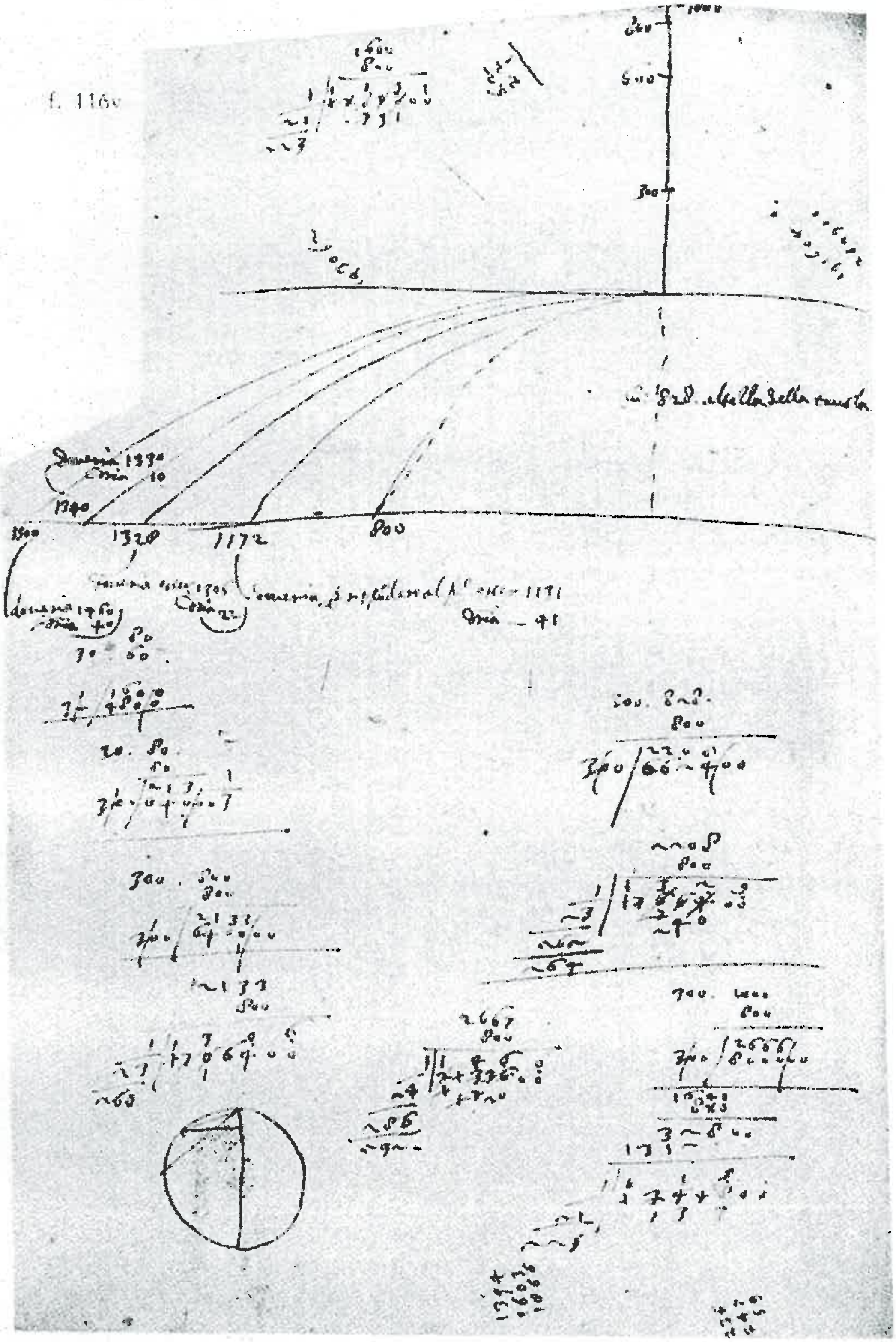
f. 116v



Galileo's "Ski-Jump" Experiment
Folio 116v

Fourth Day, Prop. 5, Corol.: Hence it follows that half the base, or amplitude, of the semi-parabola is a mean proportional between its altitude and the sublimity from which a falling body will describe this parabola.

i.e. $a = 2\sqrt{(hp)}$

sublimity p	altitude h	mean proportional	theoretical a	Galileo's measured a	% difference
300	828	498	996	800	19.7
600	828	705	1410	1172	16.9
800	828	814	1628	1328	18.4
828	**828**	**828**	**1656**	**1340**	**19.1**
1000	828	910	1820	1500	17.6

With p from rolling sphere in a groove of width = 4/9 diameter of sphere, % difference should be 18.3.

