TEACHER INTERVIEWS, STUDENT INTERVIEWS, AND CLASSROOM OBSERVATIONS IN COMBINATORICS: FOUR ANALYSES

A doctoral dissertation

submitted by

Mary C. Caddle

In partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Education

TUFTS UNIVERSITY

May 2012

Adviser: Dr. Bárbara M. Brizuela

Abstract

This research consists of teacher interviews, student interviews, and classroom observations, all based around the mathematical content area of combinatorics. Combinatorics is a part of discrete mathematics concerning the ordering and grouping of distinct elements.

The data are used in four separate analyses. The first provides evidence that student interviews can be a useful source of data when considering the qualities of instruction. The case analysis shows that the teacher's instruction shifted. During interviews, the student responses showed indications of the shifts. The student interviews allowed us to see things we would not have seen through classroom observations or written assessments, and these things reflected the qualities of the instruction.

The second analysis explores a framework of types of teacher knowledge in a novel way. The analysis assigns knowledge types to statements made during interviews. Not all teachers showed the same relative frequency of the different types. The implication is that with more teachers and in connection with classroom data, we may understand what these profiles suggest about a teacher's work and the types of supports that would help them.

The third analysis examines the connections between students solving problems involving the multiplication principle and solving problems involving permutations. Analysis of interviews showed that on problems involving permutations, students often incorrectly overextended the multiplication principle. Students are struggling to make the transition from multiplication principle

problems to permutation problems. This suggests that they need support to understand of how the two types of problems differ.

The fourth analysis looks at students' representations in combinatorics. Both interviews and classroom observations showed novel student representations. The analysis shows that students generate useful non-canonical representations and that we can benefit from utilizing these.

The four analyses connect to different areas of research. The first two papers consider the complex characteristics of teacher knowledge. They aim to become part of the ongoing conversation about how to prepare, evaluate, and support math teachers. The third and fourth papers focus on elements of student thinking in combinatorics. These provide examples to indicate that there is still much we do not know about this area.

Acknowledgments

I would like to extend most grateful thanks to the teachers and students who participated in this study. The students gave of their time for nothing more than snacks and (hopefully) an interesting experience. The teachers shared their time, their thoughts, their students, and their classrooms in exchange for nothing at all. They are simply kind and thoughtful practitioners who care deeply about math education and who want to help improve it as much possible.

I have been fortunate to have an amazing community of colleagues, mentors, and friends. I want to first thank my adviser, Bárbara M. Brizuela, for endless support, encouragement, and feedback. I knew that with her I would find guidance and expertise. I did not know I would also find a friend. Thanks also to Analúcia Schliemann for years of support, as well as for her infectious enthusiasm for students and mathematics, which first made me want to come to Tufts. Thanks also to the other members of my committee, Montserrat Teixidor i Bigas, Maria Blanton, and Heather Hill. They did not just read and comment on my work, but also invited me into their own.

I would also like to thank the faculty and staff of the Department of Education at Tufts. They have treated me like a colleague and their spirit of collaboration has affected my view of work and academia forever. In that same spirit, I would like to thank the Tufts Early Algebra Project and the Poincaré Institute for the research opportunities and experience, particularly David Carraher, Ana, Bárbara, and Montserrat.

Along with this, I would like to thank office mates and research project colleagues past and present (including Gabrielle, Mara, Candice, Pearl, Enrique, Alfredo, Caroline, Chunhua, Winnie, and Ying) for listening to me and for making work fun. Special thanks to my fellow students in the MSTE program, especially Kristen Wendell, Brian Gravel, and Chris Wright, whose hard work motivated me to push myself and whose friendship helped me at the start and helps me still.

I would also like to thank my family, including both Caseys and Caddles. I should thank my parents for many, many things, but I will thank them in particular for endless amounts of childcare and for putting aside worries and supporting my career change. I also have my daughter, Piper, to thank, both for being a mind-blowing miracle and for making me want to work harder so I could get home sooner. Finally, I have to thank my husband, Jim, for partnership, support, and encouragement from the beginning. His confidence held strong when mine wavered and his jokes kept coming when I needed them most. He never once asked if this was worthwhile, because he knew it was worthwhile to me. He never once asked when I was going to graduate, because he never doubted I would. That's something to be thankful for.

Table of Contents

Summary of the Dissertation 1

Introduction 1

Motivation and Framework 1

Combinatorics 6

Structure of the Dissertation 11

Methodology 13

Participants 13

Professional Development Course 14

Measures and Data Collection 15

Summary 25

Analysis 1: Continuity of Data Sources 25

Analysis 2: Using Interviews to Explore Teacher Knowledge Types 27

Analysis 3: Overextending the Multiplication Principle 28

Analysis 4: Student Representations in Combinatorics 29

References 31

Appendix A: Interview Questions 34

Appendix B: Classroom observation checklist 37

Analysis 1. Continuity of data sources: from teachers to classroom to students 39

Introduction 39

Background 42

Research on Student-Level Outcomes 42

Existing Work on Learning and Teaching Combinatorics 55

Methodology 60

Participants 60

Measures and Data Collection 62

Results 65

The Teacher Interview 65

The Classroom Lessons 68

The Student Interviews 78

Discussion 83
What Did We Learn in This Case? *84*
What Can We Learn Methodologically? *86*
References 89
Appendix A: Classroom observation checklist 94
Analysis 2: Using interviews to explore teacher knowledge types: The case of permutations 96
Introduction 96
Background 99
Defining Teacher Knowledge Types *99*
Connecting Teacher Knowledge to Student Outcomes *107*
Accessing Teacher Knowledge Types through Interviews *109*
Methodology 111
Summary *111*
Participants *111*
Professional Development Course *112*
Measures and Data Collection *113*
Analysis 115
Results 118
Discussion 124
Value and Applications *126*
References 131
Analysis 3. How many outfits can I make? Overextending the multiplication principle. 134
Introduction 134
Background Information 136
Multiplication Principle Problems and Permutation Problems *136*
Curricular Treatment of the Multiplication Principle and Permutations *138*
Connecting and Separating these Topics *141*
Methodology 143
Summary *143*
Participants *144*

Measures and Data Collection *145*
Data Analysis *148*
Analysis 149
Discussion 160
References 169
Appendix A: Classroom observation checklist 170
Appendix B: Interview Questions 172
Analysis 4. Student representations in combinatorics 173
Introduction 173
Background 174
Combinatorics Problems *174*
Combinatorics and Representations *179*
Methodology 182
Summary *182*
Participants *182*
Measures and Data Collection *183*
Data Analysis *194*
Results 195
Ordered Lines *197*
Set-to-Set Lines *202*
Discussion 206
References 211
Appendix A: Interview Questions 214
General Conclusions 215
Complete References 218

List of Tables

Table 1. *Summary of types of simple permutations and combinations.* 9
Table 2. *Data collected.* 16
Table 3. *Summary of topics covered by each data source.* 20
Table 4. *Number of classes and students.* 23
Table 5. *Studies with student-level outcomes.* 43

Table 6. *Data collected.* 62
Table 7. *Student interview response elements.* 79
Table 8. *Tasks classifiable as each knowledge type (Ball, Thames, & Phelps, 2008).* 104
Table 9. *A sample of a coded transcript statement.* 117
Table 10. *Relative frequency of statements flagged as each knowledge type.* 118
Table 11. *Percentage of use of knowledge types, by teacher* 119
Table 12. *Data collected.* 145
Table 13. *Number of classes and students.* 146
Table 14. *Students incorrectly using multiplication principle, by question.* 149
Table 15. *Numbers used with multiplication principle.* 155
Table 16. *Data collected.* 184
Table 17. *Number of classes and students.* 185
Table 18. *Classroom observation checklist.* 185
Table 19. *Student representation use in interviews.* 195

List of Figures

Figure 1. Shana's interview work. 66
Figure 2. Shana's tree diagram with 12 students. 70
Figure 3. Shana's tree diagram with five students. 74
Figure 4. Two students used labeled slots. 80
Figure 5. Two students mentioned tree diagrams. 81
Figure 6. Teacher knowledge types, based on Ball et al. (2008) 101
Figure 7. The interview question (top, 2009 version; bottom, 2010 version). 115
Figure 8. Individual teacher profiles. 120
Figure 9. Annie's work on the interview problem. 122
Figure 10. Betsy's work on the interview problem. 124
Figure 11. A multiplication principle problem (Lappan et al., 2006, p. 11). 139
Figure 12. A multiplication principle definition (Carter et al., 2010, p. P35). 140
Figure 13. Gabriel's solution for question 2. 154
Figure 14. A multiplication principle problem and a permutation problem (Carter et al., 2010, p. P35). 166

Figure 15. Tree diagram for interview problem 1. 189
Figure 16. Tree diagram for interview problem 2. 191
Figure 17. Tree diagram for classroom problem. 193
Figure 18. Chris' first step creating ordered lines for Problem 1. 198
Figure 19. Chris' second step creating ordered lines for Problem 1. 198
Figure 20. Chris' work on problem 1 199
Figure 21. José's work on Problem 1 200
Figure 22. Matthew's work on Problem 2. 203
Figure 23. An example of a set-to-set lines representation. 205

Summary of the Dissertation

Introduction

This introduction gives an overview of this dissertation on teachers and students working with combinatorics. It describes the content and motivations for the study and the structure of the dissertation. The introduction is then followed by the methodology of the full study. Following that, four independent analyses of the data are presented.

This work consists of teacher interviews, student interviews, and classroom observations, all based around the mathematical content area of combinatorics. Combinatorics is a subset of discrete mathematics that concerns the ordering and grouping of distinct elements; this includes both permutations and combinations. This introductory section addresses the following questions:

i. What is the motivation and theoretical framework for this study?
ii. What is included in the mathematical content area of combinatorics?
iii. What is the structure of this dissertation?

Motivation and Framework

This study was created in response to existing research and theory on teacher knowledge and on student-level outcomes of instruction. Specifically, this investigation was designed with two tenets in mind: (a) teachers of mathematics need a special kind of professional knowledge and it is important to understand the form of this knowledge, and (b) it is more useful to know how teacher attributes or interventions affect the teachers' students, rather than just how they appear at the teacher level.

In terms of the first tenet, the discussion in this study regarding what kind of knowledge is useful to teachers was inspired and motivated by the introduction of the idea of *pedagogical content knowledge* (PCK) by Shulman (1986). This original introduction of PCK put it forth as a subset of content knowledge; that is, Shulman proposed "three categories of content knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge" (1986, p. 9). Pedagogical content knowledge is defined as the knowledge, still particular to the subject matter, that is specifically used for teaching. Inside PCK, Shulman includes representations, examples, and explanations, as well as common difficulties, common student preconceptions, and ways of addressing incorrect student conceptions.

Since this introduction, studies and theoretical papers have attempted to clarify, specify, measure, or engender Shulman's PCK. However, as pointed out by Hill, Ball, and Schilling (2008), there is still little information showing how teachers' levels of PCK relate to student-level outcomes, or even about what constitutes PCK. Since PCK is by its very nature domain specific, for each area of mathematics we require a full description of all those items put forth by Shulman in order to say we have defined the PCK for this area.

In mathematics, Hill, Ball, and Schilling (2008) and Ball, Thames, and Phelps (2008) give the most comprehensive look at teacher knowledge. They propose that PCK is part of a larger construct, *mathematical knowledge for teaching* (MKT). They separate the universe of MKT into subject matter knowledge on one side, and pedagogical content knowledge on the other.

However, for them, the subject matter knowledge side includes both *common content knowledge* (CCK) and *specialized content knowledge* (SCK). The first item, *common content knowledge*, referred to as "'common' knowledge of content" (p. 387) in Hill, Rowan, and Ball (2005), includes functional knowledge or what we might consider to be pure mathematical content; this is the knowledge of mathematics apart from the need to teach it. The example provided for this first area of content knowledge is the solution for x in the expression $10^x = 1$.

The second item is the *specialized content knowledge*, or content knowledge that would be useful only to a teacher. The authors are careful to note that this second area is still mathematical knowledge, not pedagogy. For this area, the example provided requires the teachers to evaluate three methods for multiplying two digit numbers, and determine which of the methods are always mathematically valid. The knowledge used in completing an activity of this type has commonalities with pedagogical content knowledge (Shulman, 1986), in that it requires the teacher to recognize alternative solution strategies outside the traditional algorithm, and to reflect on their mathematical legitimacy. However, in their framework, specialized content knowledge sits next to PCK but does not contain it; neither is it contained by it (Ball et al., 2008; Hill et al., 2008).

On the side of pedagogical content knowledge, they include a new term, *knowledge of content and students* (KCS), that more specifically includes "knowledge of how *students* think about, know, or learn this particular content" (Hill et al., 2008, p. 375 [italics added]). The intent is to define this area as a measurable domain of knowledge that is distinct from the specialized content knowledge in that it requires more knowledge of how students learn. The other type of knowledge contained within PCK is knowledge of *content and teaching* (KCT). This type "combines knowing about *teaching* and knowing about mathematics" (Ball et al., 2008, p. 401 [italics added]).

While work continues in defining and distinguishing these knowledge areas, both theoretically and empirically, studies that capture or address some aspect of teacher-specific mathematical knowledge do exist. And, in keeping with the second tenet above regarding the importance of capturing student-level outcomes, it is fair to at least make the conjecture that teacher knowledge is powerful. Carpenter et al. (1989) harnessed this through an intervention in which teachers were explicitly taught about students' ideas, addressing the construct of *knowledge of content and students*. Higher student scores on a written assessment provide the student-level data. Hill et al. (2005) did not complete an intervention with teachers; instead they attempted to gauge each teacher's existing level of *specialized content knowledge* through their assessments. The higher teacher scores were then correlated to higher student scores, again on a written assessment. In both of these studies, positive connections to student performance are made. Though the distinctions between the specific areas contained in or bordering on PCK are not fully specified, both Carpenter et al. (1989) and Hill et al. (2005) have shown successful results related to teachers' engagement in or response to tasks that are not purely mathematical. Instead, they ask teachers to engage in activities or questions that connect mathematics to the work of teaching it.

One other issue that arises when considering the task of defining and refining different aspects and nuances of teacher knowledge in mathematics is that mathematics itself is infinite and complex. Just as defining knowledge for teaching chemistry might not fully elucidate knowledge for teaching physics, knowledge for teaching arithmetic does not necessarily imply knowledge for teaching geometry. The knowledge is not just content-based at the level of subjects in school, but actually on concepts within that. Many of the existing studies have looked at number and operations (e.g., Carpenter et al., 1989; Cobb et al., 1991; Hill et al., 2005), which is not surprising given that these are foundations for later mathematical activity in and out of school and generally comprise a student's first exposure to mathematics. Other areas have not been addressed yet, with the exceptions of some work in algebra (Hill et al., 2005) and in fractions (Saxe, Gearhart, & Nasir, 2001). This provides additional motivation for this study in combinatorics. If we wish to ultimately define teacher knowledge in multiple sub-areas, there is initial work to be done to define each of these areas of mathematics and generate a tentative framework of what the teacher knowledge for each one might look like.

In addition, if we wish to consider student-level outcomes, another aspect of the link between student and teacher is the degree to which *understanding*, not just performance, is connected between them. We are currently dependent on test scores, which are partial measures of performance, to determine the impact on students. This is not unusual: it is consistent with the increased emphasis on standardized testing in the schools and it is the most realistic plan for looking at large numbers of teachers and students. Nevertheless, it does not generally allow us to see all relevant aspects of performance.

Written tests may certainly seek to draw out and measure understanding on a topic, rather than necessarily focusing on procedural knowledge. However, eliciting student explanations on mathematical topics elucidates the depth of their understanding (Ginsburg, 1997; Piaget, 1976/1926). A focus on depth and understanding may ultimately require smaller scale studies, allowing us to shift the question from the impact of different types of teacher knowledge on student *performance*, to the impact of teacher knowledge on student *understanding*. One of the main arguments behind this study is that there exists an opening in the field of research in mathematics teacher education for a qualitative analysis of the connections between teacher and student understanding. This, then, provides justification for the use of student interviews, instead of written assessments, in this study.

Combinatorics

This study is unique because of the subject matter within mathematics that will be examined. As more work has occurred in number and operations and algebra, the research community can and should begin to expand efforts in examining teacher knowledge to other topics within mathematics. In pre-university education, rarely is much classroom time devoted to the study of the mathematical topic that is the focus of this study – combinatorics – and the topic may be peripheral to the other mathematics taught within the same school year. In fact, it often appears as a small section of a class in algebra (e.g. Carter et al., 2010; Collins et al., 1997; Glencoe McGraw-Hill, 2010). This is not to say that the topic is unimportant; in fact, understanding of this material forms the basis for more advanced theoretical probability, which leads in turn to statistics, a field with numerous practical applications and with connections to many careers.

Combinatorics, including the combinations and permutations mentioned above, deals with the ordering and grouping of fixed numbers of items. This topic is within the field of discrete mathematics, or the mathematics of unconnected elements (Rosen, 2003). Here, I will clarify the types of counting and probability problems that were included in this study. The purpose of this discussion is to define a small segment of combinatorics for consideration; this in no way covers the breadth of these mathematical topics. As part of this purpose, I will outline the relationships and connections among the questions. This is an attempt at examining the conceptual field of combinatorics (Vergnaud, 1996). Vergnaud proposed a theory of conceptual fields based on the need to understand the mathematical area in which cognition occurs. He defines a conceptual field as, "a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations" (p. 225).

For instance, we can consider questions about permutations and combinations in this small subset of combinatorics as referring to finite arrays of objects. Within permutations, there are two initial cases. First, there is the case with n objects, where all n must be arranged. For example, if we have three different letters, how many ways can we arrange all three of them? Second, there is the case with n objects where some number less than n must be arranged. For example, given all 26 letters in the English language alphabet, how many three-letter words can be formed? Asking for the implications of allowing or disallowing repeat letters can further a case like this one. This, then, leads to the more difficult cases of permutations, in which there are non-unique objects to be arranged. For example, if we have three letters, but two are identical and cannot be differentiated, and only the third is unique, how many ways can we arrange all three of them? This is a potentially more troubling case because it requires the individual to determine which, of the $n!$ arrangements that would be present for unique items, would be duplicates in this new structure. This sort of problem can be solved by force (i.e., by listing all possible permutations and manually checking for those that appear identical) for small arrays of items, but even this technique can then be the source for conjecture on determining how many items would need to be removed.

Combinations follow a similar pattern. In this case, n objects, of which n are selected, result in one possible combination. This shift from the ordered permutations discussed above to the unordered groups here can actually be difficult to conceptualize; this first case is not trivial. From here, cases can progress to choosing an unordered subset of fewer than n items, followed by consideration of what happens when some of the items are identical. A summary of these types of items is shown in Table 1. Brute force can solve the problem for small arrays and may also lead to fruitful discussions. Combinations seem like they should be easier than permutations, when considered from a nonmathematical standpoint. The complication of ordering has been removed, which makes it seem as if we should be able to breathe more easily. However, in the formulaic calculation, in the brute force solution methods, and in the conceptualization, it can be challenging to establish the distinction between both.

Table 1. *Summary of types of simple permutations and combinations.*

	Permutations	Combinations
Given:	Determine:	Determine:
n unique objects	Number of arrangements of all n objects	Number of groups of all n objects
n unique objects	Number of arrangements of m objects for $m < n$	Number of groups of m objects for $m < n$
n objects, of which some are not unique	Number of distinguishable arrangements of all n objects	Number of distinguishable groups of m objects for $m < n$

In working with these types of problems, often the individual needs to judge whether a permutation or combination is needed. However, by requiring this type of decision as part of the question, this suggests the use of problems posed within extra-mathematical contexts. This is because problems stated in mathematical symbols and language, as seen in Table 1, specify directly whether they want the number of *arrangements* or the number of *groups*. This may be phrased differently, say by asking for the number of *permutations* or the number of *combinations*, or the number of *ordered lists* or the number of *sets*. However, if the reader has experience with this vocabulary, then the phrasing of the question betrays whether permutations or combinations are required. As a result, the need to judge which of the two to use is removed from the problem. Instead, if the goal is to require students to decide whether or not order matters, then a problem should have an extra-mathematical context. The student must then use their knowledge of extra-mathematical topics to deem whether or not order matters. For example, a question might ask about the number of possible automobile license plates given a particular format of four numeric digits followed by two letters. Cultural knowledge of license plates tells us that the plate *1234 PK* is not the same as the plate *1234 KP*. As a result, someone responding to this question might deduce that a permutation is required to reach the correct answer, and not a combination.

Several established representations of combinatorics exist, and these are used for both instruction and understanding. One possibility is a list of all the outcomes. This brute force method is effective for small sets. Tree diagrams are also commonly used, particularly for permutations. The slot method is another option, and, of course, there are established mathematical formulae for problems of this type. One potential area for exploration in teacher and student understanding in combinatorics would be the relationships between these representations. In particular, it may occur that the use of one representation leads naturally to the adoption of another. Representations may also be invented, or they may have been explicitly taught to an individual.

Probability can be included within this limited look at combinatorics, if only where it connects to permutations and combinations. That is, if we focus not on large-scale probability, but on simple cases of discrete probability, such as determining the probability of an outcome when it is necessary to use combinatorics to count all possible outcomes. For example, a permutation can be used to determine the number of possible sequences of raffle winners given a set pool of entrants. Probability could then be applied to find the likelihood that a particular person wins a prize.

Structure of the Dissertation

This dissertation has a non-traditional structure. Following this introduction, there is a full methodology section. This methodology section, and the related appendices, describes all the elements of data collection and all the instruments used. However, the resulting data are used in four separate analyses. Each analysis is incorporated into a standalone paper. That is, each analysis has its own introduction, background literature, methodology, analysis, results, and conclusions. Each can be read and understood individually, with or without having read the introduction and full methodology described here. In each, only the background literature and segments of the methodology relevant to the particular analysis are included.

The structure of this dissertation is not without precedent or advocates in educational research. Duke and Beck (1999) define a "traditional" dissertation as "a lengthy document (typically 200-400 pages in length) on a single topic presented through separate chapters for the introduction, literature review, methodology, results, and conclusions" (p. 31). Using this definition, they argue that if we see the two main purposes of a dissertation as (a) a tool to train future researchers, and (b) a means to contribute to educational research, then the traditional format of a dissertation is not the only, or most efficient, way to achieve either of these goals. On the point of contributing to educational research, they point out that few people will read a doctoral dissertation in its entirety due to its length, so the contents may not reach even those for whom they are directly relevant. While many authors subsequently rewrite their dissertations into books or shorter articles for publication, this is dependent on the individual and their career plans; many are not rewritten. On the point of training future researchers and academics, they point out that the form is markedly unlike other forms of academic writing, including journal articles, book chapters, and grant proposals. This restricts any increase of competence in writing that doctoral students will need in future careers. Krathwohl (1994), who also made an argument for non-traditional dissertations in educational research, makes this last point compellingly: "it wastes the opportunity for students to learn writing for publication under faculty tutelage. Given the usual individual dissertation supervision, faculty are in a far better position to pass on this capacity to their students than at any other time in the graduate experience" (pp. 30-31).

Duke and Beck go on to advocate for a potential alternate form of dissertation as described by Krathwohl: "*write the dissertation as an article (or a series or set of such articles) ready for publication. Use appendices for any additional information the committee may desire for pedagogical and* *examination purposes*" (p. 31, italics in original).

This suggested format is taken up here in this dissertation. As described above, this introduction and the full methodology (below) serve to provide a full account of the study to interested parties and to those responsible for validating that the work is sufficient to justify a doctoral degree. The four separate analyses are written as individual articles with the hope that the shorter length and self-contained format will allow for them to be published, distributed to, and read by any member of the field of education research interested in the subject of the particular analysis.

Methodology

Participants

All teachers who participated in a summer professional development workshop (described below) received a letter at the beginning of the workshop inviting them to participate and explaining the project. Teachers were asked if they would be willing to be interviewed for the study, and if they would be willing to allow classroom observations and to have student participants sought from within their classrooms.

Eight teachers participated; of these, only two teachers both were willing to allow student participants and were planning to teach lessons related to combinatorics during the following school year. Once these two teachers were identified, the administration of each of their schools was contacted with a letter explaining the project and asking for their participation. Both of these teachers teach at secondary schools in the same large urban school district in the state of Massachusetts.

After the administrations of the schools had consented to participate, the families of all students of the two teachers received a consent letter and explanation of the study. The students were also asked for their assent. Fourteen students assented and had their parent or guardian consent as well. Ultimately, eleven of these students were available and all eleven of these were interviewed.

Professional Development Course

The 2008 and 2009 Tufts University Problem Solving and Discrete Math Workshops were both available to teachers who teach mathematics in grades 5 through 9 in Massachusetts, with preference given to teachers from districts that are classified by the state as high needs. The 2008 workshop was a seven day summer workshop, with two full-day follow up sessions in the fall and winter of 2008. The 2009 workshop was an eight day summer workshop, also with two full-day follow up sessions in the fall and winter of 2009. The extra day of workshop time in 2009 did not include any additional time spent on teaching or discussing combinatorics (of which permutations are a part); for this reason, the 2008 and 2009 workshops can be considered here to have used identical curricula. No teacher attended more than one year of the workshop. Many teachers who participated in the workshop did not participate in this study, and they were not required to be part of this study in order to attend the workshop.

The PSDM workshop was primarily focused on mathematical content. There were approximately three hours of instruction per day on mathematical content. During this time, the teachers listened to a lecture, completed problems, worked in groups, and asked questions. No teaching methods were suggested to the participants. The teachers also had additional problems for homework that could be completed in groups or independently, and they had one to two hours of time during the workshop day to work on these. The following day, a subset of teachers would explain their solutions to completed homework problems to the full group of participants, and they would also answer any questions.

However, teachers did work in groups on curricular plans and considered how their own students would interact with the materials. The teachers spent two hours of time each afternoon working in groups to create a three-day lesson for their own students, covering one of the topics taught in the workshop. Since no methods of teaching were suggested, and guidance was provided only on the mathematical content, the time spent on planning lessons can be considered as self-directed time.

Each year of the workshop had four days of content that was directly related to this study: one day focused on simple counting problems, two days on permutations and combinations, and a fourth day on probability.

