DECOMPOSING CAT(0) CUBE COMPLEXES

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Abstract

By work of Caprace and Sageev ([CS11]), every finite-dimensional CAT(0) cube complex X has a canonical product decomposition $X = X_1 \times \cdots \times X_n$ into irreducible factors and any group G which acts on X must have a finite index subgroup which embeds in

$$\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_n).$$

We explore the natural followup questions, "If $G = G_1 \times \cdots \times G_n$ acts geometrically and essentially on a CAT(0) cube complex X, does X have product decomposition $X = X_1 \times \cdots \times X_n$? If so, how close is the action to a product action?"

In chapter 3, we answer this question when each G_i is a non-elementary hyperbolic group. We recover the canonical product decomposition $X = X_1 \times \cdots \times X_n$ and show that G has a finite index subgroup which acts with the product action on this decomposition of X.

In chapter 4, we create a generalization of the work in chapter 3 to the case when each G_i satisfies a property we call (AIP). Essentially, this property gives us a large degree of control over how locally maximal abelian subgroups can intersect. We again show that X has a decomposition $X = X_1 \times \cdots \times X_n$, though this is not the canonical decomposition. We also show that G has a finite index subgroup which acts with the product action on X.

In chapter 5, we consider groups of the form $\Gamma = G \times A$, where G satisfies (AIP) and A is free-abelian. We show that any cube complex X on which Γ acts geometrically and essentially decomposes as $X = X_G \times X_A$, and there is a finite index subgroup $G' \times A'$ of Γ which acts with a product action on $X_G \times X_A$.

To my wife, without whom this would not have been possible.

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Chapter 1

Introduction

A fundamental tenet of geometric group theory is, given a nice enough action of a group G on a metric space X, the coarse geometry of X and G are the same. A common proof technique is to take a group which is well-understood, find an action of that group on a space, and use the action to understand the space better. This thesis shows that if a nice enough direct product of groups acts on a CAT(0) cube complex, then the cube complex decomposes as a direct product of cube complexes and the action must be close to a product action.

A major inspiration for this thesis was the work of Caprace-Sageev in [CS11]. In the paper, they prove that every finite dimensional CAT(0) cube complex has a canonical decomposition as a product of cube complexes which is preserved, up to permutation of isomorphic factors, by any automorphism of the cube complex. In particular, this means that any group acting on a finite dimensional CAT(0) cube complex X with canonical decomposition $X_1 \times \cdots \times X_n$ must have a finite index subgroup that embeds in the product of the automorphism groups $\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_n)$. We have a discussion of some of the results from the paper in section 2.9.

Caprace-Sageev use information about how a cube complex decomposes to say something about a group acting on it. The question we asked is the reverse. Are there conditions on the action of a product of groups $G_1 \times G_2$ on a CAT(0) cube complex X that force X to decompose as a product $X_1 \times X_2$? Is this the same as the canonical decomposition of [CS11]? Is the action of $G_1 \times G_2$ on $X_1 \times X_2$ a product action? If not, is it close? We give an answer for finite products of non-elementary hyperbolic groups in our first theorem. A group action is *essential* if there are orbit points arbitrarily far from every hyperplane H in both of its halfspaces H^+ and H^-.

Theorem 3.1.1. *Let $G_1, \ldots, G_n$ be non-elementary hyperbolic groups. Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, and essentially, by cubical isometries* *on a CAT(0) cube complex X. Then*

- *X splits as a product of irreducible CAT(0) cube complexes $X = X_1 \times \cdots \times X_n$ and each $g \in G$ acts on $X_1 \times \cdots \times X_n$ as a product of isometries $\mu_1 \times \cdots \times \mu_n$;*
- *every G_i acts properly and cocompactly on X_i; and*
- *every G_i contains a finite-index subgroup G_i'' which acts trivially on every X_j for $i \neq j$.*

An important technique used in the proof combines ideas from [CS11] and [WW15] to control how highest free-abelian subgroups intersect. An abelian subgroup is *highest* if it does not contain a finite index subgroup that belongs to a higher rank abelian subgroup. The property we need in order to exert this control we call *(AIP)*, the abelian intersection property. A group satisfies (AIP) if it is infinite and contains a collection of highest abelian subgroups with trivial intersection. We prove a generalization of this first theorem in chapter 4.

Theorem 4.1.1. *Let $G = G_1 \times \cdots \times G_n$ satisfy (AIP) and have finite center, where each G_i is an infinite group. Suppose G acts properly, cocompactly, and essentially on a CAT(0) cube complex X. Then*

- *X decomposes as a product of CAT(0) cube complexes $X_1 \times \cdots \times X_n$ and each $g \in G$ acts on X as a product of isometries $\mu_1 \times \cdots \times \mu_n$;*
- *every factor G_i acts on X_i properly, cocompactly, and essentially; and*
- *every factor G_i contains a finite-index subgroup G_i' that acts trivially on G_j when $i \neq j$.*

In the case when $G_1 \times G_2$ satisfies (AIP), the action of $G_1 \times G_2$ on $X_1 \times X_2$ is very close to a product action. If we try to extend this to a group $G \times A$, where G satisfies (AIP) and A is free-abelian, the action can be farther from a product action. A *quasi-line* is a CAT(0) cube complex quasi-isometric to $\mathbb{R}$.

Theorem 5.2.1. *Let G be a group with finite center satisfying (AIP) and $A \cong \mathbb{Z}^p$. Suppose $\Gamma = G \times A$ acts properly, cocompactly, and essentially on a CAT(0) cube complex X. Then*

- *X decomposes as a product of CAT(0) cube complexes $X_A^\perp \times X_A$, where X_A is a product of p quasi-lines;*
- *Γ has a finite-index subgroup $\Gamma' = G' \times A'$ that acts on $X_A^\perp \times X_A$ as a product action; and*
- *furthermore, G' is isomorphic to a subgroup of G.*

An important point is that while G' is isomorphic to a finite index subgroup of G, its generators have been multiplied by elements of A in order to make its action on X_A trivial. The details can be found in chapter 5.

In [NR98], Niblo-Roller prove that there are CAT(0) groups which cannot act on finite dimensional CAT(0) cube complexes, and Bergeron-Wise proved in [BW12] that every closed, hyperbolic 3-manifold group acts properly and cocompactly on a finite-dimensional CAT(0) cube complex. However, it is possible that the cube complexes constructed by Bergeron-Wise may have a dimension much higher than 3. To measure the difference in complexity between the CAT(0) spaces on which these groups act and the CAT(0) cube complexes on which they act, we will refer to the *CAT(0) dimension* and the *CAT(0) cubical dimension* of a group. The CAT(0) dimension of a group is the minimum covering dimension of the CAT(0) spaces on which the group acts geometrically. The CAT(0) cubical dimension of a group is the minimum dimension of the CAT(0) cube complexes on which the group acts geometrically.

In [Li02], Tao Li found an infinite family of closed, hyperbolic 3-manifolds which are not homeomorphic to any 3-dimensional non-positively curved cube complex. We extend his result and apply Theorem 3.1.1 to get the following result.

Corollary 3.2.1. *Given any natural number k, there is an infinite family of groups which have CAT(0) cubical dimension at least k larger than their CAT(0) dimension.*

Chapter 2

Background

2.1 Introduction to CAT(0) Cube Complexes

We start by reviewing basic notions of cube complexes that we will need. An n-dimensional cube (n-cube) is an isometric copy of $[-\frac{1}{2}, \frac{1}{2}]^n$. We start with a collection $\mathcal{C}$ of cubes of various dimensions along with a collection $\mathcal{F}$ of isometries between their faces. A *cube complex* X is the quotient space $\mathcal{C}/\mathcal{F}$ of two such collections. For example, the number line with vertices (0-cubes) at each integer and edges (1-cubes) connecting adjacent integers is a cube complex. The integer lattice on $\mathbb{R}^2$ is a 2-dimensional cube complex. The *dimension* of a cube complex is defined to be the supremum of the dimensions of its cubes. In this thesis, we will assume that every cube complex is finite dimensional. That is to say, every cube complex will have a finite upper bound on the dimension of its component cubes.

A *local edge* is a subinterval of length $1/3$ of an edge of a cube in $\mathcal{C}$, one of whose endpoints is a vertex of $\mathcal{C}$. The local edges of X are images of the local edges of $\mathcal{C}$. The *link* of a vertex v, denoted $\mathrm{lk}(v)$, is a simplicial complex. Its vertices are the local edges of X, and a collection of vertices spans a simplex when the corresponding local edges are contained in the same cube of $\mathcal{C}$ and share a vertex.

A simplicial complex is *flag* if for every 1-skeleton of a k-simplex ($k \geq 2$), the corresponding k-simplex is in the complex. A cube complex is *non-positively curved* if the link of every vertex is a flag simplicial complex. For example, if the link of a vertex contains a triangle then the cube complex contains the 2-skeleton of a 3-cube attached to the vertex. Since the 3-cube is not filled in, this region of the cube complex will have positive curvature. A cube complex is *CAT(0)* if it is non-positively curved and simply connected.

The definition of CAT(0) for a cube complex given above is often more convenient when working with cube complexes, but the original definition of CAT(0) comes from metric geometry. Let T be a geodesic triangle in a geodesic metric space X with side lengths a, b, c. We can construct a *comparison triangle* $\overline{T}$ in $\mathbb{E}^2$ with the same side lengths. If x is a point in T, then the *comparison point* $\bar{x}$ is the point the same distance along the corresponding edge of $\overline{T}$ as x is along its edge. The triangle T satisfies the *CAT(0) inequality* if for every two points $x, y \in T$, the inequality

$$d_T(x,y) \leq d_{\overline{T}}(\bar{x},\bar{y})$$

holds. A geodesic metric space X is CAT(0) if every geodesic triangle satisfies the CAT(0) inequality.

A cube complex can be endowed with a path metric induced by the piecewise Euclidean metric of each cube. If we refer to a metric on a cube complex, it will be the path metric unless otherwise specified. It is a theorem of Gromov ([Gro87]) that the path metric is CAT(0) if and only if the cube complex is non-positively curved and simply connected.

There is a second commonly used metric for CAT(0) cube complexes that is induced by the piecewise-L_1 metric on each cube. If a subspace is convex under this metric, we call it L_1-convex. We refer to the convex hull of a subspace under this metric as the L_1-convex hull or Hull_1 to differentiate it from the L_2-convex hull.

An *isomorphism* of cube complexes is an isometry which sends n-cubes to n-cubes. We may sometimes refer to these maps simply as isometries, but it is assumed that any map between cube complexes preserves the cell structure.

A *midcube* is a subset of a cube obtained by restricting one coordinate to 0. A 2-cube has two midcubes, one horizontal and one vertical. An n-cube has n midcubes, one for each coordinate. Let X be a non-positively curved cube complex. Let $\square$ be the equivalence relation on the edges of X generated by the relation $e\square f$ if e and f are the opposite edges of a square in X. A *hyperplane* of a cube complex is the set of midcubes intersecting a $\square$-equivalence class of edges. The $\square$-equivalence class of edges which a hyperplane intersects are said to be *dual* to the hyperplane and vice versa. Studying the hyperplanes of a CAT(0) cube complex can reveal quite a lot of information about the cube complex, itself. Some important features of hyperplanes are stated in the following proposition.

Proposition 2.1.1 ([Sag95]). *Let X be a CAT(0) cube complex and H a hyperplane of X. Then*

- *H is two-sided*
- *H is convex*
- *H is a CAT(0) cube complex*
- *If a set of hyperplanes of X pairwise intersects, then the intersection of the full set of hyperplanes is nonempty.*

The proof of this proposition is somewhat involved; for more detail see [Sag95], where it was originally proven.

Let X be a finite-dimensional, locally finite CAT(0) cube complex and H a hyperplane of X. The hyperplane H separates X into two components, which we will denote H^+ and H^-. These components are called *halfspaces*, and they give X a wallspace structure. In the case that X is finite dimensional, the wall space induced by the hyperplanes and the cube complex itself are dual. That is to say, there is an operation by which one can reconstruct the cube complex with only the wallspace structure. The details of that procedure are in the following two sections.

2.2 Cubulating Wallspaces

There are several approaches to this construction in various settings described by Roller [Rol99], Nica [Nic04], Chatterji-Niblo [CN05], and Guralnik [Gur06]. We will follow the approach of Chatterji-Niblo.

A *wallspace* is a set $\mathcal{S}$, called the *underlying set*, together with a nonempty collection $\mathcal{W}$ of *walls*. A wall $W \in \mathcal{W}$ is a partition $\mathcal{S} = W^- \sqcup W^+$ into two nonempty sets called *halfspaces*. A wall W is said to *separate* two points $s, s' \in \mathcal{S}$

if s and s' are in distinct halfspaces associated to W. We will assume that all wall spaces satisfy the *finite interval condition*: for all $s, s' \in \mathcal{S}$, there are finitely many walls that separate s from s'.

Two walls V and W are said to *cross* if all four *quarterspaces*

$$W^+ \cap V^+, W^+ \cap V^-, W^- \cap V^+, W^- \cap V^-$$

are nonempty.

Let $(\mathcal{S}, \mathcal{W})$ be a wallspace and $\mathcal{W}^\pm$ be the set of halfspaces associated to $\mathcal{W}$. Let $\pi \colon \mathcal{W}^\pm \to \mathcal{W}$ be the map which takes a halfspace W^+ (or W^-) to its associated wall W. The *dual cube complex* $X(\mathcal{S}, \mathcal{W})$ to this wallspace is constructed in the following way.

The vertices X^0 of the cube complex are a subset of the sections of the map π. Each section should be thought of as choosing a preferred halfspace for each wall. Let σ be a section of π. Then σ is said to be *consistent* if for every pair of walls $V, W \in \mathcal{W}$, the intersection of the halfspaces chosen by σ, $\sigma(V) \cap \sigma(W)$, is nonempty. Note that if V and W cross, then any choice of halfspaces for V and W will be consistent, assuming they do not conflict with the choice of halfspace for another wall. If V and W are disjoint and σ is consistent, then σ cannot choose the halfspaces of V and W that are disjoint, but the other three configurations are valid.

We will construct a graph Γ, one connected component of which will become the 1-skeleton of the dual cube complex. The vertex set of Γ is the set of consistent sections. We connect two vertices σ_x, σ_y with an edge if they differ as functions by only one value. That is to say, one can change from the orientations chosen by σ_x to those chosen by σ_y by swapping the choice of exactly one halfspace. The graph with the specified vertex and edge sets is not, in general, connected.

One example is the wallspace $(\mathbb{Z}, \mathcal{W})$, where $\mathcal{W}$ consists of the partitions

$$\{\ldots, n-1, n\} \sqcup \{n+1, n+2, \ldots\}$$

for every $n \in \mathbb{Z}$. One consistent section $\sigma_{-\infty}$ chooses the halfspace with smaller numbers for every wall, and another, $\sigma_{+\infty}$, chooses the halfspace with larger numbers for every wall. One can show that the only other consistent sections σ_n are those which fix an $n \in \mathbb{Z}$ and choose the halfspace containing n for every wall. The graph Γ associated to this wallspace has three connected components: $\sigma_{+\infty}$, $\sigma_{-\infty}$, and the rest of the graph, which is generated by the σ_n.

Choose an element $s \in \mathcal{S}$. Let σ_s be the section which chooses, for each wall W, the halfspace containing s. This section is consistent, so it is a vertex of Γ. The sections constructed in this way are called *special sections*. We claim that the special sections are in the same connected component. Note that there may be consistent sections in this component that are not special. This component will be the 1-skeleton of the dual cube complex. Fix two special sections $\sigma_s, \sigma_{s'}$. Since $(\mathcal{S}, \mathcal{W})$ satisfies the finite interval condition, there are finitely many walls separating s from s'. One can check that there is a path from σ_s to $\sigma_{s'}$ achieved by iteratively flipping these separating walls in a sensible order.

Consider the path in the example above from the special section about 3 to the special section about 5. One would first flip the wall separating 3 from 4, arriving at the special section about 4. Then flip the wall separating 4 from 5, and we arrive at our destination.