(The electronic version provides precise connections with cross-references.) Many folios are filled with computations. A number of folios, 80r, 81r, 86a r, 87, 90r, 91v, 102, 107, 111, 113r, 114v, 115r, 116v, 117, 152r, and 175v among them, contain diagrams and data that suggest studies of motion. The abbreviations r and v stand for "recto" and "verso", the "front" and "back" of the sheet in question. (In the listing just given, if neither r nor v appears, then both sides of the sheet are relevant.) Some of these folios are geometric explorations of the parabola and some are records of experiments. The historians who have studied them consider it "well substantiated by the evidence" (watermarks, for example) that they stem from the later Paduan period 1604–1610. For example, see Naylor [20, p. 366].

The present article will focus on 81r, 114v and 116v. Each of these folios gives evidence of an experiment in which Galileo has placed an inclined plane on a table, lets a ball roll down the plane, and records quantitative data about the ball's flight from the table's edge to the floor. Salviati informs us on the third day of the *Discorsi* that Galieo repeated some of his experiments "a full hundred times." Thus it would seem that each recorded measurement represents a cluster of trials. The general conclusions of Drake [12, 27, 32, 36, 37], Drake-MacLachlan [16], Naylor [13, 18, 19, 20, 25, 26, 28], and Hill [33, 35] – these are the historians who have studied them most thoroughly – are in agreement:

Drake [32, p. 4] uses folios 81r and 114v to conclude that Galileo is a "skilled experimentalist capable of holding his results within a variance of four units ... ." The unit referred to here is Galileo's *punto*, or "point", a unit of length slightly less than one millimeter.

Naylor [18, pp. 168–169], reflecting about 81r, speaks of "indications that Galileo carried out meticulous, thorough-going studies of the form of projectile motion" and suggests that "Galileo had a striking talent for combining a mathematical approach to nature with a considerable mathematical technique. The simplicity and power of this particular form of experiment is quite remarkable."

Hill [35, p. 666] comments that "worksheets 81r, 114v, and 116v reveal an impressive experimental program, ingenious in structure, ambitious in concept, eminently successful in execution. This series of procedures enabled Galileo to provide powerful, perhaps empirically decisive, evidence for both the new speed law and the parabolic trajectory."

It is a fact that Galileo's record of the experiments on these folios omits important details, in reference to both the descriptive and numerical elements. Thus, an important ingredient in the studies of these folios has been the careful reconstruction of the experiments from the information that Galileo does supply. These reconstructions – both actual and mathematical – become an important part of the evidence. The numerical data that they generate is carefully compared with the analogous data from Galileo's record. These comparisons are used to inform the authors' comments about the plausibility of their reconstructions and the validity of their analyses of the experiments. Unfortunately, in terms of particulars (for example, the inclination of the inclined plane and release heights of the balls), these reconstructions as well as the conclusions drawn from them – specifically as to the purpose and precision of the experiments – differ widely.

This state of affairs calls for a sober re-examination of these folios. What aspects of his insights about motion did Galileo put to the test? How precise were his experiments? What conclusions can legitimately and compellingly be drawn from Galileo's record of them? Is there indeed convincing evidence that they were successful? The answer to these questions is the purpose of the discussion that follows. The focus will be on the folios themselves (rather than the reconstruction of the experiments) and on related aspects of the *Discorsi*. The folios 116v, 81r, and 114v and all the information on them are reproduced below. The originals can be studied at either of the websites listed above. The organization of the calculations on 116v and 114v into rectangular "frames" follows the practice of the websites.