Measures and Data Collection

Data were collected through teacher interviews, classroom observations, and student interviews. An overview of the data types is given in this introductory paragraph, with a summary shown in Table 2, followed by a detailed description of each source.

The teacher interviews were conducted first, in spring 2009 and spring 2010. Next, classroom observations were completed with a subset of the teachers, in May and June 2010. Finally, student interviews were conducted in June 2010. Student interviews took place after instruction related to these topics was complete in each classroom.

Table 2. *Data collected.*

Measure	Time of measurement	Mode
Teacher interviews	Spring 2009 (5) / Spring 2010 (3)	Videotaped interview; subsequently logged
Classroom observations	May / June 2010	Written observations; checklist; collection of handouts and instructional materials
Student interviews	June 2010	Videotaped interview; subsequently logged

Teacher interviews. The teacher interviews were conducted in spring 2009 and 2010. In all cases, the interview took place the spring following the summer in which the teacher had participated in the summer workshop. Five teachers were interviewed in spring 2009 and three teachers were interviewed in spring 2010. However, since the workshops did not differ in content related to this study and this study does not look at the impact of the workshop or make any assumptions about its effect, all eight teachers are analyzed together, without distinguishing their year of participation.

An interview was carried out and videotaped with each participating teacher. While interviews were flexible and open-ended, they had a goal of discussing mathematical situations and material that are germane to combinatorics, including questions on permutations. To do this, the teachers were given problems to solve (shown in Appendix A). After solving each problem, they were asked for an explanation of their work. They were then asked for a different way to solve the same problem and a different explanation. They were also asked about what they believed their students would do when working on the same problem. This process repeated for each mathematical problem. Interviews lasted between 45 and 90 minutes. Interview participants were given as much time as they wanted to work on and discuss each problem.

The combinatorics problems used in the teacher interviews were constructed to ensure coverage of the mathematical topics. In a previous analysis of this topic, I had looked at teacher understanding of combinatorics, as well as the conceptual field of the mathematics itself (Vergnaud, 1996). This analysis formed the basis for problem selection for the first round of teacher interviews, as shown in Table 3. Note that in this table, the question numbers for all interviews are given; the questions themselves are shown in Appendix A. In the first round, five teachers were interviewed as they completed and discussed a set of seven combinatorics problems. The problems completed by teachers covered a broad range of mathematical situations. This was valuable in evaluating the mathematical landscape. However, by examining such a range of concepts, many of the problems were classified by the participating teachers as being beyond the scope of the middle and high school curricula. While these topics are still valuable mathematics and valuable as part of the mathematical landscape, they were not part of what the teachers actually used in concert with their students. For this reason, the range of topics was reduced in the second round of teacher interviews, as shown in Table 3. The main change was to eliminate questions on permutations or combinations of objects in which some of the objects being arranged or grouped are indistinguishable from each other. For example, a problem of the type in which someone is asked for the number of arrangements of all four of the letters ABBC was not included in the second round. Teachers frequently cited these items as being outside the scope of school curricula. In addition to this, I also decided not to include a question about the number of combinations of all *n* unique objects. A question of this type appeared in the first round in order to gauge teacher reaction to this case, in which the answer is that there is only one combination of all objects. Teachers generally confirmed that this unusual question is also not part of what they teach.

In addition to these reductions, simple problems using discrete probability were introduced in the second round of interviews. The reason for adding discrete probability was straightforward: this is a common topic in the middle and high school curricula, and it allows for the examination of how participants view the number of possibilities, be they permutations or combinations, in a different light. In addition, some teachers introduce simple discrete probability either prior to or without introducing permutations and combinations. This often includes problems like determining the probability of getting, for example, a three when rolling a six-sided die. Students are led to the idea that the probability of this occurrence is 1/6 by observing that there are six possible results, but only one of these is the three. Thus, they don't need to explicitly calculate permutations and combinations to determine the number of possible outcomes or the number of sought outcomes. The first round of teacher interviews did include one problem about probability, but this was a more complex problem that required the teachers to determine the numbers of combinations for both the possible outcomes and for the sought outcomes.

Also note that in the first round of interviews (spring 2009), the objects being arranged (as shown in Appendix A) were numbers and in the second round of interviews (spring 2010) the objects were letters. During the intervening time, the question was modified because it was thought that it could be less confusing for the students to arrange letters, rather than numbers. However, during the teacher interviews, no teacher mentioned or critiqued the use of numbers or letters when discussing this question. As a result, the eight teachers will still be analyzed together when the question is the same except for this change.

Table 3. *Summary of topics covered by each data source.*					
		Round 1 Teacher interviews	Round 2 teacher interviews	Classroom observations	Student interviews
Permutations	Given n unique objects, number of arrangements of all n objects	Round 1, Question 1 Question 7	Round 2, Question 1 Question 5	X	Round 2, Question 1 Question 5
	Given n unique objects, number of arrangements of m objects for $m < n$	Round 1, Question 2	Round 2, Question 2	X	Round 2, Question 2
	Given n objects, of which some are not unique, number of distinguishable arrangements of all n objects	Round 1, Question 3			

Combinations	Given n unique objects, number of groups of all n objects	Round 1, Question 4	
	Given n unique objects, number of groups of m objects for $m < n$	Round 1, Question 4	Round 2, Question 3
	Given n objects, of which some are not unique, number of distinguishable groups of m objects for $m < n$	Round 1, Question 5	

Probability	Calculation not required to find number of possibilities		Round 2, Question 4a Question 4b	X	Round 2, Question 4a Question 4b
	Calculation of permutation or combination required to find number of possibilities	Round 1, Question 6	Round 2, Question 6		Round 2, Question 6

Classroom observations. The second source of data was classroom observations. As described above, two of the teachers who were interviewed in the second round consented to classroom observations. Classes were observed during the time that the teacher was providing instruction on combinatorics. In the secondary school curriculum in the school district in which this study took place, these topics are introduced in Algebra I at the end of the school year, in May or June.

Each of the two participating teachers was teaching two sections of Algebra I, each attended by students 14 to 16 years old (i.e., freshmen to juniors in high school), at the time of data collection. The first teacher, who we will identify by the pseudonym Shana, taught related topics during five school days;

classroom observations were carried out on all five days. Shana taught two sections of the same class; each section received the same 5 days of instruction and both sections were observed. The second teacher, Whitney (also a pseudonym), taught these topics during two school days, so the classroom observations were carried out on two days only. As with Shana, Whitney also taught two sections of the same class; each section received the same 2 days of instruction and both sections were observed. These details are summarized in Table 4.

The mathematical topics that were covered during the classroom observations are shown in Table 3. Note, though, that this information was added to this methodology after the classroom observations were completed. This provides an easy comparison between the topics covered by all data sources. This study did not suggest mathematical topics to the teachers, nor did it require teachers to cover a certain topic. This is merely a record of what was covered in these classrooms.

Table 4. *Number of classes and students.*

Teacher name	Number of classes spent on material	Number of students interviewed
Shana	5 per section	7
Whitney	2 per section	4

During classroom observations, I recorded on paper as much of the classroom activity as possible, along with recording all mathematical problems that were addressed during the class and collecting handouts when used. In addition, I used a checklist for quickly noting topics, concepts, and representations addressed during the class. A new checklist was used for each 15-minute interval (see Appendix B). The checklist provided a structure to allow quick notation of the type of mathematical problem, the type of representations being used, and the type of classroom activity, such as teacher-led discussion, group work, or independent work. This checklist was based on the classroom video coding categories described by Hill et al. (2008b) as the "mathematical quality of instruction" (MQI) measure. Note that because the study here did not use video of the classroom lessons, the actual MQI instrument could not be properly applied; its design is based on being able to review a single lesson multiple times. Therefore, while it formed the theoretical basis for the classroom observation checklist, no MQI score is assigned to these classroom observations and no comparisons can be made to other classroom lessons that were coded using the MQI metric.

Student interviews. All consenting and available students in each class were interviewed. Eleven students were interviewed, with seven students from Shana's classes and four students from Whitney's classes, as shown in Table 4. The interviews with the students followed the same pattern as the second round of interviews with the teachers, and used the same mathematical questions. As discussed above, the breadth of mathematical topics was reduced between the first and second round of teacher interviews, in order to align with the topics commonly addressed in these teachers' classrooms. However, even after the second round of teacher interviews, one additional question was eliminated from the student interviews as a result of the classroom observations. This question was on combinations, asking when given n unique objects, the number of groups of m objects for $m < n$, as shown in Table 3. This question was eliminated because neither teacher addressed combinations, or unordered groups, in the classroom.

During the interviews, the students were given problems to solve (shown in Appendix A). After solving each problem, they were asked for an explanation. They were then asked for a different way to solve the same problem and a different explanation. I attempted to elicit a full explanation of both correct and incorrect answers and techniques.

Summary

In this section, I will summarize the findings and analyses from this dissertation study. Since, as described above, the structure is four papers that can be read independently of each other, this section will provide a brief overview of each.

Analysis 1: Continuity of Data Sources

The purpose of this paper is to provide evidence for the argument that student interviews can be a useful source of data when considering the qualities of instruction in mathematics. Researchers have examined the qualities of instruction mostly through: (a) teacher-level characteristics, such as coursework or test scores, (b) student scores and gains in scores on written assessments, and

(c) descriptive or quantitatively coded observations of classroom lessons. Despite the breadth of the existing research, there is a missing piece: the consideration of student interviews in conjunction with classroom observations and teacher interviews. In this paper, I address the questions: (1) what do student interviews tell us about the qualities of the mathematics instruction; and (2) how do student interviews enrich data from classroom observations and teacher interviews? Specifically, this paper presents a case study of one teacher interview, the related classroom observation, and the related student interviews.

The case analysis shows that the teacher's instruction shifted in notable ways, even on the same school day, between the first section when she taught the material and the second section with the same material. She made adjustments in the methods she used to solve example problems, and she changed which types of student errors she addressed with the whole class, rather than with individuals. During the subsequent student interviews, four of the elements that were prominent in the responses (the use of labeled slots, mention of tree diagrams, mention of listing, and explanations of why the number of possibilities decreases for each office) showed indications of the shifts in instruction. The student interviews allowed us to see things we would not have seen at the level of classroom observations or written student assessments, and these things reflected the qualities of the instruction. This particular case shows that, in general, student interviews allow us to examine mechanisms through which instruction may affect students and to put forth researchable claims about what is valuable in classroom instruction.

Analysis 2: Using Interviews to Explore Teacher Knowledge Types

The purpose of this analysis is to explore the use of an existing framework of types of teacher knowledge put forth by Ball, Thames, and Phelps (2008) in a novel way. Specifically, the analysis assigns knowledge types to teacher statements made during an interview, focusing on all eight participating interviews and one mathematical problem. This type of detailed coding has not previously been done with teacher interviews. This analysis addresses the questions (1) what knowledge types (common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching) do teachers exhibit when answering a question about permutations in an interview setting; and (2) is it reasonable to consider knowledge types as manifested in particular statements, rather than as attributes of a teacher?

This analysis rests on the belief that it is interesting and important to be able to separate teacher knowledge into different types, both to understand the work of teaching and to understand how (and in what areas) teachers need support. Analyses of interview statements, as shown in this paper, may provide a link between how a teacher completes a task and the knowledge they use in completing the task. As such, this may begin to shed light on the ways in which different types of knowledge are exhibited in teaching practice. This argument assumes that teachers' reflections in an interview are closer to the kinds of verbalizations made in a classroom setting than what is exhibited in a written assessment.

The analysis showed that teachers, as a group, used specialized content knowledge most frequently in their statements, in comparison with the other knowledge types included. Most notably, however, not all teachers showed the same relative frequency of the different knowledge types; different teacher profiles emerged. The implication for this work is that with more teachers and in connection with classroom data, we may begin to understand what these different profiles suggest about a teacher's work and the types of supports that would be beneficial to them. Another advantage of examining these teacher profiles is that we begin to see that different profiles may complement each other. That is, perhaps teachers with different profiles would be able to each take the lead in turn in sharing teaching knowledge in a mutually beneficial way.

Analysis 3: Overextending the Multiplication Principle

The purpose of this analysis is to examine the connections between how students solve problems involving the multiplication principle and how they then solve problems involving permutations. The multiplication principle, often referred to as the "product rule" or "fundamental counting principle," states that if an event occurs in m ways and another event occurs independently in n ways, then the two events can occur in $m*n$ ways. The term "permutation" refers to an ordered arrangement of a number of objects. The data for this study consists of observations from two classrooms during a unit that included problems using the multiplication principle and problems using permutations, as well as interviews with 11 students from these classrooms.

Analysis of the student interviews showed that on each of the three interview problems involving permutations, 45-55% (5-6 of the students) incorrectly overextended the multiplication principle, multiplying two or more numbers that they described as being different "types" of things, as one would do to (correctly) find the possibilities for outfits choosing from three pairs of pants and four shirts. This suggests that students are struggling to make the transition from problems that use the multiplication principle with different categories of items (e.g., ice cream toppings and cone type) to those that permute items within a category (e.g., arranging five different cereal boxes). While no textbooks were used in the classrooms in this study, a review of some common curricular materials suggests that the existing instructional sequence may be built off the idea that problems using the multiplication principle will serve as an entry point into problems with permutations of objects. This is mathematically reasonable, but not sufficient for students who are seeing these concepts as learners. The results suggest that students need additional support to gain an understanding of how the two types of problems are similar and how they are different, and when each one is applicable.

Analysis 4: Student Representations in Combinatorics

This paper looks at students' representations in combinatorics. The analysis shows that (1) students generate useful non-canonical representations of combinatorics, (2) the production of these representations is associated with the way in which we, as instructors, present problems to them, and (3) we can benefit from recognizing and utilizing the variety of representations that students produce as tools for solving and understanding problems.

The student interviews showed evidence of novel representations, and they used these as tools to find answers and to express their thinking. The classroom observations provided additional evidence to support the existence of useful non-canonical student representations. During the interviews, two types of these were used, and both cases involved students generating lines connecting the items that were to be combined or arranged. The instances that we have available to us suggest that the way in which the problem statement is represented is a primary factor in student choice of representations. When the items to be combined or arranged were presented to the students as objects or words that were all marked on paper for them, some students made use of this presentation to create one of these non-canonical representations. Only one student explicitly represented a set of objects on which to build a representation. While we may eventually want students to extract the meaning of problems from words alone and to construct their representations from scratch, an early introduction to combinatorics should include providing students with names or notation for the set of objects they are using, upon which they can construct. By providing this scaffolding for students, we can enable them to produce their own representations that they can use as tools to explain their understanding.

References

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Carter, J.A., Cuevas, G.J., Day, R., Malloy, C., Holliday, B., & Luchin, B. (2010). *Algebra 1*. Glencoe / McGraw-Hill, 2010.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., and Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3-29.

Collins, W., Cuevas, G.J., Foster, A.G., Gordon, B., Moore-Harris, B., Rath, J., Swart, D., & Winters, L.J. (1997). *Algebra 2*. Glencoe / McGraw-Hill, 1997.

Duke, N.K. & Beck, S.W. (1999). Education should consider alternative formats for the dissertation. *Educational Researcher, 28*(3), 31-36.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.

Glencoe McGraw-Hill (2010). Algebra 2 Online Learning Center. Retrieved from http://glencoe.mcgraw-hill.com/sites/0078884829/

Hill, H.C., Ball, D.L, & Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H.C., Blunk, M.L., Charalambous, C.Y., Lewis, J.M., Phelps, G.C., Sleep, L., & Ball, D.L. (2008b). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Krathwohl, D.R. (1994). A slice of advice. *Educational Researcher, 23*(1), 29-32, 42.

Piaget, J. (1976). *The child's conception of the world*. (J. Tomlinson & A. Tomlinson, Trans.). Savage, MD: Littlefield Adams. (Original work published 1926.)

Rosen, K. (2003). *Discrete Mathematics and Its Applications*. Boston, MA: McGraw Hill.

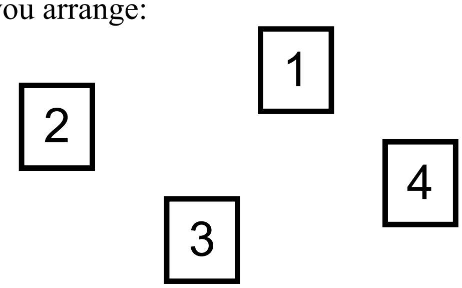
Saxe, G.B., Gearhart, M., & Nasir, N.S. (2001). Enhancing students' understanding of mathematics: A study of three contrasting approaches to professional support. *Journal of Mathematics Teacher Education, 4*, 55-79.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning* (pp. 219-239). Hillsdale, NJ: Lawrence Erlbaum.

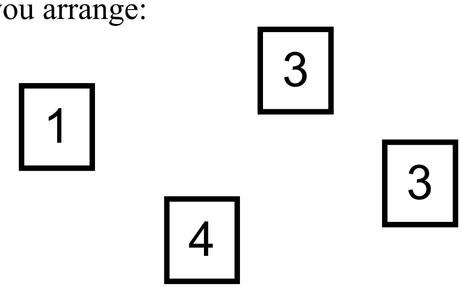
Appendix A: Interview Questions

<u>Round 1, Question 1</u>
How many ways can you arrange:

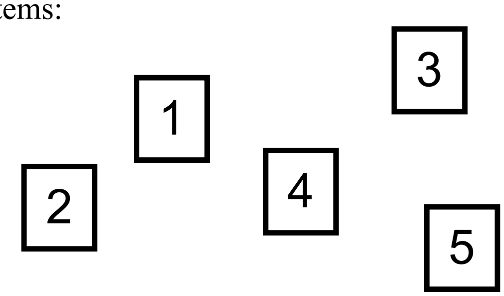


<u>Round 1, Question 2</u>
There is a basket with tickets numbered 1 through 10. How many ways can you pick 4 tickets if the order you pick them is important?

<u>Round 1, Question 3</u>
How many ways can you arrange:



<u>Round 1, Question 4</u>
If you have these 5 items:



How many different groups of 5 can you make?
How many different groups of 3 can you make?

Round 1, Question 5
If you have these 5 items:

$\boxed{1}$ $\boxed{5}$ $\boxed{2}$ $\boxed{4}$ $\boxed{4}$

How many different groups of 3 can you make?

Round 1, Question 6
If you have a bag of candy with:

8 gobstoppers
5 peppermints
6 caramels

And you pull out 3 pieces of candy, how many distinguishable groups of 3 pieces are possible?
What is the probability of getting 2 caramels and 1 gobstopper?

Round 1, Question 7
If you have a class of 20 students, you need to assign 4 jobs each day: one student to pass out calculators, one student to pass out pencils, and two students to pass out notebooks.
How many different job assignments are possible?

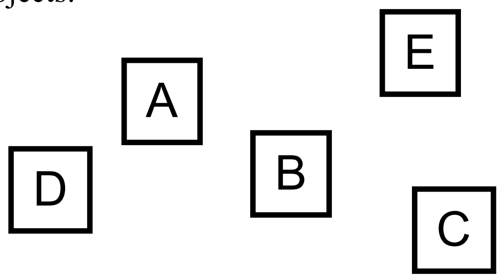
Round 2, Question 1
How many ways can you arrange these objects:

$\boxed{A}$ $\boxed{C}$ $\boxed{B}$ $\boxed{D}$

Round 2, Question 2
If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?

Round 2, Question 3 (Teacher interview only)

If you have these 5 objects:



How many different groups of 2 can you make?
How many different groups of 3 can you make?

Round 2, Question 4a

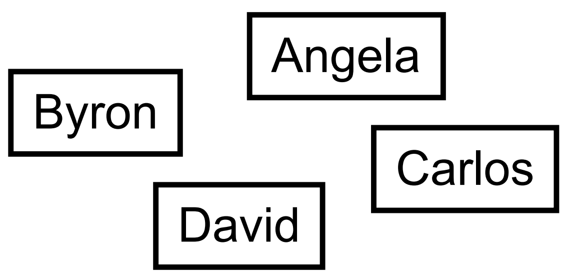
You have 6 marbles in a bag. Four marbles are blue and two marbles are yellow. If you choose one marble without looking, what is the probability that the marble you pick is yellow?

Round 2, Question 4b

If you choose one marble without looking, and then you choose a second marble without looking, what is the probability that they are both yellow?

Round 2, Question 5

There are 4 students staying after school:



How many ways can you choose 2 students to clean the board, 1 student to sharpen pencils, and 1 student to organize papers?

Round 2, Question 6

You have 10 cards, numbered 1 through 10. If you draw 2 cards, what is the probability that the sum of the numbers on the cards is even?

Appendix B: Classroom observation checklist

Category	Item	Time
Lesson format	Teacher-led instruction (with times)	
	Students address whole class (with times)	
	Individual work (with times)	
	Group work (with times)	
	Working on applied (real-world) problems	
Richness	Multiple procedures or solution methods	
	Explanations	
	Developing mathematical generalizations	
	Mathematical language	
Working with students and math	Teacher questioning	
	Remediating student difficulties	
	Uses student mathematical ideas in instruction	
Errors	Major mathematical errors or oversights	
	Imprecision in notation or math language	
Student activity	Students provide explanations	
	Student mathematical questioning and reasoning	
	Enacted task cognitive demand	
Representations	Formula	
	List	
	Tree	
	Slot	
	Explicit linking	
General topic issues	Questions about assumptions	
	Comparing methods	
	Using complement	
	Compound probability	
	Deciding whether order matters	
	Estimation of probability	
	Finding probability denominator separately	
	Language use	
	Order of introducing methods	
	Replacement of elements	
	Testing smaller cases	
	Using physical objects	
	Word analogy for identical items	
Formula method	Deriving formula	
	Division to remove identical combinations	
	Division to remove identical items	

	Finding probability denominator	
	Justifying formula	
	More than one set of identical items	
	Multiplication choice	
	Remembering formula	
List method	Knowing when all possibilities are there	
	Removing identical combinations	
	Removing identical items	
	Removing identical permutations	
	Systematic listing	
Slot method	Division to remove identical combinations	
	Division to remove identical items	
	Finding number of possibilities per slot	
	Finding number of slots	
	Multiplication choice	
	Removing identical items	
Tree method	Finding number of branches from a vertex	
	Multiplication choice	
	Removing identical combinations	

Analysis 1. Continuity of data sources: from teachers to classroom to students

Introduction

The purpose of this paper is to provide evidence for the argument that student interviews can be a useful source of data when considering the qualities of instruction in mathematics. Current concern over mathematics education has led to efforts to evaluate not only student performance, but teacher quality as well. This paper does not address the difficult questions of how to evaluate student learning or performance on a national scale, and whether or not teachers should be held accountable for student performance. Rather, this paper considers that researchers have examined the qualities of instruction mostly through: (a) teacher-level characteristics, such as coursework or test scores, (b) student scores and gains in scores on written assessments, and (c) descriptive or quantitatively coded observations of classroom lessons. Despite the breadth of the existing research, however, there is a missing piece: the consideration of student interviews in conjunction with classroom observations and teacher interviews.

Eliciting student explanations and ideas via interviews on mathematical topics is an established route to understanding student thinking (e.g., Ginsburg, 1997; Piaget, 1976/1926). This approach is better suited than written assessments to elucidate the depth and subtleties of student understanding. Using interviews, we could, on a smaller scale, shift the question from the impact of instruction on student *performance*, to the impact of instruction on student *understanding*. This indicates an opening in the field for a qualitative analysis of the connections between teacher and student understanding, and the mechanisms through which one can affect the other. In this paper, I will address the questions: (1) what do student interviews tell us about the qualities of the mathematics instruction; and (2) how do student interviews enrich data from classroom observations and teacher interviews? Specifically, this paper presents a case study of one teacher and one very specific mathematical topic, permutations. The data sources are a teacher interview, classroom observation, and student interviews. When taken together, these three different types of information provide a richer picture of the teacher's instruction. The presentation of the case study provides an example in order to argue on behalf of the collection and consideration of this type of data.

It is certainly not new to interview students to determine how they think about mathematics. In the tradition of Piaget, interviews have provided a wealth of information on cognition, and not just in mathematics. It is also not new to try to determine how to measure outcomes for students as a function of the instruction their teachers provide. However, the argument here is that educational research should include student interviews, not just as information on how students think or to determine what they have learned, but in connection with how the students are affected by their teachers. This type of case study presented here has two particular advantages. First, most studies that try to correlate student outcomes to teachers rely on written assessments in which the researcher does not have an opportunity to assess the depths and nuances of student understanding. Second, most of these studies are also quite large, requiring many participants and a great deal of funding. The intent of the work presented in this paper is to explore the potential ways in which teachers affect students through smaller scale and more in-depth investigations.

This study is relevant and unique because of the close qualitative examination of connections between student and teacher, specifically with the use of student interviews. It is also different from much existing research that considers student-level outcomes because of the subject matter within mathematics that will be examined, permutations. The literature discussed below reveals that most work on student outcomes as a function of instruction has focused on the mathematical topics of number and operations, with some expansion to ideas related to algebraic thinking. This is just and reasonable given that these topics are usually the first school-based mathematics that students undertake, and that they form a foundation for other topics.

However, as more work has occurred on these, the research community can and should begin to expand efforts in examining instruction in other topics within mathematics. In pre-university education, rarely is much classroom time devoted to the study of the mathematical topic that is the focus of this analysis — permutations — and the topic may be peripheral to the other mathematics taught within the same school year. This is not to say that the topic is unimportant; in fact, understanding of this material forms the basis for more advanced theoretical probability, which leads in turn to statistics, a field with numerous practical applications and with connections to many careers.

Background

To provide a context in which to address the questions above, and justification for examining them, existing research and theory in relevant areas will be summarized here. I will outline existing work on connections between teachers' instruction and student outcomes. Looking at this material will provide justification for looking at the student interviews as a data source for considering the qualities of classroom instruction. The intent of the present analysis is to argue in favor of including student interviews as a form of data to be considered in concert with other types of student-level outcomes when discussing how to recognize and encourage quality teaching. The intent is not specifically to illuminate the issue of teaching permutations. However, in order to ensure that the discussion of the case is clear, we will look briefly at existing work on the teaching and learning of permutations.

Research on Student-Level Outcomes

In the following section, studies that measure student-level outcomes will be explored in greater detail in order to examine how connections between teachers and students have been conceptualized or assessed.

