms of M.*

In two extreme cases the group of μ-preserving diffeomorphisms is in some sense too large to admit interesting constraints. On one extreme, if μ is a volume the group of μ-preserving diffeomorphisms is an infinite-dimensional manifold. On the other extreme, if μ is a supported on a finite set, the group of μ-preserving diffeomorphisms is a finite-codimensional manifold in the space of all diffeomorphisms of M. Thus a natural class of measures in which to first consider the above problem is the class of singular measures with full support. In this thesis, we study the above problem for families of measures supported on the 2-torus.

4.1 STATEMENT OF RESULTS

Consider a compact smooth manifold M, and a collection of Borel probability measures $\{\mu_i\}$ supported on M. For $r \in [1,\infty) \cup \{\infty\}$ we write $\mathrm{Diff}^r(M;\{\mu_i\})$ for the group of C^r diffeomorphisms $f\colon M \to M$ such that $f_*\mu_i = \mu_i$ for all i. For $\{\mathscr{F}_i\}$ a family of foliations on M, we write $\mathrm{Diff}^r(M;\{\mathscr{F}_i\})$ for the group of C^r diffeomorphisms preserving each foliation $\mathscr{F}_i$.[1]

[1] We recall that a diffeomorphism $f\colon M \to M$ is said to preserve a foliation $\mathscr{F}$ if, for any $x \in M$, the restriction $f\restriction_{\mathscr{F}(x)}$ is a diffeomorphism $f\restriction_{\mathscr{F}(x)}\colon \mathscr{F}(x) \to \mathscr{F}(f(x))$.

To state our results, fix $\theta \in (1,\infty) \cup \{\infty\}$ and a (nonlinear) C^θ Anosov diffeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$. For $\sigma \in \{s,u\}$ and $v \in E^\sigma(x) \smallsetminus \{0\}$ define the functions

$$\lambda^\sigma(x) := \lim_{n\to\pm\infty} \frac{1}{n} \log(\|Df_x^n v\|). \tag{4.1}$$

By Oseledec's Theorem [Ose68], there is a set $\Lambda \subset \mathbb{T}^2$, with $\mu(\Lambda) = 1$ for any f-invariant Borel probability measure μ, such that for every $x \in \Lambda$ the limits in (4.1) exist. If μ is f-ergodic the functions $\lambda^u(\cdot)$ and $\lambda^s(\cdot)$ are constant μ-a.e. whence we write λ^u_μ and λ^s_μ for these constants.

Theorem 4.1. *Let μ be an f-ergodic measure on $\mathbb{T}^2$ with $h_\mu(f) > 0$ and full support. If μ satisfies*

$$\lambda^u_\mu \neq -\lambda^s_\mu$$

then for $r > 1$,

1. *the set of zero-entropy diffeomorphisms $N := \{g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu) \mid h_\mu(g) = 0\}$ is a normal subgroup of $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$;*

2. *if $\mathrm{Diff}^r(\mathbb{T}^2;\mu) \neq N$ then there is an isomorphism $\mathrm{Diff}^r(\mathbb{T}^2;\mu)/N \cong \mathbb{Z}$.*

In particular, Theorem 4.1 says that, up to zero-entropy diffeomorphisms, the group $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ looks like the cyclic group $\mathbb{Z}$. Note in particular that $\mathrm{Diff}^r(\mathbb{T}^2;\mu) \neq N$ whenever $r \leq \theta$ as we have $f \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$.

For a large subclass of measures satisfying the conditions of Theorem 4.1, we are able to give a more precise description of the group $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$.

Theorem 4.2. *Let μ be an equilibrium state for a Hölder continuous potential (with respect to f) that is neither the measure of maximal entropy, nor the forwards or backwards SRB measure. Assume in addition that*

$$\lambda^u_\mu + \lambda^s_\mu \neq 0.$$

Then for any $r \geq 1$ there is an $m \in \mathbb{N}$ such that the cyclic subgroup generated by $f^m\colon \mathbb{T}^2 \to \mathbb{T}^2$ has finite index in $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$. In particular, $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ is either finite or virtually infinite cyclic.

Recall that a group G is called *virtually infinite cyclic* if there is a finite index subgroup $G' \subset G$ with $G' \cong \mathbb{Z}$. We note that for $r \leq \theta$, we can take $m = 1$ in the conclusion of Theorem 4.2. Note however that we do not rule out the possibility that $m = 0$ in the case that there are no infinite order elements in $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ when $r > \theta$.

Using similar arguments we obtain the following.

Theorem 4.2'. *Let μ, ν be two f-ergodic Borel probability measures with full support. Assume $h_\mu(f) > 0$, $h_\nu(f) > 0$, and*

$$\lambda^u_\nu + \lambda^s_\nu < 0 < \lambda^u_\mu + \lambda^s_\mu.$$

Then for any $r \geq 1$ there is an $m \in \mathbb{N}$ such that the cyclic subgroup generated by $f^m \colon \mathbb{T}^2 \to \mathbb{T}^2$ has finite index in $\mathrm{Diff}^r(\mathbb{T}^2;\{\nu,\mu\})$. In particular, $\mathrm{Diff}^r(\mathbb{T}^2;\{\nu,\mu\})$ is either finite or virtually infinite cyclic.

For instance, if $f \colon \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism that is not volume-preserving, then Theorem 4.2' applies the forwards and backwards SRB measures for f.

We emphasize that Theorem 4.2 holds for $r = 1$, whereas Theorem 4.1 requires the additional hypothesis that $r > 1$. We note that the hypothesis in all our theorems that $\lambda^u_\mu \neq -\lambda^s_\mu$ forces the dynamics $f \colon \mathbb{T}^2 \to \mathbb{T}^2$ to be nonlinear and the measure μ to be singular with respect to the Riemannian volume.

4.2 FOLIATION RIGIDITY

Let $f \colon \mathbb{T}^2 \to \mathbb{T}^2$ be as in Section 4.1 with $\mathscr{F}^s$ and $\mathscr{F}^u$ the stable and unstable foliations. Before proving the main theorems we demonstrate mechanisms by which the preservation of an f-invariant measure forces the preservation of the dynamical foliations $\mathscr{F}^u$ and $\mathscr{F}^s$.

4.2.1 **Rigidity of the slow foliation.** Consider an f-ergodic measure μ with $h_\mu(f) > 0$ and $\lambda^u_\mu \neq -\lambda^s_\mu$. By the *slow foliation* we mean the foliation whose corresponding Lyapunov exponent is smaller in absolute value. We show that, under the additional hypothesis that μ has full support, any $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ preserves the slow foliation. For simplicity we assume $|\lambda^u_\mu| < |\lambda^s_\mu|$, and show g preserves $\mathscr{F}^u$.

Proposition 4.3. *Let μ be an f-ergodic Borel probability measure with full support and $h_\mu(f) > 0$. Suppose*

$$\lambda^u_\mu + \lambda^s_\mu < 0.$$

Then $\mathrm{Diff}^r(\mathbb{T}^2;\mu) \subset \mathrm{Diff}^r(\mathbb{T}^2;\mathscr{F}^u)$ *for all* $r \geq 1$.

Proof. Let $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$. We write $\mathscr{G} = g(\mathscr{F}^u)$. If $\mathscr{G} \neq \mathscr{F}^u$ then there is some open set $V \subset \mathbb{T}^2$ such that

1. V is a bifoliation chart for $\mathscr{F}^u$ and $\mathscr{F}^s$;
2. V is a foliation chart for $\mathscr{G}$;
3. for each $x, y \in V$ the intersection $\mathscr{G}_V(x) \cap W^u_V(y)$ contains at most one point, and the intersection is transverse.[2]

For $y \in V$ we identify $W^s_V(y)$ with the quotient space $V/\mathscr{F}^u_V$. Define $\hat{\mu}_y$ to be the quotient measure on $W^s_V(y)$ given by

$$\hat{\mu}_y(B) = \mu\big(W^u_V(B)\big)$$

and define the corresponding pointwise dimension functions

$$\hat{\delta}^+(y) = \limsup_{r\to 0} \frac{\log\big(\hat{\mu}_y(W^s_r(y))\big)}{\log r} \qquad \hat{\delta}^-(y) = \liminf_{r\to 0} \frac{\log\big(\hat{\mu}_y(W^s_r(y))\big)}{\log r}.$$

Since the unstable holonomies are bi-Lipschitz, by Proposition 3.2 $\hat{\delta}^\pm(y) = \hat{\delta}^\pm(z)$ for $z \in W^u_V(y)$.