Archive for History of Exact Sciences 56 (2002), 339–361

3. The experiment of folio 116v

The statement *punti 828 altezza della tavola* tells us that Galileo recorded distances in units he calls *punti* (that is to say "points") and that he had a table 828 punti high. There is agreement among the historians already mentioned (based on evidence from folio 166r) that one *punto* is equal to approximately 0.94 millimeters. The diagram together with the computations on the folio confirm that he placed an inclined plane on the table, fixed an angle of inclination, and released a ball (likely of bronze) from the respective heights h of

$$300,\ 600,\ 800,\ 828,\ \text{and}\ 1000$$

punti above the horizontal table top. Galileo might have made use of a curved deflector to provide a smooth transition for the ball from the inclined plane to the horizontal table. His sketch on folio 175v shows that he considered such deflectors. After a short run on the table, the ball flew off to land on a horizontal floor. Galileo measured the respective distances from the point of impact of the ball to the base of the table (the point directly below the start of the ball's flight) and recorded these on the folio as

$$800,\ 1172,\ 1328,\ 1340,\ \text{and}\ 1500 \qquad \text{(a)}$$

punti. These are the experimental values that correspond to the various heights of release listed above.

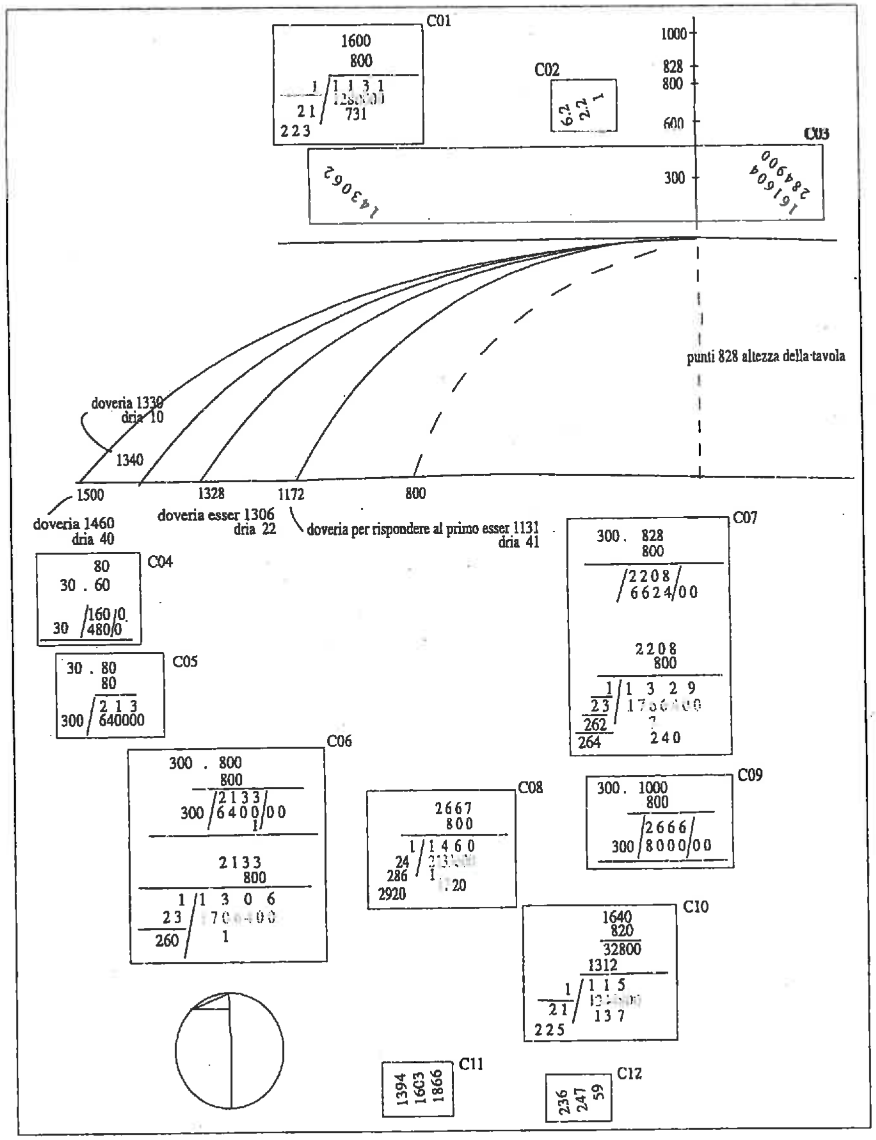
3A. *Understanding the folio*

We now turn to the analysis of the experiment as well as the computations that Galileo carried out. Consider the ball in its initial position on the inclined plane. Let

h = the height of the ball above the table, and
d = the distance from the ball to the bottom of the inclined plane.

Now release the ball and let

t = the time it takes for the ball to descend to the bottom of the plane,
v = the speed of the ball at time t. This is also the speed of the ball at the beginning of its fall from the table. Finally, let
R = the distance from the point of impact of the ball to the point on the floor precisely below the starting point of the ball's flight.



Folio 116v (size of original: 306 by 207 mm)

At the time of the experiment – before the end of the Paduan period in 1610 – Galileo had discovered, or at least wrestled with, all essential aspects of his program on motion as outlined in Sect. 1 above. In particular, he was in a position to put to the test the proportion

