Elastic Stress Analysis

Elastic Stress Analysis:

Eq-ns of elasticity

- Equilibrium
- Compatibility of strains
  - constraints on strains so that holes do not form (material remains continuous)
- Hooke's law: stress-strain relations

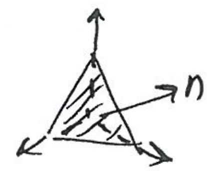
Boundary conditions

- Which ones are proper? (the problem is neither over specified nor underspecified)

  I.e. what should be given to a computer?

Equilibrium

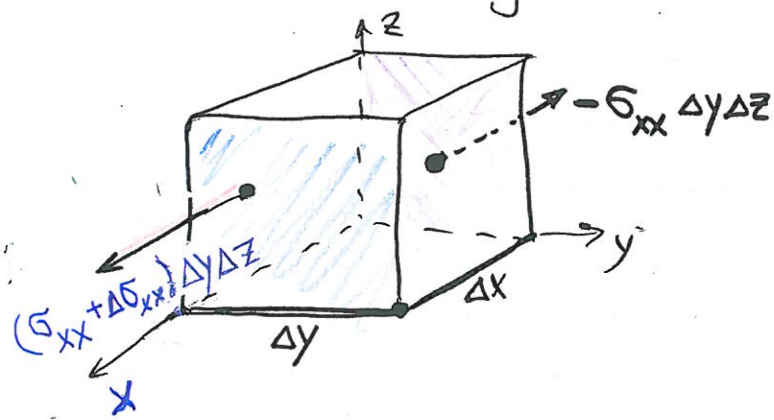
Earlier: considered equilibrium of tetrahedron



Found: orientat. dependence of tractions $\underset{\sim}{t}^{(n)} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n}$

Now: how do stresses vary from point-to-point in order to maintain equilibrium?

rectangular element:



Sum of all forces $= 0$

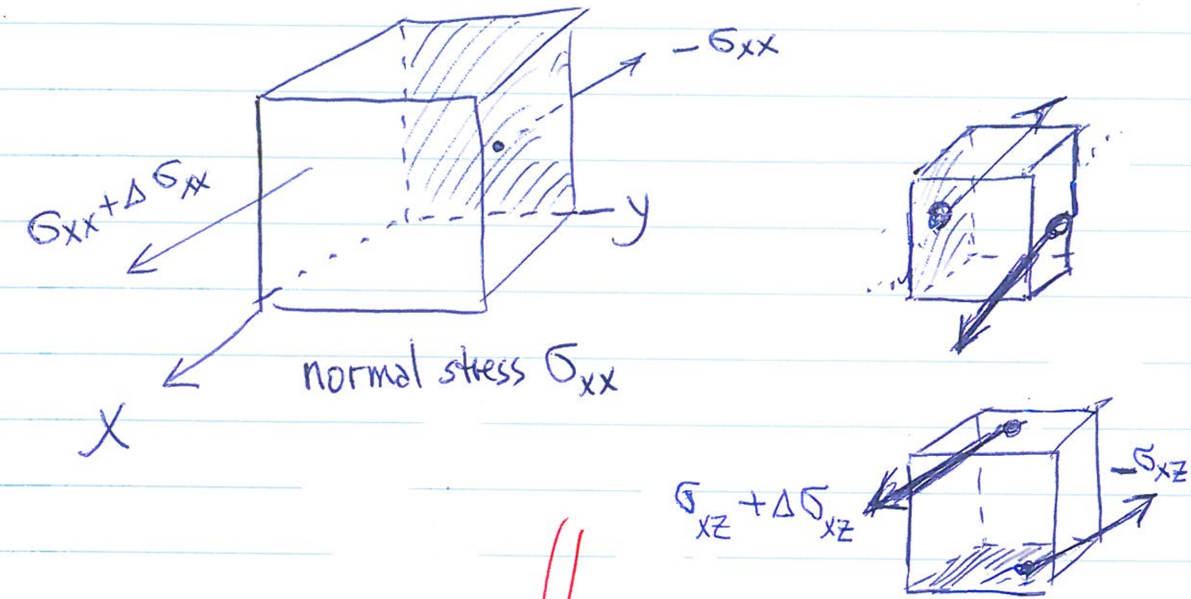
In X-direction: from pairs of opposite faces