By [LY85, Lemma 11.3.1] we have

$$\hat{\delta}^-(y) + \delta^u \leq \dim(\mu, y)$$

for μ-a.e. $y \in V$, whence, by (3.2) we conclude that

$$\hat{\delta}^-(y) \leq \delta^s \tag{4.2}$$

for μ-a.e. y.

Note that (3.3) and the hypothesis $\lambda^u_\mu + \lambda^s_\mu < 0$ implies that $\delta^u - \delta^s > 0$. Fix $0 < \eta < \delta^u - \delta^s$. We write $\{\tilde{\mu}^{\mathscr{G}}_{V,y}\}_{y\in V}$ for a family of conditional measures associated to the (measurable)

[2] See Section 2.1.2 for notation for local foliations.

partition of V by the leaves of the local foliation $\mathscr{G}_V$. Note that by Claim 3.3, the fact that g is bi-Lipschitz, and Proposition 3.2, we have $\dim(\tilde{\mu}^{\mathscr{G}}_{V,y}, y) = \delta^u$ for a.e. $y \in V$. Define

$$\Gamma^l_R := \{x \in V \mid l^{-1} r^{\delta^u + \eta} \leq \tilde{\mu}^{\mathscr{G}}_{V,x}(B(x,r)) \leq l r^{\delta^u - \eta} \quad \text{for all } 0 < r < R\}$$

and fix l and R such that $\mu(\Gamma^l_R) > 0$. On $W^s_V(y)$ define a second quotient measure $\hat{\nu}^l_R$ by

$$\hat{\nu}^l_R(B) := \mu(W^u_V(B) \cap \Gamma^l_R).$$

Clearly $\hat{\nu}^l_R \ll \hat{\mu}$ thus, by Proposition 3.2, for every $x \in V$ and $\hat{\nu}^l_R$-a.e. $y \in W^s_V(x)$ we have $\underline{\dim}(\hat{\nu}^l_R, y) = \hat{\delta}^-(y)$.

Fix such a y. Using the uniform transversality of the local foliations $\mathscr{G}_V$ and $\mathscr{F}^u_V$ and the fact that the unstable holonomies are bi-Lipschitz, we may find a $c \in (0,1)$ such that

$$W^u_V(W^s_{cr}(y)) \subset \bigcup_{z \in W^u_V(y)} B_{\mathscr{G}}(z, r)$$

for all sufficiently small $r > 0$. Here $B_{\mathscr{G}}(z,r)$ denotes ball of radius r at z in internal metric of the submanifold $\mathscr{G}(z)$. (Note $B_{\mathscr{G}}(z,r) \subset B(z,r)$, where $B(z,r)$ denotes the ambient metric ball of radius r.) Hence

$$\begin{aligned}
\hat{\nu}^l_R(W^s_{cr}(y)) &= \int_V \tilde{\mu}^{\mathscr{G}}_{V,x}\left(W^u_V(W^s_{cr}(y)) \cap \mathscr{G}_V(x) \cap \Gamma^l_R\right) d\mu(x) \\
&\leq \int_V 2 l r^{\delta^u - \eta} \, d\mu(x) \\
&= K r^{\delta^u - \eta}.
\end{aligned}$$

for some K and all sufficiently small $r > 0$. We thus conclude that

$$\underline{\dim}(\hat{\nu}^l_R, y) \geq \delta^u - \eta > \delta^s$$

and hence $\hat{\delta}^-(y) > \delta^s$ on a set of positive measure contradicting (4.2). □

4.2.2 **Rigidity of the fast foliation for equilibrium states.** In the case that μ is an equilibrium state, we are able to utilize the local product structure of μ in Theorem 3.4 to obtain a result stronger than Proposition 4.3.

Proposition 4.4. *Let μ be an equilibrium state for a Hölder continuous potential on $\mathbb{T}^2$ (with respect to the dynamics of f). Suppose that μ is neither the forwards nor backwards*

SRB measure and satisfies

$$\lambda_\mu^u \neq -\lambda_\mu^s.$$

Then

$$\mathrm{Diff}^r(\mathbb{T}^2;\mu) \subset \mathrm{Diff}^r(\mathbb{T}^2;\{\mathscr{F}^s,\mathscr{F}^u\})$$

for all $r \geq 1$.

Proof. By passing to f^{-1} if necessary we may assume that $|\lambda_\mu^u| < |\lambda_\mu^s|$ whence Proposition 4.3 implies that any $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ preserves $\mathscr{F}^u$. We use the local product structure of μ to show that g preserves $\mathscr{F}^s$ under the additional assumption that μ is not the forwards SRB measure.

Fix an $r \geq 1$ and $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$. Writing $\mathscr{G} = g(\mathscr{F}^s)$ assume for the purpose of contradiction that $\mathscr{G} \neq \mathscr{F}^s$. Then there exists an $x_0 \in \mathbb{T}^2$ such that for all sufficiently small $\delta > 0$ and for the local chart

$$V = [W_\delta^s(x_0), W_\delta^u(x_0)]$$

and all $y, z \in V$, the intersections $\mathscr{G}_V(y) \cap W_V^u(z)$ and $\mathscr{G}_V(y) \cap W_V^s(z)$ are transverse and contain at most one point. (See Figure 4.1.) Furthermore, we may choose $\delta > 0$ small enough such that $g^{-1}(V)$ is contained in a local product chart.

We denote by $\{\mu_x^u\}_{x\in V}$ the system of canonical measures along the unstable manifolds described in Theorem 3.4. Let

$$\Upsilon := \{x \in V \mid \dim(\mu_x^u, x) = \delta^u\} \subset V.$$

Proposition 4.4 follows easily from the following claim, which will be proved shortly.

Claim 4.5. *We have the following*

1. Υ *has full measure in* V*, in particular* Υ *is nonempty;*

2. Υ *contains full* $\mathscr{F}_V^s$*-leaves;*

3. Υ *contains full* $\mathscr{G}_V$*-leaves.*

The assumptions that the intersections $\mathscr{G}_V(y) \cap W_V^s(z)$ are transverse and that Υ is both $\mathscr{F}_V^s$- and $\mathscr{G}_V$-saturated imply that Υ contains an open set. Indeed for $y \in \Upsilon$, we have $W_V^s(y) \in$

Υ whence

$$W = \bigcup_{z \in W_V^s(y)} \mathscr{G}_V(z)$$

contains an open foliation chart for $\mathscr{G}$. In particular Υ contains a curve $I \subset W_\delta^u(y)$. By Proposition 3.1 we obtain that $\dim_H(I) \leq \delta^u$, where $\dim_H(I)$ denotes the Hausdorff dimension of the set I. However the Hausdorff dimension of I is 1, whereas the assumption that μ is not the forwards SRB measure implies $\delta^u < 1$. Hence we obtain a contradiction and the proposition follows. □

We now establish the assertions in Claim 4.5. We note that for all $x, y \in V$ or $x, y \in g^{-1}(V)$, we have $x \in W_\epsilon^u([x,y])$ and $x \in W_\epsilon^s([y,x])$. Let ϕ be the Hölder continuous potential function for μ. Then we may find constants $C > 0$ and $0 < \alpha < 1$ such that for any $x, y \in \mathbb{T}^2$ we have $|\phi(x) - \phi(y)| < Cd(x,y)^\alpha$. In particular, the Hölder continuity of ϕ and hyperbolicity of f ensures that for any $x, y \in V$ or $x, y \in g^{-1}(V)$ we have the uniform bound

$$|\omega_x^s(y)| \leq C\epsilon^\alpha \frac{1}{1-(\kappa^{-1})^\alpha}$$

where $\omega_x^s(y)$ is as defined in (3.11) and κ is as in (2.2). Similarly for all $x, y \in V$ and λ as in (2.1) we have

$$|\omega_x^u(y)| \leq C\epsilon^\alpha \frac{1}{1-\lambda^\alpha}.$$

Set

$$N := \exp\left(C\epsilon^\alpha \frac{1}{1-(\kappa^{-1})^\alpha} + \max\left\{|\phi(x)| \mid x \in \mathbb{T}^2\right\}\right); \tag{4.3}$$
$$M := \exp\left(C\epsilon^\alpha \frac{1}{1-\lambda^\alpha}\right).$$

We write $\{\tilde{\mu}_x^u := \exp(\omega_x^s - \phi)\mu_x^u\}$ for the *unnormalized* family of conditional measures obtained in Theorem 3.4(f) for the partition $\{\mathscr{F}_V^u(x)\}_{x \in V}$ of V.

