# GESTURES AS EMBODIED VARIABLES AND EXPRESSIONS 

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$\boldsymbol{A b s t r a c t}$. Researchers have investigated how students may represent indeterminate (i.e., variable or fixed unknown) quantities using expressions in natural language, non-numerical symbols, and external representations, implicitly treating indeterminate quantities much as if they were known quantities (Radford, 2011). The present study aims to address the role of gestures in the development of student's algebraic thinking; specifically their use of variables and expressions. This is an underexplored area on which I focus by addressing the following research question: How do sixth graders' gestures reflect their work with indeterminate quantities and the ways in which they operate on those quantities? Specifically, the present study provides evidence that sixth grade students used gestures as embodiments of indeterminate quantities, covariation, and doubling, and that they combined gestures into embodied forms of expressions.

## Introduction

The present study aims to address the role of gestures in the development of students' algebraic thinking-specifically, their use of variables and expressions. This is an underexplored area on which I focus by addressing the following research question: How do sixth graders' gestures reflect their work with indeterminate quantities (variable and fixed unknown quantities) and the ways in which they operate on these quantities?

Gestures observed in the present study can be defined as spontaneous motions of hands and arms that co-occur with speech (McNeill, 1992) and can incorporate the use of props such as pencils. Gestures mismatched with co-occurring speech, or gestures occurring in the absence of speech were not observed in this study. Gestures are semiotic resources (Arzarello, Paola, Robutti, \& Sabena, 2009; Goldin-Meadow, 1999; McNeill, 1992, 2008; Radford, 2014; Sabena,

Radford, \& Bardini, 2005) and as such convey meaning that closely relates to accompanying speech (Goldin-Meadow, 1999; McNeil, 1992, 2008), or to other components of a semiotic bundle (Arzarello et. al, 2009). Gestures and accompanying speech might convey complementary or additional information (Goldin-Meadow, 1999) while co-expressing the same idea (McNeil, 1992, 2008). Gestures are referred to as components (Radford, 2014), mediators and locations of students' mathematical thinking (Radford, Bardini, Sabena, Diallo, \& Simbagoye, 2005). Historically, algebraic thinking has been largely associated with performing manipulation on symbols written in standard mathematics notation, for example when one engages in solving an equation by applying a formally written sequence of steps. Consequently, gestures are rarely seen as ways in which students do algebra. In this study, I focus on the role of gestures as a semiotic resource in students' algebraic thinking involving variables and expressions.

Different authors have different perspectives on what characterizes thinking as algebraic and what distinguishes it from arithmetic. Arithmetical thinking is often associated with performing calculations on numbers or known quantities often leading to treating the equal sign as an operator. Algebraic thinking, on the other hand, most authors agree, requires an inquiry into and representation of structure and relationships between different kinds of quantities, operations and their inverses, use of letters and not only numbers, and a relational understanding of the equal sign (see Kieran, 2004). Kaput (1995) discusses algebra as generalizations (generalized arithmetic or generalized quantitative reasoning), manipulation of symbols, study of structures, relations, but also as functions, joint variations, and as a modeling language. Kieran (2004) discussed algebra not only as a study of structure and relationships between quantities but also as a study of change, and considers mathematical practices such as justifying, proving and
predicting necessary for algebraic thinking. What different perspectives have in common is that algebraic thinking involves working with quantities that might or might not have a known value. For example, Carraher, Schliemann, Brizuela, and Earnest (2006) provide evidence that young students, as they develop their algebraic thinking, can work with quantities that do not have a specific value. Radford's view of what constitutes algebraic thinking overlaps with the perspectives of other authors in that it involves students working with quantities whose value has not been specified. Radford (2011) describes algebraic thinking as ways of operating on unknown quantities as if they were known:

What characterizes thinking as algebraic is that it deals with indeterminate quantities conceived of in analytic ways. In other words, you consider the indeterminate quantities (e.g. unknowns or variables) as if they were known and carry out calculations with them as you do with known numbers. (p. 310)

Radford illustrated this by using an example of a second grader working on extending a geometric design in which an element at position $n$ consisted of a row of $n$ white squares plus one shaded square placed on top of another row of $n$ white squares. The pattern corresponded to the function $y=2 n+1$, where $n$ denoted the position in the ordered sequence and $y$ denoted the number of squares in the pattern. The student extended the pattern to the 25 th position by saying, "What is 25 plus 25 ? After that you add 1 !" In this example the second grader operated on the sum $25+25$ as if it was known and added 1 to it. Although she struggled to find the sum $25+25$, she described the element at the 25 th position as a rule (in analytic ways) rather than as a value of 51. In Radford's view the essence of algebraic thinking is the ways in which students operate on indeterminate quantities by performing calculations on them.

Whereas Radford (2011) identifies indeterminate quantities with students' linguistic referents to "instances of the independent variable" (p. 310) and Brizuela (2016) with a student's
use of the non-numerical inscription "?" to represent the unknown number of candy in a candy box, Cooper and Warren (2011) identify them with points on a number line representing an unknown value. The present study makes a contribution to the body of literature on students' work with indeterminate quantities in ways different than manipulation of symbols written in standard notation (Brizuela, 2016; Cooper \& Warren, 2011; Radford, 2011), by providing evidence that students do so through gestures. Specifically, in the present study I claim that 1) sixth grade students used gestures as embodied representations of known and indeterminate quantities; 2) students operated on quantities (known and indeterminate) by combining gestures into embodied expressions; and 3) students used gestures as embodiments of covariation and doubling.