While many studies have examined the impact of coursework on teachers, or compared teachers in different intervention groups or based on various background characteristics, few studies have had the resources to take the next step in the process and attempt to link teacher-level characteristics or courses to student-level outcomes. This is an extensive undertaking if quantitative comparisons are to be made, requiring the cooperation of large numbers of teachers, schools, and students. However, we can still focus here on the few research studies that do provide a look directly at students' performance as a function of their teachers. These are summarized in Table 5, but more detail about each is given in the subsequent paragraphs.

Table 5. *Studies with student-level outcomes.*

Study	Guiding principle	Mathematical topics	Student performance outcomes
Carpenter et al. (1989)	Cognitively guided instruction (CGI)	Number and operations	Gains in student performance on written assessment, as compared to control group
Cobb et al. (1991)	Constructivist perspective	Number and operations	Gains in student performance on conceptual items on written assessment, as compared to control group
Hill et al. (2005)	Mathematical knowledge for teaching (MKT)	Number concepts; operations; patterns, functions, and algebra	Greater student gains on written assessment correlated with higher teacher MKT

Hill et al. (2011)	MKT; mathematical quality of instruction (MQI)	Varied	Greater student gains on written assessment correlated with higher observation scores
Saxe et al. (2001)	Integrating Mathematics Assessment (IMA)	Skills with fractions; understanding of fractions	Gains in student performance on conceptual items on written assessment, as compared to control group
Simon and Schifter (1993)	Constructivist perspective	Varied	No change in student performance on written assessment, as compared to control group

The first of these studies is from Carpenter, Fennema, Peterson, Chiang, & Loef (1989), a group with significant research in the area of student cognition in elementary mathematics, particularly addition and subtraction (e.g., Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1982, 1984). Drawing on their data about students' approaches to addition and subtraction problems, these researchers implemented a professional development course in which in-service teachers learned about student thinking on these mathematical topics. They followed an approach they refer to as cognitively guided instruction (CGI) that includes examination of student explanations of concepts, and also the importance of existing student conceptions. No teaching methods were suggested to the teachers. The researchers then undertook an extensive assessment of the impact of this course, using the 20 teachers in the course, a control group of 20 teachers who were not in the course, and the students of both groups of teachers. Data was collected through classroom observations, teacher and student interviews and surveys, and standardized math tests for the students. A number of results came of this, but two of them are significant for the current analysis. First, the teachers who received the course on student thinking had significantly higher scores in knowledge of student strategies for students in their classes, as measured by how accurately teachers predicted the strategies particular students would use when solving problems. Second, although the students of the teachers who had been in the experimental group spent significantly less classroom time on number fact problems, they did significantly better than the students of the teachers in the control group on questions of this type on the standardized test. Carpenter et al. (1989) show that not only did the teachers in the experimental group have stronger beliefs regarding the importance of understanding and responding to student thinking, but that the experimental group teachers changed their classroom practice to reflect these beliefs. In this way, the researchers have interwoven teacher interviews, student interviews, and classroom observations. However, all three of these elements are considered as providing potential evidence for the *outcomes* of the teachers' participation in the professional development. In particular, Carpenter et al. (1989) do use student interviews, as is the case with the study described in this paper. However, in this part of their work, in which they are addressing the qualities of instruction, the analysis of the interviews does not include connections to the classroom observations. They use the student interviews as a measure of the number of correct answers, in order to complement the written student assessment, and also to determine which strategies students used. The information about student strategies, though, is used to score the *teachers'* knowledge of their students' strategies, not to connect the students' interviews to the qualities of instruction. In this way, their consideration of student interviews in concert with instruction is different from the study described in this paper.

As with the above study, Saxe, Gearhart, and Nasir (2001) implemented a professional development workshop for teachers and measured student-level outcomes as a result. In their case, they focused on the mathematical topic of fractions, including both conceptual and procedural elements. Students of participating teachers completed pre- and post-assessments consisting of items intended to test for procedural / computational skills and others intended to test for understanding. While the test items were gathered from a mixture of sources, including existing curricula, the research team validated this assessment as well as the distinction between skill-oriented and understanding-oriented items through measures of internal consistency. The participating teachers were divided into three groups; the first group served as a control and used a traditional mathematics curriculum in the classroom. The second group used a reform curriculum and received a form of professional development in which they worked with a support group of other teachers to plan and discuss lesson topics; however, they were relatively self-directed. By contrast, the third group also worked with a reform curriculum, but received a professional development program referred to as "Integrating Mathematics Assessment" (IMA). The IMA program was designed to address teachers' own mathematical understanding, their understanding of their students' work in mathematics, and their understanding of student motivation, in addition to providing a network of other teachers. Note also that in contrast to the program presented by Carpenter et al. (1989), this program worked with a specific curriculum. The findings from Saxe et al. (2001) show greater gains for the students of teachers in the IMA group, as opposed to those in the first or second group. This indicates that the difference in student gains can be attributed to the IMA professional development program, since both the teachers in this group and in the second control group used reform curricula. However, this difference was only on the subset of items considered to test conceptual knowledge or understanding, and there was no significant difference between the IMA group and either the control group or the teacher support group on the items intended to test computational skill. Since the IMA program was restricted to teachers working with a reform curriculum, it is not possible to say if the effect of the professional development would have been measurable across different curricula.

As Saxe et al. implemented their professional development course with an orientation toward reform curriculum, Simon and Schifter's (1993) professional development course focused on guiding teachers to a constructivist view of mathematics. In their case, they gauged the student outcomes by comparing those taught by the teachers after participating in the program to those taught by the same teachers prior to the program. This study also used teacher-level outcomes designed to look at classroom practices, but of particular interest here are the student-level outcomes. These included student attitudes and beliefs, the type of math activity in the classroom, as reported by the teachers for their own classrooms, as well as student performance on standardized tests appropriate for the different grade levels. Although Simon and Schifter did find changes in the student beliefs and attitudes towards mathematics, and these changes include increased perceived importance of creativity and trying new things in math, they did not find changes in the standardized test scores before and after the teacher had participated in the program. However, in keeping with the orientation of the study, the fact that beliefs changed with no accompanying decrease in test scores may be acceptable. It would require further investigation to determine whether the changes in student beliefs were matched with changes in conceptual understanding or actual approach to mathematics problems, as these student gains in mathematical ability would not necessarily have been captured by the standardized tests used by the researchers in this study.

Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, and Perlwitz (1991) also included student-level outcomes in their study. They base their work on a particular theoretical orientation, considering both a constructivist perspective and the role of social interaction. Here, teachers participated in a professional development course and then received support during the school year. As with the work of Saxe et al. (2001), the students of the participant teachers received higher scores than those of their counterparts in the control group, but again, only on the portions of the test designed to assess conceptual knowledge as opposed to computational knowledge. The other factor to consider when looking at this work is that the participant teachers also implemented a curriculum designed by the research team, while the control teachers did not. Thus, while the effort as a whole can be examined, the specific effects of the professional development activity or of any particular resulting attribute of the teachers are obscured by the stark differences in classroom curriculum between the control and intervention groups.

Another project looking directly at the impact on students is the work done by Hill, Rowan, and Ball (2005). This group has produced a significant body of research and reflection on teacher education and teaching practice. In this particular work, the researchers report on the findings of a study of first and third grade students and their teachers across 115 elementary schools. While there is work from this research initiative that includes the evaluation of professional development courses for teachers (e.g., Hill & Ball, 2004), no intervention occurred or was measured in the particular case described here. This is in contrast to the studies listed above. Instead, the mathematical performance of eight students from each participating classroom was assessed at the beginning and end of an academic year. During that year, the teachers kept a log of measures relating to their teaching practices, such as content covered and the duration of mathematics lessons. Teachers also completed a survey, once during the year, which included educational background, certification information, experience, and other potentially relevant items. In addition, each teacher survey had five to twelve multiple-choice questions that were designed to assess the mathematics needed for teaching. The researchers provide a full description of the development of these items in a separate publication (Hill, Schilling, & Ball, 2004).

In this particular study, focusing on the student outcomes, Hill et al. (2005) included items that target common content knowledge of mathematics and specialized content knowledge that would be required by teachers, referred to together at that time as "content knowledge for teaching mathematics" (CKT-M; p. 387). Since that time, the research group has used the term *mathematical knowledge for teaching* (MKT), and the MKT designation will be used here for clarity. Hill et al. (2005) found that their measure for MKT was significantly correlated with student gains in both the first and third grades. They are careful to control for other variables, including socio-economic status, the time spent on mathematics in the classroom, and mathematics courses taken by the teacher. The diligence of the researchers lends credence to their analysis of the data, and they are justified in noting the correlation between the scores on their teacher assessment and the gains for the students, and in calling for courses that are focused on content of this type for teachers. Interestingly, they do offer a potential alternate explanation for the results. They suggest that the teachers who scored well on content knowledge for teaching mathematics might have some other, unknown, factor that truly impacts the student scores. They recommend an analysis of the practice of teachers that could potentially suggest factors which, while not necessarily independent of or dependent on mathematical knowledge for teaching, may be manifestations of some sort of teacher knowledge or practice that leads directly to student understanding.

Building on the framework and the findings in the Hill et al. (2005) study, an analysis from Hill, Kapitula, and Umland (2011) linked student-level outcomes to scored observations of classroom videos and teacher assessments. In this study, the student outcomes used were value-added scores, based on state standardized tests and calculated using several different models. Other quantitative data included the teacher's score on the math assessment described in the 2005 study above and coded videotaped lessons. The analysis of the videotaped lessons used a coding tool referred to as the "mathematical quality of instruction" (MQI) instrument, the development and validity of which is described in Hill et al. (2008b). In this particular analysis, two elements were compared to the value-added scores: a rater assessment of the mathematical quality of instruction for the lesson viewed (overall MQI), and a rater assessment of the teacher's mathematical knowledge for teaching (MKT). The latter rating could be influenced by any piece of evidence of a teacher's MKT, even if that knowledge was not manifested in instruction throughout the particular lesson being coded. That is, a teacher might briefly show evidence of a high-level of MKT and be assigned a high score even if the lesson itself was not mathematically rich and did not warrant a high score for the overall MQI. While both of these overall scores necessarily were affected by rater perception, the score for MKT could increase to account for small glimpses of a high level of teacher knowledge. In contrast, the score for overall MQI needed to consider the quality of the lesson as a whole.

In the analysis, the rater scoring of the teacher's MKT (based on the videotaped lessons) showed the strongest correlation with the student value-added scores, more so than the teacher's score on the written assessment or the rater scoring of the lessons' overall MQI (based on the videotaped lessons). It is important to note that Hill et al. (2011) then use this data to critique and compare various value-added score models and to consider the consequences of using value-added scores for rewarding or penalizing teachers. They do not, in the paper cited, explicitly consider the elements of quality instruction. However, their work is relevant here for two reasons. First, the teacher's MKT score based upon videotaped classroom observations proved even more strongly correlated with student-level outcomes than a teacher's score based on a written assessment, despite the earlier findings from Hill et al. (2005), described above, that showed correlations with the written assessment. While the specific student-level outcomes, using standardized test scores, may not fully capture student understanding, these findings still support the use of classroom observations as powerful indicators of the impact of instruction. Second, when critiquing and validating the connections between the quantitative variables, the research team turned to teacher interviews in order to develop case studies and shed light on the mechanisms through which the teachers contributed to gains in student scores.

This connects to the study presented here in that it supports the perspective that teacher interviews illuminate other aspects of the data. This implicitly supports the use of teacher interviews as a means to develop work on the qualities of instruction, as is proposed in the study described in this paper. In this paper, though, the student interviews provide yet another source of information and detail that could be used to shed light on quantitative data such as that described by Hill et al. (2011).

While some progress has been made in linking the outcomes, in terms of student performance, to factors connected to the teachers, no clear consensus exists on how this would translate into practice for teachers, or into preparation and professional development for teachers. One noticeable pattern in the studies above is that those that are able to directly measure student performance are quite large in scale, and are time- and fund-intensive projects (e.g., Carpenter et al., 1989; Hill et al., 2005; Hill et al., 2011). Smaller scale studies, including many that attempt to move directly to addressing the problem by working in courses with pre-service teachers, do not have measures that tell us about student performance or understanding (e.g., Hadfield, Littleton, Steiner, & Woods, 1998; Huinker & Madison, 1997; Lowery, 2002; Lubinski & Otto, 2004; Philipp, Thanheiser, & Clement, 2002; Tirosh, 2000). The latter studies may have insights into key elements of teaching, or may describe courses for teachers that are highly beneficial to students in the long term, but we cannot assess at present what the specific and direct benefits for students are.

One other issue that arises when considering the task of assessing mathematical instruction is that mathematics itself is infinite and complex. Just as assessing instruction in chemistry might not fully elucidate how to teach physics, excellent instruction in arithmetic does not necessarily imply that the same type of instruction would be ideal for teaching geometry. The instruction is not just content-specific at the level of subjects in school, but actually on concepts within that. Many of the existing studies have looked at number and operations (e.g., Carpenter et al., 1989; Cobb et al., 1991; Hill et al., 2005), which is not surprising given that these are foundations for later mathematical activity in and out of school and generally comprise a student's first exposure to mathematics. Other areas have not been addressed yet, with the exceptions of some work in algebra (Hill et al., 2005) and in fractions (Saxe et al., 2001). Thus, the consideration of combinatorics in the work described in this paper is unique.

In addition, if we wish to consider student-level outcomes, another aspect of the link between student and teacher is the degree to which depth and mode of understanding are connected between both. To date, studies have mostly depended on test scores or written assessments, which are partial measures of performance, to determine the impact of instruction on students. This is not incorrect or unreasonable: it is consistent with the emphasis on standardized testing in the schools and it is the most realistic plan for looking at large numbers of teachers and students. Nevertheless, it does not generally allow us to see all relevant aspects of performance and understanding. Tests may certainly seek to draw out and measure understanding on a topic, rather than necessarily focus on procedural knowledge. However, no written test can illuminate the nuances in student understanding in the way that an interview can (Piaget, 1976/1926). The use of student interviews proposed in this study is not only to examine the student thought, but also to connect it to the classroom instruction provided by the teacher. In this way, we can consider not just the impact of instruction on student performance, but also on the details of student understanding. This undertaking can also help us develop claims related to the ways in which instruction and student understanding are related.

Existing Work on Learning and Teaching Combinatorics

As discussed above, there is room for exploration of teacher knowledge in different mathematical topics. One of these untouched areas is combinatorics. For instance, an examination of the types of problems given at the middle school level yields simple combinations and permutations and simple discrete probability (see Connected Mathematics 2, Grades 6, 7, 8; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). Combinatorics, including permutations as mentioned above, deals with the ordering of fixed numbers of items.

The term *permutations*, in mathematics, refers to an arrangement of some number of objects, where the order in which the objects are arranged matters. That is, the same objects presented in a different order would be a different permutation of those objects. Within permutations, there are two initial cases. First, the case with n objects, where all n must be arranged. For example, if we have three different letters, how many ways can we arrange all three of them? The number of permutations for a set of n objects where all n are arranged is $n! =$

$n*(n-1)*(n-2)*...*1$. Second, the case with n objects where some number less than n must be arranged. The number of permutations for r of the objects from a set of n objects (for $r \leq n$) is denoted $P(n,r) = n! / (n-r)!$. (We can see that the first case is really a simplification of the second, since if $n = r$, $P(n,r) = n! / (n-r)! = n! / 0! = n! / 1 = n!$.) For our example from above, given all 26 letters in the English language alphabet, asking how many three-letter passwords (with no repeating letters) can be formed, the number of permutations is $P(26,3) = 26! / (26-3)! = 26! / 23! = 26*25*24$.

Several established representations of combinatorics exist, and these are used both to find answers and to justify them. One possibility is a list of all the outcomes. This brute force method is efficient to use for small sets. Tree diagrams are also commonly used, particularly for permutations. The slot method is another option, and, of course, there are established mathematical formulae for problems of this type. Representations may also be invented.

In considering the developmental aspects of understanding combinatorics, which includes permutations, Piaget and Inhelder (1975) suggest that children and adolescents' understanding progresses through stages that correlate with other, more general developmental stages. Specifically, they suggest that young children do not appreciate the notion of chance, and instead seek causal explanations for events, both in their every day experiences and in staged scenarios of dice games and coin flips. It is only as they reach the formal operations stage (12 to 13 years of age) that they are able to consider or enumerate a set of all possible outcomes and the likelihood of these various outcomes. For instance, in creating permutations of small sets of distinct objects, Piaget and Inhelder found that pre-operational children (before seven years of age) have no system for creating different arrangements or for considering how many arrangements are possible. As they grow older and reach the concrete operations stage (between ages seven and 11) they are able to create the different permutations more readily, but still do not use a consistent system to do so and often miss items or create the same item more than once. It was only in the third stage that students used a consistent system to create permutations or could make a conjecture on how many permutations were possible.

Schliemann and Acioly (1989) interviewed bookies with different levels of formal schooling, including those with no formal schooling at all, who were accustomed to taking bets that involved the determination of the number of permutations of a fixed set of digits. While the bookies used tables listing the number of permutations for different scenarios during their work, the researchers interviewed them about permutations of colored chips and alphabetic characters, finding that some of the subjects connected this activity to the way that numeric digits are permuted in their work, while others did not make this connection and even rejected it when it was suggested. Relating the responses to the stages suggested by Piaget and Inhelder (1975) described above, they found that the level of schooling was positively and significantly related to the stage suggested by the response. In addition, while none of the bookies had formal instruction on probability, those with some formal schooling were more able to make logical probabilistic arguments. This work confirms the types of reasoning about permutations seen by Piaget and Inhelder (1975). However, the progression through stages is shown to depend on factors other than development, such as schooling, work, and cultural factors. Even without the added element of the bookies' work, an individual's level of understanding of permutations may vary across situations and contexts.

Although their analysis is focused on children's justifications and proofs, rather than the mathematics of permutations, Maher and Martino (1996) show us young children engaged in simple problems of permutations. As part of a longitudinal study, students in fourth grade were asked to build all possible towers of blocks, given the height of the tower and two different colors of cubes to use in construction. Consistent with Piaget and Inhelder's (1975) theory regarding children in the concrete operations stage (between ages seven and 11), students often did not have a foolproof system for organizing the possible permutations. However, with Maher and Martino's emphasis on students proving their answers to an interlocutor, over time some students felt the need to create organizational schemes. In doing so, students created either *patterns* of the colored towers, or *categories* of the towers. Patterns were organized visually and often led the students to count the same permutation more than once. Categories, however, enabled students to prove that they had all possible permutations, as they were able to generate all the possibilities within a category. For example, one category could be thought of as "towers three cubes high with exactly two blue cubes", and students generated all three possibilities within this category. Aside from this increased organization in thinking about permutations, students also generated the beginnings of a recursive argument about the number of possible towers as a function of tower height, recognizing that the number doubled when the height was increased by one block. Their explanation of this suggests their reasoning is close to the classic permutation representation of a tree diagram, as they consider each existing tower with a height of $n-1$ blocks to branch into two possibilities for the nth block. This example shows the richness and variety in combinatorial techniques, even for very simple problems.

The above studies speak to the challenge in learning and explaining combinatorics. Other literature has addressed common errors in the field. Some, such as Watson (1996), have looked at specific mathematical errors, such as double counting of possible cases. One author, Szydlik (2000), suggested that students should "discover" permutations through the use of problems that require a permutation in order to answer. Specifically, Szydlik describes a problem in which there are four people to be arranged in a straight line for a photograph, asking for the number of possible arrangements. She observed that some students modeled each case using a tree diagram, where others created generalized expressions. The results are described as positive, although few details are given; the author's intent is to present this approach to practitioners rather than to justify it through specific outcomes.

In summary, permutations often receive short shrift in educational treatment and may be peripheral to other mathematics taught within the same school year. There is also little connection between the literature on instructional outcomes, as discussed in the previous section, and the literature on learning and teaching permutations.

Methodology

This study presents one case of a teacher's instruction on a particular mathematical topic, permutations. The data sources for this study were an interview with the teacher, observations from the teacher's instruction, and individual interviews with seven of the teacher's students. During interviews with students, participants were asked to work on permutation problems, and also to reflect upon them, explain their solutions, and evaluate alternative solutions.

Participants

The researcher had the opportunity to meet several teachers while assisting with a professional development course; all teachers were invited to participate in this study, though participation was independent of taking part in the course. Nine teachers consented to participate; of these, only two teachers both were willing to allow student participants and were planning to teach a unit that included lessons on permutations during the following school year. Once these two teachers were identified, the administration of each of their schools was contacted with a letter explaining the project and asking for their participation. Both teachers were interviewed for the larger project, but one teacher, Shana, was selected for this case study because of the use, in her classroom instruction, of a permutation problem similar to one used in the interview.

Shana had completed a bachelor's degree in economics, but with an additional special interdisciplinary major in mathematics and education. Immediately following her undergraduate education, she enrolled in and completed a year-long program to earn a master's degree in education. As part of this program, she was an intern to a practicing mentor teacher for the duration of the school year, participating in the classroom four days per week. After completing this degree and internship, she began as an independent classroom teacher the following school year.

At the time of the study, Shana was teaching at a secondary school in a large urban school district in a large Northeastern state. During the school year in which the study took place, Shana was teaching two sections of Algebra 1, in which permutations were part of the curricular plan. The study took place at the end of her second full school year of independent classroom teaching, not including her internship.

After the administration of the school had consented to participate, the families of all of Shana's students received a consent letter and explanation of the study. The students were also asked for their assent. Ultimately, seven of Shana's students (out of approximately 30) were available and were interviewed; three students were from one section of Algebra 1 and four students were from the other section.

Measures and Data Collection

Data was collected through a teacher interview, classroom observations, and individual student interviews. A summary of the data types is shown in Table 6, followed by a detailed description of each source.

Table 6. *Data collected.*

Measure	Time of measurement	Mode
Teacher interview	April 2010	Videotaped individual interview; subsequently logged and excerpts transcribed
Classroom observations	May 2010	Written observation notes; collection of handouts and instructional materials
Student interviews	June 2010	Videotaped individual interview; subsequently logged and excerpts transcribed

The first source of data was the teacher interview with Shana. The interview was conducted in April 2010 and was videotaped. While interviews were flexible and open-ended, they had a goal of discussing mathematical situations and material that are germane to combinatorics, including questions on permutations. To do this, Shana was given problems to solve. After solving each problem, she was asked for an explanation of her work. She was then asked for a different way to solve the same problem and a different explanation. She was also asked about what she believed her students would do when working on the same problem. This process repeated for each mathematical problem. The case study presented here focuses on her responses to and about the second interview question. In this question, Shana was asked the following: “If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?” The question was presented in writing, with space below for her to write her work.

The second source of data was classroom observations. In the secondary school curriculum in the school district in which this study took place, the topic of combinatorics (including permutations) is introduced in Algebra 1 at the end of the school year, in May or June. Shana was teaching two sections of Algebra 1, which we will call Section A and Section B, each attended by students 14 to 16 years old (i.e., freshmen to juniors in high school), at the time of data collection. Shana taught related topics during five school days; classroom observations were carried out on all five days. However, problems on permutations were only explicitly considered on one day, the second day of the unit. For both sections, the instruction occurred on the same day and both sections were observed. In addition, both sections used the same printed handouts. This day, in late May, is the focus of analysis for this case study.

During classroom observations, I took detailed notes, recording on paper as much as possible of the classroom activity. Additionally, I made note of all mathematical problems that were addressed during the class and collected blank versions of all handouts and paper assignments used. In addition, I used a checklist for quickly noting topics, concepts, and representations addressed during the class. A new checklist was used for each 15-minute interval (see Appendix A). The checklist provided a structure to allow quick notation of the type of mathematical problem, the type of representations being used, and the type of classroom activity, such as teacher-led discussion, group work, or independent work. This checklist was based on the classroom video coding categories described by Hill et al. (2008b) as the "mathematical quality of instruction" (MQI) measure. Note that because the study here did not use video of the classroom lessons, the actual MQI instrument could not be properly applied; its design is based on being able to review a single lesson multiple times. Therefore, while it formed the theoretical basis for the classroom observation checklist, no MQI score is assigned to these classroom observations and no comparisons can be made to other classroom lessons that were coded using the MQI metric.

The third source of data is student interviews. All consenting and available students were interviewed individually. Seven students were interviewed, with three students from Section A and four students from Section B. Each interview was videotaped. While interviews were flexible and open-ended, they had a goal of discussing problems about combinatorics. To do this, the students were given the same problems to solve that had been used in the teacher interview. The exception to this was the removal of problems that utilized concepts that had not been addressed in Shana's class. This was possible since the classroom observations were already complete at the time of the student interviews. No new questions were added to the student interviews. After solving each problem, students were asked to explain what they had just done and tell me how they knew to do each step. They were then asked for a different way to solve the same problem and a different explanation. This process repeated for each mathematical problem. I attempted to elicit a full explanation of both correct and incorrect answers and strategies.

Results

The Teacher Interview

The teacher interview took place in April 2010, prior to the classroom observations and student interviews described here. However, Shana had taught this material previously. As mentioned above, at the time of the interview, she was teaching at a public secondary school in a large urban district. She had been teaching independently for almost two full school years, but had already taught this material in the prior year to other Algebra 1 classes. In addition, she had taught permutations earlier in the school year to a different group of students, in a class combining algebra and geometry that was classified as advanced. As a result, in her interview she talked about the content in relation to her previous experience teaching it.

The first thing Shana did in the interview was to solve the problem mentioned above. She immediately drew slots and labeled them with P (for president), VP (for vice-president), T (for treasurer) beneath them, as shown in *Figure 1*. She filled in the slots with 10 x 9 x 8, and mentally multiplied and wrote that this was equal to “720 ways”. She then immediately wrote the formula for a permutation of three of ten objects, ${}_{10}P_3$, setting this equal to 10!/7! and writing that this was equal to 720 ways. Once she had done this, the interviewer asked her if she had ever taught this kind of case, of permuting a subset of objects but not the full set (that is, in this problem only three out of ten possible people are chosen to hold an office in any particular arrangement). Shana said that she had, and that this case had resulted in the same types of difficulties as problems with permutations of a full set. She had described some of these difficulties earlier in the interview. This description was not about the president/vice-president/treasurer interview problem, but it is relevant to include here since she referred to it when talking about the problem in question.

$$\frac{10}{P} \cdot \frac{9}{VP} \cdot \frac{8}{T} = 720 \text{ ways}$$

$${}_{10}P_3 = \frac{10!}{7!} = 720 \text{ ways}$$

Figure 1. Shana's interview work.

