RESEARCH PERSPECTIVES ON THE TEACHING AND LEARNING OF ALGEBRA

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Abstract

In this paper I will discuss different approaches to algebra adopted by researchers studying teaching and learning of algebra in the last thirty years (1977-2006). For each study reviewed in this paper, the goals are: to identify the central aspects of algebra (for example, algebra as the study of functions or algebra as the study of equation-solving), the mathematical problems researchers propose to use for teaching algebra and for evaluating learning, the used theoretical concepts, and main findings. In addition, I will provide a brief history of the evolution of the field of research on the teaching and learning of algebra, will discuss notions such as the role of equation solving techniques versus meaning making in the teaching and learning of algebra, the relation between arithmetic and algebra, algebra as a modeling tool, and the relation between algebra and proof.

Introduction

This paper focuses on different approaches to algebra employed by researchers in the study of the teaching and learning of algebra in the last thirty years (1977-2006). It joins the most recent efforts in synthesizing the state of the art in the learning and teaching of algebra at the middle and high school levels. The overarching goals of the paper are to review the researchers perspectives on algebra, to group their studies taking into account their foci, and to synthesize their epistemological foundations. With the notion of didactical transposition, Chevallard (1997) enlightened the process that objects of knowledge go through in order to become objects of teaching. My interest resides on the perspectives that the research community holds regarding algebra. Among other things, a researcher perspective is shown through their choice of problems used to collect data. The selection of problems used to do research is an important step in the process of shaping *the* mathematical object subject to research, in this case algebra. In the last ten years, there have been a variety of works reviewing where we come from as a research community, where we are in the domain, and what are the next steps for the future. Along these lines, the following works can be identified as seminal: *Approaches to Algebra* (Bednarz, Kieran, & Lee, 1996), *The Future of the Teaching and Learning of Algebra* or ICMI Study, (Stacey, Chick, & Kendal, 2004), and the chapter *Research on the Teaching and Learning of Algebra*, in the *Handbook of Research on the Psychology of Mathematics Education* (Kieran, 2006). These scholastic works sum up, from different perspectives and along different axes, the evolution of the research questions the community has asked itself along the last three decades. All of these reports agree on the importance of developing further research in the learning and teaching of algebra. The ICMI Study (Stacey, Chick, & Kendal, 2004) and *Approaches to Algebra* (Bednarz, Kieran, & Lee, 1996) are volumes entirely dedicated to algebra that show not only that copious research has been done in the domain, but also reveal the emergence of numerous new research questions.

Adopting different perspectives, researchers and educators have underlined the importance of algebra at the K-12 level. Moses (2001) draws attention to the role that algebra plays in the United States' (US) school system and society. He claims that, "the idea of citizenship now requires not only literacy in reading and writing but literacy in math and science" (Moses, 2001, p. 12). Especially because we are living in a technological era where "the visible manifestation of the technological shift is the computer, the hidden culture of computers is math" (Moses, 2001, p. 13). Algebra has been assigned the role of being the place where, young people learn the necessary symbolism that eventually becomes the tool to control technology. In addition to that, Moses (2001) states that algebra is the school subject that works as a filter in US society (while in France, for instance, geometry plays that role); this filter separates the youth that will go to college from those who will not. At risk groups identified by Moses are Latino, Black-American, and poor White American students. Stacey and Chick (2004) agree with Moses (2001) when they state that:

> Algebra is often described as a gateway to higher mathematics, not least because it provides the language in which mathematics is taught. Consequently, it is important that all students be given a genuine opportunity to learn algebra. Without this, they are cut off from many occupations, either because algebra is really used there or because it is specified as a preliminary qualification. (Stacey & Chick, 2004, p. 2)

Following Moses (2001), in order to give youth access to a full citizenship, society needs to make their students learn and succeed in algebra in order to be included in economic life in an active way.

Chazan (1996) agrees with Moses in the identification of the groups of students who are at risk in algebra, as well as in the concern about equality in society and education and the role that algebra plays in this respect. Chazan and the ICMI report (Stacey, Chick, & Kendal, 2004) identifies mass schooling as a triggering factor in the emergence and identification of failure in the learning of algebra. Chazan (1996) even questions the pertinence of teaching algebra for all in the current conditions:

> School algebra policies are contested because they have implications for access to college. Although we need to act to address inequalities in our society that limits access to college, I believe it is wrong headed to force students to take a class that almost half of the students fail (Chazan, 1996, p. 475).

The target solution for Chazan seems to be a curricular reform which should take into account all students, “this curriculum must be intended for a broad range of people –those who are not college-intending, as well as those who are” (Chazan, 1996, p. 475).[1] Stacey and Chick (2004) considering that we are now faced with teaching algebra in high school to a broader spectrum of the population state that: “the challenge has been to reconceptualize algebra as a subject that does have a relevance to students and to do this in a way that the students themselves can perceive the relevance” (Stacey & Chick, 2004, p. 2). To the same goal, Kaput (1995; , 1998) proposed to “transform algebra from an engine of inequity to an engine of mathematical power” (Kaput, 1995, p. 2) by *algebrafying* the K-12 curriculum, thus starting the teaching of algebra in elementary school: “The key to algebra reform is integrating reasoning across all grades and all topics –to *algebrafy* school mathematics” (Kaput, 1995, p. 2). In Kaput’s view algebra should include the following: (1) generalizing and formalizing patterns and constraints, (2)

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[1] In his research studies, Chazan (1996) has developed an algebra curriculum for high school bearing on the notions of variable, functions, and the use of real-life contexts.

syntactically-guided manipulation of formalisms, (3) study of structures and systems abstracted from computations and relations, (4) study of functions, relations, and joint variation, and (5) cluster of (a) modeling and (b) phenomena controlling languages. As it will be discussed in the sixth section, a group of researchers in Spain and France have been working towards an *algebraization* of the mathematics curriculum (e.g., Bolea, Bosch, & Gascon, 1999, 2003; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascon, 1993-1994). A difference between these two perspectives is when to start the *algebraization* of the curriculum. For Kaput (Kaput, Carraher, & Blanton, 2007) algebra should start in the early grades of primary school in order to avoid the arithmetic-algebra transition. This seems not to be the case for the curriculum *algebraization* proposed by Bolea, Bosch & Gascón (1999, 2003) which focuses on algebra as a modeling tool, building on the notion of mathematical model developed ealier by Chevallard (1985, 1989, 1989-1990). Although Bolea et al (1999, 2003) addressed the teaching of *elementary* algebra they do not make a case for starting algebra early. Having shown the relevance of the subject and the necessity of developing broader perspectives on algebra, in the next section I will state the goals and organization of this literature review.

Goals, Structure, and Organization of the paper

I will review and synthesize a variety of studies on the teaching and learning of algebra at the middle and high school levels.

For each of the studies, the goals of this review are to provide (whenever present in the source):

(1) A brief overview;

(2) An identification of the aspects of algebra that are central to the study and the underlying epistemology, and sample mathematical problems used by the researchers;

(3) An identification of students' learning outcomes and/or theoretical concepts that are central to the study.

The selection includes papers from 1976 to 2006 directly related with algebra. As selection criteria the word "algebra" had to appear either in the title, abstract, or keywords of the article. A diversity of articles from Europe, North America, and South America are presented in the corpus material. The material includes books, articles in journals, and conference proceedings.

In this literature review, I am not including important areas of recent development such as Early Algebra (e.g., Carraher, Schliemann, & Brizuela, 2004; Davydov, 1962; Kaput & Educational Resources Information, 2000; Schliemann, Carraher, & Brizuela, 2006), and teachers' knowledge (e.g., Arcavi & Bruckheimer, 1983; Arcavi & Bruckheimer, 1984; Ball, 1988; Even, 1993; Richardson, 2001). They are beyond the purpose of this literature review that, as mentioned above, is the teaching and learning of algebra at the middle and high school levels. It might seem paradoxical that I do not include research perspectives on teachers' knowledge given that the theme of this literature review includes 'teaching'; however, I am focusing on the teaching aspects regarding the features of the activities and problems used to collect data, as well as the conditions in which the students produced knowledge. This is not directly related to teachers' knowledge as explored in the cited literature (e.g., Arcavi & Bruckheimer, 1983; Arcavi & Bruckheimer, 1984; Ball, 1988; Even, 1993; Richardson, 2001).

In the *first section* of the paper I will present a brief history of the evolution of research on the teaching and learning of algebra. The second to sixth sections address different research perspectives, and are organized following the above mentioned goals. In the *second section*, I will discuss researchers' perspectives that has to do with the relation between arithmetic and algebra. Many research studies have pointed to the conceptual change that students have to achieve in order to understand algebra, e.g.; the different meanings of the equal sign, the different ways of solving problems in arithmetic and algebra. Usually in arithmetic we apply operations to numbers and obtain results after each operation; but in algebra, we usually do not start solving a problem using the given numbers, doing calculations with them, and obtaining a numeric result. In algebra, students have to identify the unknowns, variables and relations among them, and express them symbolically in order to solve the problem (e.g., Behr, Erlwanger, & Nichols, 1976; Booth, 1984; Herscovics & Kieran, 1980; Kieran, 1979, 1981; Kuchemann, 1981; Vergnaud, 1984, 1988; Vlassis, 2004) . In the *third section*, I will cover research in the area of generalization and patterns while in the *fourth section*, I will discuss research on algebra and proof. In addition, I will concentrate on the approach to algebra from a functional perspective in the *fifth section*. Also in this section, I will address the use of computational environments in the teaching and learning of algebra, since many of these environments use a functional perspective. In the *sixth section*, I will focus on the modeling approach to algebra.

I partially grouped the studies according to the classification developed by Bednarz, Kieran, and Lee (1996), inspired by the contributions at the colloquium on *Research Perspectives on the Emergence and Development of Algebraic Thought* held in Montreal in May 1993 and organized jointly by the *Centre Interdisciplinaire de Recherche sur l'Aprentissage te le Developpement en Education* (CIRADE) and the Mathematics Department of the Universite du Quebec a Montreal.[2] The approaches to algebra that this group identifies are the following: Historical,

[2] Lee (1997) has developed another classification of studies in algebra. This classification emerged through raising the question "What is Algebra?" to researchers in the field, mathematicians, teachers, and students. The categories that were developed from this study are Algebra as a school subject, Algebra as generalized arithmetic, Algebra as a tool, Algebra is a language, Algebra is a culture, Algebra is a way of thinking, and Algebra is an activity. Given that I want to analyse the relation between the approach to algebra and the research findings, these categories are not fertile for the type of review I want to develop. Kaput (1995, 1998) defined algebra as including the following: (1) generalizing and formalizing patterns and constraints, (2) syntactically-guided manipulation of formalisms, (3) study of structures and systems abstracted from computations and relations, (4) study of functions, relations, and joint

Generalization, Problem Solving, Modeling, and Functional. I took from this classification the following perspectives: Generalization, Problem Solving, Modeling, and Functional. I will be using this classification only in part because, first, I want to adapt it to my own interests and, second, because, as reflected in the ICMI Study (2004), this categorization seems to represent approaches taken in English speaking countries but not non-English speaking countries such as Spain, France, and Hungary. Adding new categories allows considering the diversity of approaches worldwide, including the section on *Algebra and Proof* (section 4). A third reason for the chosen categories is to group the studies according to their underlying epistemology and conceptions of algebra. Research on the relation between algebra and proof is scarce. This causes us to neglect one of the main uses of algebra as a tool, which is the power entailed in transforming expressions and allowing to read information that was "hidden" in the expression before the set of transformations was applied. Algebra becomes a powerful tool when we try to find f'(2) with f(x)=x^2 using $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{2} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$. If we try to calculate this limit using the current quotient expression, we arrive to an indetermination of the type $\frac{\to 0}{\to 0}$ when $h \to 0$. Thanks to algebraic transformations we can re-write the expression $\lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$ as $\lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$ using the definition of "squaring" and distributing. However, if we now try to calculate the limit, we will again obtain that $\frac{\to 0}{\to 0}$ when $h \to 0$. Therefore we need to try something else and keep re-writing. Using arithmetic we can transform

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variation, and (5) cluster of (a) modeling and (b) phenomena controlling languages. Since these categories were created in order to define what algebra should include but not to describe approaches to algebra or a classification of studies to algebra therefore I will not be using this classification. However, since many researchers focus on (1), (4) or (5), Kaput's components are included in sections two, five and six correspondingly.

the last expression as follows: $\lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h}$. Again, we are not there yet since we have that $\frac{\to 0}{\to 0}$ when $h \to 0$. If we factor *h* in the numerator we obtain $\lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} \frac{h(4 + h)}{h}$. If we try to calculate the limit in this format we again obtain that $\frac{\to 0}{\to 0}$ when $h \to 0$. Canceling *h* in the numerator and denominator we finally obtain $\lim_{h \to 0} \frac{h(4 + h)}{h} = \lim_{h \to 0} 4 + h = 4$.