## Method

## Data Source

Data were selected from a set of 320 classroom videos collected in 2011-2012 of 56 mathematics educators in grades 5-9 from eight school districts, participating in a 3-semester long graduate-level professional development program aimed at improving teaching of mathematics. All participating teachers were asked, but not required, to allow researchers to videotape in their classrooms both at the beginning of their participation and at several points during the three semesters. A total of 41 out of 56 teachers were videotaped this way.

## Data Selection

With the goal to look into students' ways of thinking mathematically, I narrowed down my selection to 3 teachers' videos of student-centered class sessions featuring students' reasoning either in whole-class discussions or during group work. Out of those 3 teachers I focused on one teacher whose lessons incorporated group work for extended periods of time and use of manipulatives and representations. I selected this teacher's videos because they featured
students discussing and explaining their thinking as well as engaging with mathematics in ways other than written symbolic manipulation. I then focused on a single sixth grade mathematics lesson because of the prominent use of gestures among more than half of the students.

Data used in this study are two 38-minute long video recordings of that lesson, each captured by one of two different video cameras from two different angles. Each of the two cameras was operated by one of the two program researchers who recorded different aspects of the same lesson while also engaging with students and asking them to explain their thinking. The lesson aimed at algebraic generalization of a geometric design.

At the time of this lesson, the teacher was nearing the end of her second semester of participation. The analysis presented here focuses on students' gestures at a single time point, and not on the teacher's change throughout the program.

## Participants

Participants were 12 sixth grade students in a public school in New England. Eight other students who did not consent to be videotaped were also present during the lesson but were sitting at two separate tables away from the cameras.

Four out of seven students who gestured were included in the analysis: Theo, Sophia, Henry, and Alex. Theo and Sophia were selected because they used gestures in ways that were not specific to any particular position in the geometric design. Henry and Alex were selected because they used gestures in ways different than Theo or Sophia. Three other students gestured in similar ways to the 4 students and therefore were not included in this analysis. While cameras were focused on other 5 students they did not use any hand gestures and did not engage in explaining their thinking.

With the exception and Alex and Theo, the four students in this study were not working in the same group. Theo and Alex were at a table with another two students. Sophia worked with three other students at the second table. Henry worked at the third table with another student as one group. The fourth table included two other students who did not gesture.

## Framework for Interpreting Gestures

I draw on McNeill's (2008) idea of "Co-Expressiveness and Synchrony" (p. 22) of gesture and speech according to which gestures and accompanying speech are synchronous, which means they occur at the same time, as well as co-expressive which, on the other hand, means that gesture and accompanying speech represent the same idea, at the same time, but in different ways, in two different modalities. According to McNeill, "the synchrony is crucial, because it implies that, at the moment of speaking, the mind is doing the same thing in two ways, not two separate things" (pp. 22-23).

Knowing that gesture and accompanying speech represent the same idea, when making interpretations about the meaning of gesture, according to McNeill (2008) it is important to compare the content of gesture to the content of speech, in other words, to compare what gesture looks like to what is being said. According to this framework it is also important to take into a consideration the context of speech, for example the task students are working on and any progress they made so far.

## Data Analysis

I selected and transcribed the video episodes in which the four students gestured. All their gestures co-occurred with their speech, which I refer to as gesture-speech pairs.

With McNeill's (2008) framework for interpreting gestures in mind, I first looked for gesture-speech pairs that were synchronized in space and time. I then grounded my interpretations of each gesture in 1) the context of speech, and 2) content of the gesture
compared to content of speech. This allowed me to identify the single idea that was simultaneously being co-expressed in speech and gesture. I then argued my claims considering gesture as one way in which that idea was being expressed.

## Task and Lesson Flow

The teacher introduced the task as a real-life scenario asking students to make predictions for the number of tiles needed to enclose a garden of varying length and constant width of one:

Teacher: What we are looking at today is a friend who lives in the city and she wants to put a garden on the roof of her house. So, because it's on the roof [...] she has to make it a narrow garden. So her garden is always going to be a width of one. But the length can change; she is not sure what she wants for a length yet. So we are going to look at some different designs and help her [...] OK, so she needs to make sure none of the squirrels in her city get into her garden. What can she do to enclose this? How many tiles does she need to make her garden safe?

I paraphrase the task as follows: How many tiles are needed to enclose a garden of any length and a fixed width of 1 unit? The way the teacher presented the task made it clear that in the context of this task the length was being measured by the number of tiles.

During a whole-class discussion, for each of the first three positions in the geometric design (see Figure 1), the teacher, under a document camera, laid down the green tiles representing the garden spaces and then asked students to make predictions for the number of tiles needed to enclose it. Following students' predictions, she enclosed the garden with tiles as ways of testing students' predictions, and moved to the next position. Lastly, she asked students, "What [...] have you started to notice? [...] What we look for is what is changing and what is always staying the same." The first student to respond immediately took a covariation approach (Carlson, Jacobs, Coe, Larsen, \& Hsu, 2002; Confrey \& Smith, 1995) when he said, "You add a square foot to the garden, and it increases by two tiles on the outside." Another student noticed that three tiles were needed at each end to enclose any garden with a constant width of one, "one
side is always three tiles." The teacher then announced that she would refer to the sides as "ends." Theo rephrased the first student's statement while using the terminology "top" for the tiles enclosing the garden on the top, and "bottom" for the tiles enclosing the garden on the bottom, "'Cause since you add more green tiles (garden spaces), you had one more on the top and one more on the bottom."
Teacher: What patterns have you started to notice?
Student: You add a square foot to the garden [in length], and it
increases by two tiles on the outside [one on the top and one on
the bottom].
[...]
Student: Um. The side, um, the side, one side is always three tiles.
(Teacher puts her fingers on each end, and asks if the student was
calling those the sides, she then separated the sides and
announced that she will be calling those sides the "ends")
[...]
Teacher: And why is it increasing by two? Why do you see it keeps
going up by two?
Theo: 'Cause since you add more green tiles [garden spaces], you
had one more [tile] on the top and one more [tile] on the bottom.