$$-\sigma_{xx}\Delta y \Delta z + (\sigma_{xx} + \underbrace{\Delta\sigma_{xx}}_{\frac{\partial \sigma_{xx}}{\partial x}\Delta x})\Delta y \Delta z \qquad - \sigma_{xy}\Delta x \Delta z + (\sigma_{xy} + \underbrace{\Delta\sigma_{xy}}_{\frac{\partial \sigma_{xy}}{\partial y}\Delta y})\Delta x \Delta z$$

$$-\sigma_{xz}\Delta x \Delta y + (\sigma_{xz} + \underbrace{\Delta\sigma_{xz}}_{\frac{\partial \sigma_{xz}}{\partial z}\Delta z})\Delta x \Delta y = 0$$

$$\Rightarrow \boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0}$$

All the forces in the $\underline{X}$ direction:



shear stress σ_{xy}

shear stress σ_{xz}

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad \text{– equilibrium in } X\text{-direction}$$

analogously

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \quad \text{in } y\text{-direction}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \text{– in } z\text{-dir.}$$

In tensor notations:
$(x\,y\,z) \to (x_1 x_2 x_3)$

3 equations:

$$\boxed{\frac{\partial \sigma_{ij}}{\partial x_i} = 0 \qquad j = 1, 2, 3}$$

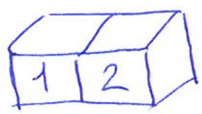
Equilibrium requirement: interrelates rates of change of stress components in different directions

All imbalances (coming from pairs of opposite faces) are balanced

Compatibility of strains

Strains ϵ_{ij} must vary from point-to-point in such way that material remains continuous

functions $\epsilon_{ij}(x_1, x_2, x_3)$ cannot be arbitrary

Example : 

before deform.


cannot put together after deform.

To prevent : $\epsilon_{ij}(x_1, x_2, x_3)$ should be derived

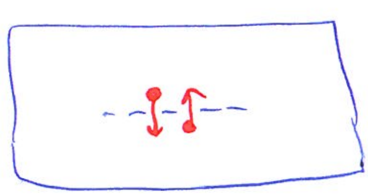
from some smooth displacem. field $u_i(x_1, x_2 x_3$

Example : forming of a crack (or a hole):

displacem. discontinuous across - - - - -

Continuous material

or,



overlapping

mathematically :

Six ϵ_{ij} are derived from three displacement components, $2\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$

$\Rightarrow$ Strains ϵ_{ij} should be interrelated in some way.

2-D case :

$$\begin{cases} \epsilon_{xx} = u_{x,x} \\ \epsilon_{yy} = u_{y,y} \\ 2\epsilon_{xy} = u_{x,y} + u_{y,x} \end{cases}$$

3 strains from 2 displacem.

$\Rightarrow$ should be interelated by one relation

To interrelate: take second derivatives

$$\begin{cases} \epsilon_{xx,yy} = u_{x,xyy} \\ \epsilon_{yy,xx} = u_{y,yxx} \\ 2\epsilon_{xy,xy} = u_{x,yxy} + u_{y,xxy} \end{cases}$$

$$\Downarrow$$

$$\boxed{\epsilon_{xx,yy} + \epsilon_{yy,xx} = 2\epsilon_{xy,xy}}$$

compatibility eq-n in 2D

follows from: ϵ_{ij} are derived from some displacement field (not prescribed arbitrarily)

- automatically satisfied if we start with displacements
- Not so if we start with strains/stresses

In 3-D: 3 compat. eq-ns (6 strains ϵ_{ij} from 3 displacem. u_i)

Eq-ns of Elastic Stress Analysis: a summary:

1. Equilibrium

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0$$ (not specifically elastic)

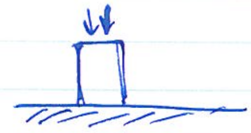
2. Hooke's law (isotropic or anisotr) (elastic solid only)

3. Compatibility of deformations – not specifically elastic

Proper boundary conditions

– of key importance when working with soft. package (Abacus, etc)