First Section. Brief history of the field

For this brief history, I used mainly two sources: *The future of the teaching and learning of algebra* (Stacey, Chick, & Kendal, 2004), and the *Handbook of Research on the Psychology of Mathematics Education* (Gutiérrez & Boero, 2006). Kieran (Gutiérrez & Boero, 2006) breaks the 1977-2006 period into three sub-periods and identifies the main themes approached by researchers in algebra. The first sub-period corresponds to the years 1977-2006, where the main themes of study were transition from arithmetic to algebra, variables and unknowns, equations and equation solving, and algebra word problems. The second sub-period corresponds to the years mid-1980s to 2006 with use of technological tools and a focus on multiple representations and generalization as main topics. The third sub-period of time matches up with the years mid-1990s to 2006 where the main subjects of studies have been algebraic thinking among elementary school students, algebra for teacher/teaching, and dynamic modeling of physical situations and other dynamic algebra environments.

From Kieran's perspective, up until the mid-1960s, "algebra was a paper-and-pencil activity, focusing primarily on transformational work" (Kieran, 2004, p. 25). At that moment, algebraic meaning-making was addressed in the introduction of algebra books and was approached by translating arithmetical sentences into algebraic expressions. In the 1970s there was a change in the way algebra was taught, given the modern math movement and the importance given to problem solving. Kieran (2004) identifies the development and introduction of technology in education as crucial factors in the mid-1980s. Research conducted in the 1970s and 1980s provided evidence that students were struggling with learning algebra, and the main issues identified related to the arithmetic-algebra transition, and the re-conceptualisation required to become proficient in algebra, e.g., the meaning of the equal sign. As a result of this, members of the research community started conducting teaching experiments in order to try alternative ways of teaching algebra, emphasizing meaning making. Following Kieran (2004), the evolution of the field seems to result in learning of algebra through meaning making activities, but with very limited attention to transformational activities:

> Indeed, in the UK for example, the search for meaning and the consequent suppression of symbolism led to a situation in the early 1990s where students were doing hardly any symbol manipulation (Sutherland, 1990). In various countries, problem solving, by whatever means, had all but replaced traditional algebra. *The hope was that, in focusing on algebraic understanding (however this might be defined), the techniques would take care of themselves*. (Kieran, 2004, p. 27. Emphasis added)

However, the techniques didn't take care of themselves. Artigue (Artigue, Defouad, Duperier, Juge, & Lagrange, 1998; M. Artigue, 2003) and her research group found compelling evidence for this dichotomy between meaning-making and technique or transformational activities in the teaching and learning of algebra. This is one of the topics in the research agenda in the area of learning and teaching of algebra: the study of activities that have a double-sided goal -understanding of algebra and construction of meaning, and proficiency of technique (Artigue, Defouad, Duperier, Juge, & Lagrange, 1998; M. Artigue, 2003; Kieran, 2004).

Analyzing the different themed-groups in the International Group for Psychology of Mathematics Education (PME) research community, one of the aspects that Kieran (2006) identifies is the broadening of sources of meaning in the teaching and learning of algebra. The author then proposes (see Figure 1) a modification of Radford's (2004) sources of meaning in mathematics education. For Radford (2004), meaning in school algebra is produced from three primary sources: (i) the algebraic structure itself, (ii) the problem context, and (iii) the exterior of the problem context (e.g., linguistic activity, gestures and bodily language, metaphors, lived experience, image building, etc.).

Kieran (2006) modified Radford's point (i) on algebraic structure itself to consider meaning from within mathematics, where the author includes algebraic structure itself, and meaning from multiple representations (see Figure 1 below).

1. Meaning from *within mathematics*: 1.a. Meaning from the algebraic structure itself, involving the letter symbolic form. 1.b. Meaning from *multiple representations.* 2. Meaning from *the problem context*. 3. Meaning derived from that which is exterior to the mathematics/context problem (e.g.: linguistic activity, gestures and bodily language, metaphors, lived experience, image building, etc.)

Figure 1: Sources of meaning in algebra (adapted from Radford [2004] by Kieran [2006, p.32].

Summarizing, from Kieran's (2006) viewpoint one of the main accomplishments of the PME research community is a broadening of sources of meaning for the teaching and learning of algebra. In addition to that, it seems that the research community now has, as a next goal to be achieved, the task of bringing together technique and meaning.

After stating the goals of this literature review and having provided a very brief history of the field as a way of framing my work, I will explore in the next sections different research studies that have been grouped by the criteria described above in the section Structure and Organization of the Paper.

Second Section. Arithmetic and Algebra

This section addresses studies on the relationship between arithmetic and algebra. I will discuss the work by Kieran (1981), Balacheff (2000), and Filloy and Rojano (1989).

Currently, given the organization of the curriculum in the majority of Western countries, students' work with arithmetic precedes their work with algebra. Students may have up to eight years of schooling before they start working on algebra. Research addressing the arithmetic-algebra relation (e.g., N Balacheff, 2000; Bednardz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1981; Wheeler, 1996) has focused mainly on two issues. The first issue can be described as the attempt to characterize both mathematical domains –arithmetic and algebra-, to identify similarities and differences, and to trace the evolution of their relationship in History. The second issue addressed by research on the transition from arithmetic to algebra, is the design and/or analysis of instructional activities. In this section, I will be exploring these two issues.

Many studies have shown that students maintain, as long as they can, their arithmetical interpretation of different mathematical objects and tools when solving problems (e.g., Filloy & Rojano, 1989; Filloy, Rojano, & Rubio, 2001; Kieran, 1981).

Within these studies, Kieran's (1981) analysis of the transition from arithmetic to algebra focuses on equations and students' interpretation of the equal sign. Kieran (1981) opens the discussion regarding the role of the equal sign in arithmetic and algebra from Kindergarten to college. The researcher emphasizes kindergarteners' interpretation of the equal sign as the "do something sign". In addition, she provides evidence that students do not conceive of the equal sign as an equivalence when they are presented with a number sentence such as "6 + 7 = 5+ 8". They do not see both sides of the equal sign as two different names for the same number or two different representations of the same number. Students have the idea that after the equal sign, the result of the calculation needs to be stated. Other students perceive the number sentence as two different problems, the problem " 6 + 7 = ", and the problem " 5 + 8 =."

Kieran (1981), who considers that the interpretation of the equal sign should be that of equivalence, points out that "the symbol which is used to show equivalence, the equal sign, is not always interpreted in terms of equivalence by the learner" (p. 317). In the analysis of the meaning of the equal sign within the context of equation solving, the researcher claims that,

> … the ability to consider an algebraic equation as an expression of equivalence because both sides have the same value does not seem to be sufficient for an adequate conceptualization of the equation solving. For not only does equation solving involve a grasp of the notion that right and left sides of the equation are equivalent expressions, but also that each equation can be replaced by an equivalent equation (p. 323).

One has to be aware, however, that the equal sign can function in different ways. Consider, for instance, the case of an equation where the equal sign is used to establish a condition on a set. Taking an example from Kieran (1981, p. 323), the problem is presented as "Solve for x: 2x+3=5+3". From a mathematical point of view, this way of asking someone to solve an equation, usually presented at school, is an incomplete request that does not specify what type of number is x. Does x belong to the set of the integers, to the natural numbers, to the complex numbers? We can think of an equation as an object that defines a subset within a set (usually referred to as solution set). We can re-write Kieran's example in the following ways:

1. Determine the set of values of x in Z such that x+1 = x+2.
2. Determine the set of values of x in Z that makes the equality (x+1 = x+2) true.
3. $\{x \in \mathrm{Z} / x+1 = x+2\}$ = S.

In this particular case, for all values of x the equality is false. This fact contradicts the notion of equivalence or equivalent expressions. In a sense, two expressions that are equivalent in a set can be interchanged for one another. In the case of the equation presented by Kieran (1981), 2x+3=5+x, the solution set is x=2. That means that the equality is true only for x=2 and not for any other value. From this it follows, for instance, that if we plug in x=3, we are going to obtain $2 \times 3 + 3 \neq 5 + 3$.

The solution set can be empty, finite, or infinite. When the solution set is strictly included in the set –the solution set is a proper set- where the unknowns are defined, we cannot talk about equivalence. It is true that the equal sign denotes an equivalence relation in the set of the real numbers, since it can be proved that the equality on the set of real numbers is reflexive, symmetric, and transitive. In my view, Kieran (1981) is overseeing the different uses and functions of the equal sign; while in the real numbers it is true that the equal sign verifies the properties of an equivalence relation, when using the equal sign in an equation-solving context it represents a condition within a number set. The meaning of the equal sign depends on, at least, the context and the task for which it is used.

Filloy and Rojano (1989) provide a different perspective from Kieran's (1981), identifying conceptual and symbolic changes which mark differences between arithmetical and algebraic thought in the individual. Some of these hallmarks are the interpretation of letters, the notion of equality, and conventions for coding operations and transformations in the solution of equations. These authors postulate the existence of a cut, "a break in the development concerning operations on the unknown" (Filloy & Rojano, 1989, p. 19). Filloy and Rojano developed this idea based on an analysis of the strategies and methods for solving equations found in the pre-symbolic algebra textbooks of the 13th, 14th, and 15th centuries. The solution strategies for equations such as $x^2 + c = 2bx$ and $x^2 = 2bx + c$ are absolutely different from each other.

> This difference would not exist if the authors had had recourse to the rule of transposing terms from one side of an equation to the other for, at the syntactical

level, the two equations would then be similar. But this facility would already imply an advanced ability to operate on the unknowns in the equations. (Filloy & Rojano, 1989, p. 19)

Considering different types of equations, Filloy and Rojano (1989) proposed a categorization for types of equations. They divide the realm of equations –linear and on one unknown- into arithmetical and non-arithmetical equations. The authors associate the "arithmetical equation" with the "arithmetical" notion of the equal sign. Filloy and Rojano (1989) agree with Kieran (1981) regarding the conception of the equal sign in arithmetic in the following way, "the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown); the right side represents the consequence of having performed such operations" (Filloy & Rojano, 1989, p. 19). The authors show evidence that this type of equation, i.e., Ax+B=C, can be solved "undoing" the operations. In this sense, if we present students only with this type of equations they can solve them using arithmetic tools. Work with such equations fails to introduce them to the algebraic world. Students need to face non-arithmetical equations, i.e., Ax+B=Cx+D, in order to promote their need to use algebraic tools. The solution of non-arithmetical equations involves operations drawn from outside of the domain of arithmetic (i.e., operations on the unknown). The authors hypothesize that the introduction of the non-arithmetical equations should be framed by problems involving different contexts. They propose the use of two contexts: the balance model and the geometrical model. Figures 2 and 3 show a geometrical representation of the equation Ax+B=Cx where A, B, and C are given positive integers and C > A.

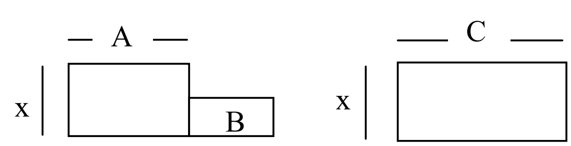

Figure 2: Representation of the equation Ax+B=Cx, using the geometrical model (from Filloy & Rojano, 1989, p. 19).