Figure 1. Initial discussion of the first three positions of the geometric design.

During the initial discussion a visualization of the tiles in the enclosure emerged as consisting of 4 parts, (1) the top (in Figure 2 the top is framed blue), enclosing the garden from the top without including the corners; (2) the tiles enclosing the garden on the bottom (also framed blue) also not including the corners; and (3) and (4) three tiles enclosing the garden on each end (those tiles are framed in red). This visualization corresponded to the function $y=2 n+6$, where $y$ represented the number of tiles needed to enclose a garden of length n . After working with garden lengths 1-10, students were asked to predict remote positions $20,25,30,100$, and 1000 before they were asked to create a rule for any garden length, $n$.

Figure 2. Visualization of the geometric design - the top, bottom, and three tiles on each end.
After the initial whole-class discussion, students were given blocks (instead of tiles) and a handout and were sent off to work in their small groups. While students worked in groups, the teacher and the two program researchers, Ana and Phil, circulated around the classroom and interacted with each group.

The handout consisted of a table with two columns: "Length of Garden" prefilled with values $1,2,3,4,5,6,7,8,9,10,20,25,30,100,1000, n(l e f t ~ c o l u m n)$, and "Number of Tiles" (right column) left blank for students to fill in. To the right of the table was extra space designated by the teacher for students to record their observations and patterns they notice. The bottom of the page contained the following prompts: "How can you find the number of tiles for any garden length?" and "Write the rule."

## Results and Claims

Four students in this study used gestures to represent a variety of ideas, for example Theo and Sophia used gestures as representations of fixed known and variable quantities, and they combined those gestures into embodied forms of expressions. On the other hand, Henry used gestures to represent covariation between two quantities, whereas Alex's gestures were visual representations of doubling. In what follows I argue and provide evidence for the following three claims: 1) sixth grade students used gestures as embodied representations of known and indeterminate quantities; 2) students operated on quantities (known and indeterminate) by combining gestures into embodied expressions; and 3) students used gestures as embodiments of covariation and doubling. Claims 1 and 3 are presented in two parts, labeled $a$ and $b$.

Data from five episodes are presented in Figures 3-7. Each episode features gesturespeech pairs of one of four students selected for analysis (Theo is presented in two episodes).

Gesture-speech pairs are labeled alphabetically with letters (a) through (f), speech fragments cooccurring with gesture are underlined, and student speech is presented in bold. Placed directly under the screen capture images are visualizations of pencil motions.

Claim 1-a. Sixth grade students used gestures as embodiments of known quantities
Figure 3 summarizes Episode 1 in which Theo used six gesture-speech pairs to describe how he calculated 46 tiles being needed to enclose a garden length of 20 .


Researcher: How did you get that? [result of 46 tiles for the garden length of 20]
Theo: 'Cause we did the twenty ${ }^{a}$ (points the pencil at the start of the top of the enclosure) for the top ${ }^{b}$ (moves the pencil alongside the blocks to the other end) and (repositions the hand) $\boldsymbol{t w e n t y}^{\text {c }}$ (points the pencil at the start of the bottom of the enclosure) for the bottom ${ }^{d}$ (moves the pencil alongside the blocks to the other end). And (repositions the hand) the three ${ }^{e}$ (gestures along one end, repositions the hand) on each side ${ }^{f}$ (gestures along the other end), which equals forty-six (brings his arm back).

Figure 3. Episode 1 - Theo describes the position 20.

Theo synchronized gesture and speech into six gesture-speech pairs, which he combined to explain how 46 tiles were needed to enclose a garden length of 20 . When the researcher asked Theo to explain how he calculated 46, the researcher was referring to the number 46 which Theo wrote in his handout in the column "Number of Tiles" associated with the row where the column "Length of Garden" was set to 20. The researcher's question focused on numbers only without an explicit reference to the lengths measured in the number of tiles. To answer the researcher's question about how he got 46 Theo described where 20, 20, 3, and 3 came from. To do so, Theo synchronized in time and space gesture and his speech in each of the six gesture-speech pairs. This synchrony, according to McNeill, implies co-expressiveness in the sense that a gesture and the co-occurring speech are different representations of the same idea.

Gesture-speech pair (a) expressed the idea that the first number 20 came from the 20 tiles. Using McNeill's framework I ground this interpretation in: 1) the context of speech "'Cause we did the twenty," thus, in the task itself of finding the number of tiles to enclose a garden of certain length, and more specifically to answer researcher's question "How did you get that [46]?" which helps me interpret that Theo was verbalizing that the number 20 in 46 came from the 20 tiles (the speech itself is not the source of this interpretation, it is the context of the task itself, but the speech helps identify which aspect of the context the student is focusing on, in this case it was the 20 tiles); and 2) the content of his gesture (pencil pointing at the start of the top of the enclosure) compared to the content of speech ("we did twenty") to determine coexpressivity of gesture and speech as a single idea that the number 20 came from 20 tiles. With this interpretation in mind I conclude that Theo used gesture (a) as an embodied representation of the idea that the number 20 came from the 20 tiles.