Proof of Claim 4.5.1. For every $x \in V$ and $y \in W_V^u(x)$ we have

$$\frac{1}{N} \leq \frac{d\mu_x^u}{d\tilde{\mu}_x^u}(y) \leq N. \tag{4.4}$$

By Proposition 3.2 applied to the identity map we have that

$$\Upsilon = \{x \in V \mid \dim(\tilde{\mu}_x^u, x) = \delta^u\}$$

and the result follows. □

Proof of Claim 4.5.2. Fix $x \in \Upsilon$ and $x' \in W^s_V(x)$. Recall that the holonomy map

$$h^s_{x,x'} \colon W^u_V(x) \to W^u_V(x')$$

defined by (3.4) is bi-Lipschitz. Furthermore, by Theorem 3.4(d) we have

$$\frac{1}{M} \leq \frac{d\mu^u_{x'}}{d((h^s_{x,x'})_* \mu^u_x)} \leq M$$

thus $x' \in \Upsilon$ by Proposition 3.2. □

Proof of Claim 4.5.3. Recall that g preserves the foliation $\mathscr{F}^u$. We consider the family of measures $\nu_x := g_*(\mu^u_{g^{-1}(x)})$ supported on unstable leaves $\{W^u_V(x)\}_{x \in V}$ and the subset

$$\Upsilon' := \{x \in V \mid \dim(\nu_x, x) = \delta^u\}.$$

Arguing as in the proof of Claim 4.5.2, we have that $g^{-1}(\Upsilon')$ is $\mathscr{F}^s_{g^{-1}(V)}$-saturated, and hence Υ' is $\mathscr{G}_V$-saturated. To establish the claim, we show $\Upsilon = \Upsilon'$.

For any $w \in V$, consider a connected open set (i.e an interval) $U \subset W^u_\delta(w)$. Let

$$T := \mathscr{G}_V(U)$$

and for $y \in V$ write $T_y := T \cap W^u_V(y)$. See Figure 4.1.

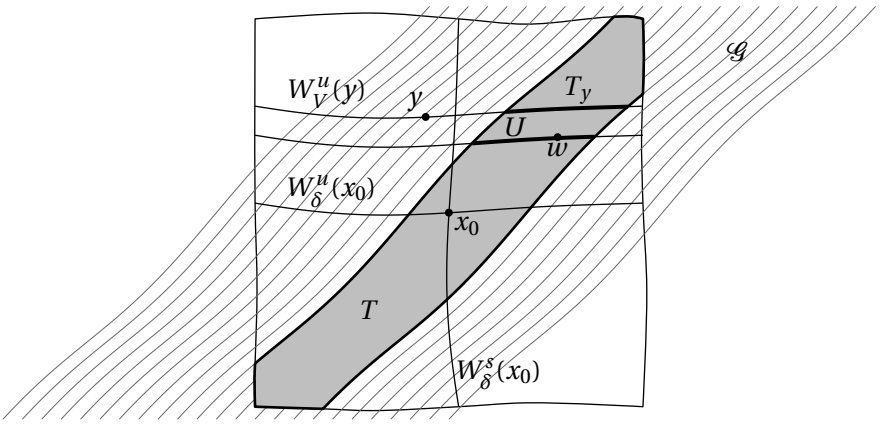

Figure 4.1: The local chart V

Consider the following real-valued functions on V.

$$
\begin{aligned}
j_{1,U}&\colon y \mapsto \mu^u_y(T_y)\\
j_{2,U}&\colon y \mapsto g_*(\mu^u_{g^{-1}(y)})(T_y)\\
c_1&\colon y \mapsto \mu^u_y(W^u_V(y))\\
c_2&\colon y \mapsto g_*(\mu^u_{g^{-1}(y)})(W^u_V(y))\\
\tilde{c}_1&\colon y \mapsto \tilde{\mu}^u_y(W^u_V(y)) = \int_{W^u_V(y)} \exp(\omega^s_y - \phi)\, d\mu^u_y\\
\tilde{c}_2&\colon y \mapsto g_*(\tilde{\mu}^u_{g^{-1}(y)})(W^u_V(y)) = \int_{W^u_V(y)} \exp(\omega^s_{g^{-1}(y)} \circ g^{-1} - \phi \circ g^{-1})\, d g_*(\mu^u_{g^{-1}(y)}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{1}{N} j_{1,U}(y) \le \tilde{\mu}^u_y(T_y) \le N j_{1,U}(y)\\
\frac{1}{N} j_{2,U}(y) \le g_*(\tilde{\mu}^u_{g^{-1}(y)})(T_y) \le N j_{2,U}(y)\\
\frac{1}{N} c_i(y) \le \tilde{c}_i(y) \le N c_i(y)
\end{aligned}
$$

Claim 4.6. *The functions $j_{1,U}, j_{2,U}, c_1$, and c_2 are continuous.*

Proof. First note that each function is invariant along $W^u_V(y)$. Secondly, since the unstable holonomies are bi-Lipschitz, there is a constant $K > 0$ such that for $y' \in W^s_V(y)$ and $z \in W^u_V(y')$ we have

$$d(z, [y, z]) \le K d(y', y)$$

and hence

$$|\omega^u_y(z)| \le C K^\alpha d(y, y')^\alpha \frac{1}{1-\lambda^\alpha}.$$

Hence for any Borel set $A \subset W^u_V(y)$ we have

$$\exp\left(-CK^\alpha d(y,y')^\alpha \frac{1}{1-\lambda^\alpha}\right) \mu^u_y(A) \le \mu^u_{y'}(h^s_{y,y'}(A)) \le \exp\left(CK^\alpha d(y,y')^\alpha \frac{1}{1-\lambda^\alpha}\right) \mu^u_y(A)$$

and hence as $y' \to y$ we have $\mu^u_{y'}(h^s_{y,y'}(A)) \to \mu^u_y(A)$. This establishes the continuity of c_1.

Let $C_1 = \max\{c_1(y) \mid y \in V\}$. For any $y, y' \in V$ and Borel $A \subset W^u_V(y)$ we have

$$\begin{aligned}
&|\mu_y(A) - \mu_{y'}(h^s_{y,y'}(A))| \\
&\qquad \leq \max_{\sigma\in\{+1,-1\}} \left\{ \left| \mu_y(A) - \exp\left(\sigma C K^\alpha d(y,y')^\alpha \frac{1}{1-\lambda^\alpha}\right)\mu_y(A) \right| \right\} \\
&\qquad = \max_{\sigma\in\{+1,-1\}} \left\{ \left| \left(1 - \exp\left(\sigma C K^\alpha d(y,y')^\alpha \frac{1}{1-\lambda^\alpha}\right)\right)\mu_y(A) \right| \right\} \\
&\qquad = \left| \left(1 - \exp\left(C K^\alpha d(y,y')^\alpha \frac{1}{1-\lambda^\alpha}\right)\right)\mu_y(A) \right| \\
&\qquad \leq \left| \left(1 - \exp\left(C K^\alpha d(y,y')^\alpha \frac{1}{1-\lambda^\alpha}\right)\right) c_1(y) \right| \\
&\qquad \leq \left| \left(1 - \exp\left(C K^\alpha d(y,y')^\alpha \frac{1}{1-\lambda^\alpha}\right)\right) C_1 \right|.
\end{aligned}$$

In particular, for any $\epsilon > 0$ we may find $\delta > 0$ such that for any $y, y' \in V$ with $d(y,y') < \delta$ and any Borel $A \subset W^u_V(y)$ we have

$$|\mu^u_y(A) - \mu^u_{y'}(h^s_{y,y'}(A))| < \epsilon/2.$$

Now consider the endpoints $\{a, b\} = \overline{T_y} \smallsetminus T_y$. There is a continuous function $r(y')$ with $r(y') \to 0$ as $y' \to y$ with the property that

$$T_y \Delta h^s_{y',y}(T_{y'}) \subset W^u_{r(y')}(a) \cup W^u_{r(y')}(b).$$

We have

$$|\mu^u_y(h^s_{y',y}(T_{y'})) - \mu^u_y(T_y)| \leq \mu^u_y(T_y \Delta h^s_{y',y}(T_{y'})) \leq \mu^u_y\big(W^u_{r(y')}(a) \cup W^u_{r(y')}(b)\big).$$

Since μ^u_y is non-atomic we have that $\mu^u_y\left(W^u_{r(y')}(a) \cup W^u_{r(y')}(b)\right) \to 0$ as $y' \to y$.