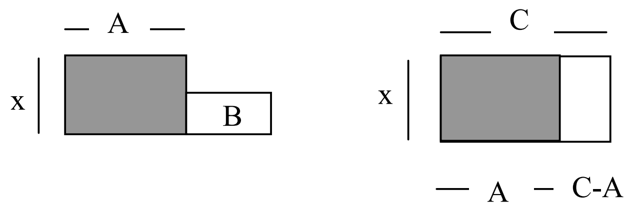

Figure 3: Representation of the equation comparison of areas as a way of giving meaning to the syntax (Filloy & Rojano, 1989, p. 19).

Figures 4 and 5 show a representation the same Ax+B=Cx equation in the context of the balance model (where A, B, and C are given positive integers and C > A).

(A) objects with equal (unknown) weight

(B) objects with equal (known) weight

(C) objects with same (unknown) weight

Figure 4. Representation of the set up of the equation AX+B=C in terms of weights (Filloy & Rojano, 1989, p. 19).

(B) objects

(C-A) objects

Figure 5. Representation of the solution of the equation AX+B=C in terms of weights (Filloy & Rojano, 1989, p. 19).

Filloy and Rojano (1989) identify different phenomena within their study results. In the first place, they conclude that "the correction of syntactic algebraic errors, and of the operational difficulties that occur in resolving complex problems or equations, cannot be left to be spontaneously resolved by children on the basis of their initial grasp of operational algebraic behaviour" (p. 24). Filloy and Rojano (1989) propose that "modeling" has two fundamental dimensions: *translation* and *separation*. Their work provides evidence that students that master the first component –translation- may be at a disadvantage at the *separation* phase. Students with mastery in translation "developed a tendency to stay and progress within the concrete context" (p. 25). In addition, students with a more syntactic tendency may develop obstacles when trying to abbreviate actions and produce intermediate codes (between the concrete and purely syntactic level). Filloy and Rojano (1989) point out the importance of the role of the teacher in "the development of the processes of detachment from, and negation of, the model, in order to lead towards the construction of the new notions" (p. 25).

Bednarz and Janvier (1996) present yet another perspective. Regarding the arithmetic-algebra transition, Bednarz and Janvier (1996) based their study on equations. They chose equations as the algebraic object to investigate differences between arithmetic and algebra, and to study students' obstacles in the equation-solving process. In their study, Bednarz and Janvier (1996) also proposed a categorization of the problems that are generally presented in algebra. Their analysis identified three types of problems: (1) problems of unequal sharing, (2) problems involving a magnitude transformation, and (3) problems involving non-homogenous magnitudes and a rate. In addition, Bednarz and Janvier (1996) analyzed features that are different in algebra and arithmetic. For instance, "arithmetic procedures are generally organized through the processing of known quantities, by attempts to create links between them in order to be able to operate on them. The unknown quantity appears at the end of the process" (Bednardz & Janvier, 1996, p. 119). Bednarz and Janvier (1996) label arithmetic problems as “connected” since a relationship can be easily established between two known data, leading to the possibility of arithmetic reasoning. On the contrary, in algebra the problems are “disconnected”, meaning that no direct bridging can be established between the known data in order to operate on them to obtain the unknown. However, in algebra, “in operating on an unknown quantity, which we directly perceive as the one which will allow us to generate all the others, and doing it as if this quantity was known, the algebraic reasoning, just as with the arithmetic reasoning, *connects* the problem in some way” (Bednardz & Janvier, 1996, p. 128. Emphasis in original). In their study, where the students (12-13 year olds) solved three problems that involved equations, the main obstacles were:

- direct generation of an equation using a single unknown;
- substitution, which requires the passage to a single unknown;
- refusal to operate on the unknown;
- the students’ symbolism used to present the relationships in the equation.

From a more theoretical perspective, Balacheff (2000) claims that arithmetic and algebra at school might be in competition for the same corpus of problems. If a student recognizes a problem as an arithmetic problem, then there is no need to use algebra when what he/she already knows and is familiar with (arithmetic) is enough to solve the problem. Therefore, this raises a need for thinking of a corpus of problems for the teaching and learning of algebra that requires algebra and can’t be solved with the resources that arithmetic offers. Regarding this point, Balacheff (2000) agrees with Bednarz and Janvier (1996) regarding what they called “connected problems”, and what Filloy and Rojano (1989) have called “arithmetical equation.”

Differing from Bednarz and Janvier (1996) and Filloy and Rojano (1989), Balacheff (2000) does not focus on equations. Balacheff (2000) proposes to focus on another essential difference between algebra and arithmetic: the system of control. Although the corpus of problems and some tools are shared by arithmetical problems and introductory algebraic problems, arithmetic and algebra differ substantially in their system of control. Balacheff (2000) conceptualizes the transition from arithmetic to algebra as a shift from emphasis on a *pragmatic control* to a *theoretical control* in the solution of problems. In this sense, the anticipatory value of algebra could be seen as one of the possible keys to open the way to algebra (as long as it is not used for trivial problems). Furthermore, Balacheff (2000) highlights another difference between arithmetic and algebra. On the one hand, in arithmetic the student can establish a parallel relationship between the computation and the referent world at all steps (Balacheff, 2000). On the other hand, algebra seems to require “the need for students to be able to associate meanings with the symbols being used, and to manipulate symbols independently of their meaning” (Balacheff, 2000, p. 253).

Avoid Summing up the different positions, research shows that students’ arithmetical interpretations of concepts are very stable. In this line, Kieran’s (Herscovics & Kieran, 1980; Kieran, 1979, 1981) analysis of the equal sign sheds light on the different roles that the equal sign plays in arithmetic and in algebra. Balacheff (2000) warns us about the choice of problems used for this purpose. The author advises us against using problems in the learning and teaching of algebra that could be solved using arithmetical tools; problems should challenge the student by requiring resources beyond arithmetic (e.g., use of letter to represent a generality, use of algebra to produce a proof, etc.). Ideally, students’ knowledge should be enough to approach the problem but insufficient to solve it, in order to promote disequilibrium, so that it is more likely that the students’ knowledge will develop to adapt to the problem. As discussed earlier in this section, Filloy and Rojano (1989), as a result of studying historical works on equation solving, proposed to divide the realm of linear equations on one unknown in two categories: arithmetical and non-arithmetical. Filloy and Rojano postulate the existence of a cognitive cut, “a break in the development concerning operations on the unknown" (1989, p. 19). In order to promote students' meaningful learning of equation solving, Filloy and Rojano (1989) developed a sequence of equations to be solved using a geometrical context (areas of squares and rectangles) and an extra-mathematical context (balancing weights). Filloy and Rojano (1989) seem to address Balacheff's (2000) concern regarding the competition between arithmetic and algebra for the same corpus of problems. Within the realm of equation solving, Filloy and Rojano (1989) identified the set of equations that could be solved with arithmetical tools, and the set that requires an algebraic set of tools. Balacheff's (2000) and Filloy and Rojano's (1989) works highlight the importance and usefulness of carrying out epistemological analyses of the mathematical domain (e.g., identification of arithmetic and algebraic equations). As mentioned before, Filloy and Rojano's (1989) contribution is in the realm of equation solving even though their analysis didn't led to a didactical solution to help children use the models in order to better understand equation solving; an interesting direction for the field would be to analyze and study sets of problems that require the use of algebra as a tool so that arithmetic and algebra problems do not compete for the same set of problems. Some of this will be discussed in *Algebra and Proof* (fourth section).

Third section. Algebra, generalization and patterns

In this section I will discuss researchers' approaches to the teaching of algebra through patterns and generalization activities. Work by Mason (1996), Lee (1996), and Radford (1996) will be the focus of my analysis since they are considered representative of the development in this area of the field.

For Mason, "generalization is the heartbeat of mathematics and appears in many forms" (1996, p. 65). A mathematics classroom, in order to be called so, should be permeated by the students' continuous expression of generality. In his work, Mason (1996) has mainly studied geometric and numeric patterns but "only to provide experiences which highlight the process [of generalization]" (p. 65). Mason (1996) analyzes many of the different forms of generalization that we can find in mathematics. Some examples of his analysis are particularly insightful. For instance, the author examines the case of theorems or propositions, in particular the statements and the way they are expressed. Mason takes the example of the sum of the interior angles of a triangle. Let's state the proposition in the usual way, "The sum of the angles in a triangle is 180 degrees." Following Mason (1996), in this proposition the most important word is "a." The word "a" is describing the whole set of triangles, each and every triangle or, in other words, any triangle. The second most important word is the modifying article "The." If the triangle is changed, the sum remains invariant. The search and identification of invariants is a typical feature of mathematical work. The fact that the invariant is 180 degrees is of relatively little importance to the fact that it is invariant. Mason states: "The essence of the angle-sum assertion, and indeed, I conjecture, of most mathematical assertions, lies in the generality which can be read in it. There is some attribute that is invariant, while something else roams around a specified or implied domain of generality" (1996, p. 68).

Another expression of generality in mathematics can be found, according to Mason (1996), in what is usually called generalized arithmetic. The structure of arithmetic, when expressed, produces algebra as generalized arithmetic.

Remainder and modular arithmetic can provide a rich context in which to practice expressing generality. In addition to what Mason mentions, modular arithmetic could be a fertile place not only to express generality but also to use algebraic notation and manipulations to obtain new information. For example, if we want to show *why* when we multiply an even number by an odd number the result obtained will always be an odd number, we can use algebraic notation to explain why. In particular, we need to move from "expressing" using the language of algebra towards problems where the use of algebra is necessary to answer a question, to solve a problem or to explain why. Mason's perspective regarding the use of algebra as a tool to solve problems when it is necessary coincides with Balacheff's (2000) perspective that we need to change the fact that arithmetic and algebra compete for the same corpus of problems in school; Balacheff (2000) agrees with Mason (1996) in proposing to teach algebra using problems where algebra is a necessary tool. Mason (1996), focusing on *how* and *what degree of generality* can be read in an algebraic expression, analyzes the expression "1+3n", where n is an integer. The expression "1+3n", where n is an integer, represents any number that divided by 3 gives a remainder of 1. At the same time, it could represent a particular number for a particular n; in a way it is a number, in other way is the structure of a number; it could also represent the rule to calculate a number. We could generalize even more departing from that expression in the following way: "r+3n" represents the numbers with a remainder "r" when divided by 3 –in the case 3>r≥0-, the expression "r+kn" denotes any number that has a remainder of r when divided by k where k>r≥0. It would be very interesting to work with generality around the notion of parameter, since in these generalizations not all letters necessarily play the same role. We could consider “n” as a variable, and “r” and “k” as parameters.

Mason also explores the role of examples in the mathematics classroom. An example is not an example in and of itself, but it is an example *of* something, of something broader. When a student is presented with an example, what he/she understands is probably a totality in itself. Until a person can see an example as an example *of* something, it remains of little meaning: “the whole notion of example depends upon and draws out the notion of generality” (Mason, 1996, p. 73). The student needs to construct an example as an example *of* something, and this can be done by reflecting on the particular aspects of the general.

In his search and study of experiences that could promote students’ generalization processes, Mason (1996) stresses the intrinsic nature, in a generalization process, of the relation between the particular and the general. In order to promote awareness of the general, the author proposes the distinction between *looking at* and *looking through,* when students are working on a sequence of exercises or problems. It is important to promote students’ awareness of seeing *the general in the particular* and seeing *the particular in the general*. One of his suggestions, in order to address the articulation between the general and the particular, is to work explicitly on a set of exercises or problems as a whole and not only working through them. Mason (1996) criticizes sets of exercises that promote students’ mindless practice. To move beyond this practice, we must study the problem as a whole.

Let’s exemplify with one activity that is representative of Mason’s approach to algebra.