Now that we have a 1-skeleton, the next step is to attach higher-dimensional cubes. In fact, this graph is the 1-skeleton of a unique CAT(0) cube complex. This cube complex is constructed by filling in an n-cube whenever the graph contains the 1-skeleton of an n-cube. The cube complex we get after filling in every skeleton is the dual cube complex to the wallspace $(\mathcal{S}, \mathcal{W})$. For a more detailed exposition, consult [CN05].

2.3 Roller Duality

Let X be a finite dimensional CAT(0) cube complex. Then $W(X)$ will denote the wallspace of the form $(X^{(0)}, \mathcal{W})$, where $X^{(0)}$ is the 0-skeleton of X and $\mathcal{W}$ is the set of hyperplanes of X. We will show why the dual cube complex $X(W(X))$ to this wallspace is isomorphic to X, the cube complex we started with.

Let X be a CAT(0) cube complex, $\mathcal{W}$ be the hyperplanes of X, $\mathcal{W}^{\pm}$ the associated set of halfspaces, and $\pi \colon \mathcal{W}^{\pm} \to \mathcal{W}$ the map that sends each halfspace to its corresponding hyperplane. Fix a 0-cube $x \in X$. Then the map σ_x which chooses the halfspace containing x is a special section. This shows that the vertices of Γ contain the 0-skeleton of X. One can check that these are the only special sections.

Let x and y be two 0-cubes connected by an edge e. Then the hyperplane H dual to e is the only hyperplane separating x from y, so we can change from σ_x to σ_y by flipping the halfspace associated to H. The only halfspaces we can flip to change σ_x to another consistent section are those corresponding to hyperplanes that intersect an edge containing x as one of its endpoints. Flipping any other halfspace would violate the consistency condition. This shows that the 1-skeleton of the dual cube complex $X(W(X))$ is graph isomorphic to $X^{(1)}$. Since two CAT(0) cube complexes with isomorphic 1-skeletons must be the isomorphic as cube complexes, we've shown that $X(W(X))$ is isomorphic to X, as desired.

We will look at a small example to see how higher-dimensional cubes are filled in and to see a vertex of $X(\mathcal{S}, \mathcal{W})$ which is not a special section. Let $(\mathcal{S}, \mathcal{W})$ be the wallspace below. **Picture**

I will follow the convention that the arrows point into the positive halfspace, so the special section σ_{x_1} is defined by the choice $\{W_1^+, W_2^+, W_3^+\}$. We can flip W_1 to get the special section σ_{x_2}, so there is an edge connecting σ_{x_1} and σ_{x_2}. Similarly, there are edges connecting σ_{x_2} to σ_{x_7}, σ_{x_7} to σ_{x_6}, and σ_{x_6} back to σ_{x_1}. This 4-cycle forms the 1-skeleton of a 2-cube, so we fill in a 2-cube with 0-cube set $\{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_6}, \sigma_{x_7}\}$. In the same way, there are 2-cubes defined by $\{\sigma_{x_2}, \sigma_{x_3}, \sigma_{x_4}, \sigma_{x_7}\}$ and $\{\sigma_{x_4}, \sigma_{x_5}, \sigma_{x_6}, \sigma_{x_7}\}$. These 2-cubes are glued together to form what appears to be three faces of a 3-cube. (**Picture**)

Note that every pair of walls crosses, so every section is consistent. The one section we haven't yet used is $\{W_1^+, W_2^-, W_3^+\}$, which is not special. We will denote this by σ_h. If we flip W_1, we get $\{W_1^-, W_2^-, W_3^+\}$, which is just σ_{x_3}. Flipping

W_2 results in σ_{x_1} and flipping W_3 gives us σ_{x_5}. Therefore σ_h is connected to each of these vertices by an edge, finishing the 1-skeleton of a 3-cube. The dual cube complex $X(\mathcal{S}, \mathcal{W})$ in this example is a 3-cube.

One notable way in which Roller duality can be used is to give a sufficient and necessary condition on the hyperplanes for a CAT(0) cube complex to decompose as a product of cube complexes. The following results come from [CS11]. We include proofs, as they are important to our main theorem.

Lemma 2.3.1 ([CS11])**.** *Let X be a CAT(0) cube complex and $\mathcal{H}$ the hyperplanes of X. X decomposes as a product of cube complexes $X_1 \times X_2$ if and only if there is a partition $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ such that every hyperplane in $\mathcal{H}_1$ intersects every hyperplane in $\mathcal{H}_2$.*

Proof. Suppose X decomposes as a product of cube complexes $X_1 \times X_2$. Then there are natural projection maps $p_i \colon X \to X_i$ and each cube in X is a product of a cube of X_1 with a cube of X_2. If two edges in X are opposite edges of a 2-cube, then they must belong to the same factor X_i. It follows that every hyperplane of X is the preimage by p_i of a hyperplane of X_i. We can use this to partition the set $\mathcal{H}$ of hyperplanes of X as $\mathcal{H}_1 \sqcup \mathcal{H}_2$, where $\mathcal{H}_i$ consists of the hyperplanes which are preimages by p_i of a hyperplane of X_i. Any hyperplane $H_1 \in \mathcal{H}_1$ must therefore be of the form $H_1' \times X_2$, where H_1' is $p_1(H_1)$. Since every hyperlane of $\mathcal{H}_2$ has the form $X_1 \times H_2'$, we know that H_1 and H_2 intersect. This concludes the proof of one direction of the biconditional.

Suppose X is a CAT(0) cube complex and the set of hyperplanes $\mathcal{H}$ of X has a partition $\mathcal{H}_1 \sqcup \mathcal{H}_2$ such that every hyperplane belonging to $\mathcal{H}_1$ intersects every hyperplane in $\mathcal{H}_2$. The space X along with a collection of hyperplanes $\mathcal{H}_i$ form a wallspace. When we cubulate these wallspaces, we get the dual cube complexes X_1 and X_2. A cube in X corresponds to a set of pairwise-intersecting hyperplanes along with a choice of orientation on the remaining hyperplanes which points towards the cube. If we "forget" the choices for the hyperplanes in $\mathcal{H}_2$, the remaining pairwise-intersecting hyperplanes and orientations define a cube in X_1 and vice versa. This means that every cube of X is a product of a cube of X_1 and a cube of X_2. In addition, since every hyperplane of $\mathcal{H}_1$ intersects every hyperplane of $\mathcal{H}_2$, any consistent section on $\mathcal{H}_1$ combined with any consistent section on $\mathcal{H}_2$ will yield a consistent section on $\mathcal{H}$. Therefore any product of a cube of X_1 and a cube of X_2 will be a cube of X, proving that $X = X_1 \times X_2$.

$\square$

A cube complex which cannot be decomposed as a product of cube complexes is called *irreducible*. A consequence is the following proposition.

Proposition 2.3.1 ([CS11]). *A finite-dimensional CAT(0) cube complex admits a canonical decomposition*

$$X = X_1 \times \cdots \times X_p$$

into a product of irreducible cube complexes X_i. Every automorphism of X preserves that decomposition, up to a permutation of possibly isomorphic factors. In particular, the image of the canonical embedding

$$\mathrm{Aut}\,(X_1) \times \cdots \times \mathrm{Aut}\,(X_p) \hookrightarrow \mathrm{Aut}(X)$$

has finite index in $\mathrm{Aut}(X)$*.*

Proof. Since X is finite-dimensional, any product decomposition can be refined into a finite product of irreducible factors. Therefore we need to show that if X admits two product decompositions $X = X_1 \times \cdots \times X_p$ and $X = X'_1 \times \cdots \times X'_q$, then $p = q$ and there is a permutation σ so that $X_i = X'_{\sigma(i)}$ for every i. By Lemma 3.1.1, the set $\mathcal{H}$ of hyperplanes of X admits partitions $\mathcal{H} = \mathcal{H}_1 \sqcup \ldots \sqcup \mathcal{H}_p$ and $\mathcal{H} = \mathcal{H}'_1 \sqcup \ldots \sqcup \mathcal{H}'_q$. The second partition induces a partition on each individual subset $\mathcal{H}_i$ of the first partition. Since each X_i is irreducible, these must all be the trivial partition. Therefore $p \leq q$. By symmetry, $q \leq p$, and so $p = q$. The desired result follows from the fact that all of the above induced partitions are trivial. $\square$

2.4 CAT(0) Axes

This section contains a discussion of CAT(0) axes, a crucial tool to the study of CAT(0) spaces and CAT(0) cube complexes. Much of the material is drawn from [BH99], the standard reference for the study of CAT(0) spaces.

Let X be a metric space and g be an isometry of X. The *displacement function* of g is the function $d_g\colon X \to \mathbb{R}$ defined by $d_g(x) = d_X(x, g \cdot x)$. The *translation length* of g is defined to be $|g| := \inf_{x\in X} d_g(x)$. The set of points where g attains its translation length is denoted $\mathrm{Min}(g)$. An isometry g is called *semi-simple* if $\mathrm{Min}(g)$ is nonempty.

Proposition 2.4.1 (II.6.2 [BH99]). *Let X be a metric space and let g be an isometry of X.*

1. $\mathrm{Min}(g)$ *is g-invariant.*
2. *If α is an isometry of X, then $|g| = |\alpha g \alpha^{-1}|$ and $\mathrm{Min}(\alpha g\alpha^{-1}) = \alpha \cdot \mathrm{Min}(g)$. In particular, if α commutes with g, then it leaves $\mathrm{Min}(g)$ invariant.*
3. *If X is a CAT(0) space, then the displacement function d_g is convex, and hence $\mathrm{Min}(g)$ is a closed, convex set.*
4. *If $C \subset X$ is nonempty, complete, convex, and g-invariant, then $|g| = |g|_C|$ and g is semi-simple if and only if $g|_C$ is semi-simple. Thus $\mathrm{Min}(g)$ is nonempty if and only if $C \cap \mathrm{Min}(g)$ is nonempty.*

We use the standard classification of isometries into three classes. An isometry g is called

1. *elliptic* if it has a fixed point,
2. *hyperbolic* if d_g attains a strictly positive minimum, and
3. *parabolic* if d_g does not attain its minimum.

The following theorem from [BH99] shows that $\mathrm{Min}(g)$ has very nice structure when g is a hyperbolic isometry of a CAT(0) space.

Theorem 2.4.1 (II.6.8 [BH99])**.** *Let X be a CAT(0) space.*

1. *An isometry g of X is hyperbolic if and only if there exists a geodesic line $c\colon \mathbb{R} \to X$ which is translated nontrivially by g, namely $g \cdot c(t) = c(t+a)$ for some $a > 0$. The set $c(\mathbb{R})$ is called an* axis *of g. For any such axis, the number a is equal to $|g|$.*

2. *If X is complete and g^m is hyperbolic for some $m \neq 0$, then g is hyperbolic.*

Let g be a hyperbolic isometry of X.

1. *The axes of g are parallel to each other and their union is* $\mathrm{Min}(g)$*.*

2. $\mathrm{Min}(g)$ *is isometric to a product $Y \times \mathbb{R}$, and the restriction of g to* $\mathrm{Min}(g)$ *is of the form $(y, t) \mapsto (y, t + |g|)$, where $y \in Y, t \in \mathbb{R}$.*

3. *Every isometry α that commutes with g leaves* $\mathrm{Min}(g) = Y \times \mathbb{R}$ *invariant, and its restriction to $Y \times \mathbb{R}$ is of the form (α', α''), where α' is an isometry of Y and α'' is a translation of $\mathbb{R}$.*

In the interest of clarity, I will refer an axis as defined above as a *CAT(0) axis*.

There is another notion of an axis for a hyperbolic isometry of a CAT(0) cube complex called a *combinatorial axis*. Let g be an isometry of a CAT(0) cube complex X. Then $\gamma\colon \mathbb{R} \to X^{(1)}$ is a combinatorial axis for g if γ is a g-invariant geodesic in $X^{(1)}$ on which g acts by translation. An automorphism is *combinatorially hyperbolic* if it has a combinatorial axis, and an automorphism is *combinatorially elliptic* if it has a fixed point in $X^{(0)}$. Although every combinatorially hyperbolic element is hyperbolic, we need some mild conditions to guarantee the converse. The same is true for elliptic elements.

An automorphism acts *without inversion* if it does not map H^- to H^+ for any hyperplane H. An automorphism acts *stably without inversion* if no power maps H^- to H^+ for any hyperplane H. A group acts *without inversion* if every element acts without inversion. Note that this implies that every element is acting stably without inversion. One can show that if an automorphism acts stably without inversion, then it is combinatorially hyperbolic if and only if it is hyperbolic and it is combinatorially elliptic if and only if it is elliptic. Note that if a group acts on a CAT(0) cube complex, then it acts without inversion on the barycentric subdivision.

The following theorem from [Hag07] classifies automorphisms which act stably without inversion.

Theorem 2.4.2 ([Hag07]). *Every automorphism of a CAT(0) cube complex acting stably without inversion is either combinatorially elliptic or combinatorially hyperbolic.*

It is an easy consequence of this theorem that every automorphism of a CAT(0) cube complex must be either elliptic or hyperbolic.

A few propositions follow which are useful for working with CAT(0) and combinatorial axes. An axis γ *crosses* a hyperplane H if γ intersects, but is not contained in, H. These propositions will show that the set of hyperplanes crossing an axis of a hyperbolic element is well defined.

Proposition 2.4.2 ([CS11]). *If γ and γ' are two CAT(0) axes of a hyperbolic element g, then γ and γ' cross the same set of hyperplanes.*

Proof. This is proven in [CS11]; we include a proof here for completeness.

By the Flat Strip Theorem (Theorem II.2.13 in [BH99]), the CAT(0) convex hull of $\gamma \cup \gamma'$ is isometric to $\mathbb{R} \times [0, D]$, where D is the Hausdorff distance between γ and γ', $\mathbb{R} \times \{0\} = \gamma$, and $\mathbb{R} \times \{D\} = \gamma'$. If γ crosses a hyperplane H, then H must intersect the convex hull of $\gamma \cup \gamma'$. In particular, H intersects γ in a point and intersects the convex hull in a Euclidean line. This line must intersect γ' when it exits the convex hull, so γ' crosses H. Conversely, if γ' crosses H, then γ must also cross H by the same logic. $\square$

Proposition 2.4.3 (Haglund [Hag07]). *If γ and γ' are two combinatorial axes of a hyperbolic element g, then γ and γ' cross the same set of hyperplanes.*

We prove that the set of hyperplanes crossed by the axis of an element is the same, regardless of whether it is a CAT(0) axis or a combinatorial axis.

Proposition 2.4.4. *Let g be a hyperbolic isometry on a locally finite, finite dimensional CAT(0) cube complex X. Then the set of hyperplanes crossed by a combinatorial axis for g is the same as the set of hyperplanes crossed by a CAT(0) axis for g.*

Proof. Let H be a hyperplane in X. An element g *skewers* a hyperplane H if there is some power g^n such that $g^n H^+ \subsetneq H^+$. From Lemma 2.3 in [CS11], H crosses a CAT(0) axis for g if and only if g *skewers* H. If we show that H intersects a combinatorial axis for g if and only if g skewers H, we will be done.

Suppose a combinatorial axis for g crosses H. Because X is finite dimensional, there must be some n so that $g^n H$ is disjoint from H. The element g does not reverse the orientation of its axis, so $g^n H^+ \subsetneq H^+$; g skewers H.