Consequently, for any $\epsilon > 0$ we may find $\delta > 0$ such that $d(y,y') < \delta$ implies

$$|\mu^u_{y'}(T_{y'}) - \mu^u_y(h^s_{y',y}(T_{y'}))| < \epsilon/2$$

and

$$|\mu^u_y(h^s_{y',y}(T_{y'})) - \mu^u_y(T_y)| \leq \epsilon/2.$$

This proves the continuity of $j_{1,U}$. Similar arguments with respect to the product structure of the local product chart containing $g^{-1}(V)$ show the continuity of c_2 and $j_{2,U}$. □

Now, consider the inequalities

$$\frac{j_{1,U}(y)}{Nc_1(y)} \le \frac{j_{1,U}(y)}{\tilde{c}_1(y)} \le N\frac{\tilde{\mu}^u_y(T_y)}{\tilde{c}_1(y)},$$
$$\frac{1}{N}\frac{g_*(\tilde{\mu}^u_{g^{-1}(y)})(T_y)}{\tilde{c}_2(y)} \le \frac{j_{2,U}(y)}{\tilde{c}_2(y)} \le N\frac{j_{2,U}(y)}{c_2(y)}$$

where N is as in (4.3). We have that $\frac{1}{\tilde{c}_1(y)}\tilde{\mu}^u_y$ and $\frac{1}{\tilde{c}_2(y)}g_*(\tilde{\mu}^u_{g^{-1}(y)})$ define families of conditional probability measures for the partition of V by local unstable manifolds, and thus

$$\frac{\tilde{\mu}^u_y(T_y)}{\tilde{c}_1(y)} = \frac{g_*(\tilde{\mu}^u_{g^{-1}(y)})(T_y)}{\tilde{c}_2(y)}$$

on a set of full measure. We then have that

$$j_{1,U}(y) \le N^4\frac{c_1(y)}{c_2(y)}j_{2,U}(y) \tag{4.5}$$

for a.e. $y \in V$. Since each side of (4.5) is continuous and the measure μ has full support, we have that (4.5) holds for *all* y in V. Similarly, we have

$$j_{1,U}(y) \ge \frac{1}{N^4}\frac{c_1(y)}{c_2(y)}j_{2,U}(y)$$

for all y. In particular, since U and w were arbitrary, for any $y \in V$ we have

$$\frac{c_1(y)}{N^4c_2(y)} \le \frac{d\mu^u_y}{dg_*\mu^u_{g^{-1}(y)}} \le N^4\frac{c_1(y)}{c_2(y)}.$$

Hence for every $y \in V$ we have that μ^u_y is equivalent to $g_*\mu^u_{g^{-1}(y)}$ with bounded Radon-Nikodym derivative. It then follows from Proposition 3.2 that $\Upsilon = \Upsilon'$. □

We finish this chapter with proofs of the main theorems.

4.3 PROOF OF THEOREM 4.1

We recall that we have fixed $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ a C^θ Anosov diffeomorphism for $\theta > 1$. For the remainder of this section, fix $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ bi-Hölder, and $A \in \mathrm{GL}(2,\mathbb{Z})$ such that

$$h \circ f \circ h^{-1} = L_A. \tag{4.6}$$

Fix μ as in Theorem 4.1, and $r \ge 1+\alpha$ for some $\alpha > 0$. By passing to f^{-1} if necessary, we assume $|\lambda^u_\mu| \le |\lambda^s_\mu|$. We continue to write $\mathscr{F}^u$ and $\mathscr{F}^s$ for the foliations of $\mathbb{T}^2$ induced by the dynamics of f. For $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$, Proposition 4.3 guarantees g preserves $\mathscr{F}^u$. The

functions r and λ_j will be as in Section 2.2 with respect to the dynamics of g. We write $E^i_g(x)$ and $\widetilde{W}^i_g(x)$ for the Lyapunov subspaces and corresponding Pesin manifolds at x under the dynamics of g.

For $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ define the following data.

1. Let $\Lambda(g)$ be the set of regular points x for g with $r(x) = 0$ and $\lambda_0(x) = 0$, or $r(x) = 1$ and $\lambda_0(x) \cdot \lambda_1(x) \leq 0$. That is, $\Lambda(g)$ is the set of regular points for g such that the Lyapunov exponents are not all positive or all negative.

2. Let $\Omega(g) \subset \Lambda(g)$ denote the set of points with one positive and one negative exponent; that is, for $x \in \Omega(g)$ we have $r(x) = 1$ and $\lambda_0(x) \cdot \lambda_1(x) < 0$.

3. Define the measurable functions χ_g and J_g on $\mathbb{T}^2$

$$\chi_g \colon x \mapsto \limsup_{n\to\infty} \frac{1}{n} \log\left(\|Dg^n_x \restriction_{T_x\mathscr{F}^u(x)}\|\right)$$

and

$$J_g \colon x \mapsto \|Dg_x \restriction_{T_x\mathscr{F}^u(x)}\|.$$

4. Define $\overline{\chi}(g) := \int \chi_g \, d\mu$.

We remark that $\mu(\Lambda(g)) = 1$. Indeed writing Υ for the set of regular points for g with all Lyapunov exponents strictly positive, we have that Υ is g-invariant and measurable. Suppose $\mu(\Upsilon) > 0$ and let ν be the probability measure $\nu(A) = \mu(A \cap \Upsilon)/\mu(\Upsilon)$. Applying (3.3) to g^{-1} we have $h_\nu(g) = 0$ (note that g^{-1} has no positive Lyapunov exponents on Υ) . On the other hand, for $x \in \Upsilon$, the unstable Pesin manifold $\widetilde{W^u(x)}$ contains a neighborhood of x in $\mathbb{T}^2$. In particular, the unstable dimension of μ at μ-a.e. point of Υ is equal to $\dim(\mu) > 0$. Applying (3.3) to g we must have $\mu(\Upsilon) = 0$. Similarly the set of points with strictly negative Lyapunov exponents is a null set.

We will see shortly that the entropy $h_\mu(g)$ is effectively computed by the dynamics of g along the foliation $\mathscr{F}^u$. Furthermore, the entropy satisfies the following 'signed additivity' property on $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$.

Proposition 4.7. *For μ as in Theorem 4.1 and $r \geq 1+\alpha$, on* $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ *the metric entropy satisfies*

$$h_\mu(g_1 \circ g_2) = \begin{cases} h_\mu(g_1) + h_\mu(g_2) & \text{if } \overline{\chi}(g_1)\cdot\overline{\chi}(g_2) \geq 0, \\ |h_\mu(g_1) - h_\mu(g_2)| & \text{otherwise.} \end{cases} \tag{4.7}$$

Proof. The proof follows in a number of claims. Fix an $r \geq 1+\alpha$ and a $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$.

Step 1: χ_g is a Lyapunov exponent for g. Indeed we have that the functions χ_g and J_g are related via the formula

$$\chi_g(x) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log(J_g(g^i(x))).$$

For $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ let $\mathscr{I}_g$ denote the σ-algebra of g-invariant sets. By the Birkhoff Ergodic Theorem[3] we have for μ-a.e. x the equalities[4]

$$\begin{aligned} \chi_g(x) &= \mathbb{E}(\log J_g \mid \mathscr{I}_g)(x) \\ &= \lim_{n\to\pm\infty} \frac{1}{n}\log\left(\|Dg_x^n v\|\right) \end{aligned} \tag{4.8}$$

establishing that $\chi(g)$ is a Lyapunov exponent.[5]

Furthermore for almost every $x \in \Lambda(g)$ with two distinct Lyapunov exponents, (i.e. with $r(x) = 1$), we have that $T_x\mathscr{F}^u(x)$ is the Lyapunov subspace associated to χ_g. Indeed if x is such a point and $0 \neq v \in T_x\mathscr{F}^u(x)$ satisfies

$$v = \alpha_0 v_0 + \alpha_1 v_1$$

where $v_j \in E_g^j(x)$ and $\alpha_j \neq 0$ then we have

$$\lambda_0(x) = \lim_{n\to-\infty} \frac{1}{n}\log\left(\|Dg_x^n v\|\right) \neq \lim_{n\to\infty} \frac{1}{n}\log\left(\|Dg_x^n v\|\right) = \lambda_1(x)$$

which can only hold on a null set by (4.8).