$$v \propto t \tag{i}$$

as well as the square law

$$d \propto t^2 \tag{ii}$$

(deduced from (i) in Proposition II. Theorem II of the *Discorsi*). From his principle of inertia he could assume that the horizontal component of the velocity is constant throughout the ball's flight and hence equal to v. (Given the relatively small velocities, distances and times, Galileo could safely assume that air resistance would not play a significant role. See [19, p. 408].) In reference to the vertical component of the ball's flight, Galileo knew that the time of fall of the horizontally projected ball from the table to the floor is independent of its starting velocity v. So this time is equal to the time t_0 that it takes for a ball to fall vertically from rest through the height of the table. Notice that these observations rely on the principle of superposition. Galileo can conclude that

$$R \propto v \tag{iii}$$

with t_0 the constant of proportionality. By similar triangles (the angle of inclination of the inclined plane is fixed) $h \propto d$. After putting the above proportions together, Galileo has

$$h \propto d \propto t^2 \propto v^2 \propto R^2 \ . \tag{iv}$$

Therefore, $R^2 \propto h$. So, if releases of the ball at the heights of h_0 and h above the table result in points of impact at the respective distances of R_0 and R from the foot of the table, then

$$\frac{R^2}{R_0^2} = \frac{h}{h_0} \ . \tag{v}$$

It is this relationship that the experiment recorded on folio 116v is designed to confirm. Galileo's next step is to insert the values $h_0 = 300$ and $R_0 = 800$ from the experiment. By doing so, he in effect determines, or at least approximates, the constants of proportionality that link R^2 and h, or equivalently, R and $\sqrt{h}$. The equation

$$R = \frac{800}{\sqrt{300}}\sqrt{h} \ . \tag{vi}$$

captures what he does. It remains for Galileo to compute R for h successively equal to 600, 800, 828, and 1000, and to compare the resulting values with the measurements for R that were provided – see (a) – by the experiment. The successive values for R that Galileo computes are (in punti)

$$—, \ 1131, \ 1306, \ 1330, \text{ and } 1460 \ . \tag{b}$$

The — refers to the value $R = 800$ that was used along with the corresponding $h = 300$ to obtain (vi).