> The picture below (see Figure 6) shows a rectangle made up of two rows of four columns and of squares outlined by matches. How many matches would be needed to make a rectangle with R rows and C columns? (1996, p. 80)

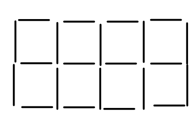

Figure 6. Rectangle made up of matches (from Mason, 1996, p. 80)

Mason (1996) claims that pattern generating and generality expressing might be more than appears at first sight. In a study with teachers where they had to solve the rectangle problem described above (see Figure 6), Mason found that there were many activities that students could learn from interacting with this problem. One of these activities is “multiple seeing,” or finding several ways to see how to count the number of matchsticks and expressing these as general formulas. Another activity is “reversing seeing,” or taking an equivalent arrangement of the expression and trying to arrange the counting process according to it. A different activity that the author mentions is “doing and undoing,” taking doing as finding the number of matchsticks for a given number of rows and columns, and undoing as deciding if a given number could arise as such an answer, and trying to characterize the form of such numbers. For Mason (1996), the key in algebra is the interplay between the particular and the general towards generalization awareness in mathematics.

Lee (1996) has also assigned a central role to generalization activities in the initiation to algebra. However, Lee (1996) understands algebra as a mini-culture and the initiation into a culture is the first step in a long process. This acculturation process requires the learning of:

> What is sayable, what we talk about and how we talk about; what we do not talk about; what level of formality is used in various writing situations; what experiences and words are untranslatable; what are the gestures and symbols, the worlds of sense around objects, dates, rites; what are the sacred cows; what is the shared history of institutions, families, communities; what are the objects of thought; what is funny and what it isn’t. (Lee, 1996, p. 89)

Lee (1996) proposes to be initiated into this culture through generalization activities. She draws conclusions based mainly on two problems; the *consecutive numbers* problem, and, the *dot rectangle* problem.

In the consecutive numbers problem students would be asked to:

> Show, using algebra, that the sum of two consecutive numbers (i.e., numbers that follow each other) is always an odd number. (Lee, 1996, p. 90)

The dots rectangle problem is expressed as:

> The drawing on the left (see Figure 7) represents a set of overlapping rectangles. The first contains 2 dots. The second contains 6 dots. The third contains 12 dots. The fourth contains 20 dots. How many dots in the fifth rectangle? How many dots in the hundredth rectangle? How do you know? How many dots in the n^{th} rectangle? How do you know?

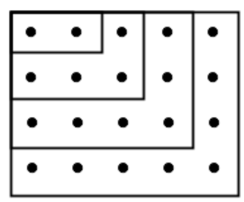

Figure 7. Overlapping Rectangles (from Lee, 1996, p. 90)

In the analysis of these two problems, Lee (1996) found that knowledge and practices that she had taken for granted were not available for a group of adult students taking the prerequisite algebra course at Concordia University. For example, when solving the dots problems, a student kept counting the dots, one by one, instead of counting by columns or rows using multiplication. For the researcher, the selection of the variables was "transparent," however, for that student it wasn't obvious. The researcher was also surprised by the fact that another student would not accept a single dot as a one by one rectangle: "she [the student] made us realize just how much our perceptions in algebra have already been trained and our potential for 'seeing' had essentially been whittled down over the past years" (Lee, 1996, p. 100).

An interesting finding in Lee's study (1996) was the fact that, after accepting that x and x+1 are any consecutive numbers, some students seemed to endow x+1 with an odd quality while x was definitively always even. Another finding was the fact that students were reluctant to use more effective strategies. For example, after one student showed that x-1, x, and x+1 were more efficient representations of three consecutive numbers for some problems, the other students did not spontaneously use this strategy. Lee (1996) shows that the students could "see" patterns perfectly; the problem was not "seeing" a pattern, but identifying a useful algebraic pattern. In addition to that, Lee (1996) claims that a generalization approach to algebra "immediately threw students into using letters as variables" (p. 105). As Mason (1996), Lee (1996) considers that the interplay between "specializing" and "generalizing" is at the core of algebra. One of the meanings that can be given to equivalent expressions, for instance, is that of generating the same pattern. The difficulties that Lee (1996) encounters in this approach are obstacles at the perceptual level (seeing the intended pattern), at the verbalizing level (expressing the pattern clearly), and at the symbolization level (using n to represent the n^{th} array or number and then representing the number of dots in terms of this). Lee (1996) agrees with Mason (1996) when he claims that flexibility must be developed, since not all pattern perceptions are equally useful and in arithmetic flexibility is almost unnecessary.

Radford (1996), in his analyses of the work by Mason (1996) and Lee (1996), reminds us of carefully thinking about the role that we attribute to generalization in the teaching and learning of algebra. Radford (1996) states that a superficial look at the history of mathematics may lead us to conceptualize mathematical activity exclusively under the process of generalization. Radford's (1996) message is that there is more to mathematics than generalization; the mathematical objects of the generalization provide distinctive features to the process of generalization. In other words, Radford (1996) claims that generalization depends on the mathematical objects that we are generalizing on. The author discusses two particular issues in geometric-numerical pattern generalizations. The first is related to the generalization of results, and the second has to do with the role that representations play in these kinds of patterns. Regarding the first issue, Radford points out that one of the goals of generalization is to produce a new result[3]; it is in this sense that generalization is not a concept but a procedure. In this generalization process, we cannot avoid the problem of validating a new result, as we usually do in mathematics. If we are working with generalization as a didactical device we should be prepared to address the validity of the result.

In terms of the external representational system, representations as mathematical symbols are not independent of the goal of the activity or problem; they require a certain anticipation of the goal. "The problem that now arises is that of knowing which facets of the object should be kept in its representation" (Radford, 1996, p. 110). This last observation somehow responds to what Mason (1996) and Lee (1996) refer to as something to be developed or learned by the students: pattern flexibility, moving among patterns, and seeing a useful algebraic pattern. In the majority of patterns presented by Mason (1996) and Lee (1996), the anticipation of the goal is related to the recognition of the variables involved in the particular pattern; i.e., in the matchsticks problem, the recognition of the rows and columns as variables is key in the development of a useful algebraic formula.

In this section, we have shown the central role that generalization activities play in the learning of algebra. As Problems that foster generalization activities have been proven fertile to promote students' work with variables and algebraic expressions (Lee, 1996; Mason, 1996). Radford (1996), however, warns us that generalization is an activity, and that the

[3] When considering the sequence of square numbers represented using points and squares, the new result is that the number of points arranged is as an square is a perfect square (square number).

product of that activity needs to be validated in mathematical terms. This brings us to the next section where the algebra-proof relation is incorporated into the mathematics research agenda regarding studies on the teaching and learning of algebra.

Fourth section. Algebra and proof

This section focuses on algebra and proof. Unlike other domains such as algebra and functions, and algebra and modeling, this area has been under-researched. In the United States, this might be a consequence of the fact that proof is almost exclusively studied in geometry courses in high school. This does not seem to be the case in England (Healy & Hoyles, 2000; Healy, Hoyles, Sowder, & Schappelle, 2002) and France (Barallobres, 2004). Many researchers (e.g., N Balacheff, 2000; Barallobres, 2004; Brousseau, 1997; Mason, 1996) have argued for the need to teach algebra using problems where algebra is a necessary tool to solve the problem.

I agree with Chevallard's (1985, 1989, 1989-1990) identification of central aspects of algebra. For Chevallard, algebra is a modeling tool for (a) setting up expressions representing the relations among variables and (b) producing equivalent expressions that allow to read and infer properties that couldn't be read in the initial expression. These are two distinct features that define part of what algebra is and that separate algebra from arithmetic and other sub-domains within mathematics. If we could incorporate these distinctive features into the design of problems, we could explore the benefits of such an approach to algebra.

Barallobres (2004) invested in this perspective towards algebra as a tool to model and to obtain new information through algebraic transformations. From an epistemological perspective, the algebra-proof approach seems promising since the resolution of problems encourages students to use algebra as a tool, at least emphasizing aspects that are central to algebra in order to solve problems. Later in this section, I will explain in detail Barallobres' (2004) vision on how to design and organize a class in order to require students to use algebra to gain insight into why something happens in a particular way in mathematics. The central issue in this perspective is to profit from two distinctive features of algebra: algebra as a modeling tool and the use of algebraic transformations to obtain information that couldn't be read in the initial expression; obtaining new information –by transforming and interpreting algebraic expressions- allows students to understand the reasons that make a proposition true. Since algebra becomes a tool to access the truth-value of a proposition and the reasons that make it true, algebra becomes a tool to prove. I will next discuss Hoyles and Healy's (2000) findings and Barallobres' (2004) work on proofs.

Hoyles and Healy (Healy & Hoyles, 2000; Healy, Hoyles, Sowder, & Schappelle, 2002) carried out a large scale study on proof conceptions in algebra in England. The study included a 70-minute survey administered to 2459 students from 94 classes in 90 schools. Students were high-attaining 14- and 15-year-old. The study also included teacher and school surveys. The student survey included multiple-choice questions where they had to choose the proof that would obtain the best mark, and the proof that would be closest to what they would do. This part of the survey was intended to provide information on students' views of what constituted a proof, its role, and its generality. In the second part of the survey, students were asked to construct their own proofs to offer insight into their competence in constructing proofs. The findings of the study are quite interesting. Regarding the results of the multiple-choice questions, they show a big difference between students' choice for their own approach and their choice for the best mark. In fact, it turned out that the arguments that were the most popular for the students' own approaches were the least popular when it came to choosing for best mark, and vice versa. Along the same lines, students judged that their teachers would reward any argument, provided it contained some "algebra." From the survey administered to teachers, they appeared to overestimate the extent to which their students would make judgments that were based on mathematical content rather than simply on form.

Following the results from the same study on students' construction of proofs, the authors found that students were much better at choosing correct mathematical proofs than at constructing them. The results also show that when students were asked to construct a proof for an unfamiliar statement, only 3% of them managed to produce a complete proof. The most popular form of argument was to provide empirical examples; if students tried to go beyond this pragmatic approach, they were more likely to give arguments expressed informally in a narrative style than to use algebra formally. These findings highlight the need to research the area of proof in algebra in order to understand why students don't use algebra when constructing their own proofs, and to develop new ways of integrating algebra and proof in the curriculum in such a way that students feel confident using algebra when proving. Hoyles and Healy (2000) state that,

> Although arguments that included algebra were the most popular among students for best mark, our results show that students knew that they would be highly unlikely to base their own arguments on similar algebraic constructions. In both multiple-choice questions, the algebraic arguments were the least frequently selected as the closest to the approach students would use, and algebra was used rarely as the language through which students attempted to write their own proofs. (p. 413)

A very interesting finding is that arguments that incorporated algebra were most likely to be viewed by the students neither as showing that the given statement was true nor as representing an easy way to explain to someone who was unsure about the truth value of a proposition. This is another aspect of the algebra-proof relation that needs to be investigated. Regarding the explanatory power of algebra, it seems that students were put off from using algebra because it offered them little in the way of explanation; they were uncomfortable with algebra arguments and found them hard to follow. More than a quarter of the student body had little or no idea of the meaning of proof and what it was for. The results show that the mathematics community needs to address the learning and teaching of algebra in relation to proof. It is surprising that we are missing one of the most important aspects of algebra that is its explanatory power. We need to develop new ways of linking algebra and proof for students to experience the usefulness of algebra as a tool.

Barallobres (2004), inspired by the French, relies heavily on a careful design of the problems used in a teaching experiment. Barallobres (2004) designed a sequence of problems in a way that when students first explore the problem, the feedback from the teaching situation contradicts students' anticipations. This type of situation is not easy to design. Barallobres proposes an introduction to algebra through a proof perspective. To develop his work, Barallobres adopts Balacheff's (1982; , 1987; , 1988) categorization of proofs as intellectual and pragmatic. Regarding the roles of proofs, the author adopts from Arsac (Arsac et al., 1992a, 1992b; Arsac & Mante, 1997) the position that proofs in a school setting can work in three different ways: (1) to decide, (2) to convince, and (3) to comprehend and know. Barallobres points to the fact that while a mathematician uses proof to convince himself about the truth value of a proposition, students can use other means to be sure but not to know *why*. Within the process of using proof to introduce algebra at school, for Barallobres (2004) there are two dimensions that should be taken into account when designing the intervention: (1) construction of a proposition, and (2) the construction of the proof itself. The first dimension helps in promoting students' ownership of the task. Barallobres' (2004) goal in his teaching experiment was for students to move from the production of pragmatic proofs towards intellectual proofs. The author designed the tasks in such a way that the search for reasons was linked to resolving a contradiction. In his work with 12 year-old students, the class was organized in fours parts. During the first part of the class, students were arranged in two groups of four students maximum. Each group had to choose two natural numbers, the second number smaller than 3000. The goal of the game was to obtain the biggest number carrying out the following set of calculations:

1. Multiply the two chosen numbers;

2. Add seven to the first number, and multiply the result by the second chosen number;
3. To the result obtained in step 2 take away the result obtained in step 1.