Let $i(x)$ be the a.e.-defined $\{0,1\}$-valued function on $\Lambda(g)$ satisfying $\chi_g(x) = \lambda_{i(x)}(x)$.

Step 2: Local Pesin manifolds associated to χ_g. We establish the following claim.

[3] See for example [KH95].

[4] Here, the right hand side of the first equality denotes a conditional expectation.

[5] More precisely, χ_g is restriction of a Lyapunov exponent to the subbundle $T\mathscr{F}^u$.

Claim 4.8. *For a.e. regular point x with $\chi_g(x) \neq 0$ we have that $\widetilde{W}_g^{i(x)}(x) \cap \mathscr{F}^u(x)$ contains a neighborhood of x in $\mathscr{F}^u(x)$.*

Proof. Fix a Riemannian metric on $\mathbb{T}^2$ and let

$$\mathrm{Exp}'_y \colon T_y\mathscr{F}^u(y) \to \mathscr{F}^u(y)$$

denote the exponential map for the restriction of the metric to $\mathscr{F}^u(y)$. We have Exp'_y is $C^{1+\alpha}$ for some $\alpha > 0$. We denote by $B_y(0,r) \subset T_y\mathscr{F}^u(y)$ the norm-ball of radius r centered at zero in $T_y\mathscr{F}^u(y)$. For a fixed r we then have that the maps $\mathrm{Exp}'_y \colon B_y(0,r) \to \mathscr{F}^u(y)$ are bi-Lipschitz and the Lipschitz constants are bounded uniformly in the variable y.

For r sufficiently small, define $\tilde{g}_y \colon B_y(0,r) \to T_{g(y)}\mathscr{F}^u(g(y))$ by

$$\tilde{g}_y = (\mathrm{Exp}'_{g(y)})^{-1} \circ g \circ \mathrm{Exp}'_y.$$

Let $\eta \colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function with $\eta(x) = 1$ for $x \leq 1/2$ and $\eta(x) = 0$ for $x \geq 1$ and $0 < \eta(x) < 1$ for $1/2 < x < 1$. Then define $G_y \colon T_y\mathscr{F}^u(y) \to T_{g(y)}\mathscr{F}^u(g(y))$

$$G_y(v) := Dg_y(v) + \eta(r^{-1}v)(\tilde{g}_y(v) - Dg_y(v)).$$

Then we have

$$G_y(v) = \begin{cases} \tilde{g}_y(v) & \|v\| \leq r/2, \\ Dg_y(v) & \|v\| \geq r. \end{cases}$$

We have that G_y is a Lipschitz perturbation of Dg_y:

$$\|(Dg_y - G_y)(v) - (Dg_y - G_y)(u)\| \leq \gamma_r \|v - u\|,$$

and, by taking r sufficiently small, we may make γ_r arbitrarily small.[6] Furthermore $Dg_y(0) = G_y(0) = 0$ by construction whence

$$\|(Dg_y - G_y)(v)\| \leq \gamma_r \|v\|.$$

We emphasize that the above bounds are uniform over all $y \in \mathbb{T}^2$. We write $G_y^n := G_{g^{n-1}(y)} \circ G_{g^{n-2}(y)} \circ \cdots \circ G_y$.

As noted earlier, we have that $\mu(\Lambda(g)) = 1$. In particular, for almost every x as in

[6] This is a standard construction and we omit the details which can be found, for instance, in [Yoc95, Section 2.4].

the claim, the Lyapunov subspace $E_g^{i(x)}(x)$ associated to $\chi_g(x)$ is 1-dimensional and hence equal to $T_x\mathscr{F}^u(x)$. Fix any x with $\chi_g(x) \neq 0$ and $E_g^{i(x)}(x) = T_x\mathscr{F}^u(x)$. By passing to g^{-1} if necessary we may assume $i(x) = 0$, that is, $\lambda_1(x) \geq 0 > \lambda_0(x)$. Fix some $0 < \epsilon < \frac{1}{4}|\lambda_0(x)|$. The nonuniform hyperbolicity of Dg along the orbit of x guarantees we may find a constant $C = C(x,\epsilon)$ (where $C(x,\epsilon)$ depends measurably on x) such that for $v \in T_x\mathscr{F}^u(x)$

$$\|Dg_x^n v\| \leq Ce^{n(\lambda_0(x)+\epsilon)}\|v\|.$$

Write $\eta = e^\epsilon - 1 > 0$. We may choose r small enough such that

$$\gamma_r < \eta \cdot \inf\{J_g(y) \mid y \in \mathbb{T}^2\}.$$

The bound on γ_r then guarantees that for any R and $y \in \mathbb{T}^2$ we have

$$G_y(B_y(0,R)) \subset Dg_y\left(B_y(0,(1+\eta)R)\right).$$

Indeed, for any $v \in B_y(0,R)$ we have

$$\begin{aligned}\|G_y(v)\| &\leq \|Dg_y(v)\| + \gamma_r\|v\| \\ &\leq J_g(y)R + \gamma_r R \\ &\leq (J_g(y) + \eta J_g(y))R\end{aligned}$$

so $G_y(v) \subset Dg_y(B_y(0,(1+\eta)R)$ for all y. Consequently, we obtain the inclusion

$$G_y^n(B_y(0,R)) \subset Dg_y^n\left(B_y(0,(1+\eta)^n R)\right).$$

(We emphasize, however, that the above arguments works because $\dim T_y\mathscr{F}^u(y) = 1$ and thus the norm and co-norm of Dg_y are equal, and all linear maps commute; any higher-dimensional argument would require far more subtle control of the geometry.)

Thus for $v \in T_x\mathscr{F}^u(x)$ we have

$$\|G_x^n(v)\| \leq Ce^{n(\lambda_0(x)+\epsilon)}(1+\eta)^n\|v\| = Ce^{n(\lambda_0(x)+2\epsilon)}\|v\|.$$

In particular there is some $r' > 0$ such that $G_x^n(B_x(0,r')) \subset B_{g^n(x)}(0,\frac{r}{2})$ for all $n \geq 0$. Note

then that $G_x^n \restriction_{B_x(0,r')} = \tilde{g}_{g^{n-1}(x)} \circ \cdots \circ \tilde{g}_{g(x)} \circ \tilde{g}_x$. Let

$$U = \mathrm{Exp}'_x(B_x(0, r')).$$

We then have that $U \subset \mathscr{F}^u(x)$ and for $y \in U$

$$d(g^n(x), g^n(y)) \leq C' e^{n(\lambda_0(x)+2\epsilon)} d(x, y)$$

for some C'. For small enough r', this characterizes[7] U as contained in a local stable Pesin manifold for $\lambda_0(x)$ at x and the claim follows. □

Step 3: Uniformity of the dynamics of g along $\mathscr{F}^u$. We assert the following regarding the dynamics of g along the foliation $\mathscr{F}^u$.

Claim 4.9.

1. *For a.e. $x \notin \Omega(g)$, we have $\chi_g(x) = 0$.*
2. *We have a dichotomy: either for every $x \in \Lambda(g)$ with $\lambda_0(x) < 0$ we have $\widetilde{W}^0(x) \subset \mathscr{F}^u(x)$ or for every $x \in \Lambda(g)$ with $\lambda_1(x) > 0$ we have $\widetilde{W}^1(x) \subset \mathscr{F}^u(x)$.*
3. *There is a set of full measure on which χ_g restricts to either a nonpositive or a nonnegative function.*

Proof. To see the first assertion, first note that χ_g is clearly zero-valued on the set of points in $\Lambda(g)$ with only zero-Lyapunov exponents. Denote by $\Upsilon \subset \Lambda(g) \setminus \Omega(g)$ the set of regular points for g with one positive and one zero-Lyapunov exponent. Note that Υ is g-invariant and measurable. Suppose $\mu(\Upsilon) > 0$ and let ν be the probability measure $\nu(A) = \mu(A \cap \Upsilon)/\mu(\Upsilon)$. Then applying (3.3) to g^{-1} we have $h_\nu(g) = 0$ since on Υ, g^{-1} has no positive Lyapunov exponents. Let

$$\Upsilon' := \{x \in \Upsilon \mid \chi_g(x) > 0\}.$$

Then applying (3.3) and Claim 4.8 we have

$$h_\nu(g) \geq \int \delta^u \chi_g(x)\, d\nu(x) = \int_{\Upsilon'} \delta^u \chi_g(x)\, d\nu(x).$$

[7] See for example the characterization of local stable manifolds in [Rue79, Theorem 6.1(a)].