Ideas represented in gestures (a) and (b) are different, yet they are related. Whereas Theo used gesture (a) to visually represent the idea that the number 20 came from 20 tiles, he used gesture (b) to further describe that these 20 tiles are arranged above the garden and referred to as "the top". This interpretation of gesture (b) is based on: 1) the context of speech ("for the top") determined by the task itself -- the tiles enclosing the garden on the top -- as well as the researcher's question ("How did you get that [46]?"); and 2) the content of his gesture (retracing faithfully the imaginary 20 tiles of enclosure on top of the garden represented with 20 blocks, from one end to the other) compared to the content of speech "for the top." Speech and gesture in the gesture-speech pair (b) conveyed a single idea that I here paraphrase as "twenty tiles enclosing the garden tiles from the top is something that we in this class refer to as the top." With this interpretation in mind, I conclude that Theo used gestures $(a)$ and $(b)$ as an embodied representation of the idea that 20 tiles are enclosing the garden from the top. Whereas gesture (a) represents the idea that number 46 is partially built from number 20, gesture (b) serves as an elaboration of gesture (a) in order to be more specific that 20 is the number of tiles at the top of the enclosure.

Theo's gestures (a) and (b) combined were therefore embodiments of a quantity that I here describe as the "tiles enclosing the garden from the top." Thompson (1994) defines quantities as "conceptual entities" (p.184) that reflect a subjective conceptualization of an object and its measurable quality:

It [quantity] is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality. (Thompson, 1994, p. 184)

More recently Thompson (2010) described quantities as conceptualizations in the mind rather than real-life entities, and clarified that the process of quantification involves conceptualizing an
object and its attribute's measure as proportional with its unit, involving understanding of that process as accumulation rather than a simple assignment of a number. Theo's conceptualization of the garden enclosure on the top involved the physical organization of tiles above the garden, but also the number of tiles " 20 " as its quality that can be measured through the process of counting the number of tiles as the units of measurement from one end to another as evident in his gesture (b). Although Theo does not explicitly say "tiles" we know from the context of the task that the top of the enclosure consists of tiles. The tiles are the units in which Theo measures the top of the enclosure. In this sense, Thompson's definition of quantity seems an appropriate choice in this case. Theo is not merely assigning the number of tiles (20) as the length but is also representing the process of accumulating the number of tiles by retracing them faithfully from one end to the other. I will also consider quantities more broadly as non-specified, continuous, generalized quantities as described by Dougherty (2008). For example, students might work with volumes or lengths without specifying any units. This approach to quantities is more suitable in cases when students describe the top and the bottom as indeterminate quantities (see Claim 2).

Using McNeill's framework I interpreted Theo's gestures $(a)$ and (b) as embodiments of the 20 tiles enclosing the garden on the top. Combining this interpretation with Thompson's definition of quantity, I argue that Theo was conceptualizing the quantity I paraphrase as the "tiles enclosing the garden from the top" as an object with the number of tiles (20) as its measurable quality, that can be accumulated (counted) in order from one end to another. His gestures (a) and (b) therefore were embodiments of a fixed known quantity.

Theo then used gesture (c), which is identical to his gesture (a) but on the opposite side of the garden, as an embodied representation of the idea that the second number 20 came from 20 tiles below the garden; and, he used gesture ( $d$ ), identical to his gesture ( $b$ ), as an embodied
representation of the 20 tiles on the bottom part of the enclosure. Combined, his gestures (c) and (d) were embodied representations of the quantity "tiles enclosing the garden from the bottom."

Finally, Theo used gesture (e) as an embodied representation of the idea that the number 3 came from the 3 tiles needed to enclose the garden at one end. This interpretation is grounded in 1) the context of his speech ("and three") being 3 tiles (we know this from the task itself); and 2) the content of his gesture (retracing an imaginary column of tiles at one of the ends) compared to the content of his speech ("and three") indicating the number 3. Theo used identical gestures (e) and $(f)$ as an embodied representation of the idea that there were two such ends. I ground this interpretation in 1) the context of his speech ("on each side") being that the garden in the task had two ends; and 2) the content of his gesture (retracing the identical column of tiles at the other end of the blocks) compared to the content of speech ("on each side") indicating that the same applied to each of the sides.

In Episode 1 Theo's thinking was arithmetical as he performed a simple calculation by adding four numbers to get 46 . As I will argue below, Theo's thinking in episode 2 was algebraic as he looked for structure and relationships between the length of the garden and the number of tiles needed to enclose it.

## Claim 1-b. Sixth grade students used gestures as embodiments of indeterminate quantities

Episode 2 in Figure 4 and Episode 3 in Figure 5 summarize the gesture-speech pairs used by Theo and Sophia to describe the pattern they observed in the general case (without a reference to any particular length of the garden, that is to a position in the geometric design). In what follows, I will argue that Theo's gestures $(a)$ and $(b)$ in Figure 4 and Sophia's gestures $(e)$ and $(f)$ in Figure 5 were embodiments of two indeterminate quantities, "tiles enclosing the garden from the top" and the "tiles enclosing the garden from the bottom."