Earlier in the interview, she said that she would normally do a permutation problem, herself, using slots or the formula. She also said that there were many ways to do it, but she would teach it to students using the slot method, and that she might teach the formula but she preferred to emphasize the slots because they had more meaning. When asked about ways other than the slot or formula method, Shana suggested listing out the options or making a tree diagram. She then corrected her initial statement to say that she actually would do a list or tree diagram first with her students, and that many of them had seen tree diagrams in middle school. She said that from the tree diagram they would determine the multiplication principle, and use that to justify the slots. The interviewer asked if the students had any areas of confusion with the permutations, and Shana responded that the hardest part was picking the "events". She suggested two problems related to this. The first was that when the first event was selecting the first item for a permutation, it was often at odds with the everyday meaning of events. She said, "[…] but it was like, 'this is the first event and we're choosing a letter!' That doesn't sound very exciting. When the events are picking a mayor and then picking a treasurer, that makes more sense as an event." The second problem she mentioned was deciding how many events there were and how to label the slots with a name for each event.

After Shana referenced this information about student difficulties that she had talked about earlier in the interview, the interviewer asked her if any of the students wanted to know why they needed to multiply. Shana said that they did ask this question, but that she would reference the tree diagram and that most students could make the leap from the tree diagram to multiplying the numbers in the slots. The interviewer then asked if students ever constructed the slots correctly but didn't know what to put in them. Shana said that this was not the most common problem, but that sometimes students would not know whether to decrease the numbers (10, 9, 8, as in this problem) versus using the same number of options for each slot (10, 10, 10, would be this [incorrect] case here). To address this, Shana said that they would do examples with passwords where you could use the same character more than once and where you could not.

The interview shows us that Shana was able to solve the problem easily and justify her answer. In addition, she was able to discuss different ways to solve the problem, as well as talk about which ways she would use in instruction. She also considered what her students might do, both in response to questions from the interviewer and in spontaneously mentioning common errors. Shana's answers suggest that she is a thoughtful practitioner with different types of specialized mathematical knowledge for teaching, including knowing about how her students work with the content and considering how to sequence the content for instruction (Ball, Thames, & Phelps, 2008).

The Classroom Lessons

As mentioned above, for both sections A and B of Shana's Algebra 1 class, one class period was spent specifically on permutations. These both occurred on the same day and both were observed. In addition, both classes used the same printed handouts. In this section of the paper, I will describe the instruction on permutations in each section of Shana's class.

Section A. Section A took place at 9 AM on an ordinary school day in May 2010. The class began with a review of the material from the previous day, which was on the multiplication principle. Students answered a problem asking: "if you toss a coin and roll a die, how many different outcomes are there?" A student explained her answer (12 outcomes) using a list, and Shana demonstrated how to find the answer using a tree diagram.

The class then moved on to the handout and read the following paragraph: "Yesterday we counted the outcomes for multiple 'events'. Today, we are going to count how many different ways we can arrange a number of things in order. This will look like doing the same event several times over." Below this description, the first example was listed: "Ex. We are choosing a class 'spirit leader', a class 'treat provider', and a class 'clown' from our class of ___ people. How many different ways can we choose these roles?" Shana then counted the number of students in the class and said there would be 12 people. She also clarified to the students that a person would not hold more than one role. On the overhead projector, Shana then displayed three "slots", with each one marked underneath with an abbreviation for the role: SL, TP, or CC. Shana asked how many people could fill the role of spirit leader (SL) and students shouted out that there could be 12. Shana wrote the 12 in the slot for spirit leader and said that next they would choose a treat provider. She asked how many people could be treat provider (TP), and some students yelled out the number 12 while others yelled out the number 11. Shana responded, "Renecia [the name of a student] is busy being spirit leader." Students responded by shouting out 11, and Shana wrote the 11 into the slot marked TP. Finally, Shana asked how many people could be class clown, and students responded with 10.

Shana then suggested she could make a "crazy tree" and began making a tree diagram with 12 branches from the starting node and the initials of the students in her class at the end of each branch, as shown in *Figure 2*. After she began to make 11 more branches to show the 11 choices for treat provider, the students began to call out for her to stop drawing the tree and just multiply the numbers instead. Shana multiplied the numbers to get an answer of 1210 outcomes, surprising some students with the magnitude of the result and prompting them to ask “are you serious?”

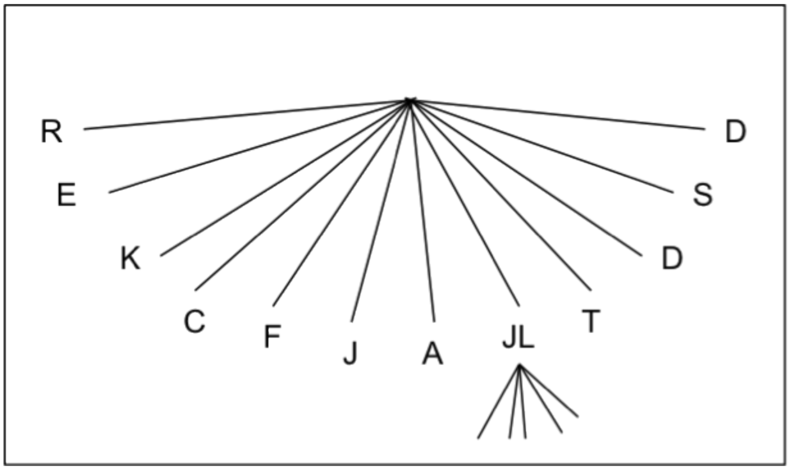


Figure 2. Shana's tree diagram with 12 students.

The class then moved on to another example: “Tamar is taking a photograph of 6 of her friends. How many different ways could they line up?” Shana asked six students to come to the front of the room and stand in a line. She asked them to then switch their order and asked the rest of the class if it looked different; they agreed that it did. The students did several repetitions of standing in different orders. After the students were seated again, Shana asked the class how many places there were in the line and they replied that there were six. When Shana then asked what we would choose first, the students suggested meaningful ordering, such as choosing the tallest person first. To clarify, Shana drew six slots on the overhead, labeling them underneath as “1st”, “2nd”, and so on through “6th”. A student then postulated that the answer would be 36, saying that there are six slots times six students. Shana asked the students to consider just how many possibilities there would be for the first position and a student answered that there would be 6, which she wrote into the slot. When Shana then asked about the number of possibilities for the second position, a student responded that there would only be 5 because there was already somebody standing in the first spot, and that then there would be 4, 3, 2, and 1 possibilities. Shana recorded this information in the slots and then wrote "720 ways / outcomes".

Students then began to work independently on another problem: "The soccer team is awarding a Most Valuable Player and a Most Improved prize. If there are 19 girls on the team, how many different ways could the players be selected (the same girl cannot get both prizes)?" Shana began to circulate to help the students, asking them to look at the examples to see how many slots there should be. After speaking with a few students, Shana then addressed the whole class and asked them to be sure they were getting the "events" clear and to use the examples they had already done to decide which one was like this new problem. Some students drew out 19 slots on their paper. To one of these students, Shana pointed to the prior example of the photograph, saying that in that case they had six people and they were taking all of them. She contrasted that with the new problem, asking if they were taking all 19 soccer players; the student responded that they were not. Shana then asked her to see which other problem the soccer player problem would be like. Shana then asked another student with an incorrect number of slots how many awards they would be giving out, to which the student responded "19 of them!" Shana assisted the student by asking her what the prizes would be (Most Valuable Player and Most Improved) and helping her to make one slot for each prize. Shana then moved to another student who had written "19 x 2 = 38", again guiding this student to the previous example of choosing 3 of 12 students to fill different roles.

Meanwhile, several students had (correctly) written 19 x 18 on their papers, with the slots labeled for the type of prize. Some of these students then moved on to other examples on the handout they had been given. One student working on a later example ("There are five finalists in the Mr. School pageant. In how many ways may the judges assign first, second, and third place?") told Shana that at first she thought it would be 1 x 2 x 3, but then she realized (correctly) that it should have been 5 x 4 x 3 because at first there were five choices, followed by four choices and then three choices. As the class was drawing to a close, Shana asked the students to complete the handout for homework. One student asked about the word "permutation" from the packet. Shana explained that "permutation" was what they had done that day, when they had one group of things and they were choosing several things from it.

Section B. Section B, Shana's other class, took place later that same day. However, there were some differences in Shana's instructional choices. The class began in the same way, with the introductory problem related to the material from the previous day ("if you toss a coin and roll a die, how many different outcomes are there?"). In Section A, a student presented a list of outcomes and Shana presented a tree diagram. This time, in Section B, Shana asked for one student to make a tree on the board and another to make a list. For these first two students, the one making the list used dot icons to show the number on the die. The student making the tree diagram drew it vertically, from the top down. At this point, a third student said that they found the dots confusing and Shana suggested they could use the digits 1 through 6 instead of dots. A fourth student wanted to draw her version of the tree diagram and represented it horizontally, from left to right. Other students agreed with the responses and with the answer of 12 total outcomes.

After this activity, students moved to the handout and read the introductory sentence, as they did in Section A ("Yesterday we counted the outcomes for multiple 'events'. Today, we are going to count how many different ways we can arrange a number of things in order. This will look like doing the same event several times over"). At this point, Shana asked for five students to volunteer to have their names listed for the problem on choosing class roles. This differed from her instruction with Section A because she chose to use a smaller number of students (five, a subset of the class) instead of the total number of students, which had been 12 students present in Section A and a similar number in Section B. In addition, she also wrote the names of the five students on the board, a step she had not done in Section A. After writing the names of the students on the board, which we will refer to here by their first initials, S, A, Y, R, and L, Shana explained that they were pretending to have five people running for the positions. She asked the class how many things they would be choosing, and the students responded that there would be three. Shana said that they would do the problem a familiar way and then a new way. She then drew a tree diagram on the board, making five branches at the initial level as shown in *Figure 3*. She then asked the students how many people would be left for the other roles and students answered "4". Shana then asked the class what the options could be if the student A was already the spirit leader (SL). She then drew four branches from A and labeled them S, Y, R, and L, saying that any of these students could be the treat provider (TP). Shana then moved to a different branch and asked what the next level could be for the class clown if S was the spirit leader and A was the treat provider. Students responded that Y, L, or R could be the class clown. Shana asked for the total number of outcomes and students responded that there would be 60, apparently able to calculate this in spite of the fact that the full tree diagram had not been constructed.

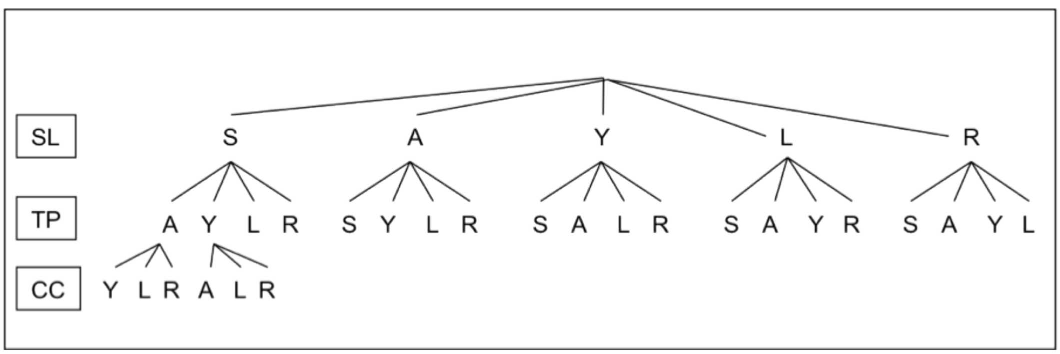


Figure 3. Shana's tree diagram with five students.

At this point, Shana said that she would demonstrate the shortcut so they would not need to do a "crazy" tree. She drew a slot and asked students what would be the first thing they would choose and how many options there would be; students answered that they would choose the spirit leader and that there would be 5 options. Shana entered this information as the label and entry in the first slot. She repeated the questions for the next choices and students responded correctly; Shana entered 4 and then 3 in the slots, along with the corresponding labels.

Shana then asked what was different about the problems they were doing today, but in the absence of student answers she told them that the choices were all from the same group of five people. At this point, the student, A, who had been named on the tree diagram, asked a question: “So, I can’t be class clown?” In this question, he was referring to the branch of the tree diagram where he had already been selected as treat provider and only the students Y, L, or R were available to be the class clown. Shana explained that it was only on that particular branch, to which Student A responded with a loud, “OH!”

Shana then moved on to the next example of arranging six people for a photograph, as she had done in Section A. This time, instead of asking six students to come to the front of the room, Shana drew six stick figure people on the board. A student immediately said that he had the answer and that it was 36, the same incorrect answer that had been suggested in Section A. Shana wrote down the suggested answer and asked the class to first tell her if the picture would look different if the people moved around, to which the students responded that it would. Shana drew six slots on the overhead, labeling them underneath as “1st”, “2nd”, and so on through “6th”, just as she had done in Section A. This time, when Shana asked how many possibilities there would be for the first slot, several students responded that it would be five, perhaps thinking of the previous example, since when they were asked why they responded that there were five students from which to choose. When Shana clarified that there were six people to be in the photograph and they hadn’t put anybody in yet, students changed their answer to six and then a girl offered, “then 5, 4, 3, 2, 1”. Shana did the multiplication to give an answer of 720 outcomes.

At this point, Shana asked the students about the differences between the two examples, a question that she did not pose at this point in the lesson with Section A. The ideas that students then shared with the class were that in the first example they chose a group of people to run for the class roles, and that one example had three spots and the other had six spots. Shana then highlighted the difference she wanted to emphasize, asking students how many people in the first example wanted to have a role (5) versus how many would actually have a role (3). She asked how many people wanted to be in the photograph (6) versus how many actually were (all 6).

The class then began to work on the same problem in the handout that was used in Section A (“The soccer team is awarding a Most Valuable Player and a Most Improved prize. If there are 19 girls on the team, how many different ways could the players be selected [the same girl cannot get both prizes]?”). In this section as well, many students began to draw 19 slots on their papers. Seeing this, Shana circulated more briefly than she had in Section A and then brought the class back together and asked them to name the events. One student immediately volunteered that there should be two slots, one labeled “MVP” for most valuable player and the other labeled “MI” for most improved. A second student suggested that they should put 19 in both slots; other students disagreed. Students volunteered both the ideas 19 x 18 and 19 x 2. A student then justified his (correct) answer of 19 x 18 by saying that there would only be 18 people left who could receive the second award. Shana then asked the whole class why they would not continue to multiply by 17, 16, and so on, to which a student quickly responded that there are only two awards. The class then moved on to work independently for the remainder of the class, with Shana circulating to check work. This went on only briefly until the end of the class.

Changes in instruction. Shana's instruction shifted in a few notable ways from Section A to Section B. First, in Section A she undertook the first permutation problem ("We are choosing a class 'spirit leader', a class 'treat provider', and a class 'clown' from our class of ___ people. How many different ways can we choose these roles?") using the full class of 12 students. In Section B, she decided to take a subset of only five students, and she wrote these student names on the board. While this seems trivial, it set her up for a shift in the way she solved the problem. In Section A, she presented the slots first and asked students how many possibilities there would be in each case. She then mentioned using a tree diagram and demonstrated it only partially, not completing any of the possibilities. Conversely, in Section B she presented the tree diagram, a familiar notation, first, and then introduced the use of labeled slots. While she still did not complete the tree diagram, one branch was carried through to the third and last student role.

A second change was the way in which Shana responded to the problem students were working on independently ("The soccer team is awarding a Most Valuable Player and a Most Improved prize. If there are 19 girls on the team, how many different ways could the players be selected [the same girl cannot get both prizes]?”). In Section A, Shana circulated and responded to student difficulties such as trying to use 19 slots. However, in Section B, Shana circulated for a much shorter time and then brought the class back together. In the ensuing discussion, she was able to address three types of student difficulties that she had observed: (1) putting 19 in both slots; (2) multiplying 19 by 2; (3) continuing to use more slots and multiply by 17, 16, 15, and so on. While the difficulties that had manifested themselves in Section B were not different from those in Section A, Shana’s return to whole class discussion allowed more students to hear her responses and those of their peers.

The Student Interviews

Note that the student interviews used the same question as had been used in the teacher interview (“If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?”). This problem is also very similar to one that had been used in the classroom instruction (“We are choosing a class ‘spirit leader’, a class ‘treat provider’, and a class ‘clown’ from our class of ___ people. How many different ways can we choose these roles?”). During the student interviews, 2 of 7 students were able to find the correct answer to the question. A summary of notable points about the responses is shown in Table 7; these points are elaborated below.

Table 7. *Student interview response elements.*						
Name	Section	Correct	Labeled slots	Tree mentioned	List mentioned	Explained why choices are decreasing
Donald	A	Yes	-	-	Yes	Yes
José	A	No	-	-	-	-
Jussara	A	No	-	-	Yes	-
Lucy	B	No	-	Yes	-	-
Matthew	B	Yes	Yes	Yes	-	Yes
Rose	B	No	-	-	-	Yes
Sandy	B	No	Yes	-	-	Yes

In the next few paragraphs, we will look at four of the elements that were prominent in the student interview responses: the use of labeled slots, mention of tree diagrams, mention of listing, and explanations of why the number of possibilities decreases for each office.

The first element is that two students, both from Section B, used labeled slots, as shown in *Figure 4*. Matthew's work is shown on the left, and Sandy's work on the right. While Matthew solved the problem correctly and Sandy did not, both students used the strategy of drawing slots and then labeling them with president, vice-president, and treasurer in order to keep track of the office they were assigning. This strategy was identical to one that Shana used in the classroom: in both Section A and Section B, she drew slots and labeled them for the example of choosing three student roles and for the example of arranging six people for a picture. In Section B, Shana modeled this one additional time, during the class discussion of awarding two prizes to two of the 19 members of a soccer team. The students in Section A completed this problem as well, but it was not discussed as a class.

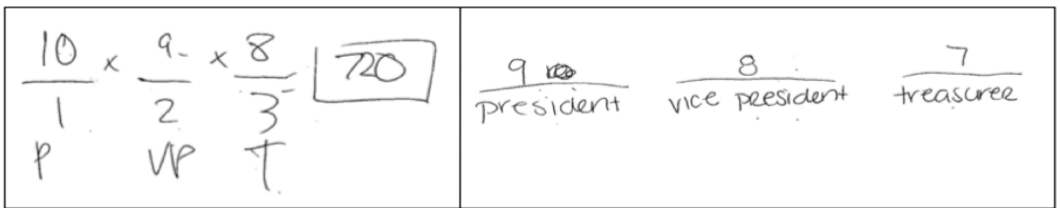


Figure 4. Two students used labeled slots.

The second element is that two students, again both from Section B, said that we would be able to solve the problem by making a tree diagram. This does not imply that the students from Section B would have made an accurate tree diagram, and in this interview neither one did, as shown in *Figure 5* with Matthew's work on the left and Lucy's work on the right. Matthew's diagram, while he called it a "tree", is something a bit different, and Lucy's diagram looks like a canonical tree diagram but does not have the correct number of branches at any of the points. However, it does suggest that, at least for these two students, a tree diagram is a viable method for finding a solution to a problem on permutations. In the classroom, Shana used a tree diagram in both sections when looking at the problem about choosing three student roles. In Section A, she first presented labeled slots and asked students how many possibilities there would be in each case. She then mentioned using a tree diagram and began to demonstrate it. Conversely, in Section B she presented the tree diagram, a familiar notation, first, and then introduced the use of labeled slots. It is also interesting to note Lucy's use of a little person at the start of the tree diagram. Shana had used this sort of figure when drawing tree diagrams in class and Lucy had apparently taken this as a feature of the representation.

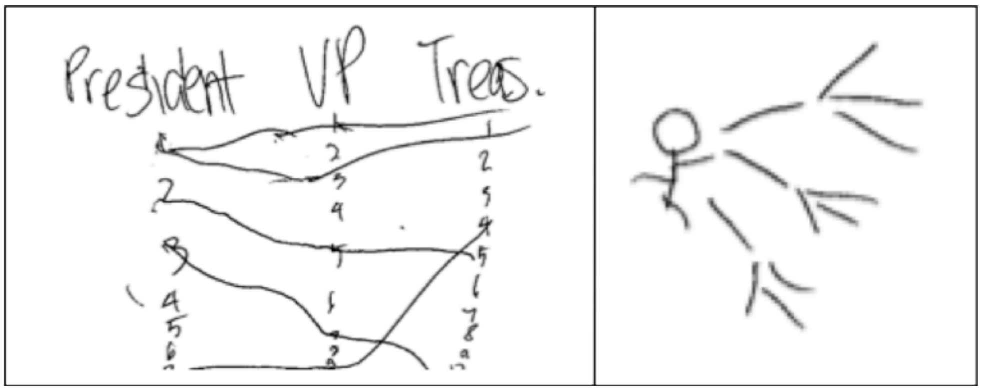


Figure 5. Two students mentioned tree diagrams.

The third element is that two students mentioned listing the possibilities; however, this time both students were in Section A. Neither student actually listed the options in order to find the number of possibilities (720), but instead mentioned that one could make a list. During Donald's interview, he explained that a list would be possible, but onerous. When asked if there would be a different way to do the problem, other than multiplying 10x9x8, he answered, "There is, but you'd need to have someone who isn't fazed by something boring. […] They would come up with this list and they would say, well, each person can be the president and they'd start writing out the list. […] They'd probably come up with the same number as me unless they made a mistake. Which they probably will since it's a long, tedious chore." In this case of the listing, there were fewer instances during the classroom observation and none specifically of permutations. In both sections, a student drew a list of possibilities on the board for the multiplication principle problem at the beginning of class (rolling one die and flipping one coin). Neither section used lists during whole class activities when discussing permutation problems.

The fourth element was the presence or absence of student explanations as to why the number of possibilities for each office decreased. One student from Section A and three students from Section B explained this. It is interesting to note that two students, Rose and Sandy, were able to explain this even though they did not solve the problem correctly. Sandy, whose work is shown on the right in *Figure 4*, did not use 10 as the first number in the slots. Rose, who did not write any numbers or slots on her paper, still gave an explanation. When asked how many people she could choose from for president, she answered “10”, and when then asked how many she could choose from for vice-president, she answered “9”. When the interviewer asked her how come she said nine, she said, “because one’s already vice-president”. (Note: it is likely that she meant “one’s already president” rather than “one’s already vice-president”.) She repeated this logic for the next office, saying “8” and explaining, “because you excluded three already”. (Note that she either misspoke, meaning we excluded only two already, or possibly she was considering the three to mean the third office we were already addressing.) Sandy gave a similar explanation: “So there’s ten students, and only one can be president so there’s nine more students left. From the nine that’s left, only one out of that nine can be vice-president so that would be eight. Out of the eight, only one can be treasurer so that would be seven.” Neither of these students found the answer of 720 possibilities, but both students explained the logic of fewer possibilities once each role had been assigned. While the awareness did not enable the students to solve the problem completely, they were aware of this aspect of permutations of objects from within a set. In the classroom, Shana addressed this issue during both sections, during the problem about choosing three student roles and the problem about arranging students for a photograph. When soliciting student suggestions about the numbers to multiply, however, it was only in Section A where students offered the same number for adjacent slots in the student roles problem (12 and 12) and Shana needed to prompt the class that one student was already occupied in the first assigned role. In Section B, students offered the correct numbers initially. The only other difference occurred when Shana discussed the problem about two awards for 19 soccer players. In Section B, a student justified his answer of 19x18 by saying that there would be only 18 people left eligible for the second award after the first was assigned. The whole class discussion of this problem did not occur in Section A.

Discussion

The discussion of this case is divided into two sections to address the two research questions: (1) what do student interviews tell us about the qualities of the mathematics instruction; and (2) how do student interviews enrich data from classroom observations and teacher interviews?. First, we will use the student interviews, supplemented by the teacher interview and classroom observations, to try to determine what we learned about the qualities of the mathematics instruction in this particular case. Second, we will use this first process and analysis to look at how the student interviews enriched the data, in order to justify this type of methodology in general.

What Did We Learn in This Case?

This case discussion focuses on the information contributed by student interviews that would not otherwise be available to us. It is first worth noting that the teacher interview and the classroom observations were worthy sources of data about Shana and her instruction. As described above, Shana's instruction changed from class to class, even on the same day. This suggests that when considering instruction, it is worthwhile to consider the shifts that occur between classes and to try to determine when and why they occur. Shana's interview and the classroom observations show that she is a thoughtful teacher with an understanding of both mathematics and her students. She anticipated that students would have difficulties knowing how many slots to use and potentially not decrementing the numbers in the successive slots. Both of these problems were manifested in the classroom and in the interview.

She did not bring up in her interview the possibility of students trying to multiply the number of objects in the set by the number being permuted, as they did when multiplying 19 x 2 in class or 10 x 3 as some students did during the interview. This suggests a next step that could inform our case even more: another interview with the teacher following the classroom instruction and the student interviews. During a later interview of this type, we could ask the teacher to review the student interview materials. This would afford them the opportunity to notice and comment on student errors that they did not anticipate previously, and also to connect the student interview responses to the responses that they encountered in the classroom. In Shana's particular case, a follow up interview would allow for discussion of and connection between the 19 x 2 response that happened in the classroom and the 10 x 3 response that happened during the student interviews. We could also solicit teacher reflections to help us understand how and why she decided to make her instructional changes between the two sections.

The student interviews, however, provide additional insights that would not have been available strictly through classroom observations or written student assessments. Most students (five of seven) were not able to solve the interview problem. Only one of three students from Section A and one of four students from Section B found the correct answer. If we were to test large numbers of students with this question, marked as right or wrong, we might or might not find differences between them that could be attributed to instruction. Certainly, looking at the split between right and wrong answers in this study tells us only that few students were able to find the correct answer.

