e. But the latus rectum is just 2/A

f. Therefore, for any conic section trajectory governed by centripetal forces aimed at the focus, the force along the trajectory varies as $1/r^2$

4. The Scholium immediately following Problem 3 has mystified people ever since -- vide Wilson

"The major planets orbit, therefore [sic!], in ellipses having a focus at the centre of the Sun, and with their radii drawn to the Sun describe areas proportional to the times, exactly as Kepler supposed"

a. To infer that the area rule holds exactly, need to establish that the only forces acting on the planets are aimed to the center of the sun

b. To infer that the orbits are exactly elliptical, then need the converse of Problem 3 -- i.e. the problem Hooke originally wanted solved, and Newton told Halley he had solved: if a body is governed by inverse-square centripetal forces, then its only closed circuit trajectory is an ellipse

c. And even then need the claim that the centripetal force is inverse-square, something that has so far been shown for the planets only under the assumption of uniform motion in perfectly circular orbits

d. The seeming logical lacuna persists right through the first edition of the *Principia*, to be filled in the second edition following some pointed comments by Johann Bernoulli

5. My suspicion is that Newton was here engaging in a type of evidential reasoning (hence appropriate to a Scholium) that he employs widely (provoking much controversy) in Book III of the *Principia*

a. From the rough phenomena alone -- e.g. circular orbit -- can conclude from Theorem 1 and 2 that, at least to a first approximation, an inverse-square centripetal force aimed at the sun is the dominant factor keeping the planets in their orbits

b. Problem 3 shows that the next level of refinement of the phenomena, to a roughly elliptical orbit sweeping equal areas vis-a-vis the focus, requires no revision of that conclusion

c. Licenses the inference that, at the second level of approximation, the planets obey Kepler's first two rules exactly, for nothing beyond forces inferred from the first approximation is required for them to do so

6. Whether such evidential reasoning is legitimate -- or more to the point, exactly what conclusion is to be drawn from it -- I leave as an open question for now

a. But the interpretation I offer at least has the virtue of not saddling Newton with a glaring blunder in logic

b. I find the idea that Newton would fall trap to such a blunder in simple 'if-then' reasoning beyond belief

B. Theorem 4: The Keplerian 3/2 Power Rule

1. Theorem 4 provides a generalization of one half of Corollary 5 of Theorem 2: Kepler's 3/2 power rule holds for bodies in elliptical orbits governed by centripetal forces aimed at (a common) focus when those forces vary as $1/r^2$ -- i.e. P^2 varies as a^3

a. Contrast the inverse-square claim here with the one in Problem 3, where the claim applied only to variations along a single trajectory
b. Here the inverse-square claim is holding throughout the space with the focus at its center

2. Geometric proof achieved by piggy-backing the case of an ellipse on that of a circle, and then using Corollary 5 of Theorem 2
   a. Consider the special case in which P is at the end of the minor axis, and compare with the circle of radius SP in which P is governed by the same inverse-square force
   b. In the limit $L*QR = QT^2$ and $2*SP*MN = MV^2$
   c. Since forces are the same, QR = MN, $QT/MV = \sqrt{(L/2*SP)} = PD/2*SP$ since $AB=2*SP$ and $L=PD^2/AB$
   d. But then the area SPQ/area SPM = $QT/MV = PD/2*SP = \frac{1}{4}*\pi*AB*PD/\pi*SP^2$ = area of ellipse/area of circle
3. But then by the area rule the time of PQ/period of ellipse = the area SPQ/the area of the ellipse, and analogously for the time PM in the circle
   a. Hence, since the time of PQ = the time of PM, the total areas of the circle and the ellipse will be completed in the same times
   b. The Theorem then follows from Corollary 5 of Theorem 2
4. In sum, if bodies moving in elliptical orbits are governed as required, then Kepler's 3/2 power rule holds exactly
   a. Thus offers at least a qualified answer to the question whether Kepler's third "law" holds exactly, as Streete, following Horrocks, claimed, or only approximately
   b. Any deviation from Kepler's rule indicates that the elliptical orbits are not governed exactly as the Theorem supposes
   c. Licenses the inference that, at the second level of approximation, the planets (can) obey Keplerian motion exactly, for nothing beyond the forces inferred from the first (circular) approximation is required for them to do so
5. Theorem 4 also provides a basis for concluding that the 3/2 power rule is nomological -- i.e. it holds in all orbital systems governed by inverse-square centripetal forces
   a. Recall that Kepler's account made the 3/2 power rule parochially contingent on specific planet densities, and hence an accidental generalization
   b. Newton by contrast removes the main source of the parochialism, thereby opening the way for the 3/2 power rule to extend to bodies orbiting the earth
   c. If this extension can be confirmed by showing that the centripetal force toward the earth is inverse-square, then have a decisive argument against the Tychonic system of the sort I suggested Newton was pursuing in the late 1660s