The group that obtained the biggest number would win. During this first part of the class, students played many different rounds until they got a sense of how the game worked. During the second part of the class, students were asked to produce a strategy that allowed winning all the time. Students were also asked to explain why this happens. A goal of this phase was to make students realize that there are infinite solutions. During the third part of the class, students were asked to explain why there are infinite solutions. During the fourth part of the class, students were asked to explore why if we assigned 2999 for the second number, and no matter what number we chose for the first one, how they could be sure that there is no other winning solution. Regarding the general findings of Barallobres' (2004) work, the mathematical task proved to be highly effective at prompting students to experience a contradiction between their expectations (finite solution vs. infinite solutions, dependence on two variables vs. dependence on one variable) and what happened when trying with specific numbers. The appearance of the contradiction proved to be a good motivator to encourage them to search for why and how this happens. Students experience with contradiction seemed to promote reflection on their actions. Barallobres found that the privileged strategy in private and public student work was using a general example (see N. Balacheff, 1982; N Balacheff, 1987; N Balacheff, 1988). The public interactions among students helped them to better understand the mathematical relations through their explicitation. Another issue that this mathematical task seemed to help accomplish is that students' work was more oriented to understanding *why* and not only centered on determining the truth-value of the proposition. It appears that students were easily convinced about the truth-value of the proposition but not about the reasons for that truth-value. The mathematical task proved effective in promoting students' experience of proof as a tool for explaining and answering an internal need of knowing.

The scarce research in the joint domain of proof and algebra described above and the poor performance of high-achievers in the UK in the intersecting domains of proof and algebra described by Hoyles and Healy (Healy & Hoyles, 2000; Healy, Hoyles, Sowder, & Schappelle, 2002) point to the need to investigate what is happening in the proof-algebra relation elsewhere. In addition, Barallobres' (2004) findings have proved that mathematical tasks involving the use of algebra and proof as tools are a fruitful context where students can learn to prove using algebra meaningfully. Approaching algebra through proof seems to offer a solution to the concern of many researchers (e.g., N Balacheff, 2000; Barallobres, 2004; Brousseau, 1997; Mason, 1996) regarding the lack of sets of problems where algebra becomes a necessary tool to solve the problem. Through such an approach, we highlighted at least two distinctive features of algebra as a tool for modeling, and as a tool to read information after applying algebraic transformations. Thus, Barallobres (2004) embraced Chevallard's (1985; , 1989; , 1989-1990) epistemological analysis of algebra where he identifies two central aspects of algebra: as a modeling tool by setting up expressions representing the relations among variables, and the production of equivalent expressions allowing to read and infer properties that couldn't be read in the initial expression.

Fifth section. Functional perspective

Here, I will explore studies that take a functional perspective to algebra. A vast majority of these studies based their proposal on the use of technology and software, sometimes specially designed for those purposes (Heid, 1996; Kieran, Boileau, & Garançon, 1996; Moschkovich, Schoenfeld, & Arcavi, 1993; Rojano & Sutherland; Yerushalmi, 2000; Yerushalmi & Schwartz, 1992; Yerushalmi & Schwartz, 1993; Yerushalmi & Shternberg, 2001). These studies emphasize the power of technology to work simultaneously with different representations of functions (tabular, algebraic expressions, and graphs). However, the functional approach is not exclusive of projects that use computational environments (the work by Chazan [2000] and Doaudy [1999] are examples of the latter). The common aspect among all these projects that focus on functions is that students' work on multiple representations is highly emphasized.

In the next sections, I will discuss Heid's (1996), Schwartz and Yerushalmy's (1992), and Kieran, Bolieau, and Garançon's works (1996), in that order. Later, I will discuss the work by Chazan (2000) and the work by Doaudy (1999).

Heid's project (1996) addresses the introduction of algebra at school using a functional approach, considering the concept of variable as central. Heid (1996) states, "what makes the study of variables interesting is the study of functions on those variables" (p. 239). In this approach, the variables are used to describe real world quantities and functions to describe the relation among those quantities. Students study families of functions, their properties, and their relation to the real world, analyzing the meaning of various rates of change, roots, maximum and minimum values, and asymptotic behavior in contextual settings. The design of the activities requires that students solve the problems using multiple representations: graphical, numerical, and tabular. For Heid, *real world* context is a main pillar in her proposal.

Heid's group designed a beginning algebra curriculum for seventh, eighth, and ninth graders. In terms of the role of technology, in this project the computing tools (function graphers, curve fitters, table generators, and symbolic manipulators) were supposed to facilitate "explorations of algebra by providing students with continual access to numerical, graphical, and symbolic representations of functions, as well as to technology-intensive procedures for reasoning about algebraic expressions" (Heid, 1996, p. 240).

The curriculum is structured in the following way: (1) variables and functions; (2) calculators, computers, and functions; (3) properties and applications of linear functions; (4) quadratic functions; (5) exponential functions; (6) rational functions; (7) algebraic systems; (8) symbolic reasoning: equivalent expressions; (9) symbolic reasoning: equations and inequalities.

The authors present as one of the positive aspects of this curriculum the fact that Computer-Intensive algebra (CIA) does not focus (or include) by-hand symbolic manipulation as a formal part of the curriculum. However, the authors should be cautioned against replacing "manipulation" with their curriculum. The ideal situation would be an integration of both manipulation and a CIA. Heid (1996) stresses the fact that the CIA curriculum is designed to help students develop a solid understanding of why such rules are needed and of graphical and numerical meanings of equivalence of expressions. In terms of the required features of the software, following Heid, what is missing from today's algebra computer systems is the ability to translate from graphs to symbolic rules.

Heid's project was evaluated in terms of the *written*, *taught*, and *learned curriculum*.

Regarding the analysis of the *written curriculum*, the CIA curriculum was compared with a popular 1960s textbook and a 1980s text used in a pilot CIA school. It was found that the current CIA curriculum asked for more complex questions than the other two texts.

Considering the *taught curriculum*, the implementation of the CIA curriculum seems to lead not only to different classroom activities but also to a different set of roles, responsibilities, and challenges for teachers (e.g., facilitator, technical assistant, catalyst) and students (e.g., new responsibilities, new goals). The CIA classes seemed to spend more time than traditional classes on conceptualizing problems, on planning solutions, and interpreting answers. In the CIA classes, different content was managed in the whole group discussion, with more talk about applied problems, more comparison of different representations, and less time spent discussing step-by-step procedures. Taking into account the *learned curriculum*, students in the CIA curriculum performed significantly better than students in traditional classes. Whereas the traditional class students developed a concept of the letter as unknown, a majority of the CIA students developed a concept of the letter as variable as well as of unknown. The study found that students in this curriculum chose and used computers with significantly greater frequency than the scientific calculator and paper-and-pencil. Students chose and used symbolic representations three times as often as they chose and used tables and graphs. Students also showed that they were able to use different strategies with a single tool and representation.

Other researchers like Schwartz and Yerushalmy (1992) also developed and used computational environments for the introduction of algebra at school. Schwartz and Yerushalmy (1992) developed three different computational environments to address the teaching and learning of algebra: the Function Analyzer, the Algebraic Supposer, and the Function Comparator. Different from Heid's work (1996), Schwartz and Yerushalmy (1992) propose many tasks within a *mathematical* context. For Heid (1996), these would be considered tasks without context and with no relation with the real world; *real world* context is a main pillar in her proposal. Schwartz and Yerushalmy's (1992) perspective is to work with functions both as process and objects. The authors link function as a process with the symbolic expression, while they link the graph of a function to the object perspective. The authors propose to accomplish this process-object duality encouraging students to carry out unary and binary operations on functions. Schwartz and Yerushalmy (1992) consider functions as the central object of algebra, and the concept of variable. Consequently, the computational environments designed by Schwartz and Yerushlamy (1992) allow students to study the effects of various unary and binary operations on functions using different representations as well as to see the consequences of their activities both symbolically and graphically. One of the limitations of other software currently available is that it is not possible to directly manipulate any representation other than the symbolic representation. Schwartz and Yerushalmy (1992), similar to Heid (1996), claim that a fully symmetrical environment would allow users to both manipulate the symbolic representation (symbolically) and see the graphical consequences of their actions, and to manipulate the graphical representation (graphically), and see the symbolic consequences of their actions.

As mentioned above, it seems that Schwartz and Yerushalmy (1992) think that the symbolic representation of a function reveals its process nature, while the graphical representation helps to make the function more entity-like. While Heid (1996) uses extra-mathematical contexts[4] in problems, Schwartz and Yerushalmy (1992) focus on unary and binary operations on functions. Within the unary operations, the authors take into account: (1) translation, from f(x) to f(x+a), and from f(x) to f(x)+a; (2) dilation and contraction, from f(x) to f(ax), and, from f(x) to af(x); and (3) reflection, from f(x) to f(-x), and, from f(x) to –f(x). The computational environment designed to work on this kind of problem is the Function Analyzer. The student can appreciate the result of the operations through looking at the graph of the function together with the symbolic expression that appears below the graph.

---

[4] Extra-mathematical context is used in Chevallard's (1985, 1989) sense. If the mathematical model is constructed on a mathematical system we say that the context is intra-mathematical. In the case that the system is formed by non-mathematical objects we will refer to the context as a extra-mathematical.

In addition, Schwartz and Yerushalmy (1992) propose to approach binary symbolic operations on functions; the goal is to provide students with the experience of transforming functions so that they are not constrained to remain in the same family of functions. For this purpose, Schwartz and Yerushalmy (1992) develop the Function Supposer. The Function Supposer includes the four typical arithmetic operations (addition, subtraction, multiplication, and division) as well as the composition between functions. An interesting feature of this software is that all the manipulations can be done without having to enter any symbolic expression. In order to promote students' work on real world problems, Schwartz and Yerushalmy (1992) designed the Algebraic Supposer. Using this software, students are required to write the information of the problem in the following categories: (1) how many, (2) what, and (3) notes. Students are in charge of defining the relations among the given data, and, as a result, students produce a graph. In addition, Schwartz and Yerushalmy (1992) developed the Function Comparator. Within this software, the work on equations and inequalities is interpreted as a comparison of functions; for instance an equation is seen as $f(x)=g(x)$, and, an inequality as $f(x)>g(x)$, with all possible variations. Students specify the two functions to compare and the software provides the solution set in the x-axis. Regretfully, in their paper, Schwartz and Yerushalmy (1992) emphasize the design aspects and the pedagogical approach more than the presentation and analysis of students' work. Nonetheless, some descriptive data is presented to exemplify students' productions. The mathematics education community could enormously benefit from more research on the use of this pool of software in schools.

Like Schwartz and Yerushalmy (1992), Kieran, Bolieau, and Garançon (1996) analyzed the introduction of algebra using a computational environment they designed. Kieran, Bolieau, and Garançon's work (1996) encompasses a deep and careful analysis of the mathematical content (algebra, problem solving, functions, and variables), the CARAPACE computational environment (Contexte d'Aide a la Résolution Algorithmique de Problemes Algébriques dans un Cadre Évolutif), and students' learning (including pedagogical and psychological issues). This project (Kieran, Boileau, & Garançon, 1996) includes six studies carried out during a six year period. Kieran, Boileau, and Garançon (1996) have opened up a new avenue whereby students have been shown to be able to develop a deeper understanding for the process of translating problem situations into notational representations without the acquisition of equation solving skills. This approach is innovative since in schools, algebra is synonymous with equation solving.