Teacher (in response to Alex, another student at Theo's table, who wrote down "tiles times two"): And why do you need to multiply this (pulls her thumb and index finger close together and briefly sets them on the table) times two? What is that going to give you?
Theo: The top ${ }^{a}$ (places his arm above and alongside the garden blocks) and the bottom ${ }^{b}$ (same arm gesture below the blocks).

Teacher: The top and the bottom. So the two is coming from needing a top and a bottom.
Theo: (nods)
Teacher: And what's the other part (of the enclosure)?
Theo: You have two sides ${ }^{c}$ (places the pencil perpendicular to the garden tiles, on the right).
Teacher: The ends, good. And how many do you need for the ends?
Theo: Three
Teacher: Three on one side, and what else do you need?
Theo: Three on the other side
Teacher: A three on the other side. Perfect. And how many is that total?
Theo: Six.

Figure 4. Episode 2 - Theo describes the pattern in general case
In Episode 2, the teacher posed a question "And why do you need to multiply this (the number tiles) times two?" to another student, Alex at Theo's table who had written "tiles times
two" on his worksheet. Theo offered an answer to teacher's question by synchronizing utterances, "the top" and "the bottom," with the two identical open-palm gestures each on a different side of the blocks (garden spaces), above and below. Theo used these two identical open-palm gestures as embodiments of two identical entities, the two imaginary parts of the garden enclosures, one on the top and another on the bottom. His gestures provided spatial orientation for each part as running parallel to the garden spaces from one end of the blocks to another. These gestures were also embodiments of two equal quantities, the number of tiles in each part of the enclosure, which added together demonstrated multiplication by two. Whereas in the specific case of garden length 20 (see the analysis of Episode 1) Theo used the two gestures (gestures (b) and (d) in Figure 3) as an embodied representation of the fixed known quantity of 20 tiles enclosing the garden on the top and on the bottom, the two gestures he used in Episode 2 ((a) and (b) in Figure 4), on the other hand, were embodiments of indeterminate quantities. This interpretation is grounded in 1) the context of his speech (Theo is addressing the teacher's question about why multiplication by two is needed for the garden tile problem without a reference to any particular length of the garden) and 2) the content of his gesture (an arm gesture aligned at the top and subsequently at the bottom of the enclosure) compared to the content of his speech ("the top," "the bottom"). Theo was referring to indeterminate quantity "tiles enclosing the garden from the top" which is unknown, yet equal to the indeterminate quantity "tiles enclosing the garden from the bottom."

A different interpretation is also possible given that Episode 2 occurred within the context of Alex previously discussing the $100^{\text {th }}$ term with the teacher. The difference between the two episodes further supports the first interpretation. In Episode 1, Theo paused with the pencil while saying " 20 " before he traced the top of the enclosure, a motion which could be interpreted
to signify the process of counting 20 tiles at the top of the enclosure. In contrast, his gesture in Episode 2 had only one component, a single hand gesture, without a pause or tracing.

Additionally, when the teacher responded to Theo's gesture-speech pairs (a) and (b) in Episode 2 with "So the two is coming from needing a top and a bottom," while not specifying any particular value for the top or the bottom, Theo nodded in agreement.

In what follows, I present Episode 3 in which Sophia used gestures as embodied representations of indeterminate quantities. Sophia's two-palm gestures were visually different than Theo's and she, unlike Theo, referred to indeterminate quantities in her speech.


Sophia: I kind of noticed something - you always need like three on the side. You always need three on the side (pauses then gestures two sides with each hand) ${ }^{d}$. But - and you need however many f(eet) - however long your um, plant, your, the length of your garden is that's how many tiles you need (starts gesturing by forming two open palms pointing at each other, then stands up so that the camera can see her) on top ${ }^{e}$ (moves her hand formation forward away from her) and the bottom ${ }^{f}$ (moves her hand formation backwards towards her).

Figure 5. Episode 3 - Sophia describes the general case.

In Episode 3, without a prompt, Sophia started sharing her observations with students at her table, by saying, "I kind of noticed." Previously she had completed putting blocks together and counting 16 tiles to enclose a garden of length 5 . She was just starting to set up the blocks for the garden length of 6 when she spontaneously described the general case in which she explicitly referred to the length of the garden as an unknown variable quantity, "however many $\mathrm{f}($ eet $)$ - however long your um, plant, your, the length of your garden is" (emphasis added). She then used that to quantify how long the "top" and the "bottom" rows should be, "that's how many tiles you need on top (e) and the bottom (f)". In other words, Sophia verbally described the length of the garden as an indeterminate quantity "however long [...] the length of your garden is" (emphasis added) and then proceeded to relate that quantity to the length of the enclosure at the top and the bottom "that's how many tiles you need on top and the bottom". Simultaneous with saying "top" and "bottom" she used two identical hand gestures, (e) and $(f)$, to visually represent two identical rows of tiles running parallel to the garden spaces. Besides spatial information, her gestures also conveyed quantitative information. Namely, for Sophia, the number of tiles was a property of the enclosure as evident in her gesture-speech pairs, "that's how many tiles you need on top (e) and the bottom (f)." Although Sophia stated that the number of tiles in the top and the bottom of the enclosure was unknown, "however many," at the same time she stated that they contained the same number of tiles "that's how many tiles you need on top (e) and the bottom $(f)$." My interpretation that the two quantities are equal is also supported by the two nearly identical hand gestures.