Thus we must have $\mu(\Upsilon') = 0$. Arguing similarly on the set of points with one negative and one zero-Lyapunov exponent we obtain the first assertion. In particular, we have that the entropy is entirely concentrated on the set $\Omega(g)$.

We write $\mathscr{E}^u$ for the unstable linear foliation of $\mathbb{T}^2$ induced by the dynamics of L_A. Let $\tilde{\mathscr{E}}^u$ denote the pulled-back foliation on the universal cover $\mathbb{R}^2$. Note that the quotient space $\mathbb{R}^2/\tilde{\mathscr{E}}^u$ may naturally be identified with the 1-dimensional linear space $\mathbb{R}^2/\tilde{\mathscr{E}}^u(0) \cong \mathbb{R}$. We have that the homeomorphism $h \circ g \circ h^{-1}$ preserves the foliation $\mathscr{E}^u$. Furthermore,

Claim 4.10. *$h \circ g \circ h^{-1}$ acts as an affine map transverse to $\mathscr{E}^u$: any lift of $h \circ g \circ h^{-1}$ to $\mathbb{R}^2$ induces an affine action on the quotient $\mathbb{R}^2/\tilde{\mathscr{E}}^u \cong \mathbb{R}$.*

Proof. Let $\tilde{l}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $h \circ g \circ h^{-1}$. Choose any $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2 \smallsetminus \tilde{\mathscr{E}}^u(x)$ and let $\eta = \dfrac{\rho(\tilde{l}(\tilde{\mathscr{E}}^u(x)), \tilde{l}(\tilde{\mathscr{E}}^u(y)))}{\rho(\tilde{\mathscr{E}}^u(x), \tilde{\mathscr{E}}^u(y))}$ where ρ denotes Euclidean distance. Since the leaves of $\mathscr{E}^u$ are linear and dense in $\mathbb{T}^2$ we deduce that

$$\frac{\rho(\tilde{l}(\tilde{\mathscr{E}}^u(x')), \tilde{l}(\tilde{\mathscr{E}}^u(y')))}{\rho(\tilde{\mathscr{E}}^u(x'), \tilde{\mathscr{E}}^u(y'))} = \eta \tag{4.9}$$

for any $x', y' \in \mathbb{R}^2$ with $\rho(\tilde{\mathscr{E}}^u(x), \tilde{\mathscr{E}}^u(y)) = \rho(\tilde{\mathscr{E}}^u(x'), \tilde{\mathscr{E}}^u(y'))$. A standard argument shows (4.9) holds for any $x', y' \in \mathbb{R}^2$ and the claim follows. □

We now show Claim 4.9.2. Suppose there exist x, y in $\Lambda(g)$ with $\lambda_0(x) < 0$, $\lambda_1(y) > 0$, $\widetilde{W}^0_g(x) \not\subset \mathscr{F}^u(x)$ and $\widetilde{W}^1_g(y) \not\subset \mathscr{F}^u(y)$. Fix a lift $\tilde{l}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of $h \circ g \circ h^{-1}$ and lifts $\tilde{x}, \tilde{y}$ of $h(x)$ and $h(y)$, respectively. We may then find $\tilde{x}'$ and $\tilde{y}' \in \mathbb{R}^2$ in lifts of $h(\widetilde{W}^0_g(x))$ and $h(\widetilde{W}^1_g(y))$, respectively, with $\tilde{\mathscr{E}}^u(\tilde{x}) \neq \tilde{\mathscr{E}}^u(\tilde{x}')$, $\tilde{\mathscr{E}}^u(\tilde{y}) \neq \tilde{\mathscr{E}}^u(\tilde{y}')$,

$$\rho(\tilde{l}^n(\tilde{\mathscr{E}}^u(\tilde{x})), \tilde{l}^n(\tilde{\mathscr{E}}^u(\tilde{x}'))) \to 0 \quad \text{as } n \to \infty,$$

and

$$\rho(\tilde{l}^n(\tilde{\mathscr{E}}^u(\tilde{y})), \tilde{l}^n(\tilde{\mathscr{E}}^u(\tilde{y}'))) \to 0 \quad \text{as } n \to -\infty.$$

However, this contradicts Claim 4.10.

Finally if Claim 4.9.3 failed, we could find $x, y \in \Omega(g)$ with

$$\chi_g(x) > 0 > \chi_g(y).$$

But then $\widetilde{W}^0_g(x)$ and $\widetilde{W}^1_g(y)$ would be transverse to $\mathscr{F}^u(x)$, contradicting the above. □

Step 4: Entropy calculations. Recall that in the proof of Claim 4.9 we saw that entropy was concentrated on the set $\Omega(g)$ in the sense that for ν_1 and ν_2 defined by

$$\nu_1(A) = \frac{\mu(A \cap \Omega(g))}{\mu(\Omega(g))} \qquad \nu_2(A) = \frac{\mu(A \smallsetminus \Omega(g))}{\mu(\mathbb{T}^2 \smallsetminus \Omega(g))}$$

we have $h_{\nu_2}(g) = 0$. In particular

$$h_\mu(g) = \mu(\Omega(g)) h_{\nu_1}(g) + \mu(\mathbb{T}^2 \smallsetminus \Omega(g)) h_{\nu_2}(g) = \mu(\Omega(g)) h_{\nu_1}(g).$$

From Claim 4.9 and (3.3) it follows that

$$h_\mu(g) = \left| \int \chi_g(x) \delta^u \, d\mu(x) \right| = |\overline{\chi}(g)| \delta^u. \tag{4.10}$$

Proposition 4.7 follows by showing that $\overline{\chi}$ is a homomorphism from $(\mathrm{Diff}^r(\mathbb{T}^2;\mu), \circ)$ to $(\mathbb{R}, +)$:

$$\overline{\chi}(g_1 \circ g_2) = \overline{\chi}(g_1) + \overline{\chi}(g_2). \tag{4.11}$$

Indeed, for μ-a.e. x we have

$$\chi_g(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(J_g(g^i(x)))$$

and

$$\overline{\chi}(g) := \int \chi_g = \int \mathbb{E}(\log(J_g) \mid \mathscr{I}_g) = \int \log(J_g)$$

whence for $g_1, g_2 \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$

$$\begin{aligned}
\overline{\chi}(g_1 \circ g_2) &:= \int \chi_{g_1 \circ g_2} \\
&= \int \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\Big(J_{g_1 \circ g_2}\big((g_1 \circ g_2)^i(x)\big)\Big) \\
&= \int \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Big(\log\Big(J_{g_1} \circ g_2\big((g_1 \circ g_2)^i(x)\big)\Big) + \log\Big(J_{g_2}\big((g_1 \circ g_2)^i(x)\big)\Big)\Big) \\
&= \int \mathbb{E}(\log(J_{g_1} \circ g_2) \mid \mathscr{I}_{g_1 \circ g_2}) + \mathbb{E}(\log(J_{g_2}) \mid \mathscr{I}_{g_1 \circ g_2}) \\
&= \int \log(J_{g_1} \circ g_2) + \int \log(J_{g_2}) \\
&= \overline{\chi}(g_1) + \overline{\chi}(g_2).
\end{aligned}$$

Thus the proposition follows. □

We note that Proposition 4.7 establishes the first assertion in Theorem 4.1. Furthermore,

if $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ contains no positive entropy diffeomorphism, then Theorem 4.1 follows. We now prove the theorem under the assumption that there exists an element $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ with $h_\mu(g) > 0$.

Proof of Theorem 4.1. Note that for any two lifts of $h \circ g \circ h^{-1}$ to $\mathbb{R}^2$ the induced affine maps on $\mathbb{R}^2/\tilde{\mathscr{E}}^u$ established in Claim 4.10 differ only by a translation. For $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ we write $\Psi(g)$ for the linear component of the affine map on $\mathbb{R}^2/\tilde{\mathscr{E}}^u$ induced by a lift of $h \circ g \circ h^{-1}$.

Note that for $g_1, g_2 \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$, if the associated linear maps $\Psi(g_1)$ and $\Psi(g_2)$ are equal then the linear map $\Psi(g_2 \circ g_1^{-1})$ associated to the composition $g_2 \circ g_1^{-1}$ is the identity. Thus, all Lyapunov exponents for the composition $g_2 \circ g_1^{-1}$ whose associated subspaces are transverse to $\mathscr{F}^u(x)$ are zero. In particular $\mu(\Omega(g_2 \circ g_1^{-1})) = 0$ and $h_\mu(g_2 \circ g_1^{-1}) = 0$. By Proposition 4.7 we have $h_\mu(g_1) = h_\mu(g_2)$. Thus Theorem 4.1 reduces to studying the linear maps $\Psi(g)$ for $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$.