Regarding their perspective on using a functional approach to algebra, Kieran, Boileau, and Garançon (1996) claim that their approach does not necessarily entail the study of functions. However, it does entail the use of letters as variables in opposition to the use of letters as unknowns. Nonetheless, the authors acknowledge that a functional approach comprises more than that. It includes viewing a function from the perspective of the relationship among the x- and the y-values. This seems very subtle and could be read as a contradiction, but the authors are referring to the dual perspective of function as an object and as a process. Kieran, Boileau, and Garançon (1996) explicitly state that while the study by Schwartz and Yerushalmy (1992) focuses on functions as object in addition to functions as process, in Kieran, Boileau, and Garançon's (1996) curriculum the main perspective is to deal with functions as process. The authors' emphasis is on the process, as the set of mathematical operations to be applied to the independent variable in order to produce the dependent variable. Kieran, Boileau, and Garançon (1996) give a clear example of this approach, intended to emphasize the process. When students confront problems of the type: "The price of an object after a tax of 15% is $23; what was its price before tax?", their first attempt is to perform an arithmetic operation. In contrast, when they are faced with a slightly different version of the same problem, for instance: "If we know the price of an object before tax, describe how to calculate its price after a tax of 15%", a definite improvement in the subsequent solution attempts by the student is observed. Students shift from an arithmetical-unknown version of the problem to an algebraic-functional process-oriented perspective.

Regarding their conception of algebra, Kieran, Boileau, and Garançon (1996) were inspired in their work by the history of algebraic writing, taking into account different periods and stages, going from a rhetorical language in the solution of problems to a modern symbolic representation. The software design is inspired to support the students' development of this wide range of algebraic writing. Within the functional approach, problems involving the letter as an unknown can be re-conceptualized constructing equations with the functions at play. Another aspect that the authors consider intrinsic to algebra is the role of generalization. Consequently, setting up the general functional relation in the form of an algorithm involves a certain amount of generalizing for beginning algebra students.

In this section, I will describe the CARAPACE environment and provide a sample problem since one of the goals of this literature review is to provide the aspects of algebra that each work highlights; in addition, it seems fruitful to illustrate the approach through sample mathematical problems. The software's goal in terms of students' learning is to strength the algorithmic aspects of their algebraic language. In this context, the student must represent a functional situation in the form of a program that tells the computer how to perform certain arithmetic calculations. This environment (see Kieran, Boileau, and Garançon, 1996) has been used by 12 to 16 years old students during a period of six years. For example, students are presented with the following problem:

> "Carine works part time in the neighbourhood. She sells subscriptions to a magazine. She earns $20 a week, plus a bonus of $4 for each subscription sold."

Within CARAPACE, students find a first screen where they are asked for the input values, operations and nature of output values (see Figure 8). In Figure 9, a potential student answer for the subscriptions magazine problem stated in Figure 8 is shown.

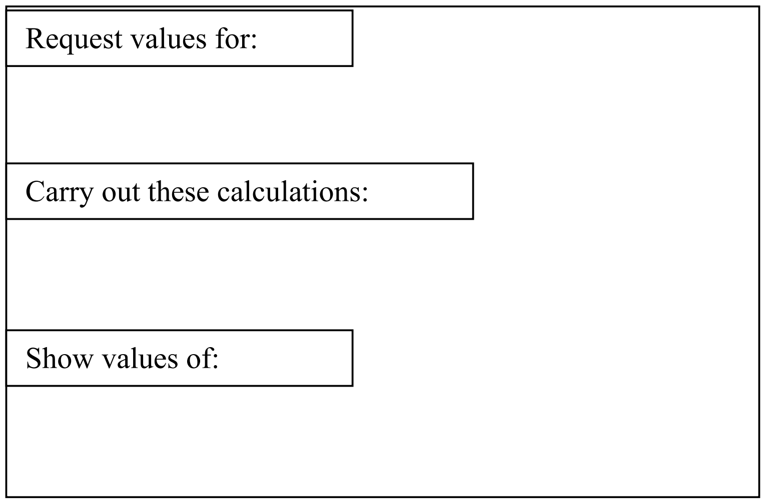

Figure 8. First screen where students are asked for the input values, operations, and nature of the output values.

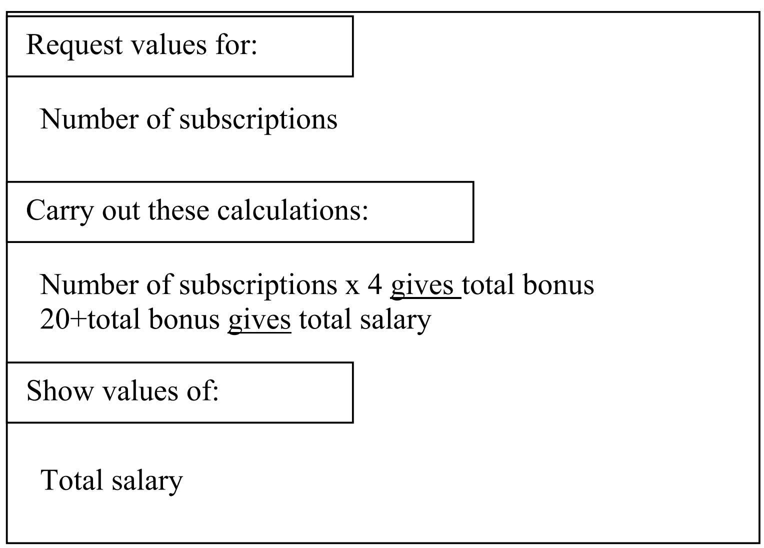

Figure 9. Potential student answer for the subscriptions magazine problem.

Regarding other features of the software, the environment does not allow the use of the equal sign. After completing the first screen, the student can try with numerical examples that are kept in a table and a graph. The highest syntactical level that CARAPACE accepts is the usual set of expressions used in elementary algebra, such as: "ax^2+bx+c gives y", without incorporating the use of the equal sign. The absence of the use of the equal sign might present an obstacle in students' understanding of algebra and later learning. The equal sign is a foundational symbol in mathematics that conveys different meanings as discussed before in the second section of this paper. Avoiding the use of the equal sign evades the arousal of the difficulties about the different meanings of the equal sign; it doesn't solve the obstacles students face when learning about the equal sign.

Regarding the graphical interphase, Kieran, Boileau, and Garançon (1996) chose to show the graphs in a discrete domain, not in a continuous way like most other software. The CARPACE environment allows the user to plot points (it doesn't plot the function) belonging to the graph of the functions; that is, the user must specify the points to be plotted, and must assist the computer in plotting the points in question. One of the main goals of the use of Cartesian graphs in CARAPACE is not only to help students in the solving of problems but also to allow for the discussion of particular questions, such as the number of solutions to a problem. In this environment, the cursor is an "adapted" cursor, it looks like another pair of axis and it is where students choose the components of the points to plot, see the scale of the graph, and modify it. A feature that this environment shares with others is the use of several representations as an integral part of a functional approach to algebra problem solving. In CARAPACE, the input is always the algorithm entered by the student; based upon the input the environment makes the graph and the tabular representation available. The user cannot change the graph or the tabular representation and cannot observe the changes in the algorithmic representation. This is a big disadvantage of this computational environment. I believe that one of the main advantages of using software in a functional perspective is that software facilitates working with multiple representations at once; it is very powerful to change one of the representations and see how this change impacts the other representations (we have seen that this is not the case for the suite of software developed by Schwartz & Yerushlamy [1989])

Kieran, Boileau, and Garançon (1996) found that the substitutions of numerical test values into algebraic representations of problems allows beginning algebra students to construct meaning for problem representations that may be different from those they have experienced in the past, while at the same time using solution methods based on familiar arithmetic techniques. The results showed that when students moved to the tabular display, they tended to forget the contextual information they had previously been relying upon. In addition, students exhibited an immediate ease in taking a functional approach and writing the algorithmic representation using forward operations.[5] The learning of algebra with CARAPACE seems to improve the transition from a more "natural" language to more standard algebraic representations. Carrying out initial numerical trials would appear to be extremely helpful for students to make sense of the word problem, to represent it, and to solve it. The use of significant names for variables can assist students in retaining the sense of a problem and in performing operations such as substitutions. However, this study uncovered the lack of any real motivation to simplify expressions in a computer-supported environment like CARAPACE. Students do not feel the desire to shorten their procedural representations because of the speed of the computer. The computer tool CARAPACE has supported the potential of algebra as a problem-solving tool. Not all technology-supported roads that intended to be algebraic lead to developing meaning for traditional algebraic representations

[5] Forward as opposed to inverse arithmetical operations that are usually used when solving an equation of this type (i.e., $ax+b=c$). Students first subtract b and then divide by a. In forward operations, required by the algebraic way of writing, we have to write multiplying by a and adding b.

and transformations. Introducing algebra with this kind of environment provides students with only part of the picture of algebra. The students' heuristic search for multiple solutions was much more effective when they used only a table of values representing the functions. Students gradually understood that there can be more than one solution to a problem. Students acquired ease in changing and thinking of different scales in the axis. Students rapidly improved in the development of successive approximation strategies in the graphical context. In cognitive psychology, thinking about variables is usually considered more complex than a single-value conception, as if it were a matter of development (Kieran, 1996). Kieran, Boileau, and Garançon's (1996) study clearly shows that beginning algebra instruction with a variable interpretation of the letter and later including single-valued situations seems to avoid the cognitive obstacles that can be encountered when one begins with a single-valued conception of letters and attempts to move on to a multi-valued interpretation. Consequently, Kieran, Boileau, and Garançon's (1996) work encouraged seeing the particular in the general and the general in the particular; in this aspect the authors agree with Mason (1996).

So far, in this section I have discussed a variety of works (i.e., Kieran, Boileau, & Garançon [1996], Schwartz & Yerushlamy [1989], and Heid, [1996]) within the functional approach to algebra that share the use of a computational environment to teach algebra at school. Next, I will discuss Chazan's (2000) work, and Douady's (1999) study. Both researchers used a functional approach to algebra, but without centering on the use of software.

In this section, I will discuss the work that Chazan (2000) developed in the Holt school[6] using a functional approach to algebra. Regarding the notion of function, Chazan (2000) chooses to emphasize, "that functions are relationships between quantities where output variables depend unambiguously on input variables" (p. 84). Moreover, following Comte's definition,

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[6] Holt was described as a suburban setting and Chazan taught a lower-track Algebra One course.

> By his definition, functions are the mathematization of our theories about the relationships of dependence, causation, interaction, and correlation between quantities. Auguste Comte, in his early study of the nature of different sorts of knowledge, sees such theories as methods for determining the values of quantities inaccessible to measurement. (Chazan, 2000, p. 84)

The structure of the curriculum was as follows: (1) reading sketches of relationships between quantities, rather than the traditional introduction to graphing, which emphasizes that each point has exact coordinates; (2) working on number recipes; (3) making traditional coordinate graphs from tables of values and reading computer displays from *The Function Supposer* (J. Schwartz, Yerushalmy, & EDC, 1989); (4) working on families of functions; (5) linear functions; and (6) standard algebraic manipulations. Regarding the results of his work, Chazan (2000) found that even though students may have come to recognize relationships between quantities as important mathematical objects, understanding their relevance, that does not mean students saw why facility with one representation of these objects such as algebraic symbols was considered so important. Chazan (2000) presents results concerning problem about a linear function (Figure 10) given to students at Holt as part of their final exam. Students found this task quite challenging and they did not perform as strong as it was expected. After the exam some students were interviewed about this problem. The results were mixed. Students could do thoughtful work about the problem beyond what they have written in their exams. However, students couldn't produce the right answer. Students made progress understanding the slope-y-intercept form and treat mathematical tasks meaningful. However, we still have much work to do in tasks involving the writing of linear expressions or equations. Students didn't appreciate the purpose of mastering the writing of linear expressions. "Thus, while a relationships-between-quantities approach provides a way of telling students what Algebra One might be about, we have not completed the task of psychologizing the subject matter. Because there are other canonical representations of functions, the issue of justifying to students (in the present tense) the importance of work with symbolic representation must continue to be addressed" (Chazan, 2000, p. 107).