This only shows a part of the picture, though. There are elements of student understanding, not just student responses, that should be considered in connection with the teacher's instruction. For example, two students suggested a tree diagram. While we cannot make quantitative claims of causation or even correlation, it is notable that both of these students were in the section, Section B, where Shana used the tree diagram as the initial method to solve the first example of a permutation problem. It is reasonable to suggest that these students may have taken from Shana's instruction the understanding of a tree diagram as a means to solving permutation problems.

Consider the two students who did not find the correct answer to the problem, but still explained why one would multiply by decreasing integers. They understood something about permutations and how they worked. They did not execute a procedure for finding the solution, nor did they develop a full enough conceptual understanding to allow them to work out the answer. However, Shana's instruction was not meaningless. These student interviews reveal partial understandings and illuminate the effects of the instruction, which once again are not captured in right/wrong categorizations.

What Can We Learn Methodologically?

The example of this case demonstrates that we should not neglect student interviews when we want to look at student-level outcomes of the instruction provided by teachers. The interviews let us see things we wouldn't see at the level of classroom observations. We can and should use these to complement quantitative information about student level outcomes. We might look at the number of correct answers on the interview questions (1 out of 3 students from Section A; 1 out of 4 students from Section B) and say that the shifts Shana made in her instruction didn't change anything. And of course, with such a small sample we would not be able to make significant claims regarding differences.

However, we can use the student interviews to provide a detailed analysis of student understanding, more so than a large-scale student assessment can allow.

We should also use this type of case study, with multiple sources of information, to consider the mechanisms through which students and teachers interact and how student understandings may change. In this case, we have a teacher interview before instruction, classroom observation, and student interviews after instruction. This enables us to gain a picture of not just *whether* or not student and teacher understandings are connected, but *how* they may be or become connected. Other studies of instruction (e.g., Hill et al., 2008; Hill, Kapitula, & Umland, 2011; Hill, Umland, Litke, & Kapitula, under review) have examined classroom observations with care and reliable metrics, as well as correlated these metrics to student performance and validated them through teacher interviews. However, the student interviews may add still more information to this type of intensive analysis. They provide more information about the detail of student understanding and the ways in which students solve problems. As shown above, this information is richer than knowing only the number of correct or incorrect responses.

A next step, in addition to replicating this type of case study in other areas of mathematics, could be to interview teachers again after the student interviews. This could take the collection and use of this type of information even further. This would enable the participating teachers to gain even more exposure to student thinking, as shown to be beneficial by Carpenter et al. (1989). The use of video for the classroom observations would also enrich this type of study, first by allowing for a more detailed analysis of the classroom through multiple viewings, and second by making this rich source of information available to the teachers as well (Sherin, 2002).

This type of information is not likely to be as attractive to policymakers, who must consider large numbers of students, as statistical correlations would be. However, this should not preclude researchers, teachers, and teacher educators from seeking out and using this information. Certainly, a case study such as this does not tell us unarguably that the instruction led to particular student understandings, as we cannot say that Shana's changes from Section A to Section B led students to consider a tree diagram as a viable method or helped them to understand the decreasing numbers to be multiplied in a permutation. But it allows us to examine mechanisms through which instruction may affect students and to put forth researchable claims about what is valuable in classroom instruction.

References

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Carpenter, T.P., Hiebert, J., Moser, J.M. (1981). Problem Structure and First-Grade Children's Initial Solution Processes for Simple Addition and Subtraction Problems. *Journal for Research in Mathematics Education, 12*, 27-39.

Carpenter, T.P. & Moser, J.M. (1982). The development of addition and subtraction problem-solving skills. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and Subtraction: A Cognitive Perspective* (pp. 9-24). Hillsdale, NJ: Lawrence Erlbaum.

Carpenter, T.P., & Moser, J.M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15,* 179-202.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., and Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3-29.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.

Hadfield, O. D., Littleton, C. E., Steiner, R. L., & Woods, E. S. (1998). Predictors of preservice elementary teacher effectiveness in the micro-teaching of mathematics lessons. *Journal of Instructional Psychology, 25*(1), 34-47.

Hill, H.C., Ball, D.L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351.

Hill, H.C., Ball, D.L, & Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H.C., Blunk, M.L., Charalambous, C.Y., Lewis, J.M., Phelps, G.C., Sleep, L., & Ball, D.L. (2008b). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H.C., Schilling, S.G., & Ball, D.L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Hill, H.C., Kapitula, L. Umland, K.U. (2011). A validity argument approach to evaluating teacher value-added scores. *American Educational Research Journal, 48*(3), 794-831.

Hill, H.C., Umland, K.U., Litke, E., & Kapitula, L. (under review). Teacher quality and quality teaching: Examining the relationship of a teacher assessment to practice. *American Journal of Education.*

Huinker, D., & Madison, S. K. (1997). Preparing efficacious elementary teachers in science and mathematics: The influence of methods courses. *Journal of Science Teacher Education, 8*(2), 107-126.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (2006). *Connected Mathematics 2, Grades 6, 7, 8*. Prentice Hall, 2006.

Lowery, N. V. (2002). Construction of teacher knowledge in context: Preparing elementary teachers to teach mathematics and science. *School Science and Mathematics, 102*(2), 68-83.

Lubinski, C. A., & Otto, A. D. (2004). Preparing K-8 preservice teachers in a content course for standards-based mathematics pedagogy. *School Science and Mathematics, 104*(7), 336.

Maher, C.A. & Martino, A.M. (1996). Young children invent methods of proof: The gang of four. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning* (pp. 431-447). Hillsdale, NJ: Lawrence Erlbaum.

Piaget, J. (1976). *The child's conception of the world*. (J. Tomlinson & A. Tomlinson, Trans.). Savage, MD: Littlefield Adams. (Original work published 1926.)

Piaget, J. & Inhelder, B. (1975). *The origin of the idea of chance in children*. New York: W. W. Norton & Company Inc.

Philipp, R. A., Thanheiser, E., & Clement, L. (2002). The role of children's mathematical thinking experience in the preparation of prospective elementary school teachers. *International Journal of Educational Research, 37*(2), 195-210.

Saxe, G.B., Gearhart, M., & Nasir, N.S. (2001). Enhancing students' understanding of mathematics: A study of three contrasting approaches to professional support. *Journal of Mathematics Teacher Education, 4*, 55-79.

Schliemann, A.D., & Acioly, N.M. (1989). Mathematical knowledge developed at work: The contribution of schooling. *Cognition and Instruction, 6*(3), 185-221.

Simon, M.A., & Schifter, D. (1993). Toward a constructivist perspective: The impact of a mathematics inservice program on students. *Educational Studies in Mathematics, 25*, 331-340.

Szydlik, J. E. (2000). Photographs and committees: Activities that help students discover permutations and combinations. *Mathematics Teacher, 93*(2), 93-99.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Watson, R. (1996). Students' combinatorial strategies. *Teaching Mathematics and Its Applications, 15*(1), 27-32.

Appendix A: Classroom observation checklist

Category	Item	Time
Lesson format	Teacher-led instruction (with times)	
	Students address whole class (with times)	
	Individual work (with times)	
	Group work (with times)	
	Working on applied (real-world) problems	
Richness	Multiple procedures or solution methods	
	Explanations	
	Developing mathematical generalizations	
	Mathematical language	
Working with students and math	Teacher questioning	
	Remediating student difficulties	
	Uses student mathematical ideas in instruction	
Errors	Major mathematical errors or oversights	
	Imprecision in notation or math language	
Student activity	Students provide explanations	
	Student mathematical questioning and reasoning	
	Enacted task cognitive demand	
Representations	Formula	
	List	
	Tree	
	Slot	
	Explicit linking	
General topic issues	Questions about assumptions	
	Comparing methods	
	Using complement	
	Compound probability	
	Deciding whether order matters	
	Estimation of probability	
	Finding probability denominator separately	
	Language use	
	Order of introducing methods	
	Replacement of elements	
	Testing smaller cases	
	Using physical objects	
	Word analogy for identical items	
Formula method	Deriving formula	
	Division to remove identical combinations	
	Division to remove identical items	

	Finding probability denominator	
	Justifying formula	
	More than one set of identical items	
	Multiplication choice	
	Remembering formula	
List method	Knowing when all possibilities are there	
	Removing identical combinations	
	Removing identical items	
	Removing identical permutations	
	Systematic listing	
Slot method	Division to remove identical combinations	
	Division to remove identical items	
	Finding number of possibilities per slot	
	Finding number of slots	
	Multiplication choice	
	Removing identical items	
Tree method	Finding number of branches from a vertex	
	Multiplication choice	
	Removing identical combinations	

Analysis 2: Using interviews to explore teacher knowledge types: The case of permutations

Introduction

The purpose of this study is to explore the use of an existing framework of types of teacher knowledge put forth by Ball, Thames, and Phelps (2008) in a novel way. Specifically, the analysis presented here assigns knowledge types to teacher statements made during an interview. This type of detailed coding has not previously been done with teacher interviews. In performing this coding and analyzing both the process and the results, this study addresses the following questions:

(i) What knowledge types (common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching) do teachers exhibit most frequently when answering a question about permutations in an interview setting?

(ii) Is it reasonable to consider knowledge types as manifested in particular statements, rather than as attributes of a teacher?

The study summarizes the results of analyzing eight teacher interviews in this way. In the interviews, the teachers were presented with a mathematical problem and were asked to complete the problem, and also to reflect upon it, explain their solutions, and provide alternative strategies. The mathematical problem asked participants how many ways there would be to arrange four distinct objects, thus the mathematical content is combinatorics, and specifically a permutation of all n

of n objects [$P(n,n) = n!$]. The mathematical content and its influence on participant responses cannot be disregarded, so this study does not make claims about what teachers would do when faced with a different type of mathematics problem that involved a different type of mathematical content. However, in this analysis, the objective is to consider the teacher knowledge types in the interview setting. Thus, the many interesting aspects of the teachers' mathematical responses that are specific to this particular mathematical content are not analyzed here.

This analysis rests on the belief that it is interesting and important to be able to separate teacher knowledge into different types. The other option is to attempt to consider teacher knowledge as a single element, without distinguishing categories or nuances in the type of knowledge. However, to do so would limit our ability to work with teachers on specific areas. While the theoretical constructions of types of knowledge may be called by different names, and the final map may look different than that elaborated below or have more or fewer types of knowledge, knowing *which* things are correlated to student learning or positive outcomes in the classroom may help us to enrich those things for teachers. Ball et al. (2008) state this elegantly: "We hypothesized that teachers' opportunities to learn mathematics for teaching could be better tuned if we could identify those types [of mathematical knowledge and skill] more clearly" (p. 399).

In addition, some knowledge types have been connected to positive student outcomes. Specifically, mathematical knowledge for teaching, as measured by a written assessment, has been correlated with greater student gains in the classroom (Hill, Rowan, & Ball, 2005). However, not all knowledge types have been measured independently or shown to be independent constructs (Hill, Dean, & Goffney, 2007; Hill, Ball, & Schilling, 2008). Thus they remain, in part, theoretical distinctions. It is useful to ask how these distinctions can, or cannot, be delineated in interviews like the ones described in this paper, as opposed to a written assessment, since this setting provides teachers with more freedom in their responses. Because of this freedom, and the theoretical underpinning that teachers do, in general, possess and use different types of mathematical knowledge, this study does not ask whether or not teachers use different types of mathematical knowledge in the interview. We would expect that they would, given the opportunity. This study asks about the frequency of the different knowledge types and how we can use this information.

In addition, research has not given us a definitive link between the mathematical tasks carried out or described in a teacher's interview and the way in which knowledge types are tapped in the interview setting. Analyses of interview statements, as shown in this paper, may provide a link between how a teacher completes a task and the knowledge they use in completing the task. As such, this analysis may begin to shed light on the ways in which different types of knowledge, as manifested on a written assessment, are exhibited in teaching practice. This argument assumes that teachers' reflections in an interview are closer to the kinds of verbalizations made in a classroom setting than what is exhibited in a written assessment. If research continues to show that some distinguishable knowledge types are connected to student outcomes, we should investigate how these are manifested in ways other than in a written assessment. This would not only illuminate the pathway through which these affect teaching, but would also allow us to consider ways to enhance teacher growth through professional development, and consider how to use interviews to determine the value of our efforts to provide development opportunities to teachers.

Background

Defining Teacher Knowledge Types

This study relies on previous work mapping and distinguishing between different types of teacher knowledge. Since the introduction of the idea of *pedagogical content knowledge* (PCK; Shulman, 1986), studies and theoretical papers have attempted to clarify, specify, measure, or engender Shulman's PCK. However, as pointed out by Hill, Ball, and Schilling (2008), there is still little information showing how teachers' PCK relates to student-level outcomes, or even about what constitutes PCK. The intent of this section is to look at the current theoretical and empirical view regarding teacher knowledge in mathematics. This view will be used to justify this study's examination of knowledge types through interviews.

Shulman introduced PCK in response to research and standards on what teachers needed to know that heavily emphasized pedagogical procedures. The requirements for teachers were fully divorced from specific content areas, such as math or reading, leading Shulman (1986) to ask, "Where did the subject matter go? What happened to the content?" (p. 5). The emphasis on pedagogy was in contrast to qualifying examinations from the 1800s that rigorously tested prospective teachers mainly on the content itself. Rejecting this dichotomy, Shulman proposed that teachers needed not content-free pedagogy, nor pedagogy-free content, but a particular kind of professional expertise that went "beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching"* (p. 9).

The original introduction of pedagogical content knowledge put it forth as a subset of content knowledge; that is, Shulman proposed "three categories of content knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge" (Shulman, 1986, p. 9). Pedagogical content knowledge was defined as the knowledge, still particular to the content, that is specifically used for teaching. Inside PCK, Shulman included representations, examples, and explanations, as well as common difficulties, common student preconceptions, and ways of changing incorrect student conceptions. Knowledge of the curriculum, though, including knowledge of the range of available materials, was not included in this initial outline of PCK.

What constitutes PCK in mathematics has not been fully specified or agreed upon by the research community (Hill et al., 2008). Note that the process of mapping PCK is domain-specific, so while work in mathematics continues as discussed here, the same process is underway in other teaching disciplines, such as for teaching science (e.g. Gess-Newsome, 1999), or for teaching teachers (e.g. Strauss, 1993). In mathematics, Hill, Ball, and Schilling (2008) give the most comprehensive look at PCK. Moreover, they also propose that PCK is part of a larger construct, *mathematical knowledge for teaching* (MKT). They separate the universe of MKT into subject matter knowledge on one side, and pedagogical content knowledge on the other, as shown in *Figure 6*. For them, the subject matter knowledge side includes both *common content knowledge* (CCK) and *specialized content knowledge* (SCK).

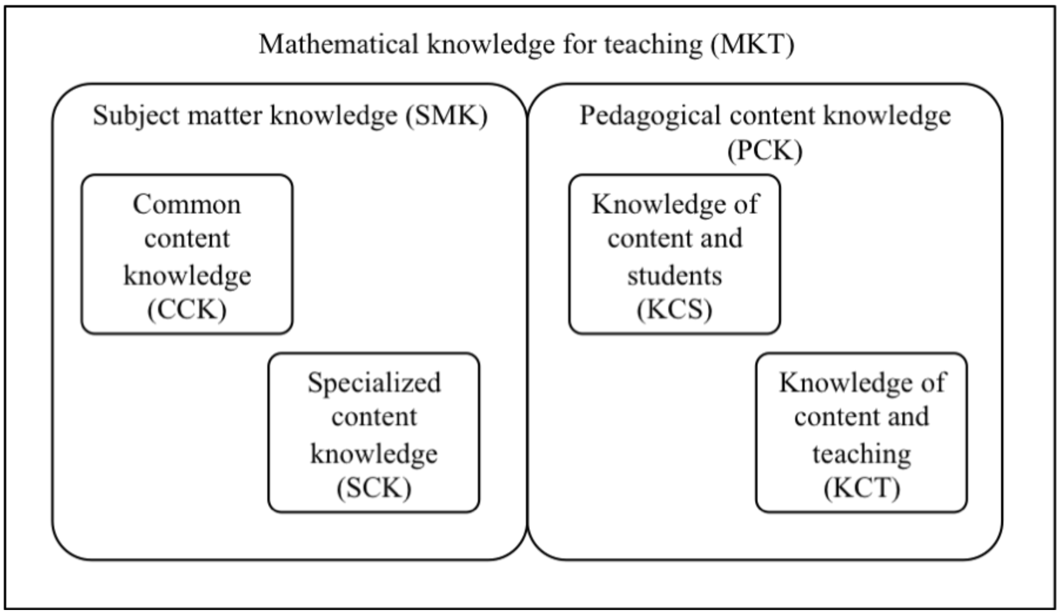


Figure 6. Teacher knowledge types, based on Ball et al. (2008).

The first item, *common content knowledge*, referred to as "'common' knowledge of content" (p. 387) in Hill et al. (2005), includes what we might consider to be pure mathematical content; this is the knowledge of mathematics apart from the need to teach it. The example provided for this first area of content knowledge is the solution for x in the expression $10^x = 1$.

The second item is the *specialized content knowledge*, or content knowledge that would be useful only to a teacher. The authors are careful to note that this second area is still mathematical knowledge, not pedagogy. For this area, the example provided requires the teachers to evaluate three methods for multiplying two digit numbers, and determine which of the methods are always mathematically valid. The knowledge used in completing an activity of this type has commonalities with pedagogical content knowledge (Shulman, 1986), in that it requires the teacher to recognize alternative solution strategies outside the traditional algorithm, and to reflect on the legitimacy of these mathematically. This specialized content knowledge sits next to PCK but does not contain it; neither is it contained by it (Hill et al., 2008). It is knowledge that would be useful while engaged in teaching, but does not require one to know anything about students or teaching.

Within pedagogical content knowledge, they include two additional types of knowledge; the first is *knowledge of content and students* (KCS) that more specifically includes "knowledge of how *students* think about, know, or learn this particular content" (Hill et al., 2008, p. 375 [italics added]). The intent is to define this area as a measurable domain of knowledge that is distinct from the specialized content knowledge (which is considered purely mathematical) in that it requires some knowledge of students. The other type of knowledge contained within PCK is knowledge of *content and teaching* (KCT). This type "combines knowing about teaching and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401). That is, knowledge of instructional strategies, choosing examples, and other elements that link the mathematics to the practice of classroom teaching.

It is important to note that Ball et al. (2008) do not limit the types of teacher mathematical knowledge to those described here (CCK, SCK, KCS, and

KCT). They leave room in their model for future discovery and definition of knowledge types, particularly as relates to knowledge of the mathematical horizon and knowledge of curriculum. However, the analysis presented here is restricted to these four relatively well-defined knowledge types.

Ball et al. (2008) and Hill et al. (2008) acknowledge the difficulty and subtlety in these distinctions, even at a theoretical level. In particular, the current work (Hill et al., 2007; Hill et al., 2008) supports the theoretical construct of *knowledge of content and students*, but has not demonstrated that this is empirically separable from specialized content knowledge through the forms of assessment used by the researchers. Specialized content knowledge as conceived of in earlier work (see Hill et al., 2005) requires making judgments about the validity of alternative solution strategies. While this activity is undoubtedly mathematical, it sits tight against knowledge of how students think about the content, which is thought to be KCS. The distinction that led to the separation between common content knowledge and specialized content knowledge also makes more difficult the measurable distinction between specialized content knowledge and knowledge of content and students. However, Ball et al. (2008) do make the theoretical dividing line more clear by marking KCS as requiring some knowledge of students, while specialized content knowledge for teaching does not require knowledge of students.

In order to clarify the theory behind these distinctions, Ball et al. (2008) give a large number of examples of tasks in which teachers may engage that would be manifestations of a particular type of knowledge. By listing these concrete activities, it is easier for us to conceptualize the boundaries between knowledge types, and this also paves the way for the type of analysis proposed in this study, as discussed below. A full list of all tasks taken literally from Ball et al. (2008) is provided in Table 8.

Table 8. *Tasks classifiable as each knowledge type (Ball, Thames, & Phelps, 2008).*

Knowledge type	Task	Justification
Common content knowledge (CCK)	Calculating an answer	p. 399
	Correctly solving mathematics problems	p. 399
	Recognizing a wrong answer	p. 399; p. 401
	Recognizing an inaccurate definition	p. 399
	Use terms and notation correctly	p. 399
Specialized content knowledge (SCK)	Presenting mathematical ideas	Figure 3; p. 400
	Responding to students' "why" questions	Figure 3; p. 400
	Finding an example to make a specific mathematical point	Figure 3; p. 400
	Recognizing what is involved in using a particular representation	Figure 3; p. 400

	Linking representations to underlying ideas and to other representations	Figure 3; p. 400
	Connecting a topic being taught to topics from prior or future years	Figure 3; p. 400
	Explaining mathematical goals and purposes to parents	Figure 3; p. 400
	Appraising and adapting the mathematical content of textbooks	Figure 3; p. 400
	Modifying tasks to be either easier or harder	Figure 3; p. 400
	Evaluating the plausibility of students' claims (often quickly)	Figure 3; p. 400
	Giving or evaluating mathematical explanations	Figure 3; p. 400
	Choosing and developing useable definitions	Figure 3; p. 400
	Using mathematical notation and language and critiquing its use	Figure 3; p. 400
	Asking productive mathematical questions	Figure 3; p. 400
	Selecting representations for particular purposes	Figure 3; p. 400

	Inspecting equivalencies	Figure 3; p. 400
	Looking for patterns in student errors	p. 400
	Sizing up whether a nonstandard approach would work in general	p. 400; p. 401
	Sizing up the nature of an error	p. 401
Knowledge of content and students (KCS)	Anticipate what students are likely to think	p. 401
	Anticipate what students will find confusing	p. 401
	Choose examples that students will find interesting and motivating	p. 401
	Anticipate what students are likely to do with a task	p. 401
	Anticipate whether students will find a task easy or hard	p. 401
	Hear and interpret students' emerging and incomplete thinking	p. 401
	Knowledge of common student conceptions and misconceptions	p. 401
	Deciding which of several errors students are most likely to make	p. 401

Knowledge of content and teaching (KCT)	Sequence particular content for instruction	p. 401
	Choose which examples to start with	p. 401
	Choose which examples to use to go deeper	p. 401
	Evaluate instructional advantages and disadvantages of representations	p. 401
	Identify what different methods and procedures afford instructionally	p. 401
	Decide which student contributions to pursue and which to ignore or save for later	p. 401

Connecting Teacher Knowledge to Student Outcomes

It is important to note that Hill, Rowan, and Ball (2005) do examine not just the separability of the knowledge types, but also the impact on students. In this particular work, the researchers report on the findings of a study of first and third grade students and their teachers across 115 elementary schools. While their research initiative includes the evaluation of professional development courses for teachers (e.g. Hill & Ball, 2004), no intervention occurred or was measured in the particular case described here. Instead, the mathematical performance of eight students from each participating classroom was assessed at the beginning and end of an academic year. While one year of student data was collected, for the teachers of the students, three years of data was collected. During the course of data collection, the teachers kept a log of measures relating to their teaching practices, such as content covered and the duration of mathematics lessons.

Teachers also completed a survey once during each year that included educational background, certification information, experience, and other potentially relevant items. In addition, each teacher survey had five to twelve multiple-choice questions that were designed to assess the mathematics needed for teaching. The researchers provide a full description of the development of these items in a separate publication (Hill, Schilling, & Ball, 2004).

In this particular study, focusing on the student outcomes, Hill et al. (2005) included items that target common content knowledge and specialized content knowledge, referred to together at that time as "content knowledge for teaching mathematics" (CKT-M; p. 387). Since that time, the research group has used the term *mathematical knowledge for teaching* (MKT), and the MKT designation will be used here for clarity. Hill et al. (2005) found that their measure for MKT was significantly correlated with student gains in both the first and third grades. They were careful to control for other variables, including socio-economic status, the time spent on mathematics in the classroom, and mathematics courses taken by the teacher in the past. The diligence of the researchers lends credence to their analysis of the data, and they are justified in noting the correlation between the scores on their teacher assessment and the gains for the students, and in calling for courses that are focused on mathematical knowledge specifically for teachers. Interestingly, they do offer a potential alternate explanation for the results. They suggest that the teachers who scored well on the MKT questions might have some other, unknown, factor that truly impacts the student scores. They recommend an analysis of the practice of teachers that could potentially suggest factors which, while not necessarily independent of or dependent on mathematical knowledge for teaching, may be manifestations of some sort of teacher knowledge or practice that leads directly to student understanding.

While some progress has been made in linking the outcomes, in terms of student performance, to factors connected to teachers, no clear consensus exists on how this would translate into practice for teachers, or into preparation and professional development for teachers. One pattern in studies on teacher knowledge throughout educational research is that those that are able to directly measure student performance are quite large in scale, and are time- and fund-intensive projects (e.g., Carpenter et al., 1989; Hill et al., 2005). Smaller scale studies, including many that attempt to move directly to addressing the problem by working in courses with pre-service teachers, do not include measures that tell us about student performance (e.g., Hadfield, Littleton, Steiner, & Woods, 1998; Huinker & Madison, 1997; Lowery, 2002; Lubinski & Otto, 2004; Philipp, Thanheiser, & Clement, 2002; Tirosh, 2000). These studies may have insights into key elements of teaching, or may describe courses for teachers that would prove to be highly beneficial to students in the long term, but we cannot assess at present what the specific benefits to students are.

Accessing Teacher Knowledge Types through Interviews

As discussed above, we have a need to find out more about how teacher knowledge affects their classroom practice and the students. Certainly teachers may shift back and forth between knowledge types while teaching, and may even hold knowledge in complexes, as proposed by Sherin (2002). However, this does not mean we cannot attempt to disentangle the types of knowledge used, even as they work together. Since Ball et al. (2008) have elaborated specific tasks connected to each knowledge type (as shown in Table 8), this enables the study design and analysis presented here. Tasks, being concrete, can be identified in statements or actions. By combining a search for these tasks in teacher statements with an understanding of the theoretical distinctions, the coding activity in this study can help judge whether or not it is useful to seek knowledge types through teacher interviews.