Suppose g skewers H. Then there is some n so that $g^n H^+ \subsetneq H^+$. Let γ be a combinatorial axis for g and p a 0-cube in γ. Without loss of generality, assume $p \in H^-$. Let m be the number of hyperplanes separating p from H. Then $g^{n(m+1)}H$ is separated from H by at least m hyperplanes. Since $g^{n(m+1)}p$ is separated from $g^{n(m+1)}H$ by m hyperplanes, it must be contained in H^+. The combinatorial axis γ is $\langle g \rangle$-invariant, so $g^{n(m+1)}p$ is also on γ. This shows that γ contains a point in H^- and a point in H^+, so γ crosses H. $\square$

2.5 Hyperbolic Spaces and the Visual Boundary

Gromov developed a class of spaces capturing many useful properties of classical hyperbolic geometry and of trees called δ-hyperbolic spaces in his seminal paper [Gro87]. The concept has proved useful in many disciplines and is considered to have started the field of geometric group theory. In this section we will cover some useful ideas from the study of δ-hyperbolic groups. In the interest of space, we will not always provide definitions in their full generality, and we omit some proofs. For a more full exposition, see [Gro87] or [GdlH90].

Let X be a geodesic metric space. A geodesic triangle in X is called *δ-slim* if each side of the triangle is contained in the δ-neighborhood of the union of the other two sides, where δ is some nonnegative constant. A geodesic metric space X is called *δ-hyperbolic* if there is a uniform δ for which every geodesic triangle in X is δ-slim. Examples of hyperbolic spaces include $\mathbb{H}^n$ and metric trees. The Euclidean plane is a space which is not δ-hyperbolic.

The class of δ-hyperbolic spaces is invariant under an important type of map: the quasi-isometry. Let $f\colon X \to Y$ be a map between metric spaces. Then f is a *(K, C)-quasi-isometric embedding* if for every two points x_1, x_2 in X we have the following bounds on the distance between their images:

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C.$$

The map f is called a *quasi-onto* if there is a constant C such that for every point y in Y, we can find a point x in X so that $f(x)$ is within a C-neighborhood of y. A map is a *(K, C)-quasi-isometry* if it is a (K, C) quasi-isometric embedding and quasi-onto. Often, we drop the constants and call a map a *quasi-isometry* if there are some constants for which it is a (K, C)-quasi-isometry. Two metric spaces are *quasi-isometric* if there exists a quasi-isometry between them, and one can show that this is an equivalence relation.

An important property of δ-hyperbolic spaces is that they are invariant under quasi-isometries. This allows us to extend the notion of δ-hyperbolicity from metric spaces to finitely generated groups. A finitely generated group is a *δ-hyperbolic group* if the Cayley graph associated to a fixed generating set and equipped with the word metric is a δ-hyperbolic space. This is well-defined because two Cayley graphs for a group associated to different generating sets are quasi-isometric. (In fact, they are bi-Lipschitz.)

We can use the Svarc-Milnor Lemma to extend this further, which we need a few definitions to state. A metric space is *proper* if every closed metric ball is compact. Let G be a group and X a proper, geodesic metric space. An action of G on X is *properly discontinuous* if for any basepoint $x_0 \in X$ and every compact subset $K \subset X$, there are finitely many elements $g \in G$ with $g \cdot x_0 \in K$. An action of G

on X is *cocompact* if the orbit space X/G equipped with the quotient topology is compact. For short, a group action that is properly discontinuous, cocompact, and by isometries is called *geometric*. The Svarc-Milnor Lemma states:

Lemma 2.5.1 (Svarc-Milnor Lemma)**.** *Let G be a group acting geometrically on a proper geodesic metric space X. Then the group G is finitely generated and for every finite generating set S of G and every point $x_0 \in X$, the orbit map*

$$f_{x_0} \colon \Gamma(G, S) \to X$$

is a quasi-isometry, where $\Gamma(G, S)$ denotes the Cayley graph of G associated to the generating set S.

This shows that if a group acts geometrically on a metric space X, then X is δ-hyperbolic if and only if G is δ-hyperbolic.

An invariant of δ-hyperbolic groups that has proved extremely useful is the *visual boundary*, sometimes called the Gromov boundary. We will denote the visual boundary of a space X by ∂X. The visual boundary is formed from equivalence classes of geodesic rays. Let X be a δ-hyperbolic space. Two geodesic rays γ_1, γ_2 in X are equivalent if there is a constant K such that

$$d(\gamma_1(t), \gamma_2(t)) \leq K$$

for all t.

The visual boundary is equipped with what is called the cone topology. Essentially, geodesic rays that stay close for a long time represent points that are close in the visual boundary. For a more formal definition, we need the Gromov product. Given three points x, y, z in a metric space X, the *Gromov product* of y and z based at x is

$$(y, z)_x = \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

The Gromov product measures how long the geodesic segments from x to y and from x to z stay close. Let x_0 be a basepoint in a δ-hyperbolic space X. Let

$p \in \partial X$. We define the set $V(p,r)$ to be the boundary points q such that there exist representatives γ_1 and γ_2 for p and q, resp., satisfying

$$\liminf_{t\to\infty}(\gamma_1(t),\gamma_2(t))_{x_0} \geq r.$$

These are the geodesics which stay close to γ_1 for time r before diverging. The *cone topology* is the topology with neighborhood basis the collection of $V(p,r)$ with $r \geq 0$. One can show that the topology does not depend on the choice of basepoint and it can be extended to a topology on $X \cup \partial X$.

Furthermore, if X and Y are δ-hyperbolic spaces and $f\colon X \to Y$ is a quasi-isometry, then we can extend f to a canonical homeomorphism $\hat{F}\colon \partial X \to \partial Y$ ([GdlH90]). Since the visual boundary of a δ-hyperbolic space is a quasi-isometry invariant, we can define the visual boundary of a δ-hyperbolic group to be the visual boundary of its Cayley graph with respect to any finite generating set.

2.6 The CAT(0) Boundary

In the previous section, we introduce δ-hyperbolic spaces and associate to each space X a visual boundary ∂X. We will describe a related construction for CAT(0) spaces called the CAT(0) boundary. In fact, if X is a δ-hyperbolic CAT(0) space, then its visual boundary and CAT(0) boundary are homeomorphic in a natural way. For this reason, we will often abuse notation and simply refer to the visual boundary when we mean the CAT(0) boundary.

Let x_0 be a basepoint in a complete CAT(0) space X. The CAT(0) boundary ∂X consists of geodesic rays with the same equivalence relation as in the visual boundary above. Two rays γ_1 and γ_2 are equivalent if there is a constant K such that

$$d(\gamma_1(t),\gamma_2(t)) \leq K$$

for all t. Let $U(\gamma, R, \epsilon)$ denote the set of equivalence classes of geodesic rays where the representative γ' based at $\gamma(0)$ satisfies

$$d(\gamma(R), \gamma'(R)) \leq \epsilon.$$

Then the *cone topology* for ∂X has neighborhood basis $U(\gamma, R, \epsilon)$, where $R > 0$ and $\epsilon > 0$.

In Proposition III.3.7 from [BH99], it is shown that in the special case where X is a proper, δ-hyperbolic CAT(0) space, the cone topologies on the visual boundary and CAT(0) boundary coincide.

2.7 Least Area Surfaces

This section discusses some background on least area surfaces and the 4-plane property, most of which comes from [FHS83] and [Li02]. First, we define some necessary terminology. We then discuss results from [FHS83] regarding least area surfaces. Last, we'll see how least area surfaces relate to cube complexes by reviewing a theorem from [Li02].

Throughout this section, M will be assumed to be a Riemannian manifold and F a closed surface. A map $f \colon F \to M$ is *incompressible* if it induces an injection on the fundamental group. A map f is *two-sided* if its normal bundle is trivial. An immersion f is *self-transverse* if given two points x and x' with $f(x) = f(x')$, there exist small disks about x and x' which are embedded by f and intersect transversely. A map $f \colon F \to M$ is *least area* if the area of f is less than the area of any homotopic map from F to M.

A natural initial question is which homotopy classes of maps contain least area representatives. Schoen and Yau resolved this in [SY79], in which they prove the following theorem.

Theorem 2.7.1 ([SY79]). *Let M be a P^2-irreducible Riemannian 3-manifold and F a closed, orientable surface not S^2. If $g \colon F \to M$ is an incompressible map, then there is a least area map $f \colon F \to M$ which is homotopic to g, any of which can be* *parametrized as an immersion.*

The general philosophy of least area surfaces is that they intersect least, meaning that the intersections and self-intersections of least area immersions are as small as their homotopy classes allow. The main result of [FHS83] formalizes this.

Theorem 2.7.2 (Theorem 5.1 from [FHS83]). *Let M be a closed, P^2-irreducible Riemannian 3-manifold and let F be a closed surface, not S^2 nor P^2. Let $f\colon F \to M$ be a least area immersion such that $f_*\colon \pi_1(F) \to \pi_1(M)$ is injective and such that f is homotopic to a two-sided embedding g. Then either*

1. *f is an embedding, or*

2. *f double covers a one-sided surface K embedded in M and $g(F)$ bounds a submanifold of M which is a twisted I-bundle over a surface isotopic to K.*

An example of case (ii) can be obtained by taking M to be the product of a flat Mobius band with a circle and F to be a torus. There is a family of totally geodesic tori homotopic to the boundary of this 3-manifold, each of which has the same area, minimal in its homotopy class. Exactly one member of this family is an immersion and double covers a one-sided, embedded torus.

Lemma 6.5 of [FHS83] shows how we can leverage algebraic intersection information into geometric intersection information. Let M be a compact, P^2-irreducible Riemannian 3-manifold. Let F_1 and F_2 be closed surfaces, not S^2 or P^2, and let $f_i\colon F_i \to M$ be a two-sided, least area, incompressible map for $i = 1, 2$. Pick a subgroup $\pi_1(F_i)$ of $\pi_1(M)$ from the conjugacy class of subgroups determined by f_i. Let M_i be the cover of M with fundamental group $\pi_1(F_i)$. We fix a lift of f_i in M_i and call this fixed lift F_i, abusing notation. It follows from earlier work in [FHS83] that this copy of F_i is embedded and two-sided. Let $\tilde{M}$ denote the universal cover of M and $\tilde{F}_i$ be the pre-image in $\tilde{M}$ of F_i in M_i. Since F_i is an embedded surface in M_i, $\tilde{F}_i$ is an embedded plane.

Lemma 2.7.1 (Lemma 6.5 from [FHS83]). *If $G = \pi_1(F_i) \cap \gamma\pi_1(F_j)\gamma^{-1}$ is infinite cyclic, where $\gamma \in \pi_1(M)$, then either $\tilde{F}_i$ and $\gamma\tilde{F}_j$ are disjoint or they intersect* *transversely in a line whose stabilizer contains G.*

In particular, if F is a two-sided, least area surface and two intersecting lifts $\tilde{F}$ and $\gamma\tilde{F}$ have $\pi_1(F) \cap \gamma\pi_1(F)\gamma^{-1} \cong \mathbb{Z}$, the two lifts must intersect exactly in a line.

In [Li02], Li found a large class of closed 3-manifolds which are not homeomorphic to a non-positively curved cube complex. The theorem from his paper follows.

Theorem 2.7.3 (Theorem 3 from [Li02])**.** *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus. Suppose that M does not contain any closed, nonperipheral, embedded, incompressible surfaces. Then only finitely many Dehn fillings on M can yield 3-manifolds that are homeomorphic to 3-dimensional non-positively curved cube complexes.*

We need one definition for a sketch of Li's proof. Let g be an incompressible surface. Then $g\colon F \to M$ has the *4-plane property* if for any least area map f homotopic to g, any four lifts of f contains a disjoint pair. Recall that these lifts must be embedded planes by work in [FHS83].

Li proves this theorem by contradiction, supposing that a Dehn filling of M with slope s is homeomorphic to a 3-dimensional non-positively curved cube complex. He then shows that M must contain an incompressible surface with boundary slope s that satisfies the 4-plane property. However, he shows in an earlier theorem that the collection of slopes of incompressible surfaces satisfying the 4-plane property is finite. Therefore the Dehn fillings along the infinitely many other slopes cannot be homeomorphic to a non-positively curved cube complex. In fact, these Dehn fillings cannot contain any incompressible surfaces satisfying the 4-plane property.

We prove a stronger result in chapter 3. In fact, the fundamental groups of the 3-manifolds obtained by Dehn filling cannot act geometrically on any 3-dimensional CAT(0) cube complex, and so the 3-manifolds are not homotopy equivalent to any 3-dimensional non-positively curved cube complex.

2.8 The Cubical Flat Torus Theorem

An abelian subgroup is *highest* if it contains no finite index subgroup that is contained in a higher rank free-abelian subgroup. In this section, we will assume that a group Γ containing a highest free-abelian subgroup A of rank p generated by $\{a_1, \ldots, a_p\}$ acts properly and cocompactly by isometries on a CAT(0) cube complex X.

By the Flat Torus Theorem (Theorem II.7.1 in [BH99]),

$$\mathrm{Min}(A) \cong Y \times \mathbb{E}^p.$$

Let E be $\{y\} \times \mathbb{E}^p$ for some $y \in Y$. If E is contained in a hyperplane, the $\frac{1}{2}$-neighborhood about E contains a region isometric to $\mathbb{E}^p \times (-\frac{1}{2}, \frac{1}{2})$, where the interval lies orthogonal to the hyperplane. In this case, let E be a level set of $\mathbb{E}^p$ that is not a hyperplane.

If a hyperplane intersects E, it intersects E in a subspace isometric to $\mathbb{E}^{p-1}$. Two hyperplanes are said to be *parallel* if their intersections with E do not intersect. Let $P_1, \ldots, P_q$ be the parallelism classes of hyperplanes in E. The only proper and cocompact action of $\mathbb{Z}^p$ on $\mathbb{E}^p$ is by translation, so each a_i acts on E by translations. An action is *disjoint* if every pair of hyperplanes in the same orbit do not intersect in X. By Lemma 2.3 of [WW15], each parallelism class P_i in E admits a disjoint action by some element $g_i \in A$.

Let H and H' be two hyperplanes that intersect E. If H and H' are not parallel, they are said to be *crossing.* The rest of the definitions will assume that H and H' are both in the parallelism class P_i. Two hyperplanes H and H' are said to be *crossing* if $g_i^k H$ intersects $g_i^{k'} H'$ for every integer k and every integer k'. The hyperplanes H and H' are said to be *aligned* if $g_i^k H$ intersects H' for only finitely many k. Otherwise, there exists an $N \in \mathbb{Z}$ such that H intersects $g_i^k H'$ for all $k > N$ or H intersects $g_i^k H'$ for all $k < N$. In this case, H and H' are said to be *semi-crossing.* One can show that these definitions are independent of the choice of g_i.

A *quasi-line* is a CAT(0) cube complex that is quasi-isometric to a line.

Theorem 2.8.1 (Wise-Woodhouse [WW15], Cubical Flat Torus Theorem)**.** *Let G act properly and cocompactly on a CAT(0) cube complex X. Let A be a highest virtually abelian subgroup of G and let $p = \mathrm{rank}(A)$. Then A acts properly and cocompactly on a convex subcomplex $Y \subseteq X$ such that $Y \cong \prod_{i=1}^{p} C_i$, where each C_i is a quasi-line.*

Proof. The Cubical Flat Torus Theorem is important to the results of this work, so we will provide a proof sketch.

From Corollary II.7.2 of [BH99], X contains an A-invariant, p-dimensional flat E. Most of the work of proving the Cubical Flat Torus Theorem is in an earlier theorem of [WW15], which states that if a (not necessarily highest) virtually abelian subgroup A of rank p acts properly and cocompactly on a CAT(0) flat E, then either $\mathrm{Hull}_1(E)$ is a product of p quasi-lines or there exists a finite index subgroup $B < A$ so that $\mathrm{Min}(B) \cap \mathrm{Hull}_1(E)$ is not B-cocompact. In order to prove the Cubical Flat Torus Theorem, they show that such a finite index subgroup cannot exist when A is highest.

Let $B < A$ be a finite index subgroup whose action on the hyperplanes intersecting E is disjoint, meaning that distinct hyperplanes in the same B-orbit are disjoint. For each parallelism class P_i, fix an element z_i which acts disjointly on the hyperplanes of P_i. It is not hard to show that alignment of $\langle z_i \rangle$-orbits is an equivalence relation.