These interpretations are grounded in 1) the context of her speech (Sophia describing what she noticed about the garden tiles task, saying that however long your garden is that is how many tiles you need for the top and the bottom) and 2) the content of her gesture (two identical
arm formations running parallel to the blocks) compared to the content of her speech ("the top," "the bottom").

In summary, Sophia's gestures $((e)$ and $(f))$ in Figure 5 were embodiments of two equal variable quantities.

## Claim 2. Sixth grade students operated on quantities (fixed and indeterminate) by combining gestures into embodied expressions

To provide evidence for Claim 2, I will now discuss the ways in which Theo and Sophia operated on indeterminate quantities by combining gestures as if they were known quantities, thus in "analytic ways" as per Radford (2011), and that these were the ways in which they embodied expressions.

Theo's gestures in Figure 3 (Episode 1) within each of the pairs (a) and (c); (b) and (d); or, (e) and (f), are nearly identical. They only appear to differ in the physical location in space and are accompanied by different speech. These nearly identical gestures appear related, and thus combined, although they belong to different gesture-speech pairs. In addition, the way Theo combined them does not seem random or accidental. Although McNeill's (2008) framework helps make interpretations for an individual gesture that is accompanied by speech, it does not offer guidance on interpreting a sequence of related gesture-speech pairs. Interpretations I made using McNeill's framework seem sufficient to make further interpretations of how Theo combined these gestures and for what purposes. He used two identical gestures (a) and (c) to visually represent the idea that 20 and 20 came from 20 tiles on each of the two sides of the garden, top and the bottom. He used two identical gestures $(b)$ and $(d)$ as embodiments of two identical entities, the two imaginary physical rows of 20 tiles enclosing the garden from the top and the bottom. These gestures (b) and (d) provided spatial orientation for each row of tiles as running parallel to the garden spaces from one end of the blocks to another but were also
embodiments of two equal quantities, 20 tiles on the top and 20 tiles on the bottom. Combining them using the word "and" accompanied by repositioning of the hand, Theo demonstrated repeated addition of 20 tiles in a top row and 20 tiles in the bottom row. Similarly, he used gesture (e) as an embodiment of the idea that there were 3 tiles at one end of the garden and gesture $(f)$ as an embodied representation of the idea that there were two such ends. He combined gestures (e) and (f) by repositioning of the hand while continuing the same utterance as an embodied explanation of where 3 and 3 came from. All six gestures combined were an embodied modality in which Theo answered the researcher's question regarding how he got the answer 46 . Theo, thus, in an embodied way represented the equation $20+20+3+3=46$. In conclusion, Theo used a combination of six gestures (a)-(e) in Figure 3 as an embodied representation of an equation.

As argued in Claim 1-b. Theo's open-palm gestures ((a) and $(b)$ ) in Episode 2, for the "top" and "bottom," were embodiments of two equal indeterminate quantities. Theo used the conjunction "and" in "the top and the bottom" (emphasis added) synchronized with repositioning of the hand, to combine the two indeterminate quantities represented by gestures, equal in size. This, I argue, is an embodied way of showing the expressions $n+n$. Although this was his response to his teacher's question on why multiplication by two was needed, it is not clear if at that point Theo simultaneously engaged in multiplicative thinking in terms of doubling (corresponding to embodied expression $2 n$ or perhaps $n \times 2$, which he later indeed described in words as " n times two"), or if he stayed in the realm of additive thinking through repeated addition. Theo might only be thinking of teacher's question about multiplication by 2 strictly in additive terms as repeated addition rather than doubling or other form of multiplicative reasoning.

Theo went along with his teacher's linguistic bid "and" to connect the "top" and "bottom" to the "ends" in "And what's the other part (of the enclosure)?" (emphasis added) and described the "ends" with another gesture-speech pair, "You have two sides (c) (places the pencil perpendicular to the garden tiles, on the right)." His pencil gesture (c) suggests that he might be thinking of two ends as one entity rather than as two different sides and served as an embodied representation of two equal fixed quantities, which he then verbally described as "Three" on one side, "Three on the other side," totaling "Six." Theo, therefore, operated on the indeterminate quantities as if they were known by adding 6 to their sum, thus in analytic ways. To do so he combined gestures $(a),(b)$ and (c). This was his embodied way of showing the expression $n+n+6$. We do not have enough evidence to claim that Theo at the same time also embodied expression $2 n+6$, which he eventually stated more explicitly when prompted to fill in the last row in the table for $n$ number of tiles:

Teacher: So what are you going to do with the $n$ now?
Theo: $n$ times two plus six.
In Claim 1-b I also argued that Sophia's gestures ((e) and (f) in Episode 3), just like Theo's, were embodiments of two equal indeterminate quantities ("top" and "bottom"). Sophia used the word "and" in her speech, "on top and on the bottom," (emphasis added) combined with alternating her hand formation, forward (e) then backward $(f)$ to show the top and the bottom, and therefore connected her two gestures into an embodied representation of the expression $n+n$. She represented the "ends" with another gesture-speech pair (d), "three on each side," a value that "always" stays the same. She used the word "but" combined with the repositioning of the hands, as a way to combine the "ends" (each of fixed length 3), with the "top" and "bottom" (each of an unknown length). The way she combined gestures in Episode 3 is an embodied way of showing the expression $3+3+n+n$ and evidence that Sophia operated on two equal
indeterminate quantities as if they were known by adding their combined sum to the sum of the "ends", thus in analytic ways.