While teacher knowledge types are theoretical distinctions, they have been *described* through tasks (see Table 8) and *measured* through written assessments (Hill et al., 2005), as discussed above. As described in Hill et al. (2004), the written assessment questions attempt to engage respondents in the same types of activities that they would be doing as teaching professionals. Thus, implicit in their extensive work on writing and testing assessment items is the idea that some *tasks* of teaching can be performed implicitly in response to a prompt from a written assessment. That is, we can theorize that if we were to analyze the written items, we could specify the tasks one would need to carry out in order to respond in one way or another. Similarly, the interviews described in this study took place outside the classroom, but we will see in the analysis that teachers both describe and carry out the mathematical tasks of teaching in an interview setting.

In fact, Hill et al. (2007) and Hill et al. (2008) describe conducting interviews to look at expressions of teacher KCS. In examining the interviews, however, their primary goal was to follow up on a written assessment in order to corroborate their belief that teachers were indeed using knowledge of students to respond to the written questions that were intended to gauge KCS. The fact that these interviews were used as evidence of a teacher knowledge type tacitly lends support to the methodological plan in this study, which is precisely to use interviews for this purpose. However, since the interviews in this study will be analyzed statement by statement and for four different types of teacher knowledge, the plan and results presented here are novel. The interviews elicit teacher knowledge of all types, and the question is not whether or not teachers possess these types of knowledge at all. The question is how much they use each type, and what this information affords the field of research in mathematics education.

Methodology

Summary

The data for this study is an interview question presented to eight participating teachers. During interviews, participants were asked to complete mathematics problems, and also to reflect upon them, explain their solutions, and provide alternative strategies.

Participants

All teachers who participated in a summer professional development workshop (described below) received a letter at the beginning of the workshop explaining this study and inviting them to participate. Teachers were asked if they would be willing to be interviewed for the study. All participants were teachers at secondary schools in a large Northeast city or a nearby urban rim community.

Professional Development Course

The study described here is independent of the professional develoopment course, and teachers were not required to take part in the study in order to participate in the course. The study does not make claims about the influence of the course, but it is described here as background information. The 2008 and 2009 Tufts University Problem Solving and Discrete Math Workshops were both available to teachers who teach mathematics in grades 5 through 9 in Massachusetts, with preference given to teachers from districts that are classified by the state as high needs. The 2008 workshop was a seven day summer workshop, with two full-day follow up sessions in the fall and winter of 2008. The 2009 workshop was an eight day summer workshop, also with two full-day follow up sessions in the fall and winter of 2009. The extra day of workshop time in 2009 did not include any additional time spent on teaching or discussing the content addressed in this study, that is, combinatorics (of which permutations are a part); for this reason, the 2008 and 2009 workshops can be considered here to have used identical curricula. No teacher attended more than one year of the workshop. Many teachers who participated in the workshop did not participate in this study, and they were not required to be part of this study in order to attend the workshop.

The PSDM workshop was primarily focused on mathematical content. There were approximately three hours of instruction per day on mathematical content. During this time, the teachers listened to a lecture, completed problems, worked in groups, and asked questions. No teaching methods were suggested to the participants. The teachers also had additional problems for homework that could be completed in groups or independently, and they had one to two hours of time during the workshop day to work on these. The following day, a subset of teachers would explain their solutions to completed homework problems to the full group of participants, and they would also answer any questions.

However, teachers did work in groups on curricular plans and considered how their own students would interact with the materials. The teachers spent two hours of time each afternoon working in groups to create a three-day lesson for their own students, covering one of the topics taught in the workshop. Since no methods of teaching were suggested, and guidance was provided only on the mathematical content, the time spent on planning lessons can be considered as self-directed time.

Each year of the workshop had four days of content that was directly related to this study: one day focused on simple counting problems, two days on permutations and combinations, and a fourth day on probability.

Measures and Data Collection

Data was collected through teacher interviews. The teacher interviews were conducted in spring 2009 and 2010. In all cases, the interview took place the spring following the summer in which the teacher had participated in the workshop. Five teachers were interviewed in spring 2009 and three teachers were interviewed in spring 2010. However, since the workshops did not differ in content related to this study and this study does not look at the impact of the workshop or make any assumptions about its effect, all eight teachers are analyzed together, without distinguishing their year of participation.

An interview was carried out and videotaped with each participating teacher. While interviews were flexible and open-ended, they had a goal of discussing mathematical situations and material that are germane to combinatorics, including questions on permutations. To do this, the teachers were given problems to solve. After solving each problem, they were asked for an explanation of their work. They were then asked for a different way to solve the same problem and a different explanation. They were also asked about what they believed their students would do when working on the same problem. This process repeated for each mathematical problem. The analysis presented here focuses on teacher responses to and about the first interview question, presented in *Figure 7*.

In this question, teachers were asked how many ways they could arrange four objects. The four objects were presented as characters in boxes, as shown in *Figure 7*. Note that in the first round of interviews (spring 2009), the objects were numbers and in the second round of interviews (spring 2010) the objects were letters. During the intervening time, the question was modified because a related study was planned with middle and high school students and it was thought that it could be less confusing for the students to arrange letters, rather than numbers. However, during the teacher interviews, no teacher mentioned or critiqued the use of numbers or letters when discussing this question. As a result, the answers by all eight teachers will still be analyzed together.

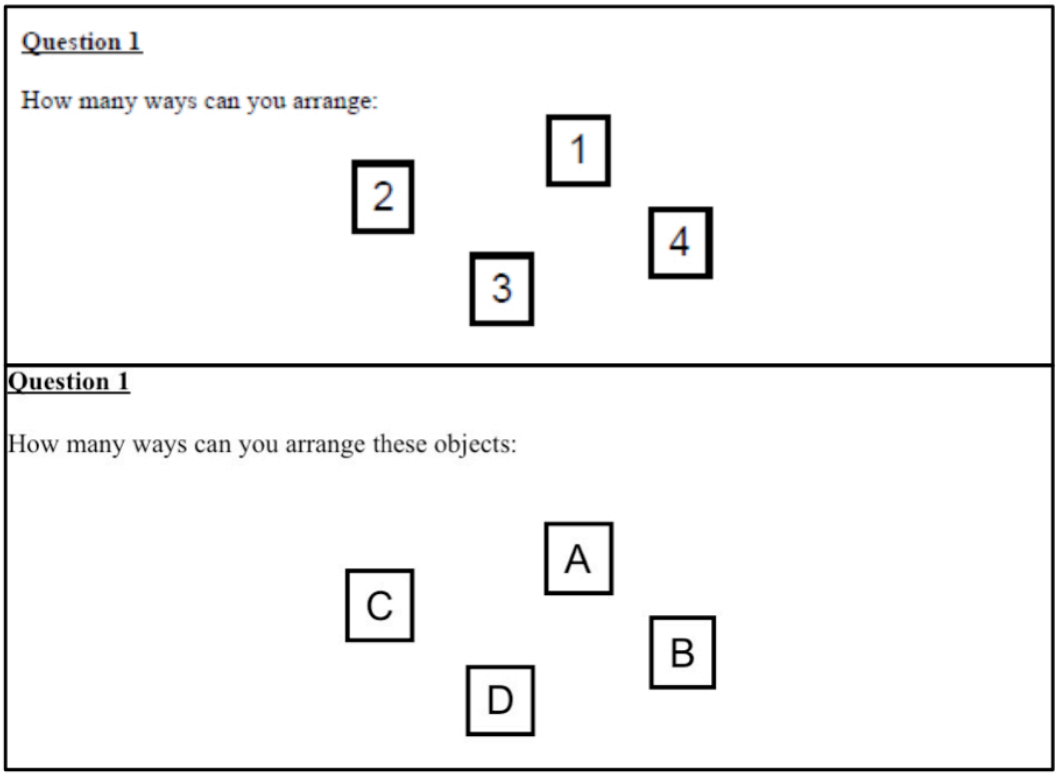


Figure 7. The interview question (top, 2009 version; bottom, 2010 version).

Analysis

To address the first research question, asking what knowledge types (CCK, SCK, KCS, KCT) teachers exhibit when answering a question about permutations in an interview setting, the first step was transcribing all eight teacher interviews. The description and listing of teaching tasks in Ball et al. (2008) was then examined and a detailed list of the tasks linked to each knowledge type was constructed, as shown in Table 8.

Using the list in Table 8, each teacher statement from the transcript was classified as related to or representative of as many of the pre-determined types of tasks as applicable. In each teacher statement, particular tasks were *carried out*, *described*, or *referred to*. A statement was defined as the full length of what a teacher said without response or interruption from the interviewer. The interviewer's statements were not coded. For example, in the teacher statement shown in Table 9, we can see that a single teacher statement included four of the tasks listed in Table 8. As many codes as were relevant were allowed, so more than one knowledge type may have been applied to a single statement, as was the case in this example.

During coding, only the tasks that corresponded to each statement were chosen. After choosing the tasks, the corresponding knowledge types were populated automatically. This was done so that the coding would not be based on the potentially more subjective view of different knowledge types, but just solely on the more concrete tasks. An example of a coded teacher statement is shown in Table 9; in this particular example, the statement showed evidence of KCT, SCK, and KCS. The transcripts and codes thus obtained were then reviewed to look for any cases where the literal coding of statements, via tasks, resulted in the presence or absence of particular knowledge types in a way that was unexpected considering the theoretical definitions of each type. Any discrepancies were noted, and are discussed below.

Once this coding was complete, individual teacher profiles were analyzed. In addition, the data were collapsed to look at the percentage of statements exhibiting each knowledge type, out of the total number of statements made by all eight teachers (168 statements). The data were reviewed to look at the percentages of statements that were coded with each knowledge type, both at the individual teacher level and for all teachers combined.

Table 9. *A sample of a coded transcript statement.*

Name	Statement	Task	Knowledge type
Sarah:	Yes – after we do a few, most of them see the pattern is that you multiply. So after we do the tree diagram, typically I go into the fundamental counting principle.	Sequence particular content for instruction	KCT
Sarah:		Linking representations to underlying ideas and to other representations	SCK
Sarah:		Anticipate what students are likely to think	KCS
Sarah:		Anticipate whether students will find a task easy or hard	KCS

Results

In reference to the first research question, regarding what knowledge types (CCK, SCK, KCS, KCT) teachers exhibit when answering a question about permutations in an interview setting, it is important to note that all 8 participating teachers made statements that could be classified under all four knowledge types considered here. Even within this single mathematical task, teachers moved between all types of mathematical knowledge. When comparing the knowledge types to each other, considering all eight teachers together, the most frequently used was specialized content knowledge (SCK), as shown in Table 10, appearing in 55% of statements. All other knowledge types (CCK, KCS, and KCT) were displayed in approximately the same proportion, each in 33% to 35% of statements.

Table 10. *Relative frequency of statements flagged as each knowledge type.*

Knowledge type	# of statements flagged with this knowledge type (out of 168 statements)	% of statements flagged with this knowledge type
CCK	59	35%
SCK	92	55%
KCS	55	33%
KCT	58	35%

Not all teachers used SCK more, however. As shown in Table 11, the dominance of SCK and relative equity of the other knowledge types in the aggregate data obscures widely variable individual teacher profiles. In fact, SCK, notably more frequent in the aggregate data, was the most commonly used knowledge type for only four of the eight teachers. In addition, none of the teachers precisely mirrored the profile of the study population, with a high percentage of SCK and nearly equal percentages of the other types. The percentage data from Table 11 is also shown in *Figure 8*, in order to provide a visual image of the differing profiles. Note that the data points for each teacher are connected in *Figure 8*. This does not imply any continuity or progression between the knowledge types; in fact, the order of these could be changed along the abscissa, resulting in a figure that looks quite different. The points are connected only for ease in reading the figure and enabling us to see all the points corresponding to a single teacher.

Table 11. *Percentage of use of knowledge types, by teacher*

Name	# CCK	CCK %	# SCK	SCK %	# KCS	KCS %	# KCT	KCT %	HIGHEST %
Jessica	8 of 14	57%	7 of 14	50%	2 of 14	14%	2 of 14	14%	CCK
Anna	7 of 14	50%	6 of 14	43%	4 of 14	29%	5 of 14	36%	CCK
Sarah	8 of 23	35%	16 of 23	70%	4 of 23	17%	9 of 23	39%	SCK
Whitney	8 of 17	47%	9 of 17	53%	4 of 17	24%	3 of 17	18%	SCK
Annie	6 of 20	30%	16 of 20	80%	7 of 20	35%	3 of 20	15%	SCK
Laura	10 of 35	29%	22 of 35	63%	15 of 35	43%	18 of 35	51%	SCK

Betsy	3 of 18	17%	6 of 18	33%	10 of 18	56%	4 of 18	22%	KCS
Shana	9 of 27	33%	10 of 27	37%	9 of 27	33%	14 of 27	52%	KCT
Total	**59 of 168**	**35%**	**92 of 168**	**55%**	**55 of 168**	**33%**	**58 of 168**	**35%**	**SCK**

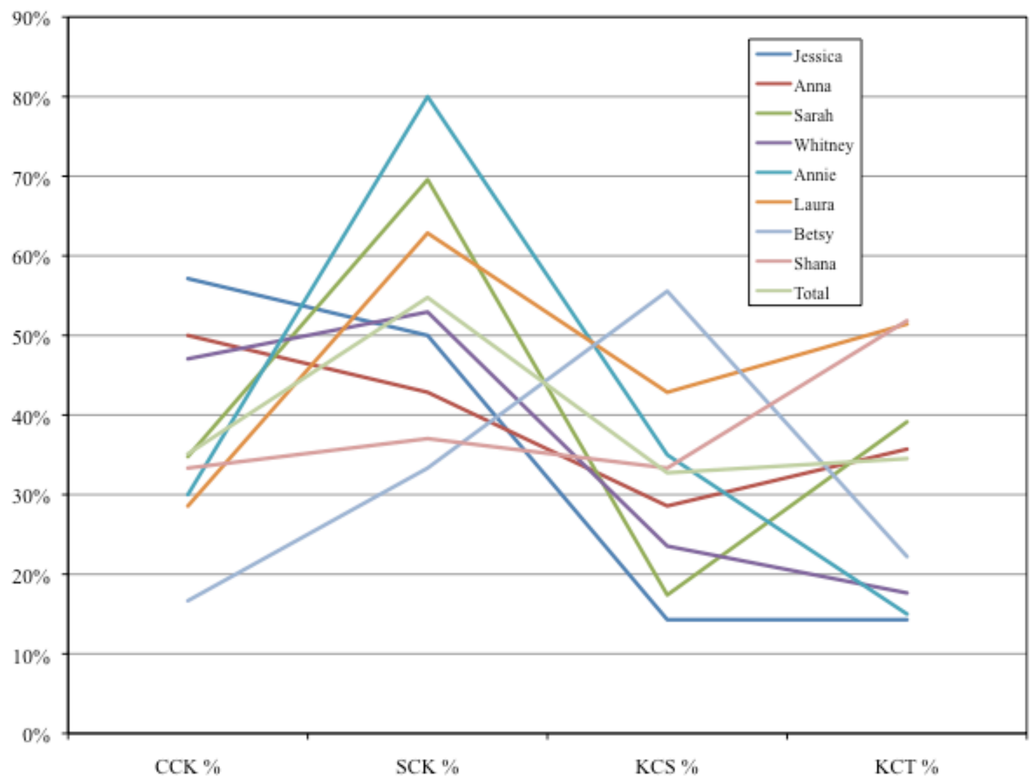


Figure 8. Individual teacher profiles.

With a larger sample, the relative percentages of each knowledge type could be used to begin to construct profiles of teachers similar to those described in Hill et al. (2008b). The eight teachers shown here are not sufficient to generalize such profiles; however, a more detailed description of some notable profiles may help us begin to construct additional ideas about the differences that could emerge when looking at the teacher knowledge types revealed in interviews.

Two cases will be described here, those of Annie and Betsy. Annie was chosen for two reasons: one, because she exhibited SCK more than any other knowledge type and in this way mirrored the aggregate data. Two, because she had the greatest percentage difference between any two knowledge types, with SCK used in 80% of statements and KCT used in only 15%, for a difference of 65%. The combination of these attributes makes her an extreme example of the prevalence of SCK among teachers when responding to the interview question.

Betsy was chosen because her profile is at the opposite extreme: she was the only teacher to exhibit KCS more frequently than any other knowledge type, and the only teacher to use KCS more frequently than SCK – all seven of the other teachers displayed SCK more frequently than KCS, even if SCK was not their most frequent type.

Annie, whose work is shown in *Figure 9*, was a young teacher with a strong background in mathematics. She had been teaching for less than five years, but her teaching had always been in secondary school mathematics. When presented with the interview task, she quickly found the (correct) answer and explained the procedure she had used. She was able to also talk about two different methods for finding the solution and discuss which one she preferred and why. The use and critique of different representations was a major factor in her high percentage of statements exhibiting SCK, as two of the tasks linked to SCK are “Recognizing what is involved in using a particular representation”, and

"Linking representations to underlying ideas and to other representations". Annie referred to one (or both) of these tasks in 8 of the 20 statements she made about this interview question, as was the case in this statement:

> And we start off with listing them all out, and then do the tree diagram, we can do the tree diagram for it, and then we came up with the formula, so they can see how many choices do they have. And I eventually show them the slots. Like think of 4 chairs that you have and then one person sits here there's only 3 people left, so… you take one out.

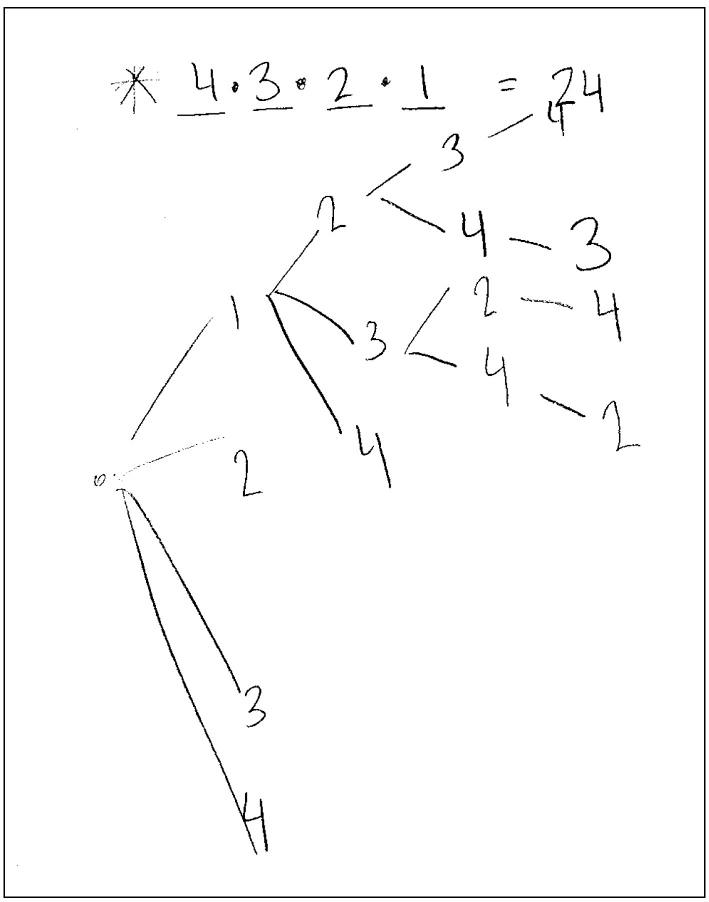


Figure 9. Annie's work on the interview problem.

Other than discussion and use of different representations, the other major factor in Annie's high percentage of statements exhibiting SCK referred to the task, “Using mathematical notation and language and critiquing its use”, which was identified in 8 of her 20 statements about the problem. In her case, all of the instances of this task sprung from discussion of how her students would struggle with knowing to multiply the numbers, rather than add them, because of the use of the word “and”. In her words, she clarified that students would have trouble “just with the ‘and’ and the ‘or.’ Because doing this stuff [permutations], it means different things.” She went on to elaborate cases when this would occur and how her students would react. The notable element here is that these instances of SCK occurred *only* because she was engaged in a task associated with knowledge of content and students (KCS), namely “anticipate what students are likely to think”.

Betsy had been teaching for much longer than Annie had, more than 20 years, but she had not always been a teacher of mathematics. In fact, she had started by working with special needs students in different subject areas, and then had begun to focus on teaching mathematics with the same population of students at the secondary school level. When Betsy was given the interview problem, she was able to solve it quickly and correctly, as shown in *Figure 10*, but she was more tentative in her work than Annie was, saying, “Okay, this is the factorial. And granted, I don’t do that too much, but what I understand is you go 4, 3, 2, 1?” When asked about other methods, Betsy was not able to spontaneously think of an alternative, so she did not refer to the same tasks in SCK that Annie had, but she had no difficulty describing how her students would react to the problem and what they would do when faced with similar problems, referring to the following tasks “anticipate what students will find confusing” and “anticipate whether students will find a task easy or hard". For example, when talking about what students would do, Betsy said, "and the other one [kind of problem] that they have trouble with too is replacement and without replacement. I mean, some of the kids got it but others just really struggled with it." Note that by "replacement" and "without replacement", Betsy meant whether an item could be used again in a permutation once it had already been used once. These terms are common in secondary school classrooms, where they often talk about pulling items from a bag and either replacing or not replacing the selected item before choosing the next.

$4 \cdot 3 \cdot 2 \cdot 1$ 24 ways

Figure 10. Betsy's work on the interview problem.

These two different teacher profiles will enable a more detailed discussion below about the potential for this type of analysis.

Discussion

When looking at the first research question (*What knowledge types [CCK, SCK, KCS, and KCT] do teachers exhibit most frequently when answering a question about permutations in an interview setting?*), the analysis of teachers' statements reveals that they all demonstrated all types of knowledge during the part of the interview analyzed in this paper. It is striking that SCK was exhibited so frequently. However, SCK also has the largest number of specific tasks attached to it in the literature, as can be seen in Table 8. Could it be that this knowledge type is just better defined and thus easier to recognize?

The coding experience, though, suggests that this is not the case. During coding, the tasks were selected as literally as possible. After selecting the tasks, the knowledge types were populated, as discussed above. All teacher statements and corresponding knowledge types were subsequently reviewed to look for consistency between the theoretical definition of the knowledge type and the values that had been assigned through the coding. For the vast majority (158 of 168; 94%) of teacher statements, the coded result seemed consistent with the theoretical construction of each knowledge type.

While coding the teacher statements was a straightforward process, the exception to this was a collection of 10 out of 168 teacher statements (6%) where it did not seem that selecting the appropriate tasks from the existing list resulted in a list of knowledge types that fully reflected the teacher's statement. In all 10 cases, the problem was that the teacher statement, upon examination, seemed to exhibit knowledge of content and teaching (KCT), but no task from the list of those identified in the Ball, Thames, and Phelps (2008) paper captured the teacher's work. In all of these cases, the teacher was either (a) describing a mathematical instructional strategy, either past or future, or (b) choosing, but not explicitly justifying, a representation for instructional purposes. Examples are:

> *"I could have also visually arranged them to show and demonstrate all possible ways of arranging the cubes for my students."*
>
> *"I had a student who volunteered to go on the board, and he presented his idea and his method, and then other students compared with him and discussed it."*

"If a student was having trouble, with a concept like this, I would take out physical objects and have them play with them."

These cases seem to fit the definition of KCT as something that "combines knowing about teaching and knowing about mathematics" (Ball et al, 2008, p. 401), yet were not precisely captured by the tasks already detailed by Ball et al themselves. Perhaps as the field's understanding of KCT develops, the list of tasks that are thought to be indicators of it will expand.

This leads to the second research question: *Is it reasonable to consider knowledge types as manifested in particular statements, rather than as attributes of a teacher?* The process shown in this study indicates that this effort is reasonable. It was neither difficult nor unnatural to choose the relevant tasks corresponding to the teacher statements. We do not know now whether we would be able to do this easily with classroom video. For example, would only teacher statements be classified? Would there be coding for other types of teacher actions? And, of course, we do not yet know what can be gained from this type of classification, either of interviews or of classroom footage, as discussed below.

Value and Applications

The analysis described here suggests next steps that are extensive and not easily achieved. First, I will revisit the premise of finding value in distinguishing teacher knowledge types. Second, I will consider the potential for use of this particular type of interview analysis.

One overarching goal of the field is to understand what types of teacher knowledge exist and how they are useful, and then to begin to build ways to help teachers construct and enrich their knowledge in these areas. While mapping and coding knowledge types may begin as a theoretical exercise, it is one with a practical end goal. As discussed above, the dichotomous view of teacher knowledge as pure content or as pure pedagogy omits the importance of integrating these elements. Teachers unite these in their practice every day, yet we know little about how to determine which tasks of teaching they need help with and how to provide this help. More information about the complex tasks of teaching can only help.

A striking element of these results is that teachers did not all use specialized content knowledge (SCK) most frequently; four did so, while two exhibited CCK most often, one KCS, and one KCT. No teacher had an individual profile that closely matched the profile of the combined data from all eight research participants. What can we make of these wide variations? While making decisions based on these differences now would be unwise, if we were to apply this technique to a larger sample, we might see a set of teacher profiles emerge. In connection with classroom data, we could begin to understand what these different profiles suggest about the teacher's work of teaching.

This is illustrated by the two profiles, Betsy and Annie, described above. Let me be clear that I am not naming one profile as superior to the other or preferable for helping students to learn. However, the differences between these two cases illuminate the breadth of experience in mathematics and the variety of perspectives that exist in the teaching force. The analysis of different knowledge types highlights and clarifies the differences between the profiles, and could ultimately help to provide professional support to the teachers. For example, Betsy made relatively few statements showing evidence of specialized content knowledge (SCK). This might lead us to infer that *for this particular mathematical area* (permutations), Betsy could benefit from working in professional development activities related to SCK, such as working with and connecting a variety of representations. Conversely, Annie made few statements that showed evidence of knowledge of content and teaching (KCT). She might be better supported, then, by professional development that focused on the teaching aspect, such as choosing examples or deciding how to respond to student contributions. Another advantage of examining these teacher profiles is that we begin to see that different profiles may complement each other. That is, perhaps Betsy and Annie would be able to each take the lead in turn in sharing teaching knowledge with each other in a mutually beneficial way.

Second, we also need to consider how examining teacher knowledge types works and is or is not valuable in a format other than a written assessment. This particular interview analysis is different from previous work on distinguishing teacher knowledge types. The analysis of individual statements in interviews is based upon Ball et al. (2008), but not recommended or endorsed by them. While teacher performance on written assessment questions associated with some knowledge types has been correlated with student gains (Hill et al., 2005), no corresponding information exists on connections between this type of teacher interview analysis and outcomes either in the act of teaching or at the student level. The question remains exactly what we will gain from analyzing knowledge types in interviews.