First, consider the case when there are no semi-crossing orbits of hyperplanes, so every pair of orbits is either aligned or crossing. Let $\{A_i\}_{i=1}^{m}$ be an enumeration of the alignment classes of orbits. Since any hyperplane in A_i must intersect any hyperplane in A_j when $i \neq j$, we can use Roller duality to show that $\mathrm{Hull}_1(E) \cong \prod_{i=1}^{m} X(A_i)$. In the case that the number of alignment classes is the same as the rank of A, we get that $\mathrm{Hull}_1(E)$ is a product of p quasi-lines. Otherwise, $m > p$ and $\mathrm{Min}(B) \cap \mathrm{Hull}_1(E)$ is not B-cocompact.

Now suppose there are at least two semi-crossing orbits of hyperplanes. Wise-Woodhouse construct a series of B-equivariant injective cubical maps ϕ_k that map E to a flat distance k from E. Each image $\phi_k(E)$ is contained in $\mathrm{Min}(B) \cap \mathrm{Hull}_1(E)$ and invariant under B, so $\mathrm{Hull}_1(E) \cap \mathrm{Min}(B)$ cannot be B-cocompact. □

From Corollary II.7.2 of [BH99], X contains an A-invariant, p-dimensional flat E. In [WW15], Wise and Woodhouse show that the cubical convex hull of E is a product of p quasi-lines. Furthermore, they show that in the case that A is free-abelian, there is a preferred set S of generators of a finite-index subgroup of A. The intersection of A with another highest abelian subgroup A' must occur "along" a subgroup generated by a subset of S. This provides a high degree of control over intersections of highest abelian subgroups.

Lemma 2.8.1 (Wise-Woodhouse [WW15], Lemma 4.2)**.** *Let A be a rank p virtually abelian group acting properly and cocompactly on a CAT(0) cube complex $\prod_{i=1}^{p} C_i$, where each C_i is a quasi-line. Then there exists a finite index free-abelian subgroup $\hat{A} \leq A$ with basis $S = \{\hat{a_1}, \dots, \hat{a_p}\}$ such that $\hat{a_i} \cdot (c_1, \dots, c_i, \dots, c_p) = (c_1, \dots, \hat{a_i} \cdot c_i, \dots, c_p)$ for each i.*

Proof. This proof is pulled directly from [WW15]. By Proposition 2.3.1 ([CS11]), there is a finite index subgroup $B < A$ which acts on $\prod C_i$ as a product of automorphisms of the factors. For each i, there is a subgroup $B_i < B$ which acts on an invariant line $l_i \subset C_i$ by translation. Let $\hat{A} = \cap_i B_i$. There is a homomorphism $\varphi\colon \hat{A} \to \mathbb{Z}^p$ induced by the action of $\hat{A}$ on $\prod l_i$. Since $\hat{A}$ acts cocompactly on $\prod l_i$, $\varphi(\hat{A})$ must be finite index in $\mathbb{Z}^p$. Therefore for every i there is an $a_i \in \hat{A}$ such that $\varphi(a_i) = (0, \dots, 0, m_i, 0, \dots, 0)$, where $m_i \neq 0$ is the ith entry. □

Theorem 2.8.2 (Wise-Woodhouse [WW15], Theorem 4.1)**.** *Let G act properly and cocompactly on a CAT(0) cube complex X. Let $A \leq G$ be a highest free-abelian subgroup, and let $p = \mathrm{rank}(A)$. There is a set $S = \{\hat{a_1}, \dots, \hat{a_p}\} \subseteq A$ such that the following holds: For any highest rank free-abelian subgroup $A'' \leq G$, the intersection $A' \cap A$ is commensurable to a subgroup generated by a subset of S.*

Proof. This is a sketch of the proof of Theorem 4.1 from [WW15].

Let $\tilde{Y} = \prod C_i$ and $\tilde{Y}' = \prod C_i'$ be the products of quasi-lines on which A and A' act, resp., by the Cubical Flat Torus Theorem. Denote by $\tilde{Y}^{+r}$ the cubes c_r in X contained in a chain of cubes $c_0, c_1, \ldots, c_r$ where $c_0 \subset \tilde{Y}$ and $c_i \cap c_{i+1} \neq \emptyset$ for every i. This forms an L_1-convex, A-cocompact subcomplex. Choose r so that the intersection $E = \tilde{Y} \cap \tilde{Y}'^{+r}$ is nonempty. Then E is an L_1-convex subcomplex of a product of cube complexes, so E must also be an L_1-convex product of cube complexes $E = \prod E_i$ where $E_i \subset C_i$. Let $\hat{A}$ and $\hat{A}'$ be the finite index subgroups of A and A', resp., we get from Lemma 2.8.1. The action of $\hat{A} \cap \hat{A}'$ on E must be cocompact since $\tilde{Y}$ is $\hat{A}$-cocompact and $\tilde{Y}'^{+r}$ is $\hat{A}'$-cocompact.

For each i, the stabilizer of E_i in $\hat{A}$ must be either $\langle \hat{a_1}, \ldots, \hat{a_{i-1}}, \hat{a_i}^{n_i}, \hat{a_{i+1}}, \ldots, \hat{a_p} \rangle$ or $\langle \hat{a_1}, \ldots, \hat{a_{i-1}}, \hat{a_{i+1}}, \ldots, \hat{a_p} \rangle$. By construction, each $\hat{a_j}$ with $j \neq i$ acts trivially on E_i. If some $\hat{a_i}^{n_i}$ stabilizes E_i, then E_i is a quasi-line. Otherwise E_i must be compact, since $\hat{A} \cap \hat{A}'$ acts cocompactly on E. Let S_0 be the subset of S where $i \in S_0$ if E_i is a quasi-line.

We can construct a subcomplex $E' = \tilde{Y}^{+r} \cap \tilde{Y}'$ as above. Wise-Woodhouse show that E and E' lie within bounded neighborhoods of each other. Let Γ be the Cayley graph of G with respect to some finite generating set and $\phi\colon \Gamma \to X$ the orbit map with respect to some basepoint $x_0 \in X$. Let $B = \mathrm{stab}_{\hat{A}}(E)$ and $B' = \mathrm{stab}_{\hat{A}'}(E')$. Since $\phi(B)$ and $\phi(B')$ lie within finite neighborhoods of each other, so must B and B' in Γ. The action of B stabilizes a finite collection of cosets of B', so B and B' must be commensurable.

Let $H = B \cap B'$, which we've shown is a finite index subgroup of both B and B'. Since $\hat{A} \cap \hat{A}' < B < \hat{A}$ and $\hat{A} \cap \hat{A}' < B' < \hat{A}'$, we have $H = \hat{A} \cap \hat{A}'$ is a finite index subgroup of B. The claim then follows from the fact that B is commensurable to the subgroup generated by S_0 and $\hat{A} \cap \hat{A}'$ is finite index in $A \cap A'$. $\square$

Following the proofs of Lemma 2.8.1 and Theorem 2.8.2 from [WW15], you will note that the generating sets S in each are the same.

2.9 Rank Rigidity for CAT(0) Cube Complexes

In this section, we will provide a brief overview of some key definitions and theorems from [CS11]. Most of the results in the paper require essential cube complexes or group actions, so we will review that concept first. Then we will review a few results that will be used later.

Let X be a CAT(0) cube complex. A halfspace $H^\pm$ is *deep* if it contains arbitrarily large balls in X and *shallow* otherwise. Then we can categorize hyperplanes into three categories. A hyperplane H is *trivial* if both H^+ and H^- are shallow. A hyperplane H is *half-essential* if H^+ is deep and H^- is shallow or vice versa. Lastly, a hyperplane H is *essential* if both H^+ and H^- are deep. We let $\mathrm{Ess}(X)$, $\mathrm{Hess}(X)$, and $\mathrm{Triv}(X)$ denote the essential, half-essential, and trivial hyperplanes of X, respectively.

A CAT(0) cube complex is *essential* if all its hyperplanes are essential. The *core* of X is defined to be the dual cube complex $X(\mathrm{Ess}(X) \cup \mathrm{Triv}(X))$. The *essential core* of X is $X(\mathrm{Ess}(X))$. Note that the essential core is always an essential CAT(0) cube complex. In addition, because the property of being trivial, half-essential, or essential is preserved by automorphisms of the cube complex, there is an induced action of $\mathrm{Aut}(X)$ on the essential core of X.

Let G be a group acting by automorphisms on a CAT(0) cube complex X. Choose a vertex $x \in X$. A halfspace $H^\pm$ is G-deep if it contains orbit points of x arbitrarily far from H. A hyperplane is G-essential if both H^+ and H^- are G-deep. We can similarly define G-half-essential and G-trivial hyperplanes and denote the sets of hyperplanes $\mathrm{Ess}(X, G)$, $\mathrm{Hess}(X, G)$, and $\mathrm{Triv}(X, G)$. The action of G on X is essential if every hyperplane in X is G-essential. Similar to above, the G-core and G-essential cores are $X(\mathrm{Ess}(X, G) \cup \mathrm{Triv}(X, G))$ and $X(\mathrm{Ess}(X, G))$, resp.

Proposition 3.12 from [CS11] neatly summarizes some useful properties of the essential core:

Proposition 2.9.1 (Proposition 3.12 of [CS11])**.** *Let G be a finitely generated group acting properly discontinuously on a finite-dimensional, locally finite CAT(0) cube* *complex X. Let Y denote the G-essential core of X. Then*

1. *G has finitely many orbits on $\mathcal{H}(Y) = \mathrm{Ess}(X, G)$.*

2. *Y is unbounded if and only if G has no global fixed point on X.*

3. *Y embeds as a convex, G-invariant subcomplex of X.*

4. *Every hyperplane of Y is skewered by some element of G.*

Of particular use is the fact that if a group acts properly discontinuously and essentially on a CAT(0) cube complex X, then for every hyperplane H of X, there is some hyperbolic element of G whose axis crosses H.

Two hyperplanes H and H' are *strongly separated* if there is no hyperplane which intersects both H and H'. One exciting result from the paper is that a CAT(0) cube complex is irreducible if and only if it contains a pair of strongly separated hyperplanes.

Proposition 2.9.2 (Proposition 5.1 from [CS11])**.** *Let X be a finite-dimensional unbounded CAT(0) cube complex such that* $\mathrm{Aut}(X)$ *acts essentially without a fixed point at infinity. Then the following conditions are equivalent.*

1. *X is irreducible.*

2. *X contains a pair of strongly separated hyperplanes.*

3. *For each halfspace H^+, there is a pair of halfspaces U^+ and V^+ such that $U^+ \subset H^+ \subset V^+$ and U and V are strongly separated.*

Lastly, we will use Lemma 6.1, the key lemma for proving rank rigidity for CAT(0) cube complexes.

Lemma 2.9.1 (Lemma 6.1 from [CS11])**.** *Let X be a finite-dimensional CAT(0) cube complex. Let H^+ be a halfspace and let $g \in \mathrm{Aut}(X)$ be a hyperbolic isometry with axis γ such that $gH^+ \subsetneq H^+$. Assume that the hyperplanes H and gH are strongly separated.*

Then there is a constant C, depending only on g, such that each geodesic segment crossing at least three hyperplanes in the orbit $\langle g\rangle H$ has a non-empty intersection with the C-neighborhood of γ.

In particular, since the CAT(0) axes for an element are all parallel, it follows that any element g with an axis γ crossing a pair of strongly separated hyperplanes must have its axes contained in a bounded neighborhood of γ. In order to find such a pair, we can make use of their Double Skewering Lemma.

Lemma 2.9.2 (Double Skewering Lemma from [CS11]). *Let X be a finite-dimensional CAT(0) cube complex and $G < \mathrm{Aut}(X)$ a group acting essentially without fixed point at infinity. Then for any two halfspaces $H^+ \subset V^+$, there exists an element $g \in G$ such that*

$$gV^+ \subsetneq H^+ \subset V^+.$$

Caprace-Sageev use this lemma to show that such an element is rank-one and prove their rank rigidity result for CAT(0) cube complexes. One formulation of their rank rigidity theorem follows.

Theorem 2.9.1 (Theorem 6.3 in [CS11]). *Let X be a finite-dimensional CAT(0) cube complex and $G < \mathrm{Aut}(X)$ a group acting essentially without fixed point at infinity. Then X is a product of two cube subcomplexes or every hyperplane of X is skewered by a contracting isometry in G. If in addition X is locally compact and G acts cocompactly, then the same conclusion holds even if G fixes a point at infinity.*

With some work they prove the following corollary, useful for determining an upper bound on the number of irreducible factors a cube complex can have.

Corollary 2.9.1 (Corollary D in [CS11]). *Let X be a locally compact CAT(0) cube complex and G be a discrete group acting cocompactly on X. If X is a product of n unbounded cube complexes, then G contains a subgroup isomorphic to $\mathbb{Z}^n$.*

As a consequence, if an unbounded CAT(0) cube complex X admits a cocompact action by a hyperbolic group, X must be irreducible.

2.10 Cubical Minset Decomposition

The cubical minsets introduced in this section are a generalization of the *characteristic sets* introduced in [FFT16]. Many propositions and proofs in this section have analogues in [FFT16].

Let $G \times A$ act essentially on a CAT(0) cube complex X, where A is free-abelian. The main idea of the section is to decompose X as a product $X_A^\perp \times X_A$, where $X_A^\perp$ represents the directions orthogonal to the axes of elements in A and X_A represents the directions parallel to to the axes of elements in A. We will also show that A has a global fixed point in $X_A^\perp$. In order to get a decomposition of X, we only need A to be central. That will be our assumption in this section, though all of our applications will have groups of the form $G \times A$.

Let $\mathcal{H}$ denote the set of hyperplanes in X. We divide $\mathcal{H}$ into two sets:

$$\mathcal{H}_A = \{H \in \mathcal{H} \mid \text{a combinatorial axis for some } g \in A \text{ crosses } H\}$$

$$\mathcal{H}_A^\perp = \{H \in \mathcal{H} \mid H \text{ separates two combinatorial axes for every } g \in A\}$$

Proposition 2.10.1. *Let G act essentially on a finite dimensional CAT(0) cube complex X and $A \leq G$ be a central free-abelian subgroup. Then every hyperplane of X is in $\mathcal{H}_A$ or $\mathcal{H}_A^\perp$.*

Proof. Let $H \in \mathcal{H} \setminus \mathcal{H}_A$ be a hyperplane of X. Then there is no H does not intersect a combinatorial axis for any $g \in A$. We'll show H separates two combinatorial axes for every $g \in A$, so $H \in \mathcal{H}_A^\perp$.

Fix $g \in A$, a combinatorial axis γ for g, and a 0-cube $p \in \gamma$. The geodesic γ does not intersect H, so without loss of generality $\gamma \subset H^-$. The action of G is essential, so there is some $g' \in G$ such that $g'p \in H^+$. Since A is central in G, $g'\gamma$ is another combinatorial axis for g. Any two combinatorial axes for g cross the same set of hyperplanes, so $g'\gamma$ does not cross H. Therefore $g'\gamma \subset H^+$; H separates two combinatorial axes for g. $\square$

Let X_A be the dual cube complex $X(\mathcal{H}_A)$ and $X_A^{\perp}$ be the dual cube complex $X(\mathcal{H}_A^{\perp})$. We will show that $X \cong X_A^{\perp} \times X_A$.

Before proving that X decomposes as a product, let's examine a motivating example. Suppose the product of groups $F_2 \times \mathbb{Z}$ is acting on the product of its standard Cayley graphs, $T \times \mathbb{R}$. Then $\{1_{F_2}\} \times \mathbb{Z}$ is a central free-abelian subgroup; this is our A. There are two types of hyperplanes in this cube complex. There are hyperplanes parallel to the $\mathbb{R}$ direction, of the form $\{p\} \times \mathbb{R}$. One can see that, given one of these hyperplanes H, we can find two combinatorial axes of any element of A separated by H. Therefore the hyperplanes of the form $\{p\} \times \mathbb{R}$ belong to $\mathcal{H}_A^{\perp}$. There are also hyperplanes orthogonal to the $\mathbb{R}$ direction, of the form $T \times \{p\}$. One can see that a combinatorial axis of any element of A will intersect all of these hyperplanes, so they belong to $\mathcal{H}_A$. In this example the dual cube complex $X(\mathcal{H}_A^{\perp}) = X_A^{\perp}$ is the tree factor T, and the dual cube complex $X(\mathcal{H}_A) = X_A$ is the $\mathbb{R}$ factor.