## Claim 3-a. Students used gestures as embodiments of covariation

Figure 6 depicts Henry using gestures when talking about how the number of tiles
increases by two for each new garden space.


Teacher: Why do you multiply by two?
Henry: Um, so you can, um (pauses), oh yeah, 'cause, um, each number of (reaches for a block) garden spaces you have (holds a green block with two fingers of the left hand), um, (moves the block to his right hand) once you put one more on, one more garden space (places the block on the paper), you have to add two, to, you have to add one ${ }^{\text {a }}$ (puts the left index finger on top and the left thumb on the bottom of the block), um, more to the side, to the side here ${ }^{b}$ (lifts the left thumb, so only the left index finger touches the paper), and one more to this side ${ }^{c}$ (puts the left thumb down, lifts the left index finger up, only the thumb of his left hand is touching the paper).

Figure 6. Episode 4 - Henry describes covariation.

Henry used three gesture-speech pairs as embodiments of a rate of change, which can be described as: for every new garden space the number of tiles needed to enclose it increases by two, one on the top and one on the bottom. Henry gestured with his fingers around a block representing a single garden space to show where two enclosing tiles would go, one on top and one on the bottom.

These interpretations are grounded in 1) the context of his speech (the task itself and the teacher asking why he multiplied by two) and 2) the content of his gesture (one finger on top and
one finger on the bottom of the block that is representing the garden space, alternating finger lifting and touching the desk) compared to the content of his speech ("you have to add one more [...] to the side here and one more to this side").

In summary, Henry combined gestures as an embodied representation of covariation understood as step-wise increments of 2 tiles in the enclosure every time a new garden tile is added.

## Claim 3-b. Students used gestures as embodiments of doubling

Figure 7 depicts Alex using a single gesture when talking about a result of multiplication by two for the 100th term.


Figure 7. Episode 5 - Alex describes doubling.

Alex used a single hand gesture in which his fingers surrounded a row of blocks while saying "Automatically, you have two hundred." Alex used this gesture to visually represent doubling. This interpretation is grounded in 1) the context of his speech (Alex working on the task on the
$100^{\text {th }}$ term) and 2) the content of his gesture (hand gesture enclosing the row of blocks from the top and the bottom) compared to the content of his speech ("Automatically, you have two hundred"). In contrast to Theo and Sophia who engaged in additive thinking by repeatedly adding two identical quantities, Alex, on the other hand engaged in multiplicative thinking through a single gesture-speech pair (a) co-expressing the idea of doubling. He then also explained doubling through repeated addition as evident subsequently in gesture speech pairs (c) and (d).

In summary, Alex's gesture (a) was an embodied representation of doubling.

## Summary and Discussion

This study aims at exploring the role of gestures in students' algebraic thinking by focusing on the ways in which four sixth grade students represented and used indeterminate quantities (Radford, 2011). I have described a variety of ways in which these students used gestures to think algebraically. For example, students used gestures as embodied representations of indeterminate quantities (Theo and Sophia), covariation (Henry), or doubling (Alex). Students also combined gestures into embodiments of expressions.

For the conceptual framework I draw, on the one hand, on the literature on gestures as semiotic resources (McNeill, 2008 in particular), and, on the other hand, on the literature on students' use of indeterminate quantities in ways different than manipulation of symbols in standard notation (Brizuela, 2016; Cooper \& Warren, 2011; Radford, 2011).

Gestures as semiotic resources (Arzarello, Paola, Robutti, \& Sabena, 2009; GoldinMeadow, 1999; McNeill, 1992, 2008; Radford, 2014; Sabena, Radford, \& Bardini, 2005) in mathematics classroom have been underexplored. Gestures are viewed as components (Radford, 2014), mediators and locations of students' mathematical thinking (Radford, Bardini, Sabena,

Diallo, \& Simbagoye, 2005). These studies portray gestures as one of the many modalities in which students convey meaning, and that the meaning conveyed in gesture closely relates to the meaning simultaneously conveyed in other modalities. However, these studies lack explicitness about how interpretations of gestures are being made. Unlike speech, the same gesture might mean one thing in one situation and something different in another, therefore interpreting gestures requires a systematic approach. In this study I use McNeill's (2008) framework for interpreting gestures that co-occur with speech which views gestures and accompanying speech as two representations of the same idea but in two different modalities. McNeill does not interpret the meaning of gesture as a standalone component and in isolation from the meaning of speech or from the context of speaking. In my study, Sophia's gesture visually resembled the top part of the garden enclosure and therefore provided spatial information to accompany her speech "however long [..] garden is, that's how many tiles you need on top". To make this interpretation I compared the content of her gesture, the shape and location in space of her hand formation with the content of her speech, all placed in the context of the task students were working on. Although Sophia's gesture provided spatial information complementary to her speech (GoldinMeadow, 1999), her gesture and her speech, according to McNeill (2008), represented the same idea, in two different modalities, that the number of tiles enclosing the garden on the top is the same as the length of the garden and can be anything. Thus, Sophia used both gesture and speech to simultaneously, in two modalities, represent one indeterminate quantity, the number of tiles in the top part of the enclosure. It is the McNeill's framework that extends the Goldin-Meadow's (1999) work in ways that allow us to see gestures -- beyond providing mere spatial information to complement her speech -- as a semiotic resource capable of conveying abstract mathematical concepts. My study provides evidence that gestures convey, and thus are embodiments of,
mathematical meaning related to algebraic reasoning. More specifically, in this paper I presented evidence and argued that sixth graders used gestures as embodiments of fixed known (Claim 1-a) and indeterminate quantities (Claim 1-b), covariation (Claim 3-a) and doubling (Claim 3-b), and that students combined these gestures into embodied expressions (Claim 2). Namely, Theo embodied the equation $20+20+3+3=46$ and the expression $n+n+6$ through a sequence of his gestures; Sophia's combined her gestures as an embodiment of the expression $3+3+n+n$; Alex' single gesture was an embodied representation of doubling of a quantity; whereas Henry used gestures to represent the abstract mathematical idea of covariation of two quantities.