It is important to consider that the freedom of an interview may make it more likely that teachers will elaborate on the elements that interest them the most. In doing so, they may move back and forth quite fluidly between knowledge types. This is supported by the findings above that all eight teachers exhibited all four knowledge types. In fact, Sherin (2002) suggests that teachers may access "content knowledge complexes" (p. 124), where the teachers' past experience results in a link between the content and the pedagogy that results in accessing these types of knowledge together. I agree that teachers link these different types of knowledge and may access them together; in fact, the way that the teachers in this study moved easily between knowledge types lends support to Sherin's theory. However, using the terms put forth by Shulman (1986), she says, "I claim that there are larger elements of teacher knowledge that cannot be categorized either as subject matter knowledge or as pedagogical content knowledge" (Sherin, 2002, p. 124-125.) I would suggest instead that it is not that a complex exhibited by a teacher can be classified as *neither* type of knowledge, but rather that it can be classified as *more than one* type of knowledge. The idea of content knowledge complexes gives us a view of how different knowledge is called forth by a teacher, but it does not preclude us from categorizing teacher statements more specifically.

The analysis presented here shows the types of knowledge the teachers exhibit in an interview and begins to create the foundation for describing different teacher profiles and using them to help in professional development. One of the most important elements, though, is that it takes the classification of knowledge types out of the context of a written assessment and into an interview setting. While not as easy to administer or score, the interview allows for a more descriptive view of a teacher's varied knowledge. This may help us not only to understand the different teacher profiles, but also to begin to see how they complement each other, as in the case described above.

In addition, written assessments that can claim to measure a particular type of teacher knowledge, like that described by Hill et al. (2005), need to be developed and tested extensively. By necessity, they can only cover a finite number of mathematical topics. If we want to know more about teacher knowledge about something specific, like the permutation question analyzed above, a coded interview allows this targeted examination. Interested researchers and those who work on professional development could look at teacher knowledge in their particular mathematical domain, even when they do not possess the resources that would be required to develop a written assessment.

Another implication for this work is that it provides a bridge to looking at teacher knowledge types in a classroom setting. The interview has more in common with classroom instruction than a written assessment because its open nature allows us to better see, through teacher statements, the process that may occur as one works at the job of teaching. Thus, looking at knowledge types through an interview may help us to begin to look at the mechanism by which teacher knowledge is manifested in the classroom, and thus how it helps students.

References

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Gess-Newsome, J. (1999). Pedagogical content knowledge: An introduction and orientation. In J. Gess-Newsome & N.G. Lederman (Eds.), *Examining Pedagogical Content Knowledge: The Construct and its Implications for Science Education* (pp. 3-17). Norwell, MA: Kluwer Academic Publishers.

Hadfield, O. D., Littleton, C. E., Steiner, R. L., & Woods, E. S. (1998). Predictors of preservice elementary teacher effectiveness in the micro-teaching of mathematics lessons. *Journal of Instructional Psychology, 25*(1), 34-47.

Hill, H.C., Ball, D.L, & Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H.C., Blunk, M.L., Charalambous, C.Y., Lewis, J.M., Phelps, G.C., Sleep, L., & Ball, D.L. (2008b). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Dean, C., & Goffney, I. M. (2007). Assessing elemental and structural validity: Data from teachers, non-teachers, and mathematicians. *Measurement: Interdisciplinary Research and Perspectives*, *5*(2), 81–92.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H.C., Schilling, S.G., & Ball, D.L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Huinker, D., & Madison, S. K. (1997). Preparing efficacious elementary teachers in science and mathematics: The influence of methods courses. *Journal of Science Teacher Education, 8*(2), 107-126.

Lowery, N. V. (2002). Construction of teacher knowledge in context: Preparing elementary teachers to teach mathematics and science. *School Science and Mathematics, 102*(2), 68-83.

Lubinski, C. A., & Otto, A. D. (2004). Preparing K-8 preservice teachers in a content course for standards-based mathematics pedagogy. *School Science and Mathematics, 104*(7), 336.

Philipp, R. A., Thanheiser, E., & Clement, L. (2002). The role of children's mathematical thinking experience in the preparation of prospective elementary school teachers. *International Journal of Educational Research, 37*(2), 195-210.

Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction, 20*(2), 119-150.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Strauss, S. (1993). Teachers' pedagogical content knowledge about children's minds and learning: Implications for teacher education. *Educational Psychologist, 28*(3), 279-290.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Analysis 3. How many outfits can I make? Overextending the multiplication principle.

Introduction

The purpose of this paper is to examine how students solve problems involving the multiplication principle, how they solve problems involving permutations, and possible explanations for their strategies and difficulties. The multiplication principle, often referred to as the "product rule" or "fundamental counting principle," states that if an event occurs in m ways and another event occurs independently in n ways, then the two events can occur in $m*n$ ways. The term "permutation" refers to an ordered arrangement of a set or subset of objects. In particular, the study presented here has uncovered an error students make while solving permutation problems, and this error seems attributable to the overextension of strategies used in multiplication principle problems. In this analysis, we will look at the following questions:

i. How do students come to understand the multiplication principle?
ii. How do students extend the multiplication principle to permutation problems?

For background in order to address these questions, we will first define the types of mathematical problems that are included in this work as "multiplication principle problems" and "permutation problems." Next, we will look at the mathematical connections between these two types of problems. Since the analysis attempts to address sources of confusion for students, we will also look at curricular treatments of these problems from textbooks at different levels.

Finally, we will clarify the idea of students "overextending" concepts and procedures by looking at existing research in this area. All of this background information will support the analysis of the data.

The data for this study consist of observations from two classrooms during a unit that included problems using the multiplication principle and problems using permutations, as well as interviews with 11 students from these classrooms. In looking at this data, we will analyze where students had difficulties with permutation problems and how this may relate to their use of the multiplication principle.

This topic is important because the multiplication principle is a concept that recurs in K-12 schooling and is used as an introduction or entry point to other topics within combinatorics. Combinatorics is a broad field of mathematics that includes many types of problems where discrete objects are counted. This field encompasses both the multiplication principle and permutations. Understanding of this material forms the basis for more advanced theoretical probability, which leads in turn to statistics, a field with numerous practical applications and with connections to many careers.

Multiplication principle problems are often students' first introduction to the ideas and representations of combinatorics, including (1) organized lists; (2) tree diagrams; and (3) the slot method. As a result, issues that arise when first studying this topic may hinder students as they progress further with this material. The multiplication principle has a direct connection to permutations, both with and without extra-mathematical contexts, yet students may misinterpret the connections to permutations. In this case, as we will show here, students may (a) be unable to solve permutation problems, and (b) still feel certain that they have correctly solved these problems.

Background Information

Multiplication Principle Problems and Permutation Problems

The multiplication principle states that if an event occurs in m ways and another event occurs independently in n ways, then the two events can occur in $m*n$ ways, as stated above. For example, suppose one has 4 shirts and 5 pairs of pants, and one must choose to wear one shirt and one pair of pants. Using the multiplication principle, we have 4 ways to choose a shirt and 5 ways to choose pants, so we have 4*5 = 20 ways to choose both. Because this outfit choice question is a typical example, I will refer to these types of multiplication principle questions as “shirts times pants” problems.

The term *permutations*, in mathematics, refers to an arrangement of some number of objects, where the order in which the objects are arranged matters. That is, the same objects presented in a different order would be a different permutation of those objects. Within permutations, there are two initial cases. First, the case with n objects, where all n must be arranged. For example, if we have three different letters, how many ways can we arrange all three of them? The number of permutations for a set of n objects where all n are arranged is $n! = n*(n-1)*(n-2)*...*1$. Second, the case with n objects where some number less than n must be arranged. For example, given all 26 letters in the English language alphabet, how many three-letter "words" (with no repeating letters) can be formed? The number of permutations for r of the objects from a set of n objects (for $r \leq n$) is denoted $P(n,r) = n! / (n-r)!$.

For our example problem of having three different letters and arranging all three, the number of permutations is $3! = 3*2*1 = 6$. For our example of arranging three out of 26 letters (with no repeating letters), the number of permutations is $P(26,3) = 26! / (26-3)! = 26! / 23! = 26*25*24$. We can see that the first case of arranging all n objects in a set is really a simplification of the second, since if $n = r$, $P(n,r) = n! / (n-r)! = n! / 0! = n! / 1 = n!$.

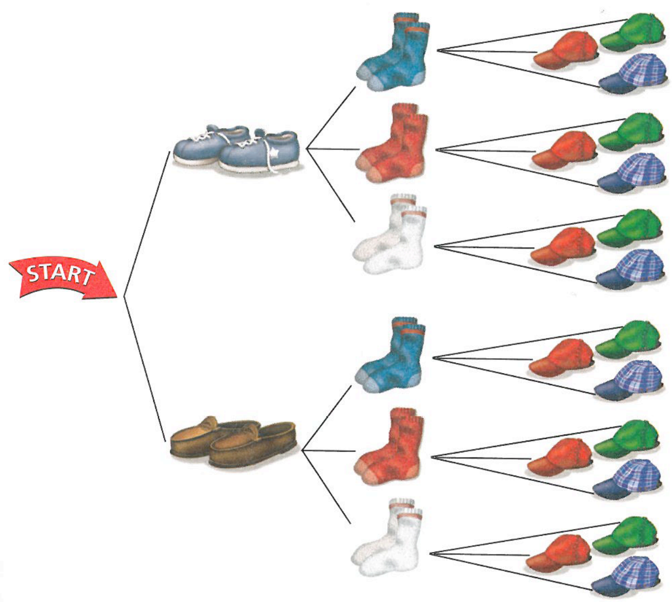
We can see that mathematically, multiplication principle problems and permutation problems are connected. When using the multiplication principle, one multiples the number of outcomes for each individual "event," as described above with the shirts times pants, in order to find the total number of outcomes. Permutations use this multiplication principle as their basis. However, making this connection requires thinking very carefully about the "events" and how many outcomes there are for each. If we take our example of arranging three different letters, we can define the first event as putting a letter in the first position of the arrangement. There are three possible outcomes for this event. We then think of a second event as putting a letter in the second position; there are two possible outcomes as there are only two letters remaining from which to choose. Our third event is defined along the same lines, as putting a letter in the third position, with only one possible outcome as there is only one letter left given that the others have been taken up already. In this way, we come around again to our permutation formula, or 3!.

While we can make the mathematical connections in this way, the differences and similarities between problems of these two types may not be apparent to learners. Problems that I will refer to as “multiplication principle problems” call for a use of the multiplication principle that does *not* result in a permutation of items from within a set. In the pants times shirts example, then, I am considering pants as one set of items and choosing a pair of pants as one event. The shirts are a different set of items and the choice of shirt a different event. I will reserve the term “permutation problems” for questions that specifically call for an ordered arrangement of items from within a single set. Examples of this type of permutation problem, from above, are the problems where we arranged three letters from a set of just three letters [P(3,3)], and arranged three letters from a set of 26 letters [P(26,3)].

Curricular Treatment of the Multiplication Principle and Permutations

Multiplication principle problems are particularly ubiquitous in K-12 education and recur throughout the curriculum. In particular, the middle school curriculum used in the school district in which this study took place makes use of tree diagrams and organized lists to illustrate problems using the multiplication principle, mostly to find a total number of possibilities in order to solve probability questions (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). They do not, however, explicitly name or generalize the multiplication principle. An example of this, from the seventh grade textbook, is shown in *Figure 11*.

4. José is going to a party. He decides to wear his jeans and a sweater, but he hasn't decided what else to wear. The tree diagram shows the possible outfits he can make if he chooses sneakers or loafers; a pair of blue, red, or white socks; and a green, red, or plaid cap, at random.



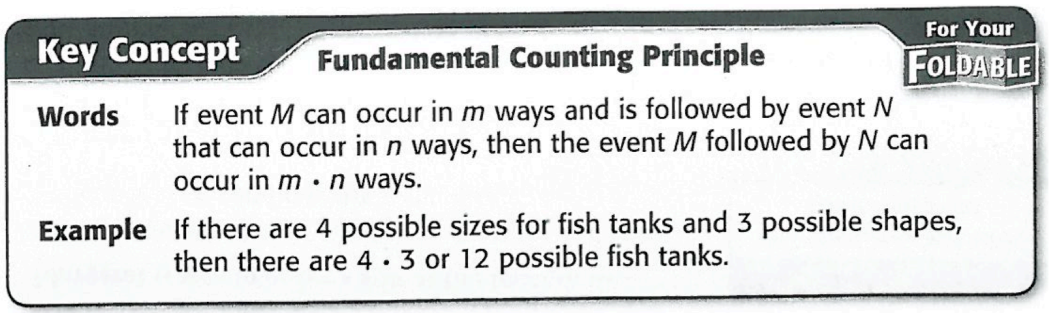
a. What is the probability that José will wear loafers, blue socks, and a plaid cap?

b. What is the probability that José will wear sneakers, either red or blue socks, and a green cap?

c. What is the probability that José will wear neither red socks nor a red cap?

Figure 11. A multiplication principle problem (Lappan et al., 2006, p. 11).

The multiplication principle is commonly used again, and defined more precisely, in high school classes in Algebra 1 and Algebra 2. While the classrooms that will be described in this study did not use student textbooks, an example of the treatment of this topic from a popular Algebra 1 textbook (Carter et al., 2010) is shown in *Figure 12* (note that it is referred to as the "fundamental counting principle"). At this level, the text defines the multiplication principle. In a subsequent section, this text also addresses permutations, defining them as follows: "The list of all the people or objects in a group is called the sample space. When the objects are arranged so that order is important and every possible order of the objects is provided, the arrangement is called a permutation" (Carter et al., 2010, p. 764, bold in original). This sequence and structure of introducing these two topics is repeated again in the Algebra 2 textbooks from the same series (Collins et al., 1997; Glencoe McGraw-Hill, 2010).



Key Concept **Fundamental Counting Principle** For Your FOLDABLE

Words If event M can occur in m ways and is followed by event N that can occur in n ways, then the event M followed by N can occur in $m \cdot n$ ways.

Example If there are 4 possible sizes for fish tanks and 3 possible shapes, then there are $4 \cdot 3$ or 12 possible fish tanks.

Figure 12. A multiplication principle definition (Carter et al., 2010, p. P35).

University level discrete mathematics instructional materials treat both the multiplication principle and permutations with rigor; however, the connections between the two may still not be clear to students. For example, Rosen (2003) defines the multiplication principle (called in his text "the product rule") as follows: "Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are n_1n_2 ways to do the procedure" (p. 302). One example in this text for using this rule is having a certain number of computers, each with a set number of ports. While the context may differ from pants and shirts, this follows our guidelines (above) as a type of problem with a non-permutation application of the multiplication principle. Tree diagrams are also introduced in the same section of the text, although after the rule has been defined. One example given for this representation is making t-shirts with different size and color options, which would continue to follow the n_1n_2 rule.

A subsequent section of the chapter introduces permutations, defining them as follows: “A permutation of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an r-permutation” (p. 321). In a similar order of instruction to that used for the multiplication principle, the text gives the theorem first: “The number of r-permutations of a set with n distinct elements is $P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$” (p. 321). The proof of this theorem calls on the multiplication principle and follows (and formalizes) the same logic we used above to describe the connections between the two mathematical topics. This section also uses contextualized examples that would not seem unfamiliar to younger students used to pants times shirts types of problems: prize winners in a lottery, runners in a race, and arrangements of letters.

In this university level text, then, there is a mathematical connection between the two types of problems, in that the multiplication principle is used to justify the formula for permutations. However, this connection is only made in the formal context of the proof of the theorem, and may not be accessible to learners. There is no explicit treatment of how questions of the two different types are similar and different, and how to choose a solution strategy based on the type of problem.

Connecting and Separating these Topics

It is important to note that while we can connect multiplication principle problems and permutation problems with mathematical ease here, these same connections may not exist for students or be made as clear to teachers of K-12 curricula. As we will see in the data below, classroom observations show that students may work on both types of problems within the same unit but realize neither the *similarities* nor the *differences* in the problem types. When this occurs, what are students to do? One possibility is that they may apply a solution method and explanation that they use for one type of problem to the other type of problem. In doing so, they would be overextending a theory about why one method, say, the use of the multiplication principle, works and is valid. This is similar to the children attempting to balance blocks described by Karmiloff-Smith and Inhelder (1974). The children they describe hold theories about what will and will not balance, and how, and these theories are not easily changed. In fact, they overgeneralize their theories to encompass new situations. This is not unreasonable; in fact, the authors state that, "Overgeneralization, a sometimes derogatory term, can be looked upon as the *creative simplification* of a problem by ignoring some of the complicating factors [...] Overgeneralization is not only a means to simplify but also to unify" (p. 209).

Karmiloff-Smith and Inhelder's work considers children interacting with the physical world, but researchers in mathematics education have observed the same type of overgeneralization. One example of this is the conception that the result of multiplication will be a bigger number and that the result of division will be a smaller number. This was observed by Bell, Swan, and Taylor (1981) in student interviews where students were asked to choose an arithmetic operation; when students judged from the problem context that the answer should be smaller, an end that would have been achieved by multiplication by a number between 0 and 1 as presented in the problem, they chose division. The widespread use of this overgeneralization was confirmed with 12 and 13-year-old students by Bell, Fischbein, and Greer (1984) and with pre-service teachers by Tirosh and Graeber (1989). Bell et al. (1984) point out the potential source of the incorrect extension: students initially work with whole numbers in the early years of schooling, thus enabling them to form a theory about the effects of multiplication and division that is correct and useful to them at the time. It is only later, when working with non-integer numbers, that the overgeneralization of their theories to the new situation leads to incorrect conclusions.

We can use this lens of overgeneralization of a theory, as presented by Karmiloff-Smith and Inhelder (1974), as we look at the classroom and interview work of the students in this study. The information found in examining the data suggests that they, too, use a theory that they have developed while working with one type of problem (multiplication principle problems) and overextend this theory to solve a different type of problem (permutation problems).

Methodology

Summary

The data sources for this study were observations from two different classrooms, as well as interviews with 11 students from these classrooms. During interviews with students, participants were asked to complete mathematics problems, and also to reflect upon them, explain their solutions, and evaluate alternative solutions. Classroom observations took place in the classes of two teachers, while the teacher was covering a topic within combinatorics.

The interviews were logged and then coded according to an *a priori* scheme based on the literature and data from a pilot study. During the coding, the author of this paper noticed a pattern of similar incorrect student responses. A new code was defined to account for incorrect application of the multiplication principle; all student interviews were then re-coded. Logs from classroom observations were reviewed for instances of the same type of responses or explanations.

Participants

The author had the opportunity to meet several teachers while assisting with a professional development workshop; all teachers were invited to participate in this study, though participation was independent of taking part in the workshop. All teachers who participated in the workshop received a letter inviting them to participate and explaining the project. Teachers were asked if they would be willing to allow classroom observations and to have student participants sought from within their classrooms.

Nine teachers consented to participate; of these, only two teachers both were willing to allow student participants and were planning to teach a combinatorics unit that included lessons on permutations during the following school year. Once these two teachers were identified, the administration of each of their schools was contacted with a letter explaining the project and asking for their participation. Both participating teachers teach at secondary schools in a large urban school district in the state of Massachusetts.

After the administrations of the schools had consented to participate, the families of all students of the two teachers received a consent letter and explanation of the study. The students were also asked for their assent. Fourteen students assented and had their parent or guardian consent as well. Ultimately, eleven of these students were available and were interviewed; seven students were from one class and four students were from the other.

Measures and Data Collection

Data was collected through classroom observations and student interviews. An overview of the data types is given in this section, with a summary shown in Table 12, followed by a detailed description of each source.

Classroom observations were completed in May and June 2010. After the classroom observations had taken place, student interviews were done in June 2010. Student interviews were conducted after instruction related to these topics was complete in each classroom.

Table 12. *Data collected.*

Measure	Time of measurement	Mode
Classroom observations	May / June 2010	Written observations; handouts and instructional materials
Student interviews	June 2010	Videotaped interview; subsequently logged

Classroom observations. The first source of data was classroom observations. Classes were observed during the time that the teacher was providing instruction on combinatorics. In the secondary school curriculum in the school district in which this study took place, these topics are introduced in Algebra I at the end of the school year, in May or June. Each of the two participating teachers was teaching two sections of Algebra I, each attended by students 14 to 16 years old (i.e., freshmen to juniors in high school), at the time of data collection. The first teacher, who we will identify by the pseudonym Shana, taught related topics during five school days; classroom observations were carried out on all five days. The second teacher, Whitney (also a pseudonym), taught these topics during two school days, so the classroom observations were carried out on two days only. These details are summarized in Table 13.

Table 13. *Number of classes and students.*

Teacher name	Number of classes spent on material	Number of students interviewed
Shana	5	7
Whitney	2	4

During classroom observations, I recorded on paper as much of the classroom activity as possible, along with recording all mathematical problems that were addressed during the class and collecting handouts when used. In addition, I used a checklist for quickly noting topics and concepts addressed during the class. A new checklist was used for each 15-minute interval (see Appendix A). The checklist provided a structure to allow quick notation of the type of mathematical problem, the type of representations being used, and the type of classroom activity, such as teacher-led discussion, group work, or independent work.

Student interviews. All consenting and available students in each class were interviewed. Eleven students were interviewed, with seven students from Shana's classes and four students from Whitney's classes, as shown in Table 13. Each interview was videotaped. While interviews were flexible and open-ended, they had a goal of discussing problems about permutations. To do this, the students were given problems to solve (shown in Appendix B). After solving each problem, they were asked for an explanation. They were then asked for a different way to solve the same problem and a different explanation. This process repeated for each mathematical problem. I attempted to elicit a full explanation of both correct and incorrect answers and strategies.

In a previous pilot study on this topic, five teachers were interviewed and completed and discussed a set of seven problems. The problems completed by teachers covered permutations, combinations, and probability. In examining such a range of concepts, many of the problems were classified by the participating teachers as being beyond the scope of the middle and high school curricula. For this reason, the range of topics in the student interview was reduced in the study reported here. The main change was to eliminate questions on combinations of objects. These questions were removed because neither teacher addressed combinations, or unordered groups, in the classroom. Also note that there were no "multiplication principle" problems in the student interview. This is because the original focus of the study was restricted to permutations, combinations, and probability. The questions used in this study are shown in Appendix B.

Data Analysis

Each student interview was logged and then coded according to an *a priori* scheme that included flagging which method or methods students were using to solve each problem. While coding, this researcher noticed that many students used incorrect methods, and corresponding explanations, that relied on multiplying two or more numbers extracted from the problem at hand. Careful examination of a few student explanations (shown below) revealed that the students were connecting their understanding of the multiplication principle to problems that required the use of permutations, yet the application was incorrect. A new code was defined to account for this and all student interviews were then re-coded to look for instances of this reasoning. This became the basis for the analysis presented here. Comparisons were made across questions on the percentage of student participants who incorrectly applied the multiplication principle.

Logs from classroom observations were reviewed to look for instances of the same type of responses or explanations. These data sources provide qualitative support for the observation that this type of reasoning occurs. They are used to enrich the analysis.

Analysis

When all student interview responses had been examined, the results showed that some questions had remarkable percentages of students using an incorrect strategy based on the multiplication principle, as shown in Table 14. For each of the three interview problems involving permutations, 45-55% (5-6 of the 11 students) incorrectly overextended the multiplication principle. On the three interview problems involving probability, 0-9% (0-1 of the 11 students) made this type of error. All questions are visible in Appendix B; in addition, the text of the questions and the mathematical classification of each are shown here, in Table 14.

For both questions 1 and 5, 6 of 11 students (55%) used an incorrect multiplication principle premise. For question 2, 5 of 11 students (45%) used an incorrect multiplication principle premise. All of these questions (1, 2, and 5) involve permutations. For questions 4a and 6, only one student (9%) incorrectly used the multiplication principle on each question. For question 4b, no students (0%) incorrectly used the multiplication principle. All of these questions (4a, 4b, and 6) involve probability and do not require the use of permutations. (Note that there was no question 3 in the interview.)

Table 14. *Students incorrectly using multiplication principle, by question.*

Question number	Question text	Mathematical classification	# / % of students incorrectly using multiplication principle

1	How many ways can you arrange these objects? *(shown as A, B, C, D)*	Permutation	6 students (55%)
2	If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?	Permutation	5 students (45%)
4a	You have 6 marbles in a bag. Four marbles are blue and two marbles are yellow. If you choose one marble without looking, what is the probability that the marble you pick is yellow?	Probability	1 student (9%)
4b	If you choose one marble without looking, and then you choose a second marble without looking, what is the probability that they are both yellow?	Probability	0 students (0%)

5	There are 4 students staying after school; how many ways can you choose 2 students to clean the board, 1 student to sharpen pencils, and 1 student to organize papers? *(shown with student names)*	Permutation	6 students (55%)
6	You have 10 cards, numbered 1 through 10. If you draw 2 cards, what is the probability that the sum of the numbers on the cards is even?	Probability	1 student (9%)
** only one instance per student is counted*			

Note that Table 14 counts only the number of students incorrectly using the multiplication principle in their answers and explanations, not the total number of responses using this. There were two instances where a student gave more than one response to a single question where both responses were based upon different incorrect applications of the multiplication principle. In each case, this was still counted as just one student. Also note that since one focus of the interviews was to elicit multiple methods and representations from students, the students may have gone on to correctly solve the problems using other methods.

The present analysis does not classify whether or not students were able to determine the correct answer at some point during the interview, only where the multiplication principle was (incorrectly) applied to the questions.

Given the prevalence of the use of the multiplication principle in questions 1, 2, and 5, the next step in the analysis was to closely examine student interview responses to these questions. In the subsequent paragraphs, each of these three questions will be discussed and student interview examples and excerpts from interview transcripts will be used to illustrate and justify the classification. The examples shown here are typical of those given by the other students who also used an incorrect multiplication principle strategy but are not specifically cited; a full view of the frequency of the different incorrect multiplication attempts is shown in Table 15.