For $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$, fix a lift $\tilde{l}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the homeomorphism $h \circ g \circ h^{-1}\colon \mathbb{T}^2 \to \mathbb{T}^2$. Let $v = \tilde{l}(0)$. Then the map $x \mapsto \tilde{l}(x) - v$ preserves the lattice $\mathbb{Z}^2$ and the linear foliation $\tilde{\mathscr{E}}^u$. Furthermore the linear map induced by $x \mapsto \tilde{l}(x) - v$ on $\mathbb{R}^2/\tilde{\mathscr{E}}^u$ is equal to $\Psi(g)$. Note that the restriction of $x \mapsto \tilde{l}(x) - v$ to $\mathbb{Z}^2$ is a homomorphism. We let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the unique linear map whose restriction $L\restriction_{\mathbb{Z}^2}$ is equal to $(x \mapsto \tilde{l}(x) - v)\restriction_{\mathbb{Z}^2}$. By the density of leaves of $\mathscr{E}^u$ on $\mathbb{T}^2$ the linear action induced by L on $\mathbb{R}^2/\tilde{\mathscr{E}}^u$ is also equal to $\Psi(g)$.

Recall the definition of A in (4.6). We show that L and A commute. Indeed, L and A commute on the 1-dimensional linear space $\tilde{\mathscr{E}}(0)$. Furthermore, since the one-dimensional subgroup $\mathscr{E}([0])$ is dense in $\mathbb{T}^2$, the actions induced by L and A on $\mathbb{T}^2$ commute. Since LA and AL lift the same map of $\mathbb{T}^2$, and since they agree at 0, we have $LA = AL$.

It is well known[8] that the centralizer of A in $\mathrm{GL}(2,\mathbb{Z})$ is of the form

$$C(A) = \{\pm M^n \mid n \in \mathbb{Z}\}$$

for some hyperbolic matrix M. Hence $L = \pm M^n$ for some n; in particular, for any $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ the linear map $\Psi(g)$ is equal to the map induced on $\mathbb{R}^2/\tilde{\mathscr{E}}$ by the matrix $\pm M^n$ for some $n \in \mathbb{Z}$.

[8] See, for example, [BR97]).

As a consequence, we obtain that there is a smallest positive entropy for all diffeomorphism in $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$. Indeed for any $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ with $h_\mu(g) > 0$ we have that $\Psi(g)$ is equivalent to the map induced by $\pm M^n$ for some n, thus a (non-strict) lower bound on the entropy of any positive entropy map in $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ is

$$\frac{1}{|n|} h_\mu(g).$$

We check the above lower bound is in fact independent of the choice of g. Let g' be such that $h_\mu(g') > 0$ and $\Psi(g')$ is equivalent to the map induced by $\pm M^{n'}$ for some n'. Then we have $h_\mu((g')^{2n}) = h_\mu(g^{2n'})$ and hence

$$\frac{1}{|n|} h_\mu(g) = \frac{1}{|n'|} h_\mu(g').$$

It then follows that the set

$$\{h_\mu(g) \mid g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)\}$$

is discrete. Indeed if there were an accumulation point, Proposition 4.7 would provide arbitrarily small positive entropies.

Let λ denote the smallest positive entropy attained by any map in $\mathrm{Diff}^r(\mathbb{T}^2;\mu)$ and let $g \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ be such that $h_\mu(g) = \lambda$. To complete the proof we show that the image of g generates the group $\mathrm{Diff}^r(\mathbb{T}^2;\mu)/N$. Indeed, suppose there is a $g' \in \mathrm{Diff}^r(\mathbb{T}^2;\mu)$ with $g' \neq g^n \circ l$ for any $n \in \mathbb{Z}$ and $l \in N$. By (4.7) we have $h_\mu(g') \neq k\lambda$ for any $k \in \mathbb{N}$. In particular, there is a $k \in \mathbb{N}$ such that $h_\mu(g^k) < h_\mu(g') < h_\mu(g^{k+1})$ whence we obtain either

$$0 < h_\mu(g^{-k} \circ g') < h_\mu(g)$$

or

$$0 < h_\mu(g^{-k} \circ (g')^{-1}) < h_\mu(g)$$

from (4.7). This contradiction establishes the second statement in Theorem 4.1. □

4.4 PROOF OF THEOREMS 4.2 AND 4.2'

We begin with a claim that reduces Theorems 4.2 and 4.2' to the case of affine transformations. Recall that we identify the torus $\mathbb{T}^n$ with the quotient group $\mathbb{R}^n/\mathbb{Z}^n$. We write $[x]$ for the equivalence class of x in $\mathbb{T}^n$. For $B \in \mathrm{GL}(n,\mathbb{Z})$ we write L_B for the induced map on

$\mathbb{T}^n$ and for $v \in \mathbb{R}^n$ we write $T(v)$ for the toral translation $[x] \mapsto [x+v]$.

By a *k-dimensional linear foliation* $\mathscr{E}$ of the torus, we mean the partition of $\mathbb{T}^n$ by cosets of H, where H is a connected k-dimensional subgroup of $\mathbb{T}^n$. We say a linear foliation is *irrational* if each leaf $\mathscr{E}([x])$ is dense in $\mathbb{T}^n$ and is the injective image of $\mathbb{R}^k$.

Claim 4.11. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be k_1- and k_2-dimensional, irrational linear foliations of $\mathbb{T}^n$ with $1 \le k_i$, $k_1 + k_2 = n$ and such that $\mathscr{E}_1([0]) \cap \mathscr{E}_2([0])$ contains no 1-dimensional subgroups. Let $g\colon \mathbb{T}^n \to \mathbb{T}^n$ be a homeomorphism preserving the foliations $\mathscr{E}_j$. Then g is affine; that is, there are $B \in \mathrm{GL}(n,\mathbb{Z})$ and $v \in \mathbb{R}^n$ such that*

$$g = T(v) \circ L_B.$$

Proof. Let $\tilde{g}$ be any lift of g to $\mathbb{R}^n$, let $v = \tilde{g}(0)$ and set $\bar{g}\colon x \mapsto \tilde{g}(x) - v$. Then $\bar{g}\restriction_{\mathbb{Z}^n}$ is a homomorphism. Write $\tilde{\mathscr{E}}_j$ for the lifts of the foliations to $\mathbb{R}^n$. We note that $\tilde{\mathscr{E}}_1(x) \cap \tilde{\mathscr{E}}_2(y)$ contains exactly one point for each $x, y \in \mathbb{R}^n$ and the set

$$\Xi := \{\tilde{\mathscr{E}}_1(n) \cap \tilde{\mathscr{E}}_2(m) \in \mathbb{R}^n \mid n, m \in \mathbb{Z}^n\}$$

is dense in $\mathbb{R}^n$. We check the following:

1. *Ξ is closed under addition in $\mathbb{R}^n$.* Indeed if $x = n + v_1 = m + v_2$ and $y = n' + v_1' = m' + v_2'$ for $v_i, v_i' \in \tilde{\mathscr{E}}_i$ then

$$x + y = (n+n') + (v_1 + v_1') = (m+m') + (v_2 + v_2') \in \tilde{\mathscr{E}}_1(n+n') \cap \tilde{\mathscr{E}}_2(m+m').$$

2. *Ξ is $\bar{g}$-invariant.* Indeed, note that $\bar{g}$ preserves leaves of the foliations $\tilde{\mathscr{E}}_j$ thus with the above notation

$$\bar{g}(x) \in \tilde{\mathscr{E}}_1(\bar{g}(n)) \cap \tilde{\mathscr{E}}_2(\bar{g}(m)).$$

3. *For $x, y \in \Xi$ we have $\bar{g}(x+y) = \bar{g}(x) + \bar{g}(y)$.* Indeed

$$\begin{aligned}\bar{g}(x+y) &\in \tilde{\mathscr{E}}_1(\bar{g}(n+n')) \cap \tilde{\mathscr{E}}_2(\bar{g}(m+m')) \\ &= \tilde{\mathscr{E}}_1(\bar{g}(n) + \bar{g}(n')) \cap \tilde{\mathscr{E}}_2(\bar{g}(m) + \bar{g}(m')) \\ &= \tilde{\mathscr{E}}_1(\bar{g}(n)) \cap \tilde{\mathscr{E}}_2(\bar{g}(m)) + \tilde{\mathscr{E}}_1(\bar{g}(n')) \cap \tilde{\mathscr{E}}_2(\bar{g}(m')).\end{aligned}$$