I looked for transitions in speech and gestures as evidence that students combined gestures to represent addition, and multiplication as repeated addition, and by doing so operated in analytic yet embodied ways on the indeterminate quantities represented by those gestures. This in turn was an embodied way in which students represented algebraic expressions. McNeill's framework did not provide any guidance on how to interpret a combination of gestures. However, I observed that students combined gesture-speech pairs by using transitions in speech and repositioning of the hand and interpreted that in the context of the task as an embodied arithmetical operation of addition. Combining gestures was a way students in this study operated on indeterminate quantities as if they were known quantities, thus in analytic ways (Radford, 2011). In other words, students combined gestures as embodied forms of their algebraic thinking.

My findings complement those by Radford (2011), Brizuela (2016), and Cooper and Warren (2011) by providing evidence that students use indeterminate quantities (Claim 1-b) and operate on them in analytic ways (Claim 2), through gestures. Whereas Radford (2011) finds evidence of students' work with indeterminate quantities in their linguistic references (e.g.,
"What is 25 plus 25? After that you add 1!"), Brizuela in non-numeric inscriptions (e.g., "?" used to represent an unknown number of candy in a candy box), and Cooper and Warren in points on the number line, I do so in students' gestures. Those studies, including the present, further contribute and extend the body of literature on the development of algebraic thinking in ways other than the manipulation of symbols written in standard algebraic notation. Algebra was traditionally taught at school as a manipulation of symbols, thus aimed towards mastery of symbolic algebraic procedures with little or no connections to the meaning behind the symbols and understanding of the relationships between quantities represented by these symbols. Algebraic symbolization was, at times, considered a necessary part of algebraic thinking (Kieran, 1989). This image of algebra has significantly changed over time to include use of tables and other representations besides symbols (Kaput, 1999). A growing body of more recent literature provides evidence and argues that algebraic thinking develops and is manifested in a variety of ways and not just in students' use of symbols (e.g., Abrahamson, Lee, Negrete, \& Gutiérrez, 2014; Kaput, 1999; Radford 2014; Zazkis \& Liljedahk, 2002). Algebraic symbolism is viewed more as a refinement of algebraic thinking (Kaput, 1999; Zazkis \& Liljedahk, 2002) than the necessary component of it. These studies provide evidence that algebraic thinking occurs in a variety of modalities (e.g., kinesthetic, verbal, imagery, tactile, etc.) besides symbolic manipulation. The present study adds to this body of literature by providing evidence that gestures are another modality in which students think algebraically in terms of known and indeterminate quantities, and their relationships in terms of doubling, covariation and expressions. Theo's embodied expression $n+n+6$, just like Sophia's $3+3+n+n$, although not represented in their formal symbolic forms, are the forms of both students' algebraic thinking about the relationship between the quantities, the length of the garden and the number of the tiles
in the enclosure. My study adds gesture as another modality through which students express how they are thinking about quantities and relationships among quantities in situations.

Implications of the present study for research and instruction involve taking students' gestures as co-expressive of the same idea that is being conveyed in their speech. Gestures discussed in this study do not just add spatial information to students' speech to further characterize the objects and locations in space. Rather, they are embodiments of complex algebraic ideas such as quantities, expressions, doubling or covariation. Moreover, gestures might signify aspects of those ideas not present in speech. For example, Theo's retracing gesture when he said " 20 for the top" further reveals the process of accumulation of 20 tiles at the top part of enclosure by accounting for garden spaces from one end to another. Process of accumulation is essential in Thompson's (2010) definition of quantity and quantification. Based on Theo's speech alone " 20 for the top" it would not be clear if Theo was thinking about the top row of tiles as a quantity in itself in terms of quantification or if he was simply assigning number 20 --which was the length of the garden-- to an entity he referred to as "the top". Another example when gestures conveyed more than spatial aspects of a row of tiles is that both Theo and Sophia's used identical gestures to represent the top and the bottom parts of the enclosure. The notion that top and bottom are identical is not present in their speech but only in their gestures. These examples suggest that students might use gestures to represent aspects of their mathematical thinking and ideas that might not be present in their accompanying speech beyond mere spatial information.

## Study Limitations

The present study has several limitations.

1. The sample was very small; it included only 12 students, 7 of which gestured.
2. All gestures observed in this study co-occurred with speech. There were no gestures that were mismatched with accompanying speech, occurred in the absence of speech, or involved other parts of the body besides hands.
3. Students in the study used gestures for communication purposes to answer an adult's question. This study does not shed light on whether students might also use gestures while sense making.
4. This study does not shed light on why some students did not gesture beyond merely noting that they also did not engage in verbal explanations of their thinking.
5. This study does not shed light on whether the type of students' gesture (openpalm vs. pointing) might or might not correspond to the type of quantities students are representing (indeterminate vs. known).
6. This study does not shed light on how teachers might relate students' gestures to their algebraic thinking.

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