Question 1 asked students for the number of ways to arrange four objects, a problem with a correct answer of $4! = 4*3*2*1 = 24$. This answer could have been determined using the factorial, the slot method, a tree diagram, a list of possibilities, or possibly some non-traditional or hybrid method. Of the 6 students incorrectly using the multiplication principle, Donald, a student of Shana, and Gabriel, a student of Whitney, offered explanations that clearly show connections both to mathematical problems in which the multiplication principle would be helpful and to their classroom experiences. Donald first answered the question by writing "16 ways" and explaining that he multiplied 4 times 4 because he had learned how to do it the "fast way". He explained that instead of listing out the possibilities, you multiply the number of objects. When asked what to multiply the 4 objects by, he said that it's by 4 outcomes, but he was not exactly sure why. He stated that he learned to do it the "fast way" early and then he just knew how to do it so he would multiply.

Gabriel also stated that the answer was 16, 4 times 4. He then made an explicit link to an example of a different problem that would correctly use the multiplication principle: "There's a formula that she taught us… what was it… remember what she gave us about the cookies and stuff? If there's 3 types, 3 what you could put on it, and 3 drinks, you multiply 3 by 3 by 3 and you get your answer." In making this link, he used the problem with cookies and drinks to justify his choice to multiply 4 times 4.

Question 2 asked students for the number of ways to choose a president, a vice president, and a treasurer for a club with 10 members. This problem, a permutation of 3 of 10 unique objects, has a correct answer of 10*9*8 = 720. As with question 1, students could use the slot method, a tree diagram, or a list, although a tree diagram or list would be onerous given the large number of possibilities. For this problem, the 5 students who incorrectly used the multiplication principle showed different types of reasoning. The most common of these is exemplified by José, a student of Shana, who simply stated that he would multiply 3 roles by 10 kids, so there would be 30 possibilities. Gabriel, Whitney's student, had a different approach. He first stated that the answer would be 10, and then justified this answer by writing out "student", "president", "vice president", and "treasurer" as his categories, as shown in *Figure 13*. He then explained that he put the 10 because there are 10 options for students, but only one option each for president, vice president, and treasurer. He referred to each of the four words as a "type" and says you multiply all the "types" together. Thus, he reached his answer by multiplying 10*1*1*1 = 10.

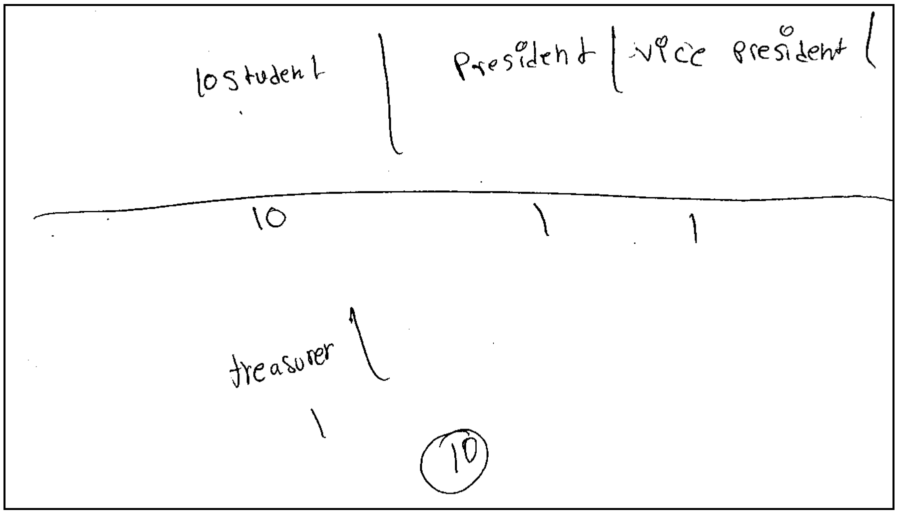


Figure 13. Gabriel's solution for question 2.

Question 5 differed from questions 1 and 2 as it was not a simple permutation. The question asked for the number of ways to assign jobs to four students, where two of the students do the same job. In this case, the answer, (4*3*2*1)/2 = 12, could be determined using the slot method, a tree diagram, or a list of possibilities. There could also be student-generated methods or hybrid methods that would yield the correct answer. However, there were 6 students who showed evidence of an overgeneralized use of the multiplication principle on this question. Sandy, a student of Shana, reasoned that there were four people and three jobs, so it would be four times three (note that this actually returns the correct answer of 12, but with an incorrect justification). She then changed her mind and thought it could be 16, with four students multiplied by four jobs. Ultimately, she decided to stay with an answer of 12, explaining that there are four students and "three different categories to choose from for all the students," so 4 times 3 equals 12. David, Whitney's student, used a different approach, writing 2*4*4=32. He explained that the 2 was for the two students cleaning the board, the 4 is for the number of students that are there, and the other 4 is for the number of choices of students that can do the pencil sharpening.

These sample explanations show that there were a number of different ways in which students overextended the use of the multiplication principle, as shown in Table 15. Note that in this table, each incorrect response of this type is included, even where some students gave more than one for a particular question. As a result, the number of instances for each question is different than in Table 14, where the number of students using the multiplication principle is shown. Although the numbers that were multiplied differed, the explanations all included a justification based on multiplying the number of things of one type by the number of things of another type, just as you would multiply the number of shirts by the number of pants in order to determine the number of outfits.

Table 15. *Numbers used with multiplication principle.*

Question	Which numbers were multiplied?	Number of explanations using these numbers
1	2*4	1
	4*4	5
2	1*1*1*10	1
	3*10	4
	3*10!	1
5	2*1*1	1

2*1*1*4	1
2*4	2
2*4*4	1
3*4	1
4*4	1

Having observed this phenomenon in so many student responses, I examined the logs and handouts from classroom observations in order to look for (1) instances of the same type of responses or explanations, and (2) the connections in the classroom between problems using the multiplication principle to find the number of things like outfits and problems involving permutations of items.

In Whitney's class, the first day of the unit on combinatorics, and thus the first day of observations, began with a problem about a variety of ice cream flavors (2), toppings (2), and cone type (2), yielding 8 possible choices. This problem is an exemplar of a "shirts times pants" multiplication principle problem. Students used a handout and first constructed a list of the possibilities and then a tree diagram. After completing this, students did a list and a tree diagram for a second problem involving pizza, at which point a student correctly suggested that you could multiply the options for each choice to get the answer. At this point, the teacher moved to an overhead display of the multiplication principle (referred to in this classroom as the fundamental counting principle) reading: "If an event M can occur in m ways and an event N can occur in n ways, then M followed by N can occur in m*n ways." The teacher then revisited the problems and solved them using the multiplication principle. Whitney asked which method the students preferred and they answered that they preferred to multiply because it was shorter and less work.

Whitney then distributed a handout that contained seven problems using the multiplication principle, similar to those they had just seen as a class, and three problems involving permutations. For example, one permutation question read, “In how many ways can you arrange 5 boxes of cereal on a shelf?” As students moved easily through the problems using the multiplication principle, several began to ask Whitney about this cereal box problem. She explained to each of them that they had 5 choices for the first box, and then asked how many choices there would be for the second box. Students were able to respond that there would be 4 choices, then 3 choices, and so on. In this way, students were able to answer the permutation questions correctly with assistance.

On the following day, Whitney’s students did three problems to begin class, one of which was a straightforward multiplication principle problem involving 5 shirts, 2 pairs of pants, and 10 pairs of shoes. The second problem was a permutation of all n of a set of n objects (the order to ride 12 roller coasters in an amusement park); the third problem was a permutation of m of n objects for $m<n$ (the number of ways to ride the roller coasters if they only had time to ride 8 of 12). Students had no questions about the multiplication principle problem; Whitney reviewed the second and third problems with the class and also introduced the language and notation for factorial. Following this, the students

played a game that included both permutation and multiplication principle problems, as well as decontextualized problems with factorials.

In Shana's class, the first lesson in the unit on combinatorics began with groups of students working at four different stations; each station had a different problem using the multiplication principle. At each station, students were to find the number of possible outcomes for rolling two dice, creating a two-digit code with a limited number of letters, creating an outfit from a collection of dresses and scarves, and building a sandwich with options for bread type and filling type. Students were allowed to use any methods or representations they found helpful; many students made lists of options, while some of the students claimed to their groups that they could just multiply the different numbers. Shana then led the whole class in a different problem of electing a president and vice-president, but using separate slates of three and two candidates, respectively, that did not overlap. To address this multiplication principle problem, the teacher led the class in building a list and then a tree. Students then worked to build lists and trees for additional multiplication principle problems.

On the second day of the unit, students began by making trees or lists for another multiplication principle problem, rolling a six-sided die and flipping a coin. They then talked about a shortcut to find the number of possibilities if they added a second coin flip to the chain of events. Students were able to state that they should multiply by two in this case and the teacher agreed and wrote on the board that one could multiply the number of options for each event. Shana then moved to a section of the handout talking about permutations, which read, "this will look like doing the same event several times over." The class worked on a problem about electing people from a class of 12 students to three different offices. Shana used a tree diagram and the slot method and explained that once a student had been elected, there would only be 11 options for the second slot. In addition, in one class section, she specified that the difference between this problem and the problems from the previous day was that all the things being arranged are from the same group. Students then worked on a mixture of multiplication principle and permutation problems. Two instances of students applying the multiplication principle incorrectly were observed: in one case, the problem asked about arranging 6 people in a line and a student stated that the answer should be 36. In the second case, a question about distributing two different awards to two of 19 students (19*18), at least two students believed the answer should be 19*2. At the close of the class, Shana stated that they had been working on permutations, and that these occur when there is one group and they are choosing several things from it.

On the third day, Shana's classes took a quiz at the beginning of the period in which the first problem used the multiplication principle to solve a problem with ice cream flavors and ice cream toppings, and the second question used a permutation to determine the number of ways to give out awards in a dog pageant. Two instances of confusion between the multiplication principle and permutations were observed. In one case, rather than using a permutation, a student multiplied 17 (the number of dogs) times 4 (the number of prizes) to get an answer of 68. In another case, a student correctly used a permutation for the dog pageant problem

(17*16*15*14), but attempted to also use this for the ice cream problem, multiplying 3*2*1 and marking beneath each number the names of the ice cream flavors; thus, this student was finding all the arrangements of the three ice cream flavors.

In Shana's class, the instruction on the third, fourth, and fifth days was part of the same unit; however, it focused on experimental and theoretical probability, as well as recording and reporting data in tables and histograms. There was no additional instruction or activities focused on the multiplication principle or permutations.

Discussion

The results from the student interviews show that a high percentage of students made errors on permutation problems that could be attributed to their overgeneralizing the application of the multiplication principle. In order to help students solve and understand permutation problems more easily, we should attempt to identify factors that could contribute to this type of student difficulty.

In looking at the classroom observations, we should consider the *transition* and *connections* between the multiplication principle and permutations. Similar to the overgeneralizations described by other researchers (e.g. Karmiloff-Smith & Inhelder 1974; Bell et al., 1981; Bell et al., 1984; Tirosh & Graeber, 1989), the students in this study appear to be using a reasonable strategy that has worked well for them in the past; however, they are applying it to problems where it no longer works. It stands to reason, then, that we might be able to help student distinguish between problems where the multiplication principle is immediately useful and problems where it is not.

In Whitney's class, the transition occurred while students worked on a set of problems independently or in small groups while the teacher circulated and answered questions. The problems moved from those that use the multiplication principle with different categories of items (e.g., ice cream toppings and cone type) to those that permute items within a category (e.g., arranging five different cereal boxes). The two types of problems were then intermingled on the assignment sheet. The teacher explained how to do one of the permutation problems, but there was no discussion of how the problems were *similar to* or *different from* the other problems.

In Shana's class, the transition began when the teacher introduced a handout that stated that they would be looking at arranging things in order and then led the class in solving one of the problems using both a tree diagram and the slot method. Then, as students began to work independently, the two types of problems were also intermingled. The handout had a question asking how the problems differed, but this was not explicitly discussed in class.

In both cases, we see that the transition occurred as problems shifted from those using the multiplication principle to those requiring permutations. In one case, the transition was addressed and an example problem done with the class before students attempted problems independently; in the other case, it was not. However, in both classrooms there is an implicit assumption that permutations will follow from the multiplication principle. The classroom observations and the review of curricular materials (Carter et al., 2010; Collins et al., 1997; Glencoe McGraw-Hill, 2010) suggest that the existing instructional sequence may be built off the idea that problems using the multiplication principle will serve as an entry point into problems with permutations of objects. This is mathematically reasonable, as the multiplication principle enables the proof of the formula for permutations (Rosen, 2003).

However, while the transition follows mathematically, the *connections* between the multiplication principle and permutations were not discussed explicitly in either classroom. In both cases, the teachers elegantly used lists and tree diagrams as the initial methods to solve multiplication principle problems about things like “dresses times scarves” and “ice cream flavors times toppings”. Because the teachers structured their lessons in this way, this process of mathematical observation enabled students to discover the multiplication principle itself, as well as justify it. The students saw the tree diagrams and lists as more onerous and, in both interviews and classroom observations, multiplying the number of options for each event was seen as a shortcut.

Having appropriated this method, it seems the students may use their newfound expertise to solve permutation problems. It is reasonable and appropriate that they would try to extend their theories. However, they did so without knowing how the two problem types differ. They used the “fast way” to identify numbers within the problem and multiply them together, not recognizing that the strategy is misapplied. This is reminiscent of the students discussed by Karmiloff-Smith and Inhelder (1974). The student responses during class and the student responses during the interviews show clear evidence of students using the multiplication principle as a shortcut, but with incorrect numbers. The explanations that they provided during the interviews suggest that they are building on multiplication principle problems, looking for "shirts" to multiply by "pants," even when they do not explicitly state this. For example, in the election problem given during the interview, when José, described above, stated that he would multiply 3 roles by 10 kids, giving 30 possibilities, he has defined "roles" as a category of things, akin to "shirts". The second category of things is "kids", of which there are 10, akin to "pants". Having determined that there are two categories of things and determined how many things are in each category, he used the multiplication principle to multiply these two numbers together.

This incorrect application of the multiplication principle is entirely reasonable for students! They know that they have previously determined a shortcut. They know that these problems come from within the same unit in their math class, and are many times, as we have seen, even dealt with in the same lesson and handouts, with no explicit distinction between the types of problems pointed out to them. So they extend what they know to solve problems with permutations. What they do not realize is where the mathematical connections are between these two types of problems.

In the college discrete mathematics text (Rosen, 2003), permutations were introduced using the multiplication principle as proof for the formula. In the classroom observations carried out for this study, permutations were introduced as an extension of "shirts times pants" problems, which is sensible given the mathematical connections between them. However, the formula for finding the number of permutations of a set of objects was not initially determined by the students through the use of lists and trees, as they had done when they discovered the multiplication principle. In addition, there was little explicit discussion through example or explanation of how these permutation problems differed from multiplication principle problems and why the students' inclination to use the shortcut (in which they felt confident) was actually incorrect.

If students connect these concepts incorrectly, this would suggest that curricular materials and instruction should be modified to take this into account, either clarifying or divorcing the concepts. It seems that students are operating from a theory that they have built, and been encouraged to build, about why the multiplication principle is valid. In the classrooms described here, the teachers led the students to construct their own understanding of the multiplication principle, rather than simply giving it to them. As a result of these instructional practices, students seemed to understand the multiplication principle well and to be able to justify it. However, this theory then became a strong part of each student's mathematical understanding and we must remember that corrections and incorrect outcomes are not necessarily enough to disrupt this (Karmiloff-Smith & Inhelder, 1974).

While the classrooms in this study did not use a particular textbook, examining the popular Glencoe McGraw-Hill Algebra 1 textbook (Carter et al., 2010) reveals that the instructional sequence is the same as that used in the classrooms, with the multiplication principle preceding permutations. In fact, the multiplication principle is addressed in a beginning chapter intended to review material and prepare students for algebra; this chapter is the source of the definition shown in *Figure 12*. The text shows tree diagrams and ordered lists applied to multiplication principle problems, and then gives examples of multiplying the number of options in each category to find the answer. However, the text then moves directly from a problem that requires students to multiply the number of options in different categories (as seen in *Figure 14*) to a problem that permutes objects from within a single set (also shown in *Figure 14*). While these problems can be considered using the multiplication principle as the mathematical basis, the text offers no information about how one type of problem is *the same as* or *different from* the other.

EXAMPLE 4

a. An ice cream shop offers one, two, or three scoops of ice cream from among 12 different flavors. The ice cream can be served in a wafer cone, a sugar cone, or in a cup. Use the Fundamental Counting Principle to determine the number of choices possible.

There are 3 ways the ice cream is served, 3 different servings, and there are 12 different flavors of ice cream.

Use the Fundamental Counting Principle to find the number of possible choices.

number of scoops		number of flavors		number of serving options		number of choices of ordering ice cream
3	•	12	•	3	=	108

So, there are 108 different ways to order ice cream.

b. Jimmy needs to make a 3-digit password for his log-on name on a Web site. The password can include any digit from 0-9, but the digits may not repeat. How many possible 3-digit passwords are there?

If the first digit is a 4, then the next digit cannot be a 4.

We can use the Fundamental Counting Principle to find the number of possible passwords.

1st digit		2nd digit		3rd digit		number of passwords
10	•	9	•	8	=	720

So, there are 720 possible 3-digit passwords.

Figure 14. A multiplication principle problem and a permutation problem (Carter et al., 2010, p. P35).

Later, in a different chapter, the text addresses permutations and states that the multiplication principle can be used. The transition is made in the context of an example: "Suppose Angie's coach has 4 players in mind for the first 4 spots in the lineup. The Fundamental Counting Principle can be used to determine the number of permutations. A batter cannot bat first and second, so once that player is chosen, she is not available for the next choice" (Carter et al., 2010, p. 764). They then show the number of permutations as P = 4 • 3 • 2 • 1 = 24. This is mathematically appropriate, and again makes sense to those already comfortable with the distinctions and connections between the two types of problems. However, there is little indication of how to choose which numbers to multiply. While the multiplication principle is cited, there is no explicit connection to the types of "shirts and pants" problems that students saw in the section of the book expressly devoted to the topic. The difference between the category of number of ice cream flavors (as shown in *Figure 14*) and the category of who will bat second (as seen in the problem described above) is left implicit. In addition, while even the brief review-oriented section on the multiplication principle included tree diagrams and ordered lists as both a method to solve problems and a way to justify the multiplication principle, these representations do not appear in the section on permutations. The implication is that these alternative methods, which enabled the teachers and students in this study to impart meaning to the multiplication principle, are divorced from the "new" material of permutations.

The study presented here shows only a small number of students, and no quantitative judgments can be made about the likelihood of this type of student error in the general population. Similarly, the small number of students prevents us from comparing the instruction by the two different teachers as it relates to these errors. However, the analysis does show that the understanding of the multiplication principle that the students gained did not transfer automatically to the correct solution of permutation problems. While, as instructors, we might see the mathematical connection, the students here did not. This shows us that we cannot assume that the multiplication principle serves as a natural entry point for problems using permutations. As a result, it seems that we need to guide students to develop an extended theory that will work for permutation problems, and to help them understand when each strategy is applicable. One possible instructional approach could be to introduce permutation problems with the same use of multiple representations, such as tree diagrams and lists, and the same view toward discovering and justifying that was exhibited by the teachers described here and that enabled students to gain facility with the multiplication principle. Another potentially valuable instructional element could be the use of an activity that requires students to explicitly map the connections between the two types of problems. In this way, students may come to a strong understanding of why permutation problems can be solved using factorials, how the two types of problems are similar and how they are different, and when each one is applicable.

Most importantly, curriculum designers and teacher educators should be aware of this potential pitfall. The results from this study suggest that common instructional sequences, as suggested by existing curricula, do not take into account the difficulty students have when making transitions between multiplication principle problems and permutation problems.

References

Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure, and context. *Educational Studies in Mathematics, 15*(2), 129-147.

Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics, 12*(4), 399-420.

Carter, J.A., Cuevas, G.J., Day, R., Malloy, C., Holliday, B., & Luchin, B. (2010). *Algebra 1*. Glencoe / McGraw-Hill, 2010.

Collins, W., Cuevas, G.J., Foster, A.G., Gordon, B., Moore-Harris, B., Rath, J., Swart, D., & Winters, L.J. (1997). *Algebra 2*. Glencoe / McGraw-Hill, 1997.

Glencoe McGraw-Hill (2010). Algebra 2 Online Learning Center. Retrieved from http://glencoe.mcgraw-hill.com/sites/0078884829/

Karmiloff-Smith, A., & Inhelder, B. (1974). "If you want to get ahead, get a theory". *Cognition, 3*(3), 195-212.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (2006). *Connected Mathematics 2, Grades 6, 7, 8*. Prentice Hall, 2006.

Rosen, K. (2003). *Discrete Mathematics and Its Applications*. Boston, MA: McGraw Hill.

Tirosh, D., & Graeber, A.O. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics, 20*(1), 79-96.

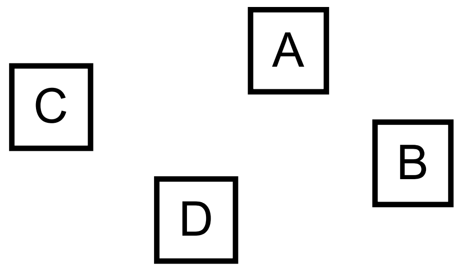
Appendix A: Classroom observation checklist

Category	Item	Time
Lesson format	Teacher-led instruction (with times)	
	Students address whole class (with times)	
	Individual work (with times)	
	Group work (with times)	
	Working on applied (real-world) problems	
Richness	Multiple procedures or solution methods	
	Explanations	
	Developing mathematical generalizations	
	Mathematical language	
Working with students and math	Teacher questioning	
	Remediating student difficulties	
	Uses student mathematical ideas in instruction	
Errors	Major mathematical errors or oversights	
	Imprecision in notation or math language	
Student activity	Students provide explanations	
	Student mathematical questioning and reasoning	
	Enacted task cognitive demand	
Representations	Formula	
	List	
	Tree	
	Slot	
	Explicit linking	
General topic issues	Questions about assumptions	
	Comparing methods	
	Using complement	
	Compound probability	
	Deciding whether order matters	
	Estimation of probability	
	Finding probability denominator separately	
	Language use	
	Order of introducing methods	
	Replacement of elements	
	Testing smaller cases	
	Using physical objects	
	Word analogy for identical items	
Formula method	Deriving formula	

	Division to remove identical combinations	
	Division to remove identical items	
	Finding probability denominator	
	Justifying formula	
	More than one set of identical items	
	Multiplication choice	
	Remembering formula	
List method	Knowing when all possibilities are there	
	Removing identical combinations	
	Removing identical items	
	Removing identical permutations	
	Systematic listing	
Slot method	Division to remove identical combinations	
	Division to remove identical items	
	Finding number of possibilities per slot	
	Finding number of slots	
	Multiplication choice	
	Removing identical items	
Tree method	Finding number of branches from a vertex	
	Multiplication choice	
	Removing identical combinations	

Appendix B: Interview Questions

Question 1
How many ways can you arrange these objects:

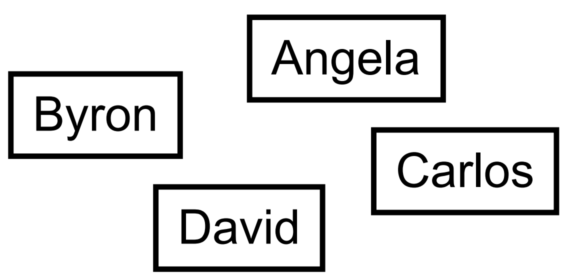


Question 2
If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?

Question 4a
You have 6 marbles in a bag. Four marbles are blue and two marbles are yellow. If you choose one marble without looking, what is the probability that the marble you pick is yellow?

Question 4b
If you choose one marble without looking, and then you choose a second marble without looking, what is the probability that they are both yellow?

Question 5
There are 4 students staying after school:



How many ways can you choose 2 students to clean the board, 1 student to sharpen pencils, and 1 student to organize papers?

Question 6
You have 10 cards, numbered 1 through 10. If you draw 2 cards, what is the probability that the sum of the numbers on the cards is even?

Analysis 4. Student representations in combinatorics

Introduction

This paper looks at students' representations in combinatorics. The data and analysis presented aims to show that (1) students generate useful non-canonical representations of combinatorics, (2) the production of these representations is associated with the way in which we, as instructors, present problems to them, and (3) we can benefit from recognizing and utilizing the variety of representations that students produce as tools for solving and understanding problems.

In this paper, the term *representation* is meant to indicate an external representation, outside the individual (Pérez Echeverría & Scheuer, 2009), and all the instances that are examined are cases of paper and pencil work. This is not to suggest that other types of external representation, such as sounds or gestures, are not useful or were not produced by the participants, but there was not enough evidence to consider them here.

We focus on the written work of eleven students who participated in interviews where they attempted to solve mathematical problems and talked about their thinking. There is also supporting evidence from classroom observations, carried out in the classes of these same students. The students produced a variety of representations, both conventional and novel, and used these as tools to find answers and to express their thinking.

Background

Combinatorics Problems

Combinatorics is an area of mathematics concerned with counting. Problems in combinatorics ask how many there are of some discrete thing. For example, we might ask how many three-letter passwords are possible if we can use any letters from the English language alphabet without repeating. We could wonder how many different lists of results there could be from tossing a coin five times. Or we could determine the number of sartorial choices we would have if we had three hats and seven sweaters. While these examples have different mathematical structures, they are part of combinatorics as they have a well-defined set of possible outcomes, and we want to count how many outcomes there are. Combinatorics has four canonical representations that are uniformly presented in curricular materials and classrooms: slots, formulae, tree diagrams, and lists. All four of these are described in detail below.

This paper addresses two types of problems in combinatorics: multiplication principle problems and permutation problems. The *multiplication principle* (sometimes referred to as the *fundamental counting principle* or the *product rule*) states that if an event occurs in m ways and another event occurs independently in n ways, then the two events can occur in $m*n$ ways. In the example from above, suppose one has 3 hats and 7 sweaters, and one must choose to wear one hat and one sweater. Using the multiplication principle, we have 3 ways to choose a hat and 7 ways to choose a sweater, so we have 3*