C. Scholium: Determining the Orbits of Planets

1. The Scholium following Theorem 4 in effect proceeds under the assumption that the planetary orbits are perfectly Keplerian in all three respects
    a. It does not as such presuppose that the planets are governed solely by an inverse-square centripetal force aimed at the center of the sun
    b. But the clear suggestion is that proceeding under strict Keplerian assumptions has gained added warrant from the preceding propositions and attendant scholia
2. The problem is to determine the elliptical orbits from observations, given the periods and the length of one semi-major axis -- e.g. the earth's
    a. One solution for the outer planets and one for the inner
    b. Both of which adopt Horrocks's and Streete's approach of first inferring the length Q of the major axis from the periods, taking Kepler's third "law" to be exact
3. In the case of the outer planets, given observations at A, B, C, etc., draw arcs of radius Q-AS around A, Q-BS around B etc., and their intersection point will determine the other focus
    a. This has the advantage of allowing a large number of observations to be taken into account, taking the mean value for the focus, thereby allowing inaccuracies in individual observations to tend to cancel one another
    b. Distances AS, BS, etc. via a method Newton attributes to Halley that presupposes the orbit of the earth (in fact, a method from Kepler's *Astronomia Nova*, showing that Newton had not read it, but unless Halley came up with the method independently, he may may well been familiar with the Kepler book); for Halley reference see *Phil*. *Trans*., 25 Sep 1676 or Rigaud
    c. Iteration, using an orbit of a planet to correct the orbit of the earth by this method, and vice versa, until achieve good agreement on the focus will help eliminate residual errors
4. In the case of the inner planets, same general idea, but using observations at maximum elongation to define tangents to the orbit of the planet about the sun
    a. Drop a perpendicular from Sun to such a tangent, then draw a circular arc of radius ½*Q about the point where the perpendicular meets the tangent, and the center of the ellipse lies on this arc
    b. Has the same advantage of allowing a large number of observations to be taken into account, with iteration as needed on the earth's orbit
5. An example of the way in which Newton, the mathematician, is not at all reluctant to think like an experimental scientist, looking for ways to exploit redundant data
    a. Methods are not historically significant in their own right -- not a major advance -- and Newton himself will shortly be concluding that the orbits are not perfectly Keplerian
    b. But still indicative of his thinking
    c. Jed Buchwald has concluded that this is one of the first places in history where averaging of

redundant data has been used as preferable to selecting the "best" observation, which was (with one exception) the practice followed in astronomy by Tycho and others following him