Galileo records these numbers on the folio with the phrase *doveria esser* (or simply *doveria*) meaning "ought to be." He also includes his calculations. For example, the calculation for $h = 600$ is carried out in frame C01. Galileo first computes

$R^2 = \frac{800 \cdot 800 \cdot 600}{300} = 1600 \cdot 800 = 1280000$ and calculates $R = \sqrt{1280000} = 1131$. For $h = 800$, this is done in C06. For $h = 1000$, the computation can be seen in frames C09 and C08. In C09, Galileo computes $1000 \times 800 = 800000$ and divides this result by 300 to get 2666. In C08, he multiplies the more accurate value 2667 of this computation (the actual value is $2666\frac{2}{3}$) by 800 to get 2133600. This is R^2. To get R, he calculates $\sqrt{2133600} = 1460$. The computation in frame C10 is analogous to that of C01 and suggests that Galileo also considered a table height of 820 punti. Note that some of the computations are only approximations and that the computation $\sqrt{1344800} = 115$ in frame C10 is incomplete. In the course of computing the square root of a number, Galileo crosses the digits of the number out. In the rendition of the folio above these numbers are entered in a lighter shade.

Galileo compares his experimental values (a) to his theoretical values (b) and records the respective differences of 41, 22, 10, and 40 punti using the abbreviation *dria* for *differentia*. The fact that the theoretical values fall short of the experimental values (from about 1 to 4 centimeters) seems contrary to expectation. After all, the experimental values are subject to the retarding effects of the imperfections in Galileo's experimental setup, whereas the theoretical values are not. The explanation is provided by the fact that Galileo's theory, as captured by equation (vi), depends on one data point from the experiment. We will see, in particular, that the measured distance of 800 punti (corresponding to the height of 300 punti) falls short of the predicted mark. So the constant $\frac{800}{\sqrt{300}}$ is too small, and thus all of Galileo's computed values are too small as well.

We turn next to the question of the precision of the experiment of folio 116v. We will test the accuracy of the experimental values (a) against the predictions of elementary mechanics. (Galileo's theory can't be used because it depends on his experiment.) We will only outline these matters here. The details are available in many texts, for example, in Chap. 9.3 of the basic calculus text [42].[2] Note that the analysis that follows goes far beyond what Galileo was familiar with.

3B. *The underlying mathematics*

Return to the ball on the inclined plane and assume that it is perfectly homogeneous and spherical. Let $t = 0$ be the instant at which it is released. For any time $t \geq 0$, let $f(t)$ be the frictional force on the rolling ball (*a priori* it depends on t). This is the force that rotates the ball. Assume that there is neither slippage (as the ball would experience on a frictionless surface) nor any additional retardation of the motion down the plane (as would be the case if the surface were "bumpy" or "sticky"). The connection between the torque produced by the frictional force, the resulting angular acceleration of the ball, and the ball's index of inertia (this connection is provided by the rotational analogue of force = mass × acceleration), leads to the equation

$$f(t) = \frac{2}{5}ma(t)$$

[2] My interest in the experiments of Galileo had its beginning in my efforts to develop applications of calculus with interesting historical connections for this book.

where m is the mass of the ball and $a(t)$ is its linear acceleration down the plane. By Newton's second law and the fact that the component of gravity down the plane is $F = mg \sin\beta$, where β is the angle of inclination of the plane, we get

$$ma(t) = F - f(t) = mg \sin\beta - \frac{2}{5}ma(t) \ ,$$

and therefore,

$$a(t) = \frac{5g}{7}\sin\beta \ .$$

This informs us in turn that the velocity of the ball at the bottom of the plane is $v = \sqrt{\frac{10}{7}gh}$. (Alternatively, this equation can be established by using the law of conservation of energy. See [17, pp. 398–399].) Combining this with one of the basic equations of projectile motion and letting y_0 be the height of the table, provides the connection

$$R = 2\sqrt{\frac{5}{7}y_0}\sqrt{h}$$

between the starting height h and the distance R from the point of impact of the ball to the foot of the table. With the substitution $y_0 = 828$ this equation becomes

$$R = 2\sqrt{\frac{5}{7}828}\sqrt{h} \ . \qquad \text{(vii)}$$

Plugging Galileo's starting heights of 300, 600, 800, 828, and 1000 into Eq. (vii) for h, we get the values (again in punti)

$$842, \ 1191, \ 1376, \ 1400, \ \text{and } 1538 \qquad \text{(c)}$$

for the corresponding distances R.