Proposition 2.10.2. *Let G act essentially on a finite dimensional CAT(0) cube complex X and $A \leq G$ be a central free-abelian subgroup. Then X decomposes as a product*

$$X \cong X_A^{\perp} \times X_A.$$

Proof. Fix $H \in \mathcal{H}_A$ and $H' \in \mathcal{H}_A^{\perp}$. By definition, H must intersect the combinatorial axes of some element $g \in A$ and H' must separate two combinatorial axes of g. Therefore H intersects the two combinatorial axes that H' separates, so H must intersect H'. Since each hyperplane in $\mathcal{H}_A$ intersects every hyperplane in $\mathcal{H}_A^{\perp}$, the subcomplex of X dual to $\mathcal{H}_A$ and $\mathcal{H}_A^{\perp}$ must be a product region. This shows that X must contain a subcomplex isomorphic to

$$X(\mathcal{H}_A^{\perp}) \times X(\mathcal{H}_A) = X_A^{\perp} \times X_A.$$

We showed in Proposition 2.10.1 that every hyperplane of X is in $\mathcal{H}_A$ or $\mathcal{H}_A^{\perp}$, so by Roller duality $X \cong X_A^{\perp} \times X_A$. □

Proposition 2.10.3. *Let G act on a CAT(0) cube complex X and $A \leq G$ be a* *central free-abelian subgroup. Every $g \in A$ acts by an elliptic isometry on $X_A^{\perp}$.*

Proof. Suppose there is a $g \in A$ that acts by a hyperbolic isometry on $X_A^{\perp}$. Then a combinatorial axis L for g in X projects to an axis for g in $X_A^{\perp}$. The projection of L must cross some hyperplanes in $X_A^{\perp}$; this is impossible, as the hyperplanes in $\mathcal{H}_A^{\perp}$ cannot intersect a combinatorial axis of g. By contradiction, g acts by an elliptic isometry on $X_A^{\perp}$. $\square$

Proposition 2.10.4. *Let G act on a CAT(0) cube complex X and $A \leq G$ be a central free-abelian subgroup where every element of A acts by an elliptic isometry. Then A has a global fixed point.*

Proof. Fix $g, g' \in A$ and let $C = \mathrm{Min}(g)$. Then C is non-empty, closed, and convex. C is also g'-invariant, since g and g' commute. By Proposition 6.2 in [BH99], $\mathrm{Min}(g) \cap \mathrm{Min}(g')$ is non-empty. Let $\{g_1, \ldots, g_n\}$ be a generating set for A. We can apply the same reasoning inductively to show that $\cap_i \mathrm{Min}(g_i)$ is non-empty; in particular, this implies that A has a global fixed point. $\square$

Chapter 3

Products of Hyperbolic Groups

In the first section of this chapter, we prove a decomposition theorem for cube complexes admitting a nice enough action by a product of non-elementary hyperbolic groups. In the second section, we provide an application of the theorem, proving the existence of an infinite family of groups with an arbitrarily large gap between their CAT(0) dimension and their CAT(0) cubical dimension.

3.1 Products of Hyperbolic Groups

This goal of this section is to prove the decomposition theorem below.

Theorem 3.1.1. *Let $G_1, \ldots, G_n$ be non-elementary hyperbolic groups. Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, and essentially, by cubical isometries on a CAT(0) cube complex X. Then*

- *X splits as a product of irreducible CAT(0) cube complexes $X = X_1 \times \cdots \times X_n$ and each $g \in G$ acts on $X_1 \times \cdots \times X_n$ as a product of isometries $\mu_1 \times \cdots \times \mu_n$;*
- *every G_i acts properly and cocompactly on X_i; and*
- *every G_i contains a finite-index subgroup G_i'' which acts trivially on every X_j for $i \neq j$.*

It follows from this theorem that G contains a finite index subgroup $G_1' \times \cdots \times G_n'$ which acts as a product action on $X_1 \times \cdots \times X_n$, where the action of G_i' on X_i is proper and cocompact.

We'll prove the theorem in a series of lemmas. First, we obtain the product decomposition of X. Then we prove that each factor in the product decomposition must be irreducible, and no element can permute isomorphic factors. We then prove that the action must be cocompact, after which a bit of work finishes the proof of the theorem.

Lemma 3.1.1. *Let $G_1, \ldots, G_n$ be non-elementary hyperbolic groups. Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, essentially, and without inversion by cubical isometries on a CAT(0) cube complex X. Then X decomposes as a product $X = X_1 \times \cdots \times X_n$.*

Proof. We will prove this lemma in three stages. First, note that by Proposition 3.12 of [CS11], every hyperplane is skewered by some element of G. We show that if a hyperplane H is skewered by $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$, then in fact H is skewered by exactly one g_i. Then we prove that if H is skewered by some other element $(g_1', \ldots, g_n')$, it must be the ith component g_i' that skewers H. Therefore we can partition the set of hyperplanes into n sets, where a hyperplane H is in the ith set $\mathcal{H}_i$ if it is skewered by some element of G_i. To finish the proof, we show that every hyperplane in $\mathcal{H}_i$ intersects every hyperplane of $\mathcal{H}_j$ where $i \neq j$.

First, we show that if a hyperplane H is skewered by $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$, then it is skewered by exactly one g_i. Let $\mathcal{H}$ denote the set of hyperplanes of X. Fix a hyperplane $H \in \mathcal{H}$. The group G is acting properly, cocompactly, and essentially on X, so by Proposition 3.12 of [CS11], H is skewered by some element $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$. Since each G_i is hyperbolic, the free-abelian subgroup A generated by $\{g_1, \ldots, g_n\}$ is highest. By Theorem 2.8.1, A acts properly and cocompactly on a convex subcomplex $Y = C_1 \times \cdots \times C_n$, where each C_i is a quasi-line. By Lemma 2.8.1, there is a finite index subgroup $A' < A$ generated by $\{\alpha_1, \ldots \alpha_n\}$ so that α_i acts trivially on C_j if $i \neq j$.

Every G_j is non-elementary hyperbolic, so for every j we can find an element $g_j' \in G_j$ such that g_j and g_j' do not commute. Fixing an index i, we note that the subgroup B_i generated by $\{g_1', \ldots, g_{i-1}', g_i, g_{i+1}', \ldots, g_n'\}$ is another highest abelian subgroup. By Theorem 2.8.2, the intersection $A \cap B_i = \langle g_i \rangle$ is commensurable to a group generated by a subset of $\{\alpha_1, \ldots, \alpha_n\}$. This means that some power $\alpha_j^{k_j}$ generates a finite index subgroup of $\langle g_i \rangle$. Without loss of generality we can relabel α_j as α_i, so we've shown that $\alpha_i^{k_i} = g_i^{n_i}$ for some n_i. Note that two components g_i and g_j cannot be powers of the same α_k. Since i was arbitrary, we've now found that the set $\{g_1^{n_1}, \ldots, g_n^{n_n}\}$ generates a finite-index subgroup $A'' < A'$, and $g_i^{n_i}$ acts trivially on each C_j with $i \neq j$.

By the Flat Torus Theorem of [BH99], there is an A-invariant n-dimensional flat E contained in X on which A acts properly and cocompactly. The flat E is constructed as the convex hull of $A \cdot x_0$, where x_0 is some point in $\mathrm{Min}(A)$. Note that E is not, in general, a subcomplex. Recall that the hyperplane H is skewered by $(g_1, \ldots, g_n)$, so H must intersect every CAT(0) axis of $(g_1, \ldots, g_n)$. In particular, H intersects the CAT(0) axis γ which contains x_0. By construction, γ is contained in the flat E, so H intersects E. The product of quasi-lines $C_1 \times \cdots \times C_n$ is constructed in [WW15] as the dual cube complex to the set of hyperplanes intersecting E, so H must be a hyperplane of C_i for some i. As $C_1 \times \cdots \times C_n$ is the dual cube complex to a set of hyperplanes of X, there is an embedding of $C_1 \times \cdots \times C_n$ into X which is convex in the combinatorial metric. We will often refer to $C_1 \times \cdots \times C_n$ as a subcomplex of X for this reason.

Suppose H is dual to C_i. We showed that $g_j^{n_j}$ acts trivially on C_i when $i \neq j$. Therefore for each g_j with $j \neq i$, no axis of g_j can intersect H. Otherwise, g_j would skewer H and $g_j^{n_j}$ could not act trivially on C_i. Since $(g_1, \ldots, g_n)$ skewers H and every g_j with $j \neq i$ does not skewer H, it follows that g_i must skewer H. We've shown that for any hyperplane H of X, if $(g_1, \ldots, g_n)$ skewers H, then exactly one g_i skewers H and every g_j for $j \neq i$ does not skewer H.

The next step is to show that no $g_j \in G_j$ skewers H when $j \neq i$. Suppose there exists an element $g_j \in G_j$ that skewers H. Let $g_i \in G_i$ be the element we've already shown skewers H. We can construct a highest abelian subgroup A containing g_i and g_j generated by $\{g_1, \ldots, g_i, \ldots, g_j, \ldots g_n\}$, where the generators other than g_i and g_j are arbitrary infinite order elements from their respective factors. By our previous argument, A acts properly and cocompactly on a product of quasi-lines $C_1 \times \cdots \times C_n$ and there is an n_k for each k such that $g_k^{n_k}$ acts trivially on C_l when $k \neq l$. An axis of g_i intersects H, so H must be a hyperplane dual to C_i. However, an axis for g_j also intersects H, so no power of g_j acts trivially on C_i, a contradiction. We've shown that for any hyperplane H of X, there is a well-defined index i such that if $g_j \in G_j$ skewers H, then $j = i$.

The last element of the proof that remains to be shown is that if $i \neq j$ then every $H_i \in \mathcal{H}_i$ intersects every $H_j \in \mathcal{H}_j$. Recall that $\mathcal{H}_i$ is defined to be the set of hyperplanes skewered by some element $g_i \in G_i$. The proof up to this point confirms that this index i is well defined.

Choose hyperplanes $H_1 \in \mathcal{H}_1, \ldots, H_n \in \mathcal{H}_n$. We will show that these hyperplanes all pairwise intersect. For each i, by the definition of $\mathcal{H}_i$, there is an element $g_i \in G_i$ that skewers H_i. The set of these skewering elements generates a rank-n free-abelian subgroup A. By the Flat Torus Theorem of [BH99], there is an A-invariant n-dimensional flat E contained in X on which A acts properly and cocompactly. The only proper, cocompact action of $\mathbb{Z}^n$ on $\mathbb{E}^n$ is by translations, so every g_i acts on E as a translation. Fix indices i and j and a point $x \in H_i$. The element g_j acts as a translation and does not skewer H_i, so the orbit $\langle g_j \rangle x$ must be contained in H_i. However, g_j does skewer H_j, so $\langle g_j \rangle x$ must contain points on both sides of H_j. Because H_i is convex, H_i must intersect H_j.

We've shown that the set of hyperplanes $\mathcal{H}$ of X can be partitioned as $\mathcal{H}_1 \sqcup \ldots \sqcup \mathcal{H}_n$, where $\mathcal{H}_i$ denotes the set of hyperplanes skewered by some element $g_i \in G_i$. In addition, for any $H_i \in \mathcal{H}_i$ and $H_j \in \mathcal{H}_j$, H_i and H_j must intersect. Therefore X splits as a product of cube complexes

$$X = X_1 \times \cdots \times X_n,$$

where X_i is the dual cube complex to $\mathcal{H}_i$. □

Lemma 3.1.2. *Let $G_1, \ldots, G_n$ be non-elementary hyperbolic groups. Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, essentially, and without inversion by cubical isometries on a CAT(0) cube complex X. From Lemma 3.1.1, X decomposes as a product $X = X_1 \times \cdots \times X_n$. Then each X_i is irreducible.*

Proof. This follows fairly directly from Corollary 2.9.1 of [CS11], which is as follows. Suppose X is a locally compact CAT(0) cube complex and G is a discrete group acting cocompactly on X. If X is a product of n unbounded cube complexes, then G contains a subgroup isomorphic to $\mathbb{Z}^n$.

Choose a hyperplane H_i belonging to X_i. Because G acts essentially on X, there must be orbit points arbitrarily far from H_i in X. This implies that there must be infinitely many hyperplanes in X that do not intersect H_i. As H_i intersects the hyperplanes dual to every factor other than X_i, all of these hyperplanes must be dual to X_i. Because X_i is finite dimensional and locally finite, we've shown that X_i is unbounded.

Suppose X_i decomposes as a product of CAT(0) cube complexes $X_i = X_i' \times X_i''$. Because the action of G is essential, X_i' and X_i'' must be unbounded. Then G acts on a product of $n+1$ unbounded cube complexes, so G contains a subgroup isomorphic to $\mathbb{Z}^{n+1}$. This is a contradiction, as G is a product of n hyperbolic groups. □

Lemma 3.1.3. *Let $G_1, \ldots, G_n$ be non-elementary hyperbolic groups. Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, essentially, and without inversion by cubical isometries on a CAT(0) cube complex X. From Lemma 3.1.1, X decomposes as a product $X = X_1 \times \cdots \times X_n$. Then every $g \in G$ acts on X as a product of cubical isometries $\mu_1 \times \cdots \times \mu_n$, where μ_i is a cubical isometry of X_i.*

Proof. First, by Lemma 3.1.2 and Lemma 3.1.1, we have that X decomposes as a product of n irreducible cube complexes

$$X = X_1 \times \cdots \times X_n.$$

In addition, from Proposition 2.6 of [CS11] every cubical isometry of X preserves this decomposition up to permutation of isomorphic factors. We will show that if the permutation of isomorphic factors unduced by any $g \in G$ sends a factor X_j to X_k, then $j = k$. It is enough to show this holds for every $g_i \in G_i$.

Choose $g_i \in G_i$ and suppose the permutation induced by g_i maps X_j to X_k. Let $g_j \in G_j$ be an element which skewers a hyperplane H_j of X_j. Then $g_i H_j$ is a hyperplane of X_k. Since g_j skewers H_j, $g_i g_j g_i^{-1}$ skewers $g_i H_j$. If $i \neq j$, then g_i and g_j commute. Thus $g_i g_j g_i^{-1} = g_j \in G_j$. If $i = j$, then $g_i g_j g_i^{-1} \in G_j$. In either case, $g_i g_j g_i^{-1}$ can only skewer hyperplanes of X_j. Since $g_i g_j g_i^{-1}$ skewers $g_i H_j$, a hyperplane of X_k, we must have $j = k$ as desired. □

Lemma 3.1.4. *Let $G_1, \ldots, G_n$ be non-elementary hyperbolic groups. Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, essentially, and without inversion by cubical isometries on a CAT(0) cube complex X. From Lemma 3.1.1, X decomposes as a product $X = X_1 \times \cdots \times X_n$. We claim that any $g_i \in G_i$ acts on $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$ as an elliptic isometry.*

Proof. Let $\hat{X}_i$ denote $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. Fix an element $g_i \in G_i$ and note that g_i has a well-defined action on $\hat{X}_i$ by Lemma 3.1.3. By Lemma 3.1.1, g_i can only skewer hyperlanes of X_i. Since g_i does not skewer any hyperplanes of $\hat{X}_i$, it must act as an elliptic isometry on $\hat{X}_i$. □

Lemma 3.1.5. *Let $G_1, \ldots, G_n$ be non-elementary hyperbolic groups. Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, essentially, and without inversion by cubical isometries on a CAT(0) cube complex X. From Lemma 3.1.1, X decomposes as a product $X = X_1 \times \cdots \times X_n$. Then G_i acts cocompactly on X_i.*

Proof. We showed that X_i is irreducible, so by Proposition 5.1 of [CS11], X_i contains a pair H, V of strongly separated hyperplanes. By the Double Skewering Lemma, also from [CS11], there must be some $g_i \in G_i$ such that

$$g_i H^+ \subsetneq V^+ \subset H^+.$$

Therefore a CAT(0) axis γ of g_i intersects H and V. Using Lemma 6.1 of [CS11], we ascertain that every axis of g_i must lie in a bounded Hausdorff neighborhood of of γ. The set $\mathrm{Min}(g_i)$ is the union of the axes of g_i and has the form $\mathrm{Min}(g_i) = Y \times \mathbb{R}$ by Theorem II.6.8 of [BH99]. Because the axes of g_i lie in a bounded Hausdorff neighborhood of γ, Y must be bounded.