D. Problem 4: Determining Inverse-Square Trajectories

1. Finally, offers a geometrical solution to the problem of determining an elliptical orbit, given a known inverse-square force and an initial velocity
    a. A geometrical solution in which the magnitude of the inverse-square force is represented by a perfectly circular orbit of given radius and period
    b. And the magnitude of the velocity is represented by the length of the tangent PR, where PR:$\rho\pi$ is taken in the ratio of the given velocity to the uniform velocity in the circle
2. Under these specifications, the latus rectum L of the ellipse is determined, making use of results in Problem 3
    a. But, given a focus and the tangent at P, know the direction of line PH on which the other focus H lies, and know the perpendicular distance to this line from S, SK
    b. A complicated series of algebra steps, exploiting properties of ellipses, then allows the length of PH to be determined, so that the other focus is established
    c. As is the transverse axis, for it is SP+PH
3. Newton here adds, without proof, that the construction will yield an ellipse so long as the initial velocity is not excessive, but it will yield a parabola when L= 2∗(SP+KP) and a hyperbola when the initial velocity is still greater -- i.e. when (SP+PH)/PH < 1
    a. In short, he fully realized that inverse-square centripetal forces yield different conic sections depending on the initial velocity, with lim(QT^2/QR) always L
    b. He may have even verified that the conic is uniquely determined -- no other curve can result
    c. Thus, the literal text of De Motu notwithstanding, he may have already solved Hooke's original problem, at least to his satisfaction, though we have no record of it now
4. As the attendant Scholium makes clear, the whole point of this problem is to allow the calculation of comet orbits under the assumption that comets are acted on by an inverse-square force directed toward the sun
    a. Can use planetary orbits to fix the strength of the inverse-square force "field" around the Sun (i.e. the strength of the inverse-square centripetal accelerative tendency around the Sun!): a^3/P^2
    b. And can obtain an estimate of the initial velocity from a sequence of observations curve-fitted to a straight line, though will have to iterate since do not know distance to the comet
    c. Specifically, take four observations to define a straight line, obtain velocity, now let inverse-square effect produce deviation from the straight line, then use comparison between observed and predicted longitude and latitudes to generate four new points, and repeat as needed
    d. Then use an approximate solution to Kepler's equation to determine the areal velocity

5. This method of determining comet orbits Newton soon discovers is inadequate, so that within weeks he has switched to another method, and even in the middle of 1686 he is still seeking an iterative method that will converge on a solution
    a. Notice how important he is taking the problem of comets to be and how resourceful he is in concocting approximate, iterative methods when faced with such a problem
    b. Notice also that Newton is openly speaking of the return of comets, via highly eccentric elliptical orbits -- something that may well have been fairly, if not totally, novel at the time
    c. Halley was the first to confirm this when in 1705 he used modified Newtonian methods to calculate the orbit of the comet of 1682, identifying it with the comet of 1531 and 1607 and predicting its return in 1758 (actually returned in 1759, after he died)
6. A further subtle, yet radical step has been taken in Theorem 4 and Problem 4 by tacitly concluding that any body at any point in space about a "force center" must experience an accelerative tendency toward this center of magnitude proportional to $[a^3/P^2]/r^2$
    a. What we now call an inverse-square centripetal acceleration field, with field strength (a^3/P^2)
    b. And no variation in either orthogonal angular direction
    c. De Motu now far removed from Huygens's restricted talk of force in his paper on uniform circular motion, though Huygensian in appealing to comets as a surprising, testable consequence
7. De Motu now also far removed from Newton's view of comets expressed to Flamsteed in 1681!

IV. Some Extended Results on Galilean Motion

A. Problem 5: Vertical Fall Under Inverse-Square Gravity

1. One virtue of the tenuous solution to the problem of determining comet orbits is that it points to a means of addressing another problem -- free-fall under inverse-square acceleration
    a. I.e. instead of uniform acceleration (along parallel lines), treat the case of inverse-square acceleration toward a center, with the goal of determining s versus t
    b. In effect, addressing the problem of fall to the center of the earth, including the special case of direct vertical fall, on the terms Hooke posed
2. Newton ends up providing a fairly simple geometrical solution to a rather nasty nonlinear differential equation: $(d^2r/dt^2 = -k/r^2)$
    a. Suppose body has a small initial velocity perpendicular to the vertical, so that (just as Hooke said), its trajectory will be an ellipse APB with focus at S, the center of the inverse-square centripetal force
    b. Circumscribe the circle ADB about this ellipse; then the time of fall is proportional to the area ASP, and hence to the area ASD
    c. As the initial perpendicular velocity approaches 0, ASD will continue to remain as the time, but in the process the orbit APB will approach AB, B will approach S, and area ABD, now = ASD, will be proportional to the time