This model applies to the ideal situation: a perfectly round and homogeneous ball; a path that is perfectly smooth and flat with no tilts other than the inclination of the plane; a force of friction that rotates the ball without slippage but provides no additional impedance; and a deflector that provides a perfectly smooth transition from the plane to the table. In addition, to conform to the situation of the model, the table as well as the floor on which the ball impacts need to be perfectly horizontal. There is, of course, no such perfection in the context of Galileo's experimental setup. In sum, the expectation is that the ball will land short of its theoretical target. A comparison of the lists of numbers (a) and (c) confirms this. We know, of course, from the discussion on the third day of the *Discorsi*, that Galileo is fully aware that his fundamental laws of motion apply only in idealized situations and that any experimental or real situation will encounter "impediments." Notice that the "bottom lines" of the analyses of Sects. 3A and 3B, namely the equations (vi) and (vii), differ only in the value of the constant, and that $\frac{800}{\sqrt{300}} \approx 46.19$ falls short of the correct value $2\sqrt{\frac{5}{7} \cdot 828} \approx 48.64$.

So far we have said nothing about the groove that guides the ball down the plane. The description of an inclined plane experiment in the *Discorsi* [2, Crew-Salvio p. 171, compare Drake p. 169] informs us that there was a channel "a little more than one finger in breadth" cut into the inclined plane, and that "having made this groove very straight, smooth, and polished, and having lined it with parchment, also as smooth and polished as possible, we rolled along it a hard, smooth, bronze ball ... ." The fact that Galileo says nothing specific about the groove presents a problem, because different configurations of the cross-section require different theoretical explanations. We now let d be the diameter of the ball and consider the most likely possibilities. If the cross-section of the groove is a circular arc of radius greater than the radius $\frac{d}{2}$ of the ball, then in the ideal situation, the ball will roll on the bottom of the groove throughout its descent. This is a situation to which the mathematical model already described applies. Asssume next that the groove has rectangular cross-section and let $w > 0$ be its width. If $d \leq w$, then the ball is supported by the bottom of the groove and rolls entirely within the groove. Again, the model already described applies. But if $d > w$ and the groove is deep enough, then the rolling ball does not touch the bottom of the groove and is instead supported by its two edges. In this case, the dynamics are different. The mathematical model of this situation (obtained by an analysis similar to that above) provides the relationship

$$R = 2\sqrt{\frac{y_0}{1+\frac{2}{5}\cdot\frac{d^2}{d^2-w^2}}}\,\sqrt{h}\,. \qquad \text{(viii)}$$