Choose some $g_j \in G_j$ with $i \neq j$. From [BH99], we know that $\mathrm{Min}(g_i)$ is non-empty, complete, and convex. Since g_i and g_j commute, $\mathrm{Min}(g_i)$ is also invariant under the action of g_j. It then follows from Proposition II.6.2(4) of [BH99] that $\mathrm{Min}(g_i) \cap \mathrm{Min}(g_j)$ must be non-empty. In particular, g_j fixes a point in $\mathrm{Min}(g_i)$, as g_j acts elliptically on X_i. From Theorem II.6.8 of [BH99], g_j acts as an isometry on each factor of $\mathrm{Min}(g_i) = Y \times \mathbb{R}$. This means that g_j acts on $\mathbb{R}$ with a fixed point, so g_j must either reflect across that fixed point or act trivially. Since g_i and g_j commute and g_i acts as a translation on $\mathbb{R}$, g_j must act trivially. We've shown that $g_j(y_0, t_0)$ must stay within $Y \times \{t_0\}$ for any $y_0 \in Y$ and $t_0 \in \mathbb{R}$. Let $\hat{G}_i = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$ denote the product excluding G_i. It follows that for any $g \in \hat{G}_i$, $g(y_0, t_0)$ must be in $Y \times \{t_0\}$, so $\hat{G}_i(y_0, t_0) \subset Y \times \{t_0\}$. Since Y is bounded, this implies that the $\hat{G}_i$-orbit of any point in $\mathrm{Min}(g_i)$ is bounded.

Suppose G_i does not act cocompactly on X_i. The cube complex X admits a proper, compact action, so it must be finite-dimensional and locally finite. As X_i is a factor of X, X_i must also be finite-dimensional and locally finite. Thus if G_i does not act cocompactly on X_i, there must exist an *unbounded* sequence of vertices (x_k) in X_i such that $x_k = g_k x_1$ for some $g_k \in \hat{G}_i$. Choose an element $g_i \in G_i$ that skewers a pair of strongly separated hyperplanes of X_i as above, and let γ be a CAT(0) axis for g_i containing a point x_0 of X_i. Since $x_0 \in \mathrm{Min}(g_i)$, we know that $\hat{G}_i x_0$ is bounded. Let K be the diameter of $\hat{G}_i x_0$ and D the distance between x_0 and x_1. Then every point in $\hat{G}_i x_1$ must stay within a $(D+K)$-neighborhood of x_0, contradicting the unboundedness of the sequence. By contradiction, G_i acts cocompactly on X_i. $\square$

Lemma 3.1.6. *Let $\Gamma = G_1 \times G_2$ act by cubical isometries on a CAT(0) cube complex X such that the action of G_1 is cocompact. Then there is a finite index subgroup of G_2 that acts trivially on X/G_1.*

Proof. The *simplicial barycentric subdivision* of a cube is obtained by taking the cubical barycentric subdivision of the cube, attaching the central vertex of this subdivision to every other vertex by an edge, and adding an n-simplex wherever there is a 1-skeleton of an n-simplex. The simplicial barycentric subdivision of a cube complex is the simplicial complex obtained by taking the simplicial barycentric subdivision of each cube simultaneously.

Let X' be the simplicial barycentric subdivision of X. Then Γ acts on X' by simplicial automorphisms and X'/G_1 is a simplicial complex. This is because every isometry of a cube preserves the cell structure of its simplicial barycentric subdivision. Since G_2 commutes with G_1, there is a well-defined action by simplicial automorphisms of G_2 on X'/G_1. This action can be represented as a homomorphism $\phi\colon G_2 \to \mathrm{Aut}(X'/G_1)$. The simplicial complex X'/G_1 is compact, so its automorphism group is finite. Therefore $\phi^{-1}(1)$ is a finite index subgroup of G_2 that acts trivially on X'/G_1. □

Lemma 3.1.7. *For every i there exists a finite index subgroup $G_i' < G_i$ such that G_i acts trivially on X_j, where $i \neq j$. In addition, G_i acts properly on X_i.*

Proof. Since G_i acts cocompactly on X_i, by Lemma 3.1.6 there is a finite index subgroup $\overline{G_{ij}} < G_j$ such that $\overline{G_{ij}}$ acts trivially on X_i/G_i. This induces a homomorphism $\varphi\colon \overline{G_{ij}} \to G_i$. Since every element of G_i commutes with every element of G_j, $\overline{G_{ij}}$ must be mapped into the center of G_i. Because G_i is non-elementary hyperbolic, it has a finite center. Therefore $G_{ij}' = \varphi^{-1}(1)$ is a finite index subgroup of $\overline{G_{ij}}$ that acts trivially on X_i. Let G_j' be $\cap_{i\neq j} G_{ij}'$. Then G_j' is a finite index subgroup of G_j which acts trivially on X_i if $i \neq j$. Since every G_j' with $j \neq i$ acts trivially on X_i and G acts properly on X, G_i must act properly on X_i.

□

Proof of Theorem 3.1.1. We know from Lemma 3.1.1 and Lemma 3.1.3 that X splits as a product $X_1 \times \cdots \times X_n$ and every element of G acts on $X_1 \times \cdots \times X_n$ as a product of isometries. Lemma 3.1.2 tells us that each X_i must be irreducible. By Lemma 3.1.5 and Lemma 3.1.7, each G_i acts properly and cocompactly on X_i. Lastly, by Lemma 3.1.7, each G_i has a finite index subgroup G_i' which acts trivially on each factor X_j with $j \neq i$. □

3.2 An Arbitrary Dimension Gap

Recall that the CAT(0) dimension of a group is the minimum covering dimension of the CAT(0) spaces on which it acts geometrically. Recall also that the CAT(0) cubical dimension of a group is the minimum dimension of the CAT(0) cube complexes on which it acts geometrically. The goal of this section is to prove the existence of an infinite family of groups which have an arbitrarily large gap between their CAT(0) dimension and their CAT(0) cubical dimension using Theorem 3.1.1.

First, we need to find a family of hyperbolic groups which has CAT(0) cubical dimension strictly larger than its CAT(0) dimension. We start with a family of 3-manifolds which are not homeomorphic to a 3-dimensional non-positively curved cube complex from [Li02]. We then prove that the fundamental groups of these 3-manifolds cannot act geometrically on a 3-dimensional CAT(0) cube complex. Having established that, we take direct products of these fundamental groups and use Theorem 3.1.1 to show that as we take products of more of these groups, the dimension gaps of the product groups increases without bound.

First, we need to show that if a closed, hyperbolic 3-manifold group acts properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex, then the hyperplane stabilizers are virtually surface subgroups.

Proposition 3.2.1. *Let G be a closed, hyperbolic 3-manifold group which acts properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex X. Then every hyperplane in X has S^1 boundary.*

Proof. Since G is a closed, hyperbolic 3-manifold group, $\partial X \cong S^2$. Let U be an arbitrary hyperplane of X. Fix a basepoint $x_0 \in U$. Every hyperplane is essential, so there are infinite geodesic rays $\gamma^+ \subset U^+$ and $\gamma^- \subset U^-$ based at x_0. Since U is convex and separates X into two halfspaces, any path in ∂X from γ^- to γ^+ must include a geodesic contained in U. In particular, this means that ∂U separates ∂X into at least two components. We will show that there are exactly two complementary components.

Since G is a closed, hyperbolic 3-manifold group, there is a G-equivariant quasi-isometry ϕ from X to $\mathbb{H}^3$ which extends to a homeomorphism on the boundary. Let $H < G$ be $\operatorname{stab} U$. Since G acts geometrically on X, H acts geometrically on U. In addition, H is quasi-convex, as it is a hyperplane stabilizer. Therefore ∂H is homeomorphic to the image $\phi(\partial U)$ in $\partial\mathbb{H}^3$. In this proof, we will use ∂X, ∂G, and

S^2 interchangeably since they are equivalent up to G-equivariant homeomorphism.

First, we'll show that ∂H must have covering dimension 1. We claim that the subset ∂H must be nowhere dense. Recall that a set is nowhere dense if the interior of its closure is empty. Suppose ∂H is not nowhere dense. Then the closure of ∂H has non-empty interior, which we call U. Since the endpoints of loxodromic elements are dense in ∂H, there must be some element $g \in H$ such that $g^\infty \in U$. Let x be an arbitrary point in ∂G. There is some $k \in \mathbb{N}$ such that $g^k x \in U$. This shows that $g^k x$ is in the closure of $g^{-k}\partial H$. However, ∂H is g-invariant, so x is in the closure of ∂H. In addition, the image of ∂H in ∂G is properly embedded, so it must be closed. Therefore $x \in \partial H$. Since an arbitrary point of ∂X is in ∂H, $\partial H = \partial X$. This is a contradiction, proving that ∂H is nowhere dense.

By Theorem 19 in [Sch12], ∂H must have covering dimension 0 or 1. If ∂H has covering dimension 0, it must be totally disconnected. But then it could not separate two points in S^2, so $\partial \operatorname{stab} H$ must have covering dimension 1.

The next step is to show that ∂H is connected. The set ∂H separates two points in $\partial G \cong S^2$, so it cannot be two-ended. Suppose ∂H is infinite-ended. Then using Stallings theorem about ends of groups and Dunwoody's accessibility theorem, we get that H is the fundamental group of a finite graph of groups with finite edge groups. The full group G is torsion-free, so H decomposes as a finite free product of groups which are one-ended or two-ended. If all of these groups are two-ended, then H has Cantor set boundary, which we already showed is impossible. Therefore at least one free factor F must be one-ended.

Using some known results, we will show that the boundary of a one-ended hyperbolic group contains an embedded circle. From the work of Bowditch, Levitt, and Swarup in [Bow99b, Bow98, Bow99a, Lev98, Swa96], we know that ∂F is locally connected. Bestvina-Mess showed in [BM91] that as ∂F is locally connected, it must have no global cut point. By Chapter X, Section II, Theorem 4 from [Kur92], Vol. 2, each connected component of $\partial X \setminus \partial F$ must be an open disk, so ∂F contains an embedded circle.

There is an embedded copy of ∂F in ∂H for each coset of F in H ([MS15]), so there must be two embedded circles S_1 and S_2 in ∂H. These circles do not intersect because the cosets do not intersect. There must be convex subsets H_1 and H_2 of $\mathbb{H}^3$ with $\partial H_i = S_i$ for $i = 1, 2$. Using the Jordan Curve Theorem, one can show that $S^2 \cong \partial G \setminus (S_1 \cup S_2)$ has three components. Since H_1 and H_2 are convex, it follows that $\mathbb{H}^3 \setminus (H_1 \cup H_2)$ must have three unbounded components corresponding to the three components of the boundary. This is a contradiction, since the hyperplane U separates X into exactly two complementary components. We've shown that ∂H cannot be two-ended or infinite-ended, so it must be one-ended.

Suppose $\partial G \setminus \partial H$ has at least three components. If we take the convex hull of ∂H in $\mathbb{H}^3$, we find that $\mathbb{H}^3 \setminus \mathrm{Hull}(\partial H)$ has at least three complementary components. As above, this is a contradiction since U separates X into two unbounded complementary components.

We have now shown that ∂X can be decomposed as the disjoint union of ∂U along with two open disks, D_1 and D_2, each of which is the interior of the boundary of a halfspace of H. Let $\mathrm{Fr}\, D$ denote the frontier of the set D, and note that $\mathrm{Fr}\, D_1 \subset \partial H$ and $\mathrm{Fr}\, D_2 \subset \partial U$. A point in ∂U is defined to be an equivalence class of geodesic rays contained in U. However, any geodesic contained in U is also contained in U^+ and U^-. Therefore $\partial U \subset \partial U^+$ and $\partial U \subset \partial U^-$. It follows that $\partial U \subset \mathrm{Fr}\, D_1$ and $\partial U \subset \mathrm{Fr}\, D_2$, proving that $\partial U \cong S^1$. $\square$

We now state Tao Li's result and begin the proof of our corollary.

Theorem 3.2.1 (Tao Li-2002, [Li02])**.** *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus. Suppose that M does not contain any closed, nonperipheral, embedded, incompressible surfaces. Then only finitely many Dehn fillings on M can yield 3-manifolds that are homeomorphic to 3-dimensional non-positively curved cube complexes.*

Corollary 3.2.1. *Given any natural number k, there is an infinite family of groups which have CAT(0) cubical dimension at least k larger than their CAT(0) dimension.*

First, we prove the existence of an infinite family of hyperbolic 3-manifold groups with a finite dimension gap between their CAT(0) dimension and their CAT(0) cubical dimension. We then use Theorem 3.1.1 to show that we can find groups with larger dimension gaps by taking direct products of our groups with finite dimension gaps.

Proposition 3.2.2. *There exists an infinite family of hyperbolic 3-manifold groups with a finite gap between their CAT(0) dimension and their CAT(0) cubical dimension.*

Proof. Let $\overline{M}$ be an orientable and irreducible 3-manifold whose boundary is an incompressible torus that does not contain any closed, nonperipheral, embedded, incompressible surfaces. For example, $\overline{M}$ could be a figure-8 knot complement. Let M be a 3-manifold obtained by a Dehn filling of $\overline{M}$ that is not homeomorphic to any 3-dimensional non-positively curved cube complex. This manifold exists by Theorem 3.2.1 from [Li02]. Let Γ be $\pi_1(M)$. Then Γ acts properly, cocompactly, and by isometries on $\mathbb{H}^3$. This shows that Γ has CAT(0) dimension 3.

By Proposition 3.2.1, $\partial H = S^1$. It follows that $\operatorname{stab} H$ is virtually Fuchsian since $\operatorname{stab} H$ acts properly and cocompactly on H. Let G be a finite-index surface subgroup of $\operatorname{stab} H$. Then by Theorem 1.1 of Freedman-Hass-Scott, there exists an immersed, least-area surface F in M such that $\pi_1 F = G$. From Tao Li, we know that every surface in M fails to satisfy the 4-plane property. That is to say, there exist four pairwise-intersecting lifts $g_1\tilde{F}, g_2\tilde{F}, g_3\tilde{F}, g_4\tilde{F}$ of F, where $g_i \in \Gamma \setminus G$. Since F is least-area, we get from Lemma 6.4 of Freedman-Hass-Scott that the stabilizers of these lifts must intersect in cyclic subgroups. If $\operatorname{stab} H \cap g \operatorname{stab} H g^{-1} \cong \mathbb{Z}$, then the boundaries ∂H and ∂gH must intersect in a cut pair. Then since $H \cup \partial H$ separates $X \cup \partial X$ into two components and there are points of ∂gH in each, H and gH must intersect.

We've found four hyperplanes which pairwise intersect. It is a basic fact that the intersection of all four hyperplanes is non-empty and contained in a cube of dimension at least four. This contradicts our assumption that X was 3-dimensional, proving that Γ cannot act properly and cocompactly on a 3-dimensional CAT(0) cube complex. Therefore Γ has a dimension gap of at least 1 between its CAT(0) dimension and its CAT(0) cubical dimension.

By the work of Bergeron-Wise in [BW12], Γ acts properly and cocompactly on a finite-dimensional CAT(0) cube complex, so Γ does not have an infinite dimension gap. $\square$

Now we use Proposition 3.2.2 to construct an infinite family of groups with an arbitrarily large gap between their CAT(0) dimension and their CAT(0) cubical dimension.

Proof of Corollary 3.2.1. Let Γ be a hyperbolic 3-manifold group with a dimension gap between its CAT(0) dimension and its CAT(0) cubical dimension. In order to prove our claim, we will show that $\oplus_{i=1}^{k}\Gamma$ has a dimension gap of at least k. Let G be $\oplus_{i=1}^{k}\Gamma$, and suppose G acts properly and cocompactly on a CAT(0) cube complex $\overline{X}$. Then G acts properly, cocompactly, and essentially on the essential core X of $\overline{X}$. Note that $\dim \overline{X} \geq \dim X$. By Theorem 3.1.1, X splits as a product of CAT(0) cube complexes $X = X_1 \times \ldots \times X_k$, and each factor of $\oplus_{i=1}^{k}\Gamma$ act properly and cocompactly on the corresponding factor of X. It follows that each factor of X must have dimension at least 4, proving that the dimension of X (and $\overline{X}$) is at least $4k$. However, G has a natural proper and cocompact action on $\oplus_{i=1}^{k}\mathbb{H}^3$, which has dimension $3k$. Therefore G has dimension gap at least k. $\square$

Chapter 4

Generalization of Hyperbolic Products

4.1 Products of (AIP) Groups

A group G has the abelian intersection property **(AIP)** if it contains a finite collection of highest abelian subgroups whose intersection is trivial. Many of the proofs in this chapter are similar to those of chapter 3. However, in this chapter we cannot guarantee a decomposition into irreducible cube complexes. Though we are able to get the nearly the same results, this adds a new layer of complexity to the proofs. While reading this chapter, it can be helpful to flip back and refer to the corresponding proof in chapter 3.

The property (AIP) is strictly weaker than hyperbolicity. For example, the free product of any two infinite groups which act on CAT(0) cube complexes satisfies (AIP). Choosing a highest rank free-abelian subgroup from each free factor results in a set of highest abelian subgroups with trivial intersection.

The main theorem of the chapter follows.

Theorem 4.1.1. *Let $G = G_1 \times \cdots \times G_n$ satisfy (AIP) and have finite center, where each G_i is an infinite group. Suppose G acts properly, cocompactly, and essentially on a CAT(0) cube complex X. Then*

- *X decomposes as a product of CAT(0) cube complexes $X_1 \times \cdots \times X_n$ and each $g \in G$ acts on X as a product of isometries $\mu_1 \times \cdots \times \mu_n$;*
- *every factor G_i acts on X_i properly, cocompactly, and essentially; and*
- *every factor G_i contains a finite-index subgroup G_i' that acts trivially on G_j when $i \neq j$.*

Note that the statement of this theorem matches Theorem 3.1.1 with the notable exception that the cubical factors $X_1, \ldots, X_n$ are not guaranteed to be irreducible.

As in chapter 2, we will prove the theorem using a series of lemmas and propositions.

Proposition 4.1.1. *Let $G = G_1 \times \cdots \times G_n$ be a group acting properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex X such that G has (AIP). Any highest abelian subgroup of G contains a finite index subgroup generated by $\{\alpha_1, \ldots, \alpha_m\}$ acting on a product of quasi-lines $C_1 \times \cdots \times C_m$ in X, where each α_i only acts nontrivially on its corresponding quasi-line C_i.*

Proof. Let A be a highest abelian subgroup of G. Note that such a subgroup must be of the form $A = A_1 \times \cdots \times A_n$, where $A_i < G_i$ is highest abelian for each i. Let $m_i = \operatorname{rank} A_i$ for each i. Since every G_i satisfies (AIP), it contains a finite collection of highest abelian subgroups $A_i, A_i', \ldots, A_i^{(l_i)}$ whose intersection is trivial. Note that we've added the aforementioned A_i into the collection, and the intersection is still trivial. It is possible that different factors have collections of highest abelian subgroups of different sizes. In order to make the proof easier, we will pad the smaller collections to make them as large as the largest collection. More formally, for every i, let $A_i^{(k)} = A_i^{(l_i)}$ when $l_i < k \leq \max l_i$.

Let

$$C_{11} \times \cdots \times C_{1m_i} \times C_{21} \times \cdots \times C_{nm_n}$$

be a product of quasi-lines on which A acts by the Cubical Flat Torus Theorem ([WW15]). Our goal is to show that A_1 has a finite index subgroup which acts trivially on every quasi-line other than those with initial index 1. Using Lemma 2.8.1 ([WW15]), there is a finite index subgroup of A with a preferred set of generators

$$S = \{\alpha_{11}, \ldots, \alpha_{1m_1}, \alpha_{21}, \ldots, \alpha_{nm_n}\},$$

where α_{ij} acts trivially on every quasi-line other than C_{ij}. By Theorem 2.8.2 ([WW15]), for every k the intersection

$$(A_1 \times \cdots \times A_n) \cap \left(A_1 \times A_2^{(k)} \times A_3^{(k)} \times \cdots \times A_n^{(k)}\right)$$

is commensurable with the subgroup generated by a subset of S. In particular, this means that each of these intersections has a finite index free-abelian subgroup generated by powers of the α_{ij}. Let B be one such intersection and $C < B$ the finite index subgroup generated by powers of the α_{ij}. Let $C' < B'$ be another such pair. Then $C \cap C'$ is finite index in $B \cap B'$. If we take a finite number of intersections, we will still have that

$$\cap C^{(k)} < \cap B^{(k)}$$

is finite index. We chose the $A_i^{(k)}$ so that the intersection of all of the $A_i^{(k)}$ for a given index i is trivial. Therefore we can take finitely many intersections to get

$$\bigcap_{k=0}^{\max l_i} A_1 \times A_2^{(k)} \times A_3^{(k)} \times \cdots \times A_n^{(k)} = A_1 \times \{1\} \times \cdots \times \{1\}, \tag{4.1}$$

which has a finite-index subgroup generated by powers of the α_{ij}. We have shown that A_1 has a finite index subgroup $\overline{A_1}$ generated by powers of, up to relabeling, $\{\alpha_{11}, \ldots, \alpha_{1m_1}\}$.

If we carry out this procedure for the rest of the A_i, we can construct the desired finite index subgroup $\overline{A} < A$ as

$$\overline{A} = \overline{A_1} \times \cdots \times \overline{A_n}.$$

□

Lemma 4.1.1. *Let $G_1, \ldots, G_n$ be finitely generated groups, where each G_i satisfies (AIP). Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, and essentially by cubical isometries on a CAT(0) cube complex X. Then X decomposes as a product of CAT(0) cube complexes $X = X_1 \times \cdots \times X_n$.*

Proof. We will prove this lemma in three stages. First, note that by Proposition 3.12 of [CS11], every hyperplane is skewered by some element of G. We show that if a hyperplane H is skewered by $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$, then in fact H is skewered by exactly one g_i. Then we prove that if H is skewered by some other element $(g_1', \ldots, g_n')$, it must be the ith component g_i' that skewers H. Therefore we can partition the set of hyperplanes into n sets, where a hyperplane H is in the ith set $\mathcal{H}_i$ if it is skewered by some element of G_i. To finish the proof, we show that every hyperplane in $\mathcal{H}_i$ intersects every hyperplane of $\mathcal{H}_j$ where $i \neq j$.

First, we show that if a hyperplane H is skewered by $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$, then it is skewered by exactly one g_i. Let $\mathcal{H}$ denote the set of hyperplanes of X. Fix a hyperplane $H \in \mathcal{H}$. The group G is acting properly, cocompactly, and essentially on X, so by Proposition 3.12 of [CS11], H is skewered by some element $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$. For each i, let A_i be a highest abelian subgroup of G_i containing a power of g_i and let $m_i = \operatorname{rank} A_i$. Let $A = A_1 \times \cdots \times A_n$. Then the free-abelian subgroup A is highest in G. By Theorem 2.8.1, A acts properly and cocompactly on a convex subcomplex $Y = C_{11} \times \cdots \times C_{1m_1} \times C_{21} \times \cdots \times C_{nm_n}$, where each C_{ij} is a quasi-line. By Proposition 4.1.1, there is a finite index subgroup $\overline{A} < A$ generated by $\{\alpha_{11}, \ldots \alpha_{nm_n}\}$ so that α_{ij} acts trivially on every quasi-line other than C_{ij}. It follows that for every i, there is a power $g_i^{k_i}$ that can be written as a product of the α_{ij} with initial index i.

By the Flat Torus Theorem of [BH99], there is an A-invariant flat E contained in X on which A acts properly and cocompactly. The flat E is constructed as the convex hull of $A \cdot x_0$, where x_0 is some point in $\operatorname{Min}(A)$. Note that E is not, in general, a subcomplex. Recall that the hyperplane H is skewered by $(g_1, \ldots, g_n)$, so every CAT(0) axis of $(g_1, \ldots, g_n)$ must cross H. In particular, the CAT(0) axis γ which contains x_0 crosses H. By construction, γ is contained in the flat E, so H intersects E. The product of quasi-lines Y is constructed in [WW15] as the dual cube complex to the set of hyperplanes intersecting E, so H must be a hyperplane of Y.

The hyperplane H is dual to some quasi-line of Y. Suppose it's dual to a quasi-line C with initial index 1. We showed that each $g_i^{k_i}$ can only act nontrivially on the quasi-lines with initial index i. Therefore for each g_i with $i \neq 1$, no axis of g_i can intersect H. Since $(g_1, \ldots, g_n)$ skewers H and every g_i with $i \neq 1$ does not skewer H, it follows that g_1 must skewer H. We've shown that for any hyperplane H of X, if $(g_1, \ldots, g_n)$ skewers H, then exactly one component g_i skewers H and every

g_j for $j \neq i$ does not skewer H.

The next step is to show that if $(g_1, \ldots, g_n) \in G$ and $(g_1', \ldots, g_n') \in G$ skewer H, then the components of each which skewer H must have the same index. We assume that g_1 skewers H, and we'll show that g_1' must also skewer H. We know that some component $g_j \in G_j$ skewers H. For the sake of contradiction, we assume that $j \neq 1$. We can construct a highest abelian subgroup A containing powers of g_1 and g_j. By our previous argument, A acts properly and cocompactly on a product of quasi-lines $C_{11} \times \cdots \times C_{nm_n}$ and there are k_1 and k_j such that $g_1^{k_1}$ and $g_j^{k_j}$ only act nontrivially on quasi-lines with initial indices 1 and j, respectively. An axis of g_1 crosses H, so H must be a hyperplane dual to a quasi-line C with initial index 1. However, an axis for g_j also crosses H, so no power of g_j can act trivially on C, a contradiction. Therefore j must be 1, and the index of components that skewer a given hyperplane in well-defined.

Recall that $\mathcal{H}_i$ is defined to be the set of hyperplanes skewered by some element $g_i \in G_i$. The last element of the proof that remains to be shown is that if $i \neq j$ then every $H_i \in \mathcal{H}_i$ intersects every $H_j \in \mathcal{H}_j$.

Choose hyperplanes $H_1 \in \mathcal{H}_1, \ldots, H_n \in \mathcal{H}_n$. We will show that these hyperplanes all pairwise intersect. For each i, by the definition of $\mathcal{H}_i$, there is an element $g_i \in G_i$ that skewers H_i. There is a highest abelian subgroup A containing a power of each of these skewering elements. Let $p = \operatorname{rank} A$. For convenience, we will rename the lowest power of g_i that is in A to g_i, as we have no need of the original g_i. By the Flat Torus Theorem of [BH99], there is a p-dimensional A-invariant flat E contained in X on which A acts properly and cocompactly. The only proper, cocompact action of $\mathbb{Z}^p$ on $\mathbb{E}^p$ is by translations, so every g_i acts on E as a translation. Fix indices i and j and a point $x \in H_i$. The element g_j acts as a translation and does not skewer H_i, so the orbit $\langle g_j \rangle x$ must be contained in H_i. However, g_j does skewer H_j, so $\langle g_j \rangle x$ must contain points on both sides of H_j. Because H_i is convex, H_i must intersect H_j.

We've shown that the set of hyperplanes $\mathcal{H}$ of X can be partitioned as $\mathcal{H}_1 \sqcup \ldots \sqcup \mathcal{H}_n$, where $\mathcal{H}_i$ denotes the set of hyperplanes skewered by some element $g_i \in G_i$. In addition, for any $H_i \in \mathcal{H}_i$ and $H_j \in \mathcal{H}_j$, H_i and H_j must intersect. Therefore X splits as a product of cube complexes

$$X = X_1 \times \cdots \times X_n,$$

where X_i is the dual cube complex to $\mathcal{H}_i$. $\square$

Lemma 4.1.2. *Let $G_1, \ldots, G_n$ be groups satisfying Property (AIP). Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex X. Following Lemma 4.1.1, X decomposes as a product of CAT(0) cube complexes $X_1 \times \cdots \times X_n$. Every $g \in G$ acts on $X_1 \times \cdots \times X_n$ as a product $\mu_1 \times \cdots \times \mu_n$ of cubical isometries.*

Proof. First, by Proposition 2.6 of [CS11] X has a unique decomposition into irreducible factors

$$X = X_{11} \times \cdots \times X_{1m_1} \times X_{21} \times \cdots \times X_{nm_n},$$

where $X_i = X_{i1} \times \cdots \times X_{im_i}$ for every i. In addition, every cubical isometry of X preserves this decomposition up to permutation of isomorphic factors. We will show that if the permutation of isomorphic factors induced by any $g \in G$ sends a factor X_j with initial index j to X_k with initial index k, then $j = k$. It is enough to show this holds for every $g_i \in G_i$.

Choose $g_i \in G_i$ and suppose the permutation induced by g_i maps a factor X_j with initial index j to a factor X_k with initial index k. Let $g_j \in G_j$ be an element which skewers a hyperplane H_j of X_j. Then g_iH_j is a hyperplane of X_k. Since g_j skewers H_j, $g_ig_jg_i^{-1}$ skewers g_iH_j. If $i \neq j$, then g_i and g_j commute. Thus $g_ig_jg_i^{-1} = g_j \in G_j$. If $i = j$, then $g_ig_jg_i^{-1} \in G_j$. In either case, $g_ig_jg_i^{-1}$ belongs to G_j and can therefore only skewer hyperplanes with initial index j. Since $g_ig_jg_i^{-1}$ skewers g_iH_j, a hyperplane belonging to a factor with initial index k, we must have $j = k$. In particular, this means that the irreducible factors of some X_i may only be interchanged with other irreducible factors of X_i. $\square$

Lemma 4.1.3. *Let $G_1, \dots, G_n$ be groups with finite center satisfying (AIP). Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex X. Following Lemma 4.1.1, X decomposes as a product of CAT(0) cube complexes $X_1 \times \cdots \times X_n$. Every $g_i \in G_i$ acts as an elliptic isometry on $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$.*

Proof. Let $\hat{X}_i$ denote $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. Fix an element $g_i \in G_i$ and note that g_i has a well-defined action on $\hat{X}_i$ by Lemma 4.1.2. By Lemma 4.1.1, g_i can only skewer hyperlanes of X_i. Since g_i does not skewer any hyperplanes of $\hat{X}_i$, it must act as an elliptic isometry on $\hat{X}_i$. □

Lemma 4.1.4. *Let $G_1, \dots, G_n$ be groups with finite center satisfying (AIP). Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex X. Following Lemma 4.1.1, X decomposes as a product of CAT(0) cube complexes $X_1 \times \cdots \times X_n$. Then G_i acts cocompactly on X_i for every i.*

Proof. Fix an index i. The factor X_i may not be irreducible, but it can be decomposed as a product of irreducible CAT(0) cube complexes

$$X_i = X_{i1} \times \cdots \times X_{im_i}.$$

Let $\overline{G} = \overline{G_1} \times \cdots \times \overline{G_n}$ be the finite index subgroup of G whose elements do not permute the irreducible factors of X and note that $\overline{G}$ acts properly, cocompactly, and essentially by automorphisms on X. Fix an index j corresponding to an irreducible factor X_{ij} of X_i. By construction, every element of $\overline{G}$ has a well-defined action on X_{ij}. For the moment, we will restrict to discussing actions on X_{ij} rather than the full cube complex X.