> In other countries, like Canada, temperature is measured in degrees Celsius. We are familiar with the meaning of temperatures in Fahrenheit. So when traveling, it is useful to have a rule for changing temperature from Celsius to Fahrenheit.
>
> The temperature at which water freezes is 0 degrees Celsius and 32 degrees Fahrenheit. The temperature at which water boils is 100 degrees Celsius and 212 degrees Fahrenheit. The relationship between temperature Celsius and temperature Fahrenheit is a linear rule.
>
> Using the information given above:
>
> a. Figure out the slope for your rule. Show your work.
>
> b. Write a rule that will do the conversion.
>
> c. Using your rule, is 40 degrees Celsius the temperature of a hot day?

Figure 10. Sample problem to address the function-algebra relation (Chazan, 2000, p. 99).

Douady (1999), also adopting a functional approach to algebra without relying on computational environments, developed a didactical sequence on polynomial functions based on the method of Didactic Engineering (M. Artigue, 1988, 1994; M. Artigue & Perrin-Glorian, 1991; Douady, 1997). Schematically, the basic principles of Didactic Engineering are: (1) chose a teaching object in the current program, (2) place the mathematical context in relation with the teaching tradition, (3) bring out hypotheses about students' difficulties and set the basis for a didactical engineering, (4) develop such an engineering, proceed to the apriori analysis, (5) implement it and make an a-posteriori analysis of the collected data, (6) reproduce the implementation, under experimental control, after possible modification in view of the previous analysis, (7) test the supposedly acquired knowledge of the students in questions for which they adapted tools, and (8) compare the output of the students and their skill with expectations, and conclude about the relevance of the didactical hypotheses. The goal in Douady's sequence was to integrate the graphical representation in the study of polynomial functions and its roots, multiplicity, and signs. Participants were 15-16 years old French students. In the traditional curriculum, this topic is taught as a mere manipulation of algebraic expressions (or chains of signs) through factoring techniques. No allusion is made to Cartesian graphs of the polynomial as a function, nor to issues related to differentiability and continuity, at least implicitly. In Douady's (1999) proposal, students deal with problems where functions, graphs, and, algebraic expressions "are put in stage dialectically" (p. 113). Sample problems given to the students are shown in Figures 11 and 12 below. The problems seem to be interesting; regretfully, the author doesn't provide information on students' work.

1. *Calculators forbidden*

Giving x numerical values, you will get numerical values for the following expression:

$$f(x) = (x - 2) (2x - 3) (x + 5) (4x + 1) (1 - x)$$

Are they always positive ? Are they always negative question
Are they sometime positive, sometime negative, sometime zero ? Compute.
When you have an answer, call your teacher.

Figure 11. Sample problem to address the function-algebra relation (Douady, 1999, p. 113).

2. *Calculators forbidden*

$$f(x) = (x - 2)(2x - 3)(x + 5)(4x + 1)(1 - x)$$

Find a way which enables you to tell, very fast and reliably, when your teacher gives you a numerical value for x, whether the expression is > 0, < 0 or $= 0$.
Orders : *Only one answer accepted. Computing the expression is not allowed : it is too long.*
When you think you have a method, call your teacher.

Figure 12. Sample problem to address the function-algebra relation (Douady, 1999, p. 113).

The above set of studies within the functional approach shows use of computational environments to introduce algebra through a functional perspective, focus on the algebra and functional polynomials relation, and use of multiple representations in order to promote a better understanding of the concepts. The works under the functional approach have advanced a deeper understanding of the core of algebra (functions and variables) and the symbolic representations (reinterpretation of the letter as a variable, instead of the letter as an unknown). The researchers' innovation is grounded in the reinterpretation of the traditional algebraic objects (unknown, equations, and inequalities) from the perspective of functions. I believe that this innovation pushes for a re-conceptualization of our understanding of algebra within the mathematics education community. We can re-conceptualize: unknowns as specific values of a variable under particular conditions, and equations/inequalities as comparisons between functions. There are two other advantages to this approach. One of them is that the functional perspective calls for a multiple representation approach, since functions traditionally are represented in tabular, graphical, and formulaic forms whenever possible. The other advantage is that this approach calls for an integration of algebra and functions, that are two of the most important topics in the K-12 mathematics curriculum.

However, as Chazan (2000) found, it is still a challenge to design activities that will successfully communicate to students the need for using algebraic tools.

Sixth section. The Modeling perspective

Regarding the modeling perspective in the research of teaching and learning of algebra, it is not a surprise that there are different perspectives and understandings of the word "modeling." On the one hand, some authors (e.g., Chazan, 1993; Janvier, 1996; Nemirovsky, 1996) that employ the word modeling with an implicitly agreed upon meaning that could be understood as creating a mathematical model of a (mostly) non-mathematical reality (extra-mathematical context[7]), better known as "real world" problem. These authors, however, emphasize different aspect/s of modeling. For instance, Nemirovsky (1996) emphasizes the work on modeling through mathematical narratives; Chazan (1993; , 2000) emphasizes modeling through a functional approach to algebra; and Janvier (1996), besides emphasizing a functional approach, takes the nature of the variables involved as a main factor ("pure" numbers and extensive and intensive magnitudes). On the other hand, other researchers (e.g., Bolea, Bosch, & Gascon, 1999, 2003; Chevallard, 1985; Chevallard, 1989, 1989-1990; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascon, 1993-1994) take the study of the concept of modeling in and of itself, as well as the meaning and implications of conceiving algebra as a modeling tool in the school curriculum, as central matters. The distinction between the two groups of researchers is not clear-cut since I believe that the authors in the first group would agree with intra-mathematical modeling activities; however, their emphasis is not on the "algebraization" of mathematical organization as it is presented in the work of the second group of authors. After discussing Chazan's (1993) proposal I will discuss the perspective that adopts the use of algebra as a modeling tool.

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[7] I will use Chevallard's (1985, 1989, 1989-1990) classification of mathematical problem contexts: intra-mathematical (mathematical reality) and extra-mathematical (non-mathematical reality) since it is more fruitful than the "real world" vs. "non-real world" distinction.

Regarding Chazan's (1993) perspective, which is a functional perspective to modeling with algebra, he mentions four benefits to such a perspective: (1) it helps students approach the solution of equations in diverse ways and suggests that students should not be limited to the traditional symbolic manipulations for solving equations; (2) the conceptualization of an equation as an equality of functions can be used to clarify the basic terminology of the algebra curriculum: "equations, identities, inequalities, and what are now called relations can all be treated as comparisons of two functions and solved with the same three solution methods" (1993, p. 23); (3) this conceptualization of equations helps clarify the utility of algebra; and (4) the dynamic dependence relationships captured by functions are accessible aspects of real world situations: "thus an approach which views equations as built up from functions which vary, instead of expressions which represent an unknown, may make algebraic modeling more natural and coherent for students" (1993, p. 23). Chazan proposes problems that are "real world problems," for instance, presenting the "bike-manufacturing company" problem illustrated in Figure 13, and adapted from materials provided by the Michigan Department of Commerce in training sessions for citizens contemplating starting a small business.

A bike-manufacturing example

The people who invested in starting this company did not know how to calculate break-even points. They began their business manufacturing bikes to order. They knew that with the shop that they had set up the average cost to make each bike was $60, including labour, materials, and everything else (except for rent and salary for the boss). The rent for the shop was $500 a month. The head of the company was paid $1000 a month salary. They decided to set the average price for their bikes at $100 dollars a bike figuring that they would make about $40 on each bike. During the first month, they sold 10 bikes.

Figure 13. Sample problem with a real world context. (Chazan, 1993, p. 23)

The example in Figure 13 illustrates that a central aspect in Chazan's modeling conception is real-world relevance. In Chazan's (2000) work, a strong Dewey influence can be found and thus understood as part of his epistemology, since one of Dewey's concerns was to provide instruction using objects and resources from the world surrounding students to promote learning with meaning. Nemirovsky coincides in this focus on real world phenomena:

> A shared goal among mathematics educators is having students come to be able to fluently use graphs and equations in the description and interpretation of events in the world. However, as currently taught, often these representations arise out of nothing—and so have to be imposed on students as notations devoid of personal meaning. To change this situation it is essential to identify and nurture the students' domains of everyday experience that may offer a fertile background for the growth of mathematical ideas (1996, p. 197).

Nemirovsky (1996) differs from Chazan (1993, 2000) in his approach in that for Nemirovsky mathematical narratives are a central aspect to his approach; as mentioned above, for Chazan a functional approach is the central perspective. A mathematical narrative is a narrative articulated with mathematical symbols. An example of a mathematical narrative is the following:

> (a) First it rained more and more and it started to become steady (pointing to a piece of the graph in Figure 14).
>
> (b) then it rained steadily (marking piece b of the graph in Figure 14).
>
> (c) then it rained more and more (pointing to piece c of the graph in Figure 14) (Nemirovsky, 1996, p. 200).

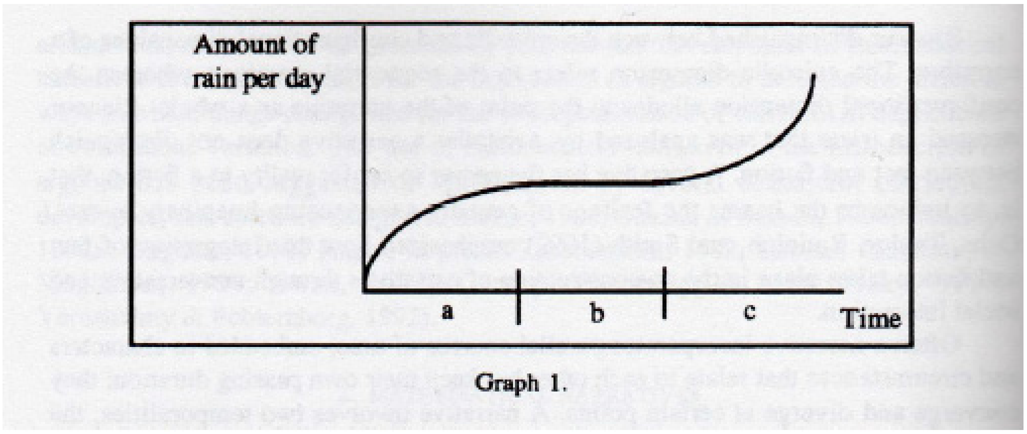

Figure 14. Graph used by Nemirovsky (1996, p. 200) to promote the work with narratives in mathematics.

Another set of studies (Bolea, Bosch, & Gascón, 1999, 2003; Chevallard, 1985, 1989, 1989-1990; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascón, 1993-1994) take a different perspective to the modeling approach to algebra.

Chevallard, Bosch, and Gascón (1997) claim that an essential characteristic of a mathematical activity consists in building a (mathematical) model about systems (intra-mathematical or extra-mathematical contexts) to be studied, to use it, and to produce an interpretation of the obtained results. In others words, the mathematical activity can be characterized as making (mathematical) models of (intra or extra- mathematical) systems. The authors underline three aspects involved in building a mathematical model: the routine utilization of pre-existing mathematical models, the learning of models as well as the way of using them, and the creation of mathematical knowledge. For example, consider the system formed by two right triangles (ignoring their position in the plane) shown in Figure 15. The system is well known in the theory of geometry. We can build a metric model from the system where: a and b are the measures of the sides of the rectangle; d is the measure of the diagonal; the measure of the area is S; u and v are the measures of the angles formed by the sides and diagonal. Building this mathematical model we have the following relationships among the variables: $S=ab$, $d^2=a^2+b^2$, $u=\text{arctg}(b/a)$, $v=\text{arctg}(a/b)$.