Proper prescription of loading conditions on boundary of the structural element

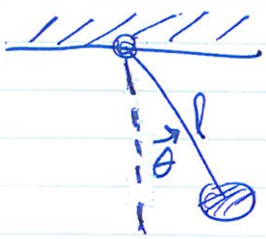


Proper: neither underspecified (then, infinitely many solutions)

nor over specified (no solution at all)

Example of over / under specification:

Pendulum:



$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad \text{for } \theta = \theta(t)$$

Initial cond: two needed (second order eq.)

for example,
$$\left.\begin{aligned} \theta(0) &= \theta_0 \\ \dot{\theta}(0) &= \dot{\theta}_0 \end{aligned}\right\} \text{prescribed}$$

or

$$\left.\begin{aligned} \theta(0) &= \theta_0 \\ \theta(t_1) &= \theta_1 \end{aligned}\right\}$$

If we prescribe just one: underspecified (∞ number of possible motions satisfying this i.c.)

— —— three: overspecified (no solution)

Proper boundary conditions

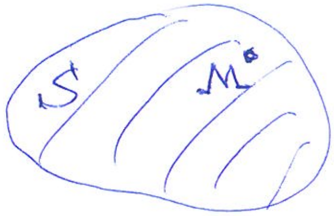
neither under-
nor over-specified

what do we prescribe on
boundary S?

- In displacements
- In tractions
- Mixed

b.c. in displacements:

Displacements are prescribed on boundary:



$$\boxed{\underset{\sim}{u} = \underset{\sim}{u}(M) \quad \text{on } S}$$

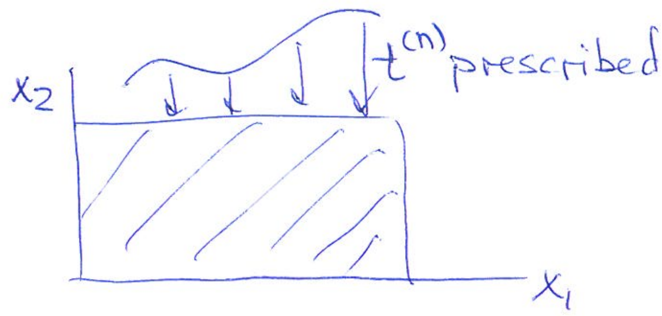
Frequently: has the form of rigid foundation:

b.c. in traction:

$$\boxed{\underset{\sim}{t}^{(n)} = f(M) \quad \text{on } S}$$

frequently: has the form $\underset{\sim}{t}^{(n)} = 0$ (traction-free surface)

Note on prescribing tractions on boundary



On boundary: $\underset{\sim}{t}^{(n)}$ is related to stresses σ_{ij} in the body by

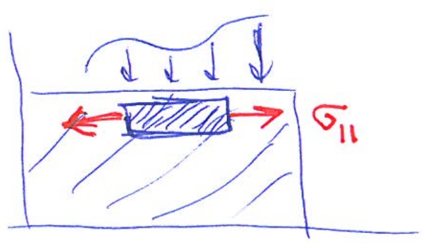
prescribed: $\underset{\sim}{t}^{(n)} = \underset{\sim}{\sigma} \cdot \underline{n} = \sigma_{i2} \underline{e}_i = \sigma_{12} \underline{e}_1 + \sigma_{22} \underline{e}_2$

$\underset{\sim}{\sigma}$: $\sigma_{ij} \underline{e}_i \underline{e}_j$ in material

$\underline{n}$: $\underline{e}_2$

$\Rightarrow$ σ_{12}, σ_{22} inside material must take prescribed values on boundary

But : σ_{11} is not prescribed ! (It does not affect $\underset{\sim}{t}^{(n)}$)

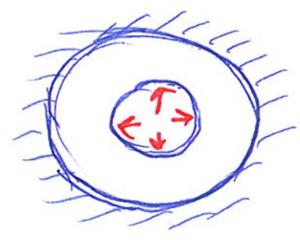


Mixed b.c:

(A) $\underset{\sim}{u}$ is prescribed on part of S

$\underset{\sim}{t}^{(n)}$ ——— on remainder of S

pressurized pipe, confined outside:

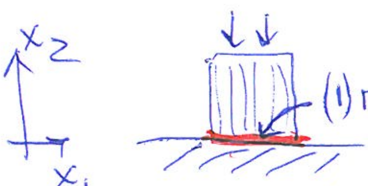


- traction (normal) prescribed inside
- zero displacement ——— outside

(B) <u>Both</u> $\underset{\sim}{u}$ and $\underset{\sim}{t}^{(n)}$ prescribed on some part of S

<u>But</u>: in this case, must have:

$$\underset{\sim}{u}_{\text{prescribed}} \perp \underset{\sim}{t}^{(n)}$$



(1) rigid foundation $\Rightarrow u_2 = 0$

(2) no friction (perfect lubrication) $\Rightarrow \sigma_{12} = 0$

<u>Cannot</u> prescribe: u_2 <u>and</u> σ_{22}

(would be overspecified: σ_{22} develops as a reaction to load, cannot prescribe

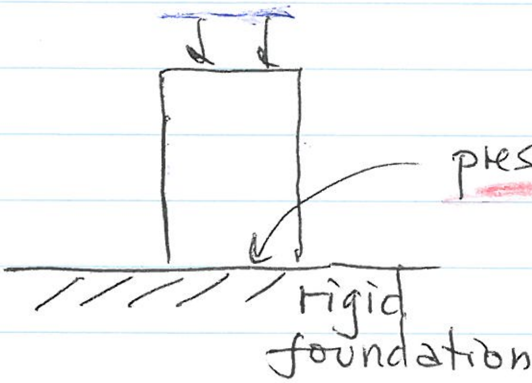
Notes on B.C:

1. Should be prescribed on entire boundary

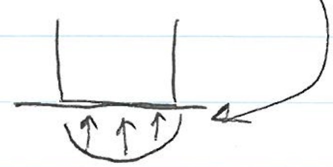
If no loads applied on certain part of boundary, it should be stated that $\underset{\sim}{t}^{(n)} = 0$ there

2. The above-described BC are relevant for elastic behavior only

Examples of incorrect BC:



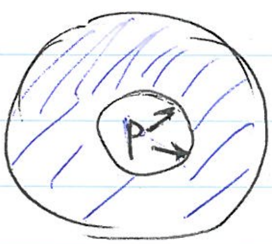
prescribed:
- $u_z = 0$ (rigid foundation)
- certain distribution of normal traction



— overspecified

(actually, traction distribution develops as a result of loading, cannot be prescribed)

pipe:



- Pressure is prescribed (p)
- Radial displac. at inner boundary prescribed

— overspecified at inner boundary

— underspecified at outer boundary

(is it free to expand, or is it constrained?)

Two simplifications in elastic stress analyses:

- Superpositions

  breaking the problem into a sum of simpler sub-problems

- Saint-Venant's principle

  replacing the problem by approximately equivalent simpler one

Principle of Superposition

$$\begin{cases} 1.\ \partial\sigma_{ij}/\partial x_i = 0 \\ 2.\ \text{Hooke's law } \epsilon_{ij} = S_{ijkl}\,\sigma_{kl} \\ 3.\ \text{Compatibility}: \epsilon_{xx,yy} + \epsilon_{yy,xx} = 2\epsilon_{xy,xy} \quad (\text{in } 2D) \end{cases}$$

all eq-ns are linear

$\Rightarrow$ sum of the solutions satisfying B.c. A and B

is a solution satisfying B.C. A + B.

$\sigma_{ij}^{(A)}(\underset{\sim}{x})$ - stress field satisfying $\underset{\sim}{t} = \underset{\sim}{t}^{(A)}$ on S

$\sigma_{ij}^{(B)}(\underset{\sim}{x})$ - " ——— $\underset{\sim}{t} = \underset{\sim}{t}^{(B)}$ on S

then: $\sigma_{ij}^{A} + \sigma_{ij}^{B}$ satisfies B.C. $\longrightarrow$ $\underset{\sim}{t} = \underset{\sim}{t}^{(A)} + \underset{\sim}{t}^{(B)}$ on S

Key point: eq-ns are linear.