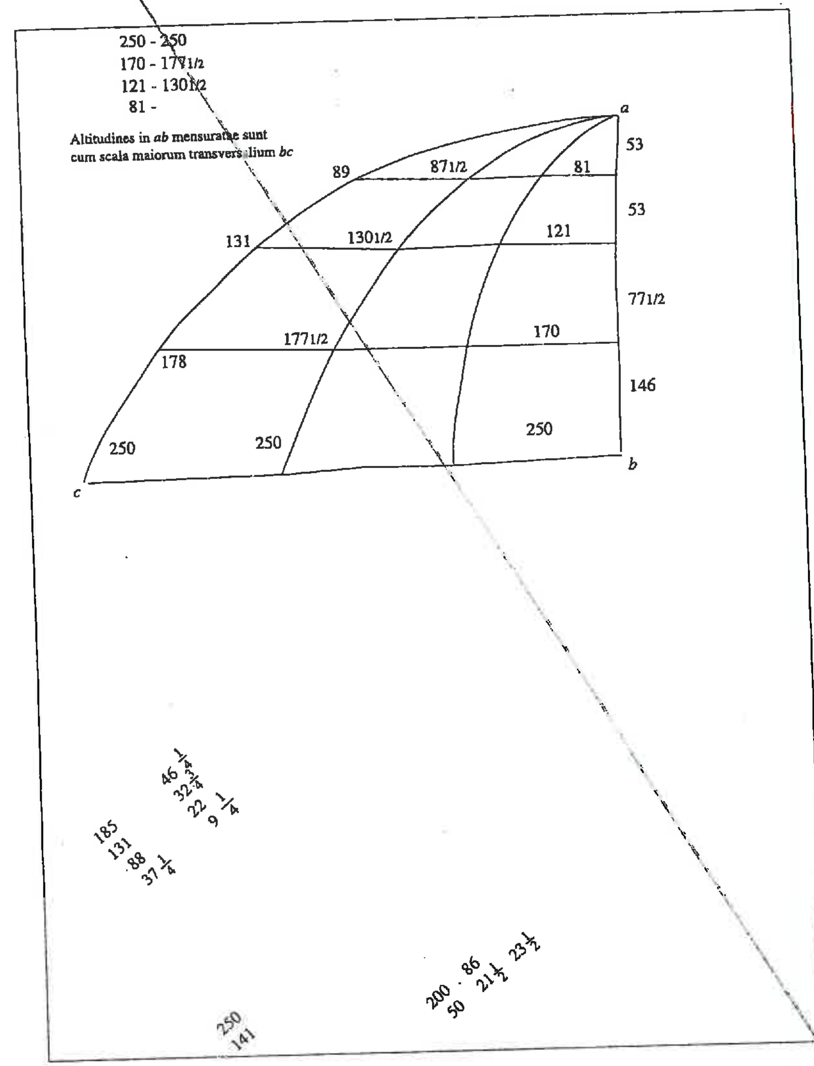
This equation also applies to a groove with a cross-section in the shape of an isosceles triangle, if w is taken to be the distance between the two points of contact of the ball with the groove. Let $y_0 = 828$ punti be the height of the table. Because $\frac{d^2}{d^2-w^2} > 1$, the value of equation (viii) is less than the value of equation (vii) for any $h > 0$. In particular, the values for R that equation (viii) supplies for the respective starting heights h equal to 300, 600, 800, 828, and 1000 are less than the values (c) supplied by equation (vii). Hence the values provided by equation (viii) will be closer to Galileo's experimental values (a).

Now to the comparison of Galileo's experimental data against the predictions of the theory. It follows from the analysis of the cross-section of the groove that the respective differences between the experimental data (a) and the predictions (c) are the largest possible. Therefore, in assessing the accuracy of the folio 116v experiment, these differences provide the worse case scenario. The differences are $-42 = 800 - 842$, $-19 = 1172 - 1191$, $-48 = 1328 - 1376$, $-60 = 1340 - 1400$ and $-38 = 1500 - 1538$ punti. In terms of percentages, this amounts to -5.0%, -1.6%, -3.5%, -4.3%, and -2.5%, respectively. What can be said about this discrepancy? While the inclined planes used by Galileo seem no longer to exist, we do know – see [34] for example – that the apparatus that Galileo used in other investigations was well crafted. The physicists Shea and Wolf [17], considering the many sources of possible experimental error in the folio 116v experiment, regard the data generated by Galileo to fall "within acceptable limits of experimental error." All indications are that this assessment is correct. For example, Naylor [13, pp. 109–111] reconstructed the folio 116v experiment with considerable care (the cross-section of the groove was a circular arc of radius greater than $\frac{d}{2}$) and obtained distance data very close to Galileo's.

4. The experiment of folio 81r

There is a consensus among historians – see [18], [35], and [37, Chap. 8] – that folio 81r focusses its attention on the trajectories of balls that are propelled obliquely into

space after having descended down an inclined plane placed on a table. In important contrast to folio 116v, the balls drop directly from the inclined plane and there is no horizontal deflection. Each of the three curves on the folio corresponds to a certain fixed angle of inclination of the plane and fixed starting height of the ball. In repeated trials Galileo intercepts the flight of the ball with horizontal planes placed at different heights and marks the points of impact. Evidently, he starts by placing the intercepting plane at a distance of $53 + 53 + 77\frac{1}{2} + 146 = 329\frac{1}{2}$ punti below the plane of the table and "calibrates" the three trajectories so that the the points of impact are at the respective horizontal distances of $250, 250+250 = 500$, and $250+250+250 = 750$ punti from the



Folio 81r (size of original: 304 by 205 mm)