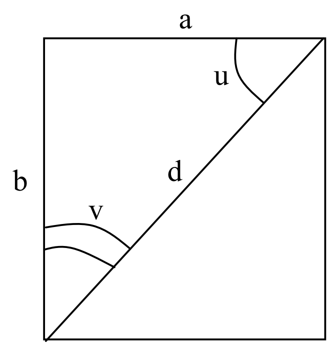

Figure 15. System formed by the two right triangles.

Very simple mathematical work on the model provides us with new information: the system that initially was parameterized by the measures of the sides a and b can be parameterized also in terms of d and u, where $a=d \cos(u)$ and $b=d \sin(u)$. In the example just shown, the model is constructed on a mathematical system, that is in the intra-mathematical context, as opposed to extra-mathematical contexts formed by non-mathematical objects,

This last perspective of mathematical activity agrees with Chevallard, Bosch, and Gascón's position (1997), characterizing mathematical activity as that of constructing (mathematical) models of (intra or extra-mathematical) systems. Inherent to the discussion of what is a mathematical activity is the discussion of what is mathematics. In particular, regarding the activity of algebraic modeling, Gascón (1993-1994) bases his proposal on Viète and Descartes' works, where the principal feature is the systematic introduction of a literal representation to designate the unknowns and the given data, since it provides the advantage of studying a general case and the structure of the problems, and not only obtaining the unknown value. The French-Spanish research group formed by Chevallard, Bolea, Bosch, and Gascón consider that as a consequence of creating an algebraic model, the structure of mathematical problems can be studied as the most important feature of algebra. Gascón (1993-1994) provides an outline of what they call a new conception of elementary algebra as having the following features: (1) elementary algebra consists in the study of a certain field of problems which contain not only arithmetic problems but also geometric construction problems[8], simple combinatorics to determine a finite set's cardinality, and level sets[9]; (2) the algebraic method provides a global symbolization of the relationships among the given data and the unknowns of the problem without distinguishing essentially among them; in other words, this method's main goal is to make explicit the formal structure of these relationships; (3) the language at play involves symbols that can be interpreted as unknowns, general numbers, variables, and parameters; (4) algebraic manipulation allows us to determine the "existence conditions" of the unknown object, as well as the form of dependence of each variable in relation to the other variables within the system; and (5) the algebraic modeling activity allows us to determine the "existence conditions" of objects other than the ones that we initially wanted to study; as any modeling activity, it allows us to invent new problems regarding the studied system. Along the same lines, Bolea, Bosch, and Gascón (, 1999; , 2003) state that,

> Elementary algebra does not appear as a self-contained mathematical work comparable to other works studied in academic core courses (such as arithmetic, geometry, statistics, etc.), but rather as a modeling tool to be (potentially) used in all mathematical curricular works and which appears to be more or less used in them. (p. 138)

In addition to the features provided by Gascón (1993-1994), mentioned above in (1) through (5), Bolea, Bosch, and Gascón (1999, 2003) offered more characteristics:

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[8] For instance, "Construct with ruler and compass a triangle ABC given the side c=AB, the height h_C from vertex C, and the median m_A from vertex A".
[9] A detailed description of these problems can be found in Gascón (1989).

> In particular, the more a mathematical work is algebraized, the more it enables us to describe the different types of problems that can be solved, as well as the necessary conditions for solutions to exist, their possible uniqueness and their structure (p.142).
>
> An indicator of the algebraization degree of a given mathematical work is linked to the possibility of considering, describing and handling the global structure of the above-mentioned problems [various types of problems that arise from a given question]. (p. 142)
>
> [I]n an algebraized work, we use parameters and variables systematically. (p.142)

So far we have discussed two different perspectives on the modeling approach to algebra; on the one hand, Nemirovsky (1996) and Chazan (1993), who consider real world (problem contexts) central to their work, and, on the other hand, the French-Spanish group (Bolea, Bosch, & Gascón, 1999, 2003; Chevallard, 1985, 1989, 1989-1990; Chevallard, Bosch, & Gascón, 1997; Combier, Guillaume, & Pressiat, 1996; Gascón, 1993-1994) that emphasizes the modeling role of algebra to create models not only from real-world phenomena (extra-mathematical) but also from within mathematics (intra-mathematical contexts). As a summary and in comparing these two different approaches mentioned above (more or less inclined towards real world contexts and/or inside mathematics) to modeling, it can be appreciated that the one proposed by the French-Spanish research group (Bolea, Bosch, & Gascon, 1999, 2003; Chevallard, 1985, 1989; Chevallard, Bosch, & Gascón, 1997; Gascon, 1993-1994) is broader and at the same time more specific; it doesn't take for granted what we understand to be the objects of algebra[10]; it incorporates parameters as a central component of algebra; it captures its potential instrumental power linked to its historical roots (Descartes and Viète). The other approaches deal with a more limited pool of mathematical concepts and

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[10] Chevallard (1997) claims that one of the main tasks of the *didactician* is to care about the principle of "epistemological vigilance" that is to analyze the *distance* between the object of knowledge and the teaching object. In order to be taught an object of knowledge goes through a process of transformations in order to become an object of teaching, this process is known as *didactical transposition*.

objects, and have the issue of making mathematical problems relevant (only) by using real world problems. As a closing remark, I will recall the description of what the French-Spanish group defined as their new conception of elementary algebra: (1) elementary algebra involves the study of a set of problems which contain arithmetic problems, geometric construction problems[11], and simple combinatorics to determine a finite set's cardinality, and level sets[12]; (2) the algebraic method's main goal is to make explicit the formal structure of the relationships among the given data and the unknows; (3) the language at play involves letters playing different roles such as unknowns, general numbers, variables, and parameters; (4) algebraic manipulation allows us to determine the "existence conditions" of the unknown object, as well as the form of dependence of each variable in relation to the other variables within the system; (5) moreover, the algebraic modeling activity allows us to determine the "existence conditions" of other objects than the ones that we initially wanted to study; as any modeling activity, it allows us to invent new problems regarding the studied system. The just given French-Spanish characterization of elementary algebra as a modeling tool synthesizes its central aspects. In this synthesis, the French-Spanish (e.g., Bolea, Bosch, & Gascon, 1999, 2003; Chevallard, 1985, 1989; Chevallard, Bosch, & Gascón, 1997; Gascon, 1993-1994) group push the mathematics education community towards a re-conceptualization of what we understand as algebra. Similarly, the researchers (e.g., Heid, 1996; Kieran, Boileau, & Garançon, 1996; Moschkovich, Schoenfeld, & Arcavi, 1993; Rojano & Sutherland; Yerushalmi, 2000; Yerushalmi & Schwartz, 1992; Yerushalmi & Schwartz, 1993; Yerushalmi & Shternberg, 2001) who argue for a functional approach (as presented in the Fifth section on this paper) to algebra pushed the field to re-conceptualize what we

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[11] For instance, "Construct with ruler and compass a triangle ABC given the side c=AB, the height h_C from vertex C, and the median m_A from vertex A".
[12] A detailed description of these problems can be found in Gascon (1989).

understand by algebra (beyond the concept of equation), how we teach it (beyond teaching equation solving), and why we do it that way. Thanks to the functional approach it was highlighted that algebra is not mainly about equation solving but also about variables and functions. In addition, we have now new technological tools that allow students to work with multiple representations of the same object simultaneously.

Concluding remarks

In this review, I attempted to characterize different perspectives regarding algebra as revealed by the work of researchers who have studied the learning and teaching of algebra. Within the research community, there has been a shift from the importance of considering and introducing the letter as an unknown towards considering the letter as a variable as well as a complementary shift from considering equations as the main object of algebra towards considering functions as its main object. In this effort, there have been enormous contributions from the functional perspective and from the use of technology. From the research on the relation between arithmetic and algebra we learnt distinctive features that characterize each domain (i.e., the meaning of the equal sign). Regarding the generalization approach, we learnt about the importance of the general in the mathematics classroom, and the different interpretations and uses of expressions. In the modeling perspective we addressed two different conceptions: the approach that emphasizes the use of real world contexts, and the approach that proposes the use of algebra as a modeling tool for any kind of context (intra- and extra-mathematical). Taking into account the algebra-proof approach, it is clear that the research community needs to investigate this relation further and evaluate how students could benefit from such an approach.

Regarding a generalization perspective, research (e.g., Lee, 1996; Mason, 1996) has shown the central role that generalization activities play in the learning of algebra. Problems that foster generalization activities have been proven fertile to promote students' work with variables and algebraic expressions (e.g., Lee, 1996; Mason, 1996). Radford (1996), however, warns us about the fact that generalization is an activity, and that the product of that activity needs to be validated in mathematical terms.

Underlying these approaches to algebra we can identify different epistemologies guiding the research. On the one hand, Bednarz and Kieran (1996) and Filloy and Rojano (1989) chose equations as a central object in algebra. Bednarz and Kieran (1996) tried to understand the notion of equation and different components (e.g., the equal sign). Filloy and Rojano (1989) tried to understand the process involved in equation-solving and developed different contexts (e.g., weights, area in geometry) to provide an opportunity to learn equation-solving embedded with meaning. On the other hand, researchers grouped under the functional perspective (e.g., Chazan, 1993; Chazan, 1996, 2000; J. L. Schwartz, 1991; J. L. Schwartz & Yerushalmy, 1992) chose functions as the central object in algebra and proposed a re-conceptualization of the letter as variable and of the equation as a comparison of functions. One of the advantages of this approach is the integration of multiple representations (e.g., tabular, formula, graphical).

The French-Spanish group proposes another re-conceptualization of algebra as a modeling tool of extra- or intra-mathematical systems with the following features: (1) problems which contain not only arithmetic problems but also geometric construction problems, simple combinatorics to determine a finite set's cardinality, and level sets; (2) the algebraic method provides a global symbolization of the relationships among the given data and the unknowns of the problem without distinguishing essentially among them; (3) the language at play involves symbols that can be interpreted as unknowns, general numbers, variables, and parameters; (4) algebraic manipulation allows us to determine the "existence conditions" of the unknown object, as well as the form of dependence of each variable in relation to the other variables within the system; (5) moreover, the algebraic modeling activity allows us to determine the "existence conditions" of other objects than the ones that we initially wanted to study; as any modeling activity, it allows us to invent new problems regarding the studied system.

After describing the different approaches to algebra, one could ask how these different approaches relate to each other: Are they complementary? Are they ordered in a hierarchy? Are they compatible?

In order to answer this question, I would like to recall Brousseau (1997) and Bell's (1995) works. From a general perspective regarding mathematics knowledge, Brousseau (1997) states:

> The meaning of a piece of mathematical knowledge is defined, not only by the set of situations in which this knowledge is realized as a mathematical theory (semantic in Carnap's sense), not only by the set of situations in which the subject has come across it as a means of solving a problem, but also by the set of conceptions, of previous choices which it rejects, of errors which it avoids, the economies it procures, the formulations that it re-uses, etc. (p. 81)

From a local perspective regarding algebra knowledge, Bell (1995) states:

> In general, the approach advocated is to learn the algebraic language in a way similar to that in which the mother tongue is learned; that is by using it in order to communicate, with oneself and with others, in the course of activities defined by the three main modes of activity already described: (a) generalizing, (b) forming and solving equations, and (c) working with functions and formulae. (p. 50)

Lastly, this paper has shown evidence regarding the scarce research on the algebra-proof relation. On the one hand, Healy and Hoyles (2000) have shown how poorly English students performed in the construction of proofs that required the use of algebra. On the other hand, Barallobres' (2004) findings showed the potential of teaching algebra in a proof production context. Consequently, further research needs to be conducted to understand better what learning problems involving both algebra and proof could entail.

Another issue discussed in this paper is the relation between meaning and technique in the teaching and learning of algebra. As argued in *Brief history of the field* (first section in this paper) there have been changes on the emphasis placed on technique and meaning on the research on the learning and teaching of algebra. After a first emphasis on the importance of technique and procedures to solve equations, there was a shift towards learning algebra meaningfully, accompanied by an underestimation of the role of technique (and formal writing of algebraic expressions). I believe that it is now time to reconcile both meaning and technique through investigations of how these two essential components of algebra can empower each other and promote a meaningful learning of algebra.

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