By the continuity of $\bar{g}$ and density of Ξ it follows that $\bar{g}$ is linear and the claim holds. □

Proof of Theorems 4.2 and 4.2'. We prove both theorems simultaneously. By Proposition 4.4 and Proposition 4.3, respectively, for any $r \geq 1$, any $g \in \mathrm{Diff}^r(\mathbb{T}^2; \mu)$ satisfying the hypotheses of Theorem 4.2 and any $g \in \mathrm{Diff}^r(\mathbb{T}^2; \{\mu, \nu\})$ satisfying the hypotheses of Theorem 4.2' preserve both foliations $\mathscr{F}^s$ and $\mathscr{F}^u$. Write

$$\Gamma = \begin{cases} \mathrm{Diff}^r(\mathbb{T}^2; \mu) & \text{in Theorem 4.2,} \\ \mathrm{Diff}^r(\mathbb{T}^2; \{\mu, \nu\}) & \text{in Theorem 4.2'.} \end{cases}$$

Recall that we have $A \in \mathrm{GL}(2, \mathbb{Z})$ and $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $L_A \circ h = h \circ f$. For any $g \in \Gamma$ we have that $h \circ g \circ h^{-1}$ preserves the linear stable and unstable foliations induced by the dynamics of L_A. These foliations satisfy the hypothesis of Claim 4.11, whence we conclude that $h \circ g \circ h^{-1} = T(v) \circ L_B$ for some $B \in \mathrm{GL}(2, \mathbb{Z})$ and $v \in \mathbb{R}^2$. We note that L_B preserves the unstable foliation of $\mathbb{T}^2$ induced by the dynamics of L_A. Arguing as in the proof of Theorem 4.1, the density of leaves of the (1-dimensional) unstable foliation for L_A, implies that L_A and L_B, and hence A and B, commute.

Note that in the case of Theorem 4.2', one of μ or ν is not the measure of maximal entropy; we assume that μ is this measure. In the case of either theorem write

$$H := \{[v] \in \mathbb{T}^2 \mid T(v)_*(h_*(\mu)) = h_*(\mu)\}.$$

Claim 4.12. *H is finite.*

Proof. Recall that $B \in \mathrm{GL}(n, \mathbb{Z})$ is said to be *irreducible* if all L_B-invariant, closed proper subgroups of $\mathbb{T}^n$ are finite. We verify that $H \subset \mathbb{T}^2$ is a closed L_A-invariant subgroup. That $H \subset \mathbb{T}^2$ is a subgroup follows by definition. We claim that H is a closed. Indeed if $[v_j] \in H$ with $v_j \to w$, then for any continuous $\phi\colon \mathbb{T}^2 \to \mathbb{R}$ we have

$$\int \phi(x)\ d(T(v_j)_* h_*(\mu))(x) = \int \phi(x + v_j)\ d(h_*(\mu))(x) \xrightarrow{j \to \infty} \int \phi(x + w)\ d(h_*(\mu))(x)$$

where the last equality follows from dominated convergence and that $\phi(x + v_j) \to \phi(x + w)$ pointwise. Thus $\int \phi(x)\ d(T(w)_* h_*(\mu))(x) = \int \phi(x)\ d(h_*(\mu))(x)$ for any continuous ϕ. This shows $h_*(\mu)$ is $T(w)$-invariant. Finally, we note that H is L_A-invariant since $T(Av) = L_A \circ T(v) \circ L_A^{-1}$.

Since the matrix A is irreducible, if H were infinite we would have $H = \mathbb{T}^2$ which would

imply $h_*(\mu)$ is the Haar measure on $\mathbb{T}^2$. However, it is well known that the only f-invariant measure μ for which $h_*(\mu)$ is the Haar measure is the measure of maximal entropy. We thus conclude that H is finite. □

Write $C(A)$ for the centralizer of A in $\mathrm{GL}(2,\mathbb{Z})$. Again we have that $C(A)$ is of the form

$$\{\pm M^n \mid n \in \mathbb{Z}\}$$

for some hyperbolic matrix M. Replacing M with $-M$ if needed we may find a k such that

$$h \circ f \circ h^{-1} = L_A = L_{M^k}.$$

Then for any $g \in \Gamma$ we may find $v \in \mathbb{R}^2$, $l \in \mathbb{Z}$, and $\sigma \in \{-1, 1\}$ such that

$$h \circ g \circ h^{-1} = T(v) \circ L_B = T(v) \circ L_{\sigma M^l}. \tag{4.12}$$

We calculate that for g as above

$$h \circ g \circ f \circ g^{-1} \circ f^{-1} \circ h^{-1}([x]) = \left[\sigma M^l (M^k (\sigma M^{-l} (M^{-k}(x) - v))) + v\right] = \left[x - M^k v + v\right]$$

thus

$$g \circ f \circ g^{-1} \circ f^{-1} = h^{-1} \circ T(v - M^k v) \circ h$$

and $v \in (L_{I-M^k})^{-1} H$ where L_{I-M^k} denotes the toral endomorphism induced by $I - M^k \colon \mathbb{R}^2 \to \mathbb{R}^2$. In particular, v has rational coordinates.

Now, if for every $g \in \Gamma$, the corresponding l in (4.12) is zero, it follows that the group Γ is finite and the conclusion of each theorem follows with $m = 0$. Indeed in this case we have that $h \circ g \circ h^{-1} \in H$ for every $g \in \Gamma$.

We thus assume the existence of a $g \in \Gamma$ with infinite order and derive the remainder of the result. Fix an infinite order $g \in \Gamma$ and corresponding B, v, σ and $l \neq 0$ as in (4.12). We claim that the orbit of $[0]$ under the map $T(v) \circ L_B$ is finite. Indeed, let $v = (v_1, v_2) \in \mathbb{Q}^2$ and let D denote the least common denominator of v_1 and v_2, when written in lowest terms. Let Σ denote the set of rational points $(p, q) \in \mathbb{Q}^2$ such that the least common denominator of p and q, when written in lowest terms, is at most D. Since B has integer entries, we have that $\Sigma \subset \mathbb{R}^n$ is invariant under the linear transformation B; furthermore Σ is invariant under the the translation $x \mapsto x + v$. Furthermore, Σ is $\mathbb{Z}^2$-periodic and discrete and thus descends to

a finite set $\bar{\Sigma} \subset \mathbb{T}^2$. Hence the map $[x] \mapsto [Bx + v]$ is a permutation of a finite set $\bar{\Sigma}$.

Thus we may find a j such that $(T(v) \circ L_B)^j([0]) = [0]$. We check from (4.12) that

$$h \circ g^m \circ h^{-1}(x) = L_{(\sigma B)^m}(x) + (T(v) \circ L_{\sigma B})^m([0])$$

for $x \in \mathbb{T}^2$, whence

$$h \circ g^{2jk} \circ h^{-1}(x) = L_{M^{2ljk}}(x) + (T(v) \circ L_B)^{2jk}([0]) = h \circ f^{2jl} \circ h^{-1}(x).$$

In particular, setting $m = 2jl$ we have $f^m \in \Gamma$. Note this follows even in the case $r > \theta$.

Write

$$\Gamma' := \{h \circ \gamma \circ h^{-1} \mid \gamma \in \Gamma\}$$

and $\Sigma \subset \mathbb{T}^2$ for the smallest L_M invariant subgroup containing $(L_{I-M^k})^{-1}H$. Note that since $\det(I - M^k) \neq 0$ and H is finite, we have that $(L_{I-M^k})^{-1}H$ is finite. We note that for a matrix M with integer entries, the orbit under L_M of a point with rational coordinates is finite; thus Σ is finite.

Write

$$G = C(A) \ltimes \Sigma$$

with multiplication

$$(B, [v]) \cdot (B', [v']) = (BB', [Bv' + v]).$$

We abuse notation and identify $T(v) \circ L_B \in \Gamma'$ with $(B, [v]) \in G$ whence we obtain a natural inclusion of subgroups

$$\langle M^{mk} \rangle \subset \Gamma' \subset G.$$

Since Σ is finite, G contains $\langle M^{mk} \rangle$ as a finite index subgroup. Consequently, Γ' contains $\langle M^{mk} \rangle$ as a finite index subgroup and the conclusion follows. □

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