For example: if we had $\epsilon = \frac{1}{E}\sigma^2$:

if σ^A corresponds to B.C. $\underset{\sim}{t}^{(A)}$

σ^B ——— $\underset{\sim}{t}^{(B)}$

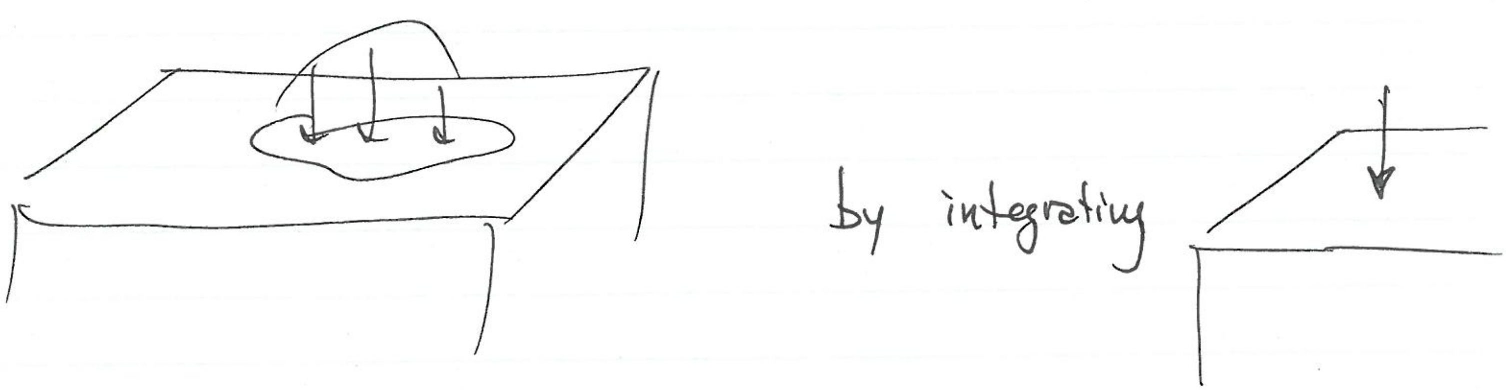
then, for the sum $\sigma^A + \sigma^B$:

$$\epsilon = \frac{1}{E}(\sigma_A+\sigma_B)^2 = \underbrace{\frac{1}{E}\sigma_A^2 + \frac{1}{E}\sigma_B^2}_{\text{corresponds to B.C. }(t^A+t^B)} + \underbrace{\frac{1}{E}2(\sigma_A+\sigma_B)}_{?}$$

Uses of superpositions :

1. Breaking a problem into a sum of simpler ones

2. Solutions for distributed loads – by summing up – integrating – point force solutions
[⇒ importance of point-force sol'ns]



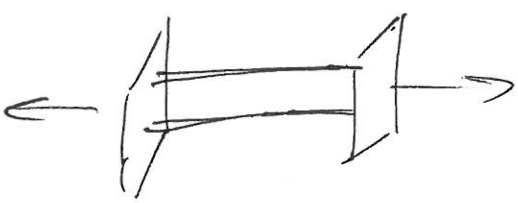
Saint-Venant's Principle

– Some traction distribution over a (small) area S

– Replace it by any other dist'n with the same principal < vector, moment ("statically equivalent")

⇒ at distances >> size of S differences in stresses
$$\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$$
will be small

Importance : distrib'n of applied loads is usually known only approximately, whereas resultants are known quite accurately



uniaxial tension

If not for this principle, could not have analyzed realistic cases

Solving problems of stress analysis:

The simplest approach : guessing

1. Make a (reasonable) guess (based on intuition)
2. Verify, whether this guess is in agreement with
   - equations of elasticity
     - equilibrium
     - compatibility
     - Hooke's law
   - boundary conditions
3. Recall : if b.c. are ~~proper~~
   then solution is unique (problem is not underspecified)

   ⇓

   solution found by guess
   is the solution

This is in contrast with

math. theory of elasticity (solving eq-ns)

when guess is not clear at all

stress state
near inclusion ?