By Proposition 5.1 of [CS11], X_{ij} contains a pair H_j, V_j of strongly separated hyperplanes. By the Double Skewering Lemma, also from [CS11], there must be some $g_j = (g_{1j}, \dots, g_{nj}) \in \overline{G}$ such that

$$g_j H_{ij}^+ \subsetneq V_{ij}^+ \subset H_{ij}^+.$$

Since g_j skewers a hyperplane of X_i, g_{ij} must skewer the same hyperplanes of X_i by Lemma 4.1.1. Therefore a CAT(0) axis γ of g_{ij} in X_{ij} intersects H_{ij} and V_{ij}. Using Lemma 6.3 of [CS11], we see that every CAT(0) axis of g_{ij} in X_{ij} must lie in a bounded Hausdorff neighborhood of γ. The set $\mathrm{Min}(g_{ij})$ is the union of the axes of g_{ij} and has the form

$$\mathrm{Min}(g_{ij}) = Y \times \mathbb{R}$$

by Theorem II.6.8 of [BH99]. Because the axes of g_{ij} lie in a bounded Hausdorff neighborhood of γ, Y must be bounded.

Choose some $g \in \overline{G_{i'}}$, where $i \neq i'$. From [BH99], we know that $\mathrm{Min}(g_{ij})$ is non-empty, complete, and convex. Since $i \neq i'$, g_{ij} and g commute. Then following Proposition II.6.2 of [BH99], $\mathrm{Min}(g_{ij})$ is invariant under the action of g and $\mathrm{Min}(g_{ij}) \cap \mathrm{Min}(g)$ must be non-empty. From Lemma 4.1.3, we know that the action of g on X_{ij} is elliptic, and so $\mathrm{Min}(g)$ consists of the fixed points of g. We've shown that g fixes some point on $\mathrm{Min}(g_{ij}) = Y \times \mathbb{R}$.

From Theorem II.6.8 of [BH99], g acts as a product of isometries $\mu_Y \times \mu_{\mathbb{R}}$ on $Y \times \mathbb{R}$. Since g has a fixed point in $\mathrm{Min}(g_{ij})$, $\mu_{\mathbb{R}}$ must have a fixed point. Therefore $\mu_{\mathbb{R}}$ must be either a reflection or the identity. Recall that we chose g so that it commutes with g_{ij}. Therefore since g_{ij} acts on $\mathbb{R}$ as a translation, so $\mu_{\mathbb{R}}$ must be the identity. We've shown that the image of $(y_0, t_0) \in \mathrm{Min}(g_{ij})$ under the action of any $g \in \overline{G_{i'}}$ with $i \neq i'$ is contained in the slice $Y \times \{t_0\}$. Since Y is bounded, we've shown that $\overline{G_1} \times \cdots \times \overline{G_{i-1}} \times \overline{G_{i+1}} \times \cdots \times \overline{G_n}$ has bounded orbits in X_{ij}.

Since $\overline{G_1} \times \cdots \times \overline{G_{i-1}} \times \overline{G_{i+1}} \times \cdots \times \overline{G_n}$ has bounded orbits in each irreducible factor X_{ij} of X_i, it must have bounded orbit in X_i. The group $\overline{G_1} \times \cdots \times \overline{G_{i-1}} \times \overline{G_{i+1}} \times \cdots \times \overline{G_n}$ is a finite index subgroup of $\hat{G}_i = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$, so $\hat{G}_i$ must also have bounded orbit in X_i. We assumed that $G_i \times \hat{G}_i$ acts cocompactly on X_i, so it follows that the action of G_i on X_i must be cocompact. □

Lemma 4.1.5. *Let $G_1, \ldots, G_n$ be groups with finite center satisfying (AIP). Suppose $G = G_1 \times \cdots \times G_n$ acts properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex X. Following Lemma 4.1.1, X decomposes as a product* *of CAT(0) cube complexes $X_1 \times \cdots \times X_n$. For every i there exists a finite index subgroup $G_i' < G_i$ such that G_i acts trivially on X_j, where $i \neq j$. In addition, G_i acts properly on X_i.*

Proof. Since G_i acts cocompactly on X_i, by Lemma 3.1.6 there is a finite index subgroup $\overline{G_{ij}} < G_j$ such that $\overline{G_{ij}}$ acts trivially on X_i/G_i. This induces a homomorphism $\varphi\colon \overline{G_{ij}} \to G_i$. Since every element of G_i commutes with every element of G_j, $\overline{G_{ij}}$ must be mapped into the center of G_i. Because G_i has finite center, $G_{ij}' = \varphi^{-1}(1)$ is a finite index subgroup of $\overline{G_{ij}}$ that acts trivially on X_i. Let G_j' be $\cap_{i \neq j} G_{ij}'$. Then G_j' is a finite index subgroup of G_j which acts trivially on X_i if $i \neq j$. Since every G_j' with $j \neq i$ acts trivially on X_i and G acts properly on X, G_i must act properly on X_i.

$\square$

Proof of Theorem 4.1.1. We know from Lemma 4.1.1 and Lemma 4.1.2 that X splits as a product $X_1 \times \cdots \times X_n$ and every element of G acts on $X_1 \times \cdots \times X_n$ as a product of isometries. By Lemma 4.1.4 and Lemma 3.1.7, each G_i acts properly and cocompactly on X_i. Lastly, by Lemma 3.1.7, each G_i has a finite index subgroup G_i' which acts trivially on each factor X_j with $j \neq i$. $\square$

Chapter 5

Product of (AIP) with Abelian

5.1 A Motivating Example

This example is from [BR96]. It illustrates some of the challenge of defining the boundary of a CAT(0) group. Unless otherwise specified, I will denote the visual boundary of a space X by ∂X. Let $\Gamma = F_2 \times \mathbb{Z} = \langle a, b\rangle \times \langle c\rangle$. There is a natural action of F_2 on its standard Cayley graph, a 4-valent tree. Let T be the standard Cayley graph of F_2, and let $\langle a, b\rangle \cong F_2$ act on it by left multiplication. We can construct an action of Γ on $X = T \times \mathbb{R}$ as follows:

$$a \circ (x, r) = (a \cdot x, r)$$

$$b \circ (x, r) = (b \cdot x, r + 2)$$

$$c \circ (x, r) = (x, r + 1)$$

Note that this action is proper and cocompact.

What happens when we try to embed ∂F_2 into ∂X? The most obvious way to do this is to pick a basepoint $x_0 \in X$ and continously extend the quasi-isometry $(F_2 \times \{0\}) \circ x_0$ to the boundary. We'll see that the map $f\colon \partial F_2 \to \partial X$ induced by this quasi-isometry is not continuous. We do this by constructing a sequence of points in $y_n \in \partial F_2$ that converge to y such that $f(y_n)$ does not converge to $f(y)$.

Let $\pi\colon T \times \mathbb{R} \to T \times \{0\}$ be the projection map onto the first factor. Consider the sequence $y_n = (a^n b^n)^\infty \in \partial F_2$. This sequence converges to $y = a^\infty$ as $n \to \infty$. For each n, the sequence of points $(a^n b^n)^k \circ x_0$ defines an infinite geodesic ray γ_n based at x_0 representing $f(y_n) \in \partial X$. The ray γ_n is a line in the Euclidean plane $\pi(\gamma) \times \mathbb{R}$. Since the action of $a^n b^n$ translates by distance $2n$ in the tree direction and $2n$ in the $\mathbb{R}$ direction, γ_n meets $T \times \{0\}$ at an angle of $\pi/4$ for every n. This implies that a geodesic representing $\lim_{n\to\infty} \gamma_n$ must also meet $T \times \{0\}$ at an angle of $\pi/4$.

However, the boundary point corresponding to a^∞ is contained in $\partial(T \times \{0\})$, so this map of ∂F_2 into ∂X is not continuous.

The previous example shows that the natural quasi-isometric embedding $(g,0) \mapsto (g,0) \circ x_0$ does not always extend continuously to a map of the boundary $\partial F_2 \to \partial X$. However, [BR96] shows that we can construct an embedding $h\colon \partial F_2 \to \partial X$ that extends to a homeomorphism between the suspension $\Sigma(\partial F_2)$ and ∂X using a minset decomposition of X.

Unfortunately, $h(\partial F_2)$ is not always $F_2 \times \{0\}$-invariant in ∂X. In fact, it is possible that the only element in $F_2 \times \mathbb{Z}$ that leaves $h(\partial F_2)$ invariant is the identity. It is an exercise to show this is the case with the action

$$
\begin{aligned}
a * (x,r) &= (a \cdot x, r+e) \\
b * (x,r) &= (b \cdot x, r+\pi) \\
c * (x,r) &= (x, r+1).
\end{aligned}
$$

While we cannot in general find a nontrivial stabilizer of $f(\partial F_2)$ in the CAT(0) space setting, we can find a "twisted" finite-index subgroup of F_2 that preserves $f(\partial F_2)$ when X is a CAT(0) cube complex.

5.2 Product Decomposition

In the previous chapter, we showed that if a product of groups with (AIP) acts properly, cocompactly, and essentially by automorphisms on a CAT(0) cube complex, we get a nice product decomposition of the cube complex. However, any CAT(0) group with an infinite center fails to satisfy (AIP). Suppose Γ is of the form $G \times A$, where G satisfies (AIP) and $A \cong \mathbb{Z}^p$. The main theorem from this section shows that we can still recover a product decomposition of a CAT(0) cube complex this group acts on, and the group is close to acting as a product action.

Theorem 5.2.1. *Let G be a group with finite center satisfying (AIP) and $A \cong \mathbb{Z}^p$. Suppose $\Gamma = G \times A$ acts properly, cocompactly, and essentially on a CAT(0) cube* *complex X. Then*

- *X decomposes as a product of CAT(0) cube complexes $X_A^{\perp} \times X_A$, where X_A is a product of p quasi-lines;*

- *Γ has a finite-index subgroup $\Gamma' = G' \times A'$ that acts on $X_A^{\perp} \times X_A$ as a product action; and*

- *furthermore, G' is isomorphic to a subgroup of G.*

An important note about the statement is that the group G' is not a subgroup of G from the original product decomposition. It is isomorphic to a subgroup of G, but we have multiplied its generators by elements of A. Consider the example action of $\langle a, b\rangle \times \langle c\rangle \cong F_2 \times \mathbb{Z}$ on its Cayley graph from the previous section defined by

$$a \circ (x, r) = (a \cdot x, r)$$

$$b \circ (x, r) = (b \cdot x, r + 2)$$

$$c \circ (x, r) = (x, r + 1).$$

The slice $T \times \{0\}$ is not $\langle a, b\rangle$-invariant. However, there is a skewed copy of F_2 generated by a and bc^{-2} that does leave the slice $T \times \{0\}$ invariant. In this example, $\Gamma' = G' \times A'$ would be the full group decomposed as $\langle a, bc^{-2}\rangle \times \langle c\rangle$.

Proposition 5.2.1. *Let G satisfy (AIP) and have finite center, and let $A \cong \mathbb{Z}^p$. Suppose $\Gamma = G \times A$ acts properly, cocompactly, and essentially on a CAT(0) cube complex X. Then X decomposes as a product of CAT(0) cube complexes $X = X_A^{\perp} \times X_A$, where X_A is a product of p quasi-lines.*

Proof. Denote by $\mathcal{H}$ the set of hyperplanes of X, $\mathcal{H}_A$ the hyperplanes skewered by some element of A, and $\mathcal{H}_G$ the hyperplanes not skewered by any element of A. Since $\mathcal{H}_G$ and $\mathcal{H}_A$ partition $\mathcal{H}$, $\mathcal{H}_G$ is the same as $\mathcal{H}_A^{\perp}$, as defined in section 2.10. Therefore by Proposition 2.10.2, $X \cong X_A^{\perp} \times X_A$.

Let $A_1, \ldots, A_n$ be a finite collection of highest abelian subgroups of G with trivial intersection. Then each product $A_i \times A$ is highest in Γ. Since $A_1 \times A$ is highest it acts on a convex subcomplex Y that is a product of quasi-lines. The same argument as in the proof of Proposition 4.1.1 implies that there is a finite index subgroup $\overline{A}$ of A which acts trivially on all but p of the quasi-lines of Y. By construction, the product of these p quasi-lines is the dual cube complex to the set of hyperplanes skewered by some element of A, proving that X_A is a product of p quasi-lines. □

Lemma 5.2.1. *Let $\Gamma = G \times A$ act properly, cocompactly, essentially, and without inversion on a CAT(0) cube complex X, where G satisfies (AIP) and $A \cong \mathbb{Z}^p$. Let $X \cong X_A^{\perp} \times X_A$ as in Proposition 2.10.2. Then there is some subgroup $A' < A$ so that A' acts trivially on $X_A^{\perp}$, and the elements of G that stabilize the A'-orbits of X_A form a finite index subgroup of G.*

Proof. The subgroup A is central, so the sets $\mathcal{H}_A$ and $\mathcal{H}_A^{\perp}$ are Γ-invariant. Therefore the actions of Γ on $X_A = X(\mathcal{H}_A)$ and $X_A^{\perp} = X(\mathcal{H}_A^{\perp})$ are well-defined.

Since A commutes with G, there is a well-defined action of A on $X_A^{\perp}/G$. By assumption, the action of Γ on $X_A^{\perp}$ is cocompact. Each element $a \in A$ acts on $X_A^{\perp}$ as an elliptic isometry, so by Proposition 2.10.4, A must have a global fixed point in $X_A^{\perp}$. The action of G on $X_A^{\perp}$ must be cocompact, or we could pick a sequence of points in $X_A^{\perp}/G$ going arbitrarily far from the global fixed point of A. This sequence of points would maintain their distance from the fixed point of A in $X_A^{\perp}/(G \times A)$, contradicting the cocompactness of the action of $G \times A$ on $X_A^{\perp}$.

By Lemma 3.1.6 there is a finite index subgroup $\overline{A} < A$ such that every $a \in \overline{A}$ acts trivially on $X_A^{\perp}/G$. We've shown that each element $a \in \overline{A}$ has the same action on $X_A^{\perp}$ as some element of G, inducing a homomorphism $\phi \colon \overline{A} \to G$. Every element of $\overline{A}$ commutes with every element of G, so the image $\phi(\overline{A})$ must be contained in the center $Z(G)$. Recall that G has finite center. It follows that $A' = \phi^{-1}(1)$ is a finite index subgroup of $\overline{A}$ which acts trivially on $X_A^{\perp}$.

Since A' is finite index in A, $G \times A'$ acts cocompactly on X_A. Therefore by Lemma 3.1.6, there is a finite index subgroup $G' < G$ that acts trivially on X_A/A', proving our claim.

$\square$

Proof of Theorem 5.2.1. From Proposition 2.10.2, X decomposes as a product $X = X_A^{\perp} \times X_A$. In Proposition 5.2.1, we showed that X_A is a product of p quasi-lines. Lemma 5.2.1 tells us that there are finite index subgroups A' of A and $\overline{G}$ of G such that A' acts trivially on $X_A^{\perp}$ and $\overline{G}$ acts trivially on X_A/A'. If $\phi\colon \Gamma \to \mathrm{Aut}(X_A)$ is the action of Γ on X_A, then for every $g \in \overline{G}$ there exists an $a \in A'$ such that $\phi(g) = \phi(a)$. Define G' to be

$$G' = \{ga^{-1} | g \in \overline{G},\ \phi(g) = \phi(a)\}.$$

Because $\overline{G}$ is finite index in G, $G' \times A'$ is a finite index subgroup of Γ. By construction, $G' \times A'$ acts with a product action on $X_A^{\perp} \times X_A$. Therefore since $G' \times A'$ acts properly and cocompactly on X, it must be that G' and A' act properly and cocompactly on $X_A^{\perp}$ and X_A, respectively. $\square$

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