Stresses

Stresses

Body forces and surface forces (tractions)

Body force : applied to elements of mass (gravity)

$\underset{\sim}{b}\,\Delta m$ – force on Δm

$\underset{\sim}{b}$ – force / unit mass (for example, gravity: $-g\underset{\sim}{e}_3$)

switch to volume (for integration)

$= \underset{\sim}{b}\rho\,\Delta V$

$\underset{\sim}{b}\rho$ – force / unit volume

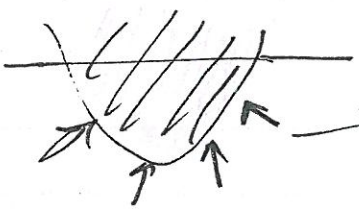
Total force on volume V : $\displaystyle\int_V \underset{\sim}{b}\rho\,dV$

Surface force : applied to elements of surface

$\underset{\sim}{t}\,\Delta S$

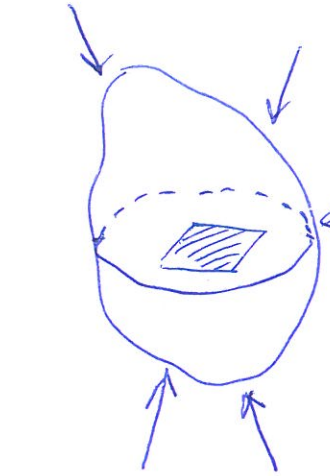
$\underset{\sim}{t}$ – traction (force per unit area)

Example : bouyancy force



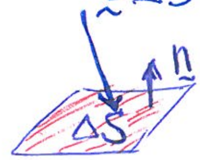
pressure from fluid (normal to surface)

<u>Stressed Solid</u>



in this cross-section:

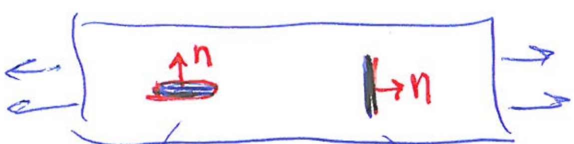
$\underset{\sim}{t}^{(n)}\Delta S$ - vector of force transmitted through ΔS



$\underset{\sim}{t}^{(n)}$ - traction vector (force per unit area)

indicates: surface element with <u>outward</u> unit normal n

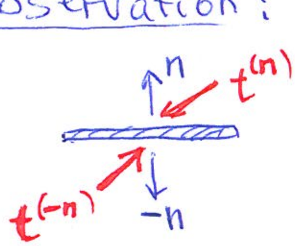
Clearly, $t^{(n)}$ depends on orientation $\underset{\sim}{n}$:



$\underset{\sim}{t}^{(n)} = 0$ $\qquad$ $t^{(n)}$ is maximal

<u>Examine</u> : the dependence $t^{(n)}(n)$

<u>Observation</u> : from equilibrium of a slice:

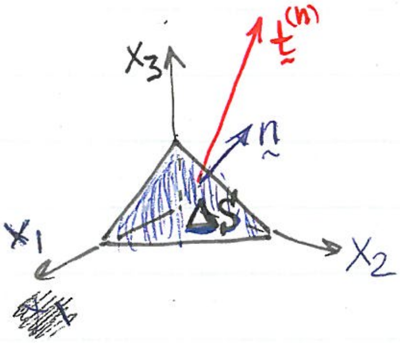


$$\underset{\sim}{t}^{(-n)} = -\underset{\sim}{t}^{(n)}$$

Equilibrium of volume element

To examine orient. dependence of traction $\underset{\sim}{t}^{(n)}$: consider small tetrahedron.

- ($\underset{\sim}{n}$ - unit normal to inclined plane)



bounded by: - coordinate planes:

$$\Delta S_i = \Delta S \, \underset{\sim}{n} \cdot \underset{\sim}{e}_i$$

- inclined plane ΔS (unit normal $\underset{\sim}{n}$)

Principal vector of all forces appl. to tetrahedron:

$$\underset{\sim}{t}^{(n)} \Delta S + \underset{\sim}{t}^{(-i)} \Delta S (\underset{\sim}{e}_i \cdot \underset{\sim}{n}) + \underset{\sim}{b} \rho \Delta V = 0$$

Divide over ΔS and use $\underset{\sim}{t}^{-(i)} = -\underset{\sim}{t}^{(i)}$

$$t^{(n)} = \underset{\sim}{t}^{(i)} (\underset{\sim}{e}_i \cdot \underset{\sim}{n}) - b\rho \underbrace{\frac{\Delta V}{\Delta S}}_{\to 0 \text{ for small element}}$$

$$\underset{\sim}{t}^{(n)} = \underbrace{\left(\underset{\sim}{t}^{(1)} \underset{\sim}{e}_1 + \dots\right)}_{\underset{\sim}{\sigma} \text{ (as a sum of three dyads)}} \cdot \underset{\sim}{n}$$

$$\therefore \quad \underset{\sim}{t}^{(n)} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n}$$ - linear vector f-n of vector arg.

traction vector

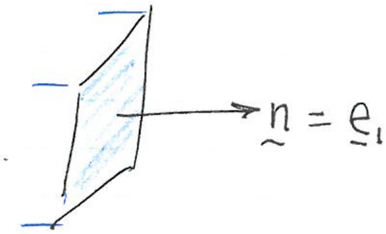
stress tensor

Meaning of stress components

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = t^{(n)}$$

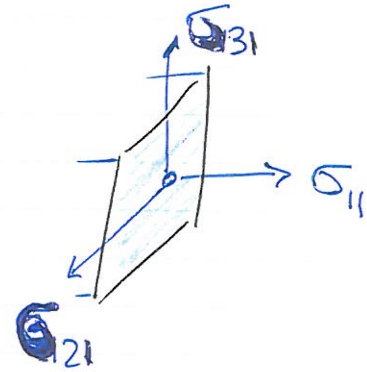
$\sigma_{ij}\, \underline{e}_i \underline{e}_j$

Choose $\underset{\sim}{n} = \underline{e}_1$

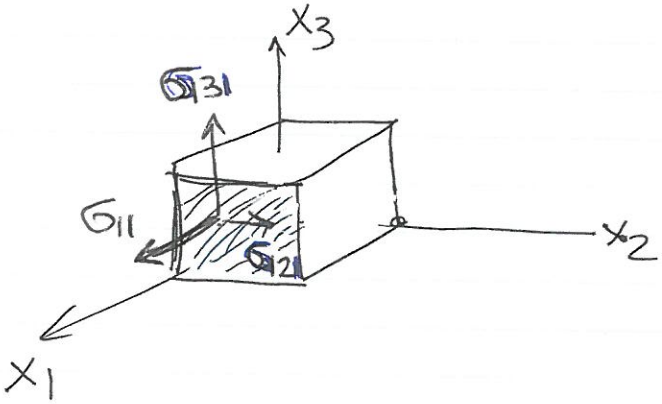


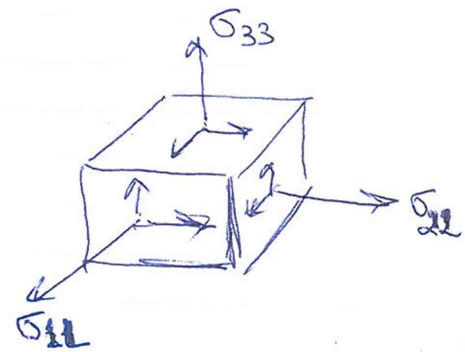
$$t^{(n)} = \sigma_{ij}\, e_i \underbrace{e_j \cdot \underline{e}_1}_{\delta_{1j}}$$

$$= \sigma_{i1}\, \underline{e}_i$$



∴ Stress components, in a given coord. system:





<u>Diagonal</u>: $\sigma_{11}, \sigma_{22}, \sigma_{33}$ >0 tensile; <0 compressive — normal stresses

<u>Off-diagonal</u>: $\sigma_{12}, \sigma_{23}, \sigma_{31}$ — shear stresses

$$p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$ – ave. hydrostatic stress

In fluids (either non-viscous (ideal) fluid: or any fluid in statics)

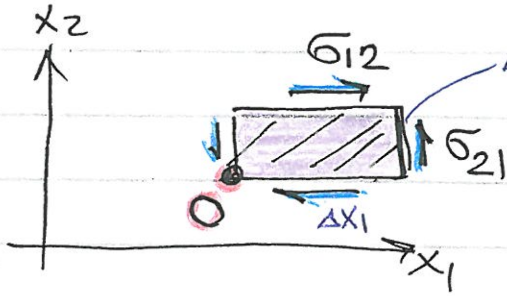
$$\underset{\sim}{\sigma} = -p\,\underset{\sim}{I}$$

fluid pressure

Implies: $\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = -p\,\underset{\sim}{n}$ (Pascal' law)

same normal pressure in all directions

Second equilibrium eq: total angular mom = 0



← planar element

Taking Mom_O :

$$\sigma_{12}\,\Delta x_1\,\Delta x_2 = \sigma_{21}\,\Delta x_1\,\Delta x_2$$

$$\Rightarrow \quad \sigma_{12} = \sigma_{21}$$

Generally :

$$\sigma_{ij} = \sigma_{ji}$$ – stress tensor is symm

→ 3 real eigenvalues

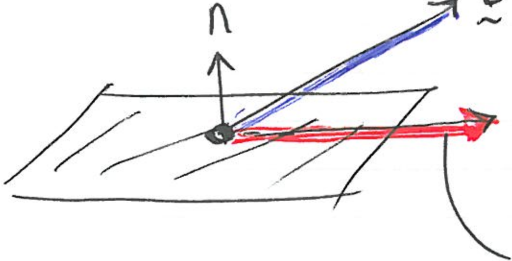
$$\Rightarrow \quad \underset{\sim}{\sigma} = \sigma_{I}\,\underline{e}_{I}\,\underline{e}_{I} + \sigma_{II}\,\underline{e}_{2}\,\underline{e}_{2} + \sigma_{III}\,\underline{e}_{III}\,\underline{e}_{III}$$

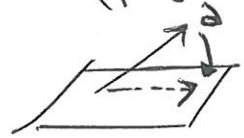
← normal stresses only in princ. directions

<u>Note</u>: shear traction in terms of projection tensor

$\underset{\sim}{\Theta} = \underline{I} - \underline{n}\,\underline{n}$

(projects $\underset{\sim}{a}$ onto plane)



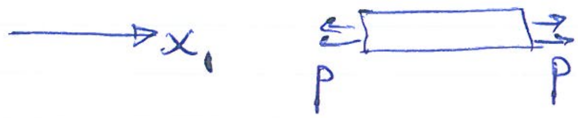


$\underset{\sim}{t}^{(n)} = \underline{\sigma} \cdot \underline{n}$

$\underline{\Theta} \cdot \underline{t}^{(n)} = \underline{\sigma} \cdot \underline{n} - (\underline{n} \cdot \underline{\sigma} \cdot \underline{n})\, \underline{n}$

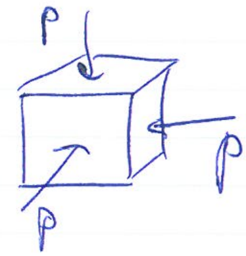
Examples

Uniaxial tension



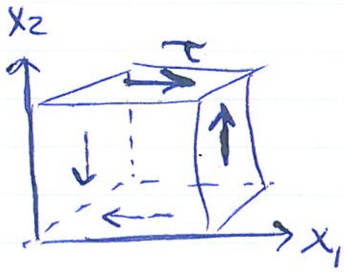
$$\underset{\sim}{\sigma} = \overset{\sigma_{11}}{p}\, \underline{e}_1 \underline{e}_1$$

Hydrostatic compression p



$$\underset{\sim}{\sigma} = -p\, \underbrace{\underset{\sim}{I}}_{\underline{e}_1\underline{e}_1 + \underline{e}_2\underline{e}_2 + \underline{e}_3\underline{e}_3}$$

Shear



$$\underset{\sim}{\sigma} = \tau(\underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_1)$$

Eigenvalue problem (mathematically identical to shear ϵ)

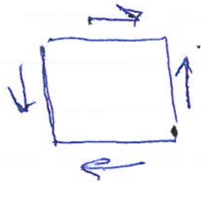
$$\det \begin{vmatrix} -\lambda & \tau \\ \tau & -\lambda \end{vmatrix} = 0$$

$$\lambda_{1,2} = \pm \tau$$

Principal representation

$$\underset{\sim}{\sigma} = \tau(e_I e_I - e_{II} e_{II})$$

(tension & compression)



princ. orientation

since stresses are local

equilibrium of small element:

princ. vector = 0 $\Downarrow$ stress tensor introduced

$$\underline{t}^{(n)} = \underline{\sigma} \cdot \underline{n}$$

orient. dependence of $\underline{t}^{(n)}$

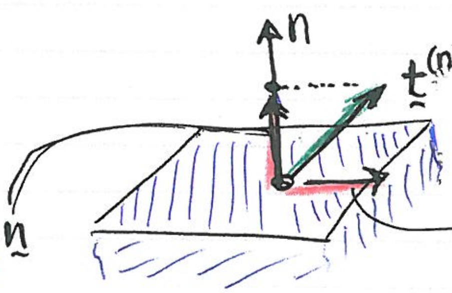
$$t_i^{(n)} = \sigma_{ij} n_j$$

ang. mom. = 0 $\Downarrow$

$$\sigma_{ij} = \sigma_{ji}$$

stress tensor is symm; three real eigenvalues (principal stresses)

Normal and shear components of $\underline{t}^{(n)}$ (traction vector)



$\underline{t}^{(n)} = \underline{\sigma} \cdot \underline{n}$

$\underbrace{(\underline{t}^{(n)} \cdot \underline{n})}\,\underline{n}$

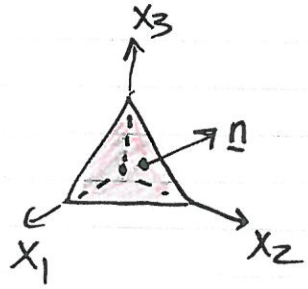
$(\underline{\sigma} \cdot \underline{n}) \cdot \underline{n} = \underline{n} \cdot \underline{\sigma} \cdot \underline{n}$

$$\underline{t}^{(n)} - (\underline{t}^{(n)} \cdot \underline{n})\,\underline{n}$$
$$= \underline{\sigma} \cdot \underline{n} - (\underline{n} \cdot \underline{\sigma} \cdot \underline{n})\,\underline{n}$$

In components :

$$\begin{cases} \underline{n} \cdot \underline{\sigma} \cdot \underline{n} = \sigma_{ij} n_i n_j \\ \underline{t}^{(n)} = \sigma_{ij} n_j \underline{e}_i \\ \underline{t}^{(n)} \cdot \underline{t}^{(n)} = \sigma_{ij} \sigma_{jk} n_i n_k = \underline{n} \cdot \underline{\sigma} \cdot \underline{\sigma} \cdot \underline{n} \end{cases}$$

Numerical example on normal & shear tractions



Shear stress: $\quad \underset{\sim}{\sigma} = \tau\,(\underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_1)$

Tractions induced on the plane forming equal angles with $x_1\ x_2\ x_3$

not $45°$! $\quad \arccos 1/\sqrt{3} \approx 55°$

$$\underset{\sim}{n} = \frac{1}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2 + \underline{e}_3)$$

$$\underline{t}^{(n)} = \underline{\sigma}\cdot\underline{n} = \frac{\tau}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2) = \underbrace{\tau\sqrt{\tfrac{2}{3}}}_{\text{magnitude}}\ \underbrace{\left(\frac{\underline{e}_1}{\sqrt{2}} + \frac{\underline{e}_2}{\sqrt{2}}\right)}_{\text{unit vector}}$$

Normal comp:

$$\underset{\sim}{t}^{(n)}\cdot\underline{n} = \frac{\tau}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2)\cdot\frac{1}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2 + \underline{e}_3) = \frac{2}{3}\tau$$

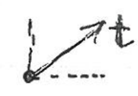
Shear

$$\underset{\sim}{t}^{(n)} - (\underline{t}^{(n)}\cdot\underline{n})\underset{\sim}{n} = \frac{\tau}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2) - \frac{2}{3}\tau\cdot\frac{1}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2 + \underline{e}_3)$$

$$= \underbrace{\tau\frac{\sqrt{2}}{3}}_{\text{magn.}}\ \underbrace{\left(\frac{1}{\sqrt{6}}\underline{e}_1 + \frac{1}{\sqrt{6}}\underline{e}_2 - \frac{2}{\sqrt{6}}\underline{e}_3\right)}_{\text{unit vector}}$$

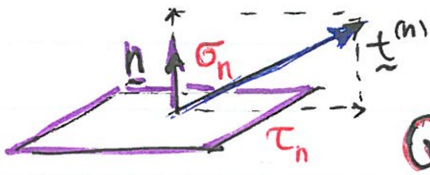
Check

- $\underset{\sim}{t}^{(n)}\cdot\underset{\sim}{t}^{(n)} = (\text{must be}) = \left(\frac{2}{3}\tau\right)^2 + \left(\tau\cdot\frac{\sqrt{2}}{3}\right)^2$ (Pithagorean th.)
- shear traction must be $\perp \underset{\sim}{n}$



$$\left(\frac{1}{\sqrt{6}}\underline{e}_1 + \frac{1}{\sqrt{6}}\underline{e}_2 - \frac{2}{\sqrt{6}}\underline{e}_3\right)\cdot n = 0$$

Mohr's Circles



Q: For given princ. stresses σ_I σ_{II} σ_{III}:
what are: possible combinations (σ_n, τ_n)
(as $\underset{\sim}{n}$ varies)

in principal axes:

$$\underset{\sim}{t}^{(n)} \cdot \underset{\sim}{n} = \sigma_n = \sigma_I n_I^2 + \sigma_{II} n_{II}^2 + \sigma_{III} n_{III}^2$$

$$\underset{\sim}{t}^{(n)} \cdot \underset{\sim}{t}^{(n)} = \sigma_n^2 + \tau_n^2 = \sigma_I^2 n_I^2 + \sigma_{II}^2 n_{II}^2 + \sigma_{III}^2 n_{III}^2$$

Pithagor. th

$$n_I^2 + n_{II}^2 + n_{III}^2 = 1$$

Given $(\sigma_I, \sigma_{II}, \sigma_{III})$: find orientation $\underset{\sim}{n}$ where certain combination (σ_n, τ_n) occurs

$\Rightarrow$ system as 3 eq-ns for n_I^2, n_{II}^2, n_{III}^2. Solving:

$$n_I^2 = \frac{\tau_n^2 + (\sigma_n - \sigma_{II})(\sigma_n - \sigma_{III})}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})}$$

$$n_{II}^2 = \frac{\tau_n^2 + (\sigma_n - \sigma_{III})(\sigma_n - \sigma_I)}{(\sigma_{II} - \sigma_{III})(\sigma_{II} - \sigma_I)}$$

$$n_{III}^2 = \frac{\tau_n^2 + (\sigma_n - \sigma_I)(\sigma_n - \sigma_{II})}{(\sigma_{III} - \sigma_I)(\sigma_{III} - \sigma_{II})}$$

For these eq-ns to be meaningful:
All right-hand parts ≥ 0

the 1^{st} inequality:

$$n_1^2 = \frac{\tau_n^2 + (\sigma_n - \sigma_{II})(\sigma_n - \sigma_{III})}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})} \geq 0$$

Denominator ≥ 0

$\Rightarrow$ Numerator must be ≥ 0

$$\tau_n^2 + (\sigma_n - \sigma_{II})(\sigma_n - \sigma_{III}) \geq 0$$

in case of equality: circle in (τ_n, σ_n) plane

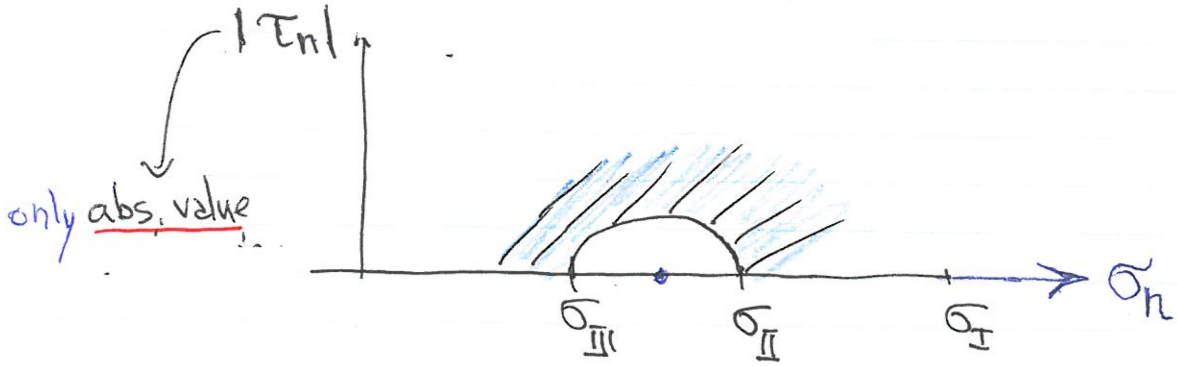
Indeed:

$$(\sigma_n - \sigma_{II})(\sigma_n - \sigma_{III}) = \left(\sigma_n - \frac{\sigma_{II}+\sigma_{III}}{2}\right)^2 + \sigma_{II}\sigma_{III} - \frac{1}{4}(\sigma_{II}+\sigma_{III})^2$$

$$\Rightarrow \quad \tau_n^2 + \Big(\sigma_n - \underbrace{\frac{\sigma_{II}+\sigma_{III}}{2}}_{\text{center of circle}}\Big)^2 = \underbrace{\frac{1}{4}(\sigma_{II} + \sigma_{III})^2 - \sigma_{II}\sigma_{III}}_{R^2}$$

$$R^2 = \frac{1}{4}(\sigma_{II} - \sigma_{III})^2$$

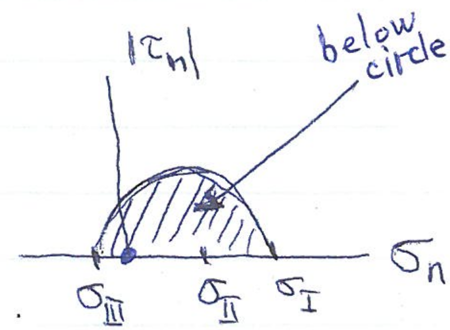
Inequality: above the circle



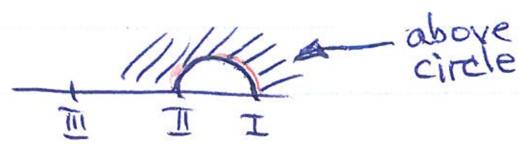
Similarly, the 2-nd inequality :

$$(n_{II}^2 =) \quad \frac{\tau_n^2 + (\sigma_n - \sigma_{III})(\sigma_n - \sigma_{I})}{(\sigma_{II} - \sigma_{III})(\sigma_{II} - \sigma_{I})} \geq 0$$

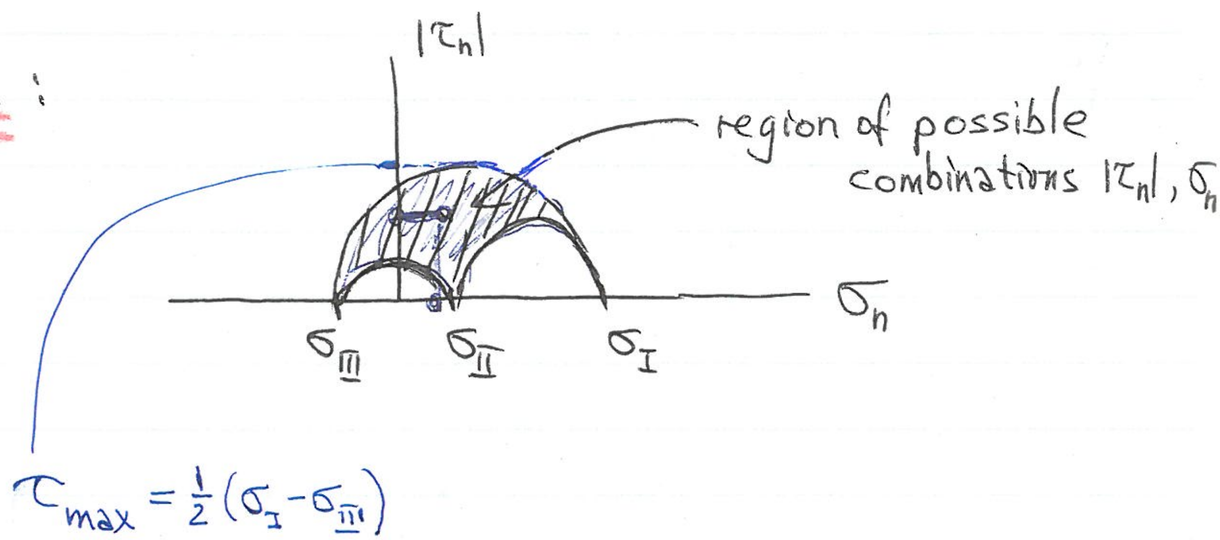
denominator < 0 $\Rightarrow$ numerator < 0



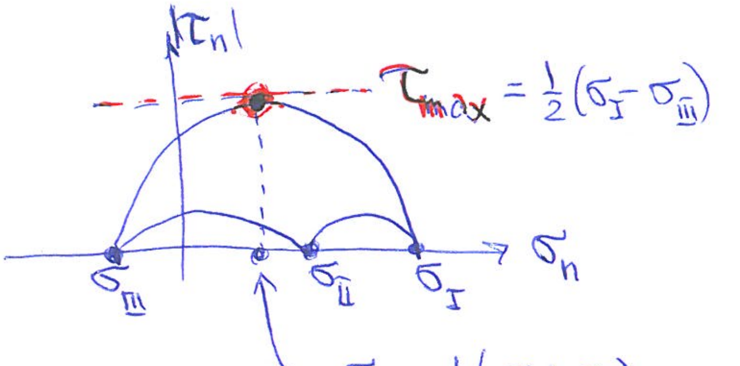
The 3rd inequality : positive numerator



Overall :



Orientation of planes where $\tau_n = \tau_{max}$?



$\sigma_n = \frac{1}{2}(\sigma_I + \sigma_{III})$

Substitute into

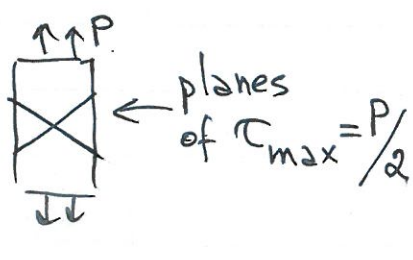
$$\begin{cases} n_I^2 = \dfrac{\tau_n^2 + (\sigma_n - \sigma_{II})(\sigma_n - \sigma_{III})}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})} \\ n_{II}^2 = \cdots \\ n_{III}^2 = \cdots \end{cases}$$

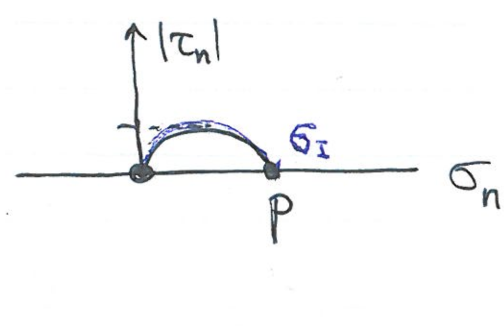
obtain $\quad n_I^2 = \frac{1}{2} \qquad n_{II}^2 = 0 \qquad n_{III}^2 = \frac{1}{2}$

$$n_I = \pm\frac{1}{\sqrt{2}} \qquad\qquad n_{III} = \pm\frac{1}{\sqrt{2}} \qquad (\cos 45^\circ)$$

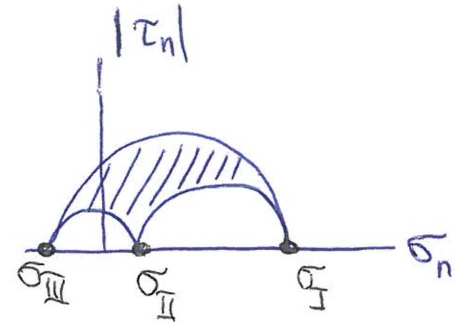
$\Rightarrow$ The plane of τ_{max} bysects the angle between directions of max / min princ. stresses, σ_I and σ_{III}

Example: Uniaxial tension





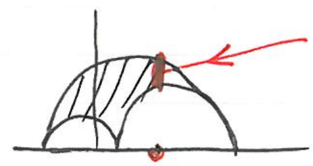
Utility of Mohr' circles



Visualize possible combinations

$(\sigma_n, |\tau_n|)$

- range of possible τ_n for given σ_n?



- range of possible σ_n for given τ_n?

Also: use eq-ns for n_I^2, n_{II}^2, n_{III}^2 to identify orientation n of the plane where a given combination (σ_n, τ_n) occurs

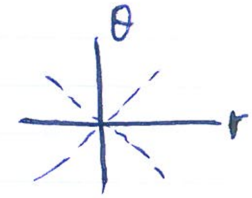
Not useful for: finding tractions on a given plane

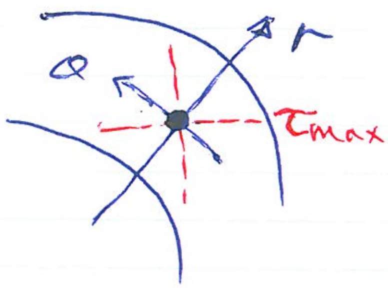
Use $\underset{\sim}{t}^{(n)} = \underline{\sigma} \cdot \underline{n}$

(and decompose into normal & shear if needed)

$$
\begin{cases}
\sigma_{rr} < 0 & \equiv \sigma_{III} \\
\sigma_{\theta\theta} > 0 & \equiv \sigma_{I} \\
\sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right) & \equiv \sigma_{II} \quad \text{– intermediate}
\end{cases}
$$

$\Rightarrow$ τ_{max} is reached on planes bysecting



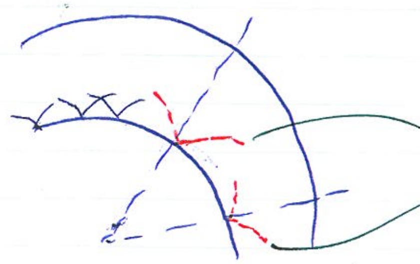


$$|\tau_{max}| = -\frac{1}{2}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = p\,\frac{a^2}{r^2} \cdot \frac{2b^2}{b^2 - a^2}$$

max. of τ_{max} – when r is minimal

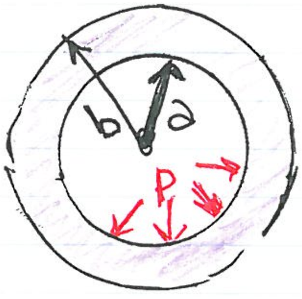
$\Rightarrow$ $r = a$ (inner wall).

Stresses are highest near inner boundary (see the solution) $\Rightarrow$



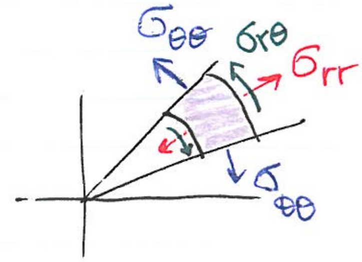
this is where plastic flow starts (ductile mat'l)

Example on using Mohr' circles



pipe
(pressurized)

Stresses in cylindr. coord. system:



Solution:
(from elasticity theory)

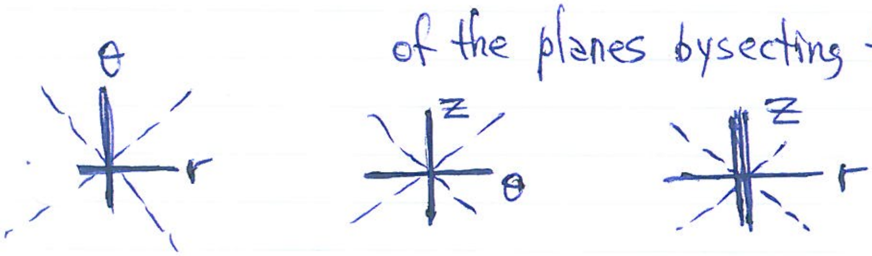
$$\begin{cases} \sigma_{rr} = p\,\dfrac{a^2}{r^2}\,\dfrac{r^2-b^2}{b^2-a^2} & < 0 \quad \text{compr.} \\[2ex] \sigma_{\theta\theta} = p\,\dfrac{a^2}{r^2}\,\dfrac{r^2+b^2}{b^2-a^2} & > 0 \quad \text{tensile} \\[2ex] \sigma_{r\theta} = 0\,, \quad \sigma_{rz} = \sigma_{\theta z} = 0 & \text{(symm.)} \end{cases}$$

plane strain (long cylinder, far from edges):

$$\sigma_{zz} = \nu\,(\sigma_{rr} + \sigma_{\theta\theta}) \qquad > 0$$

Q: What is τ_{max}, where does it occur?

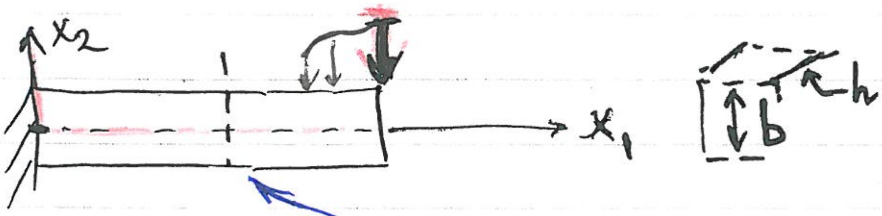
observation: r, θ, z - principal directions $\Rightarrow$ τ_{max} is reached on one of the planes bysecting the angles between them



which one?

Using $\underline{t}^{(n)} = \underline{\sigma} \cdot \underline{n}$

beam in bending



look at this cross-section,

Stress is a sum:

due to — bend. mom. M
— shear force V

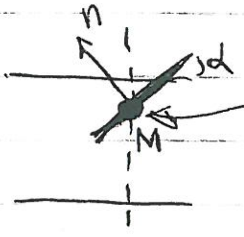
From M: $\underset{\sim}{\sigma} = \dfrac{M}{EI} x_2 \, \underline{e}_1 \underline{e}_1$ (normal axial stress σ_{11})

From V: parabolic shear traction distribution, tapers off at edges

(tapered off at edges: since $\sigma_{12} = \sigma_{21}$ and upper boundary is traction free)

$$\underset{\sim}{\sigma} = C \cdot \left(\frac{x_2}{b} + \frac{1}{2}\right)\left(\frac{x_2}{b} - \frac{1}{2}\right) (\underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_1)$$

$C = \dfrac{3V}{bh}$ — from $h \displaystyle\int_{-b/2}^{b/2} \sigma_{12} \, dx_2 = V$



traction on this plane, at (•) M (x_1, x_2)

$$\underline{n} = -\sin\alpha \, \underline{e}_1 + \cos\alpha \, \underline{e}_2$$

$$\underline{t}^{(n)} = \underline{\sigma} \cdot \underline{n} = \left\{ -\frac{M}{EI} x_2 \sin\alpha + \frac{3V}{hb}\left[\left(\frac{x_2}{b}\right)^2 - \frac{1}{4}\right]\cos\alpha \right\} \underline{e}_1 - \frac{3V}{bh}\left[\;\right] \sin\alpha \, \underline{e}_2$$

$$\sigma_n = \underline{t}^{(n)} \cdot \underline{n}, \text{ etc}$$

for example, if this is the line of glue connection, can examine normal & shear tractions, to check the strength of the connection

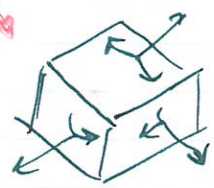
Change of stress components if coord. system is rotated:

$$\underline{\sigma} = \sigma_{ij}\, \underline{e}_i \underline{e}_j = \sigma'_{ij}\, \underline{e}'_i \underline{e}'_j$$

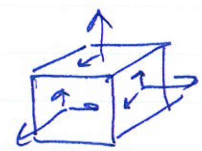
interrelate $\sigma_{ij} \leftrightarrow \sigma'_{ij}$

Similarly to strains: express $\underline{e}_i$ in terms of $\underline{e}'_i$, etc

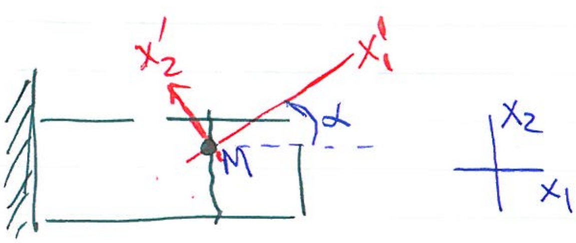
This allows to find stresses on this set of planes



provided this is known:



Example: in the beam problem:



Stress tensor at (·) M: has the form

$$\underline{\sigma} = A\, \underline{e}_1 \underline{e}_1 + B(\underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_1)$$

Express:
$$\begin{cases} \underline{e}_1 = \cos\alpha\, \underline{e}'_1 + \sin\alpha\, \underline{e}'_2 \\ \underline{e}_2 = -\sin\alpha\, \underline{e}'_1 + \cos\alpha\, \underline{e}'_2 \end{cases}$$

plug in. Then: σ'_{12} is coeff. at $\underline{e}'_1 \underline{e}'_2 + \underline{e}'_2 \underline{e}'_1$

σ'_{22} – coeff at $\underline{e}'_2 \underline{e}'_2$

σ'_{11} – coeff at $\underline{e}'_1 \underline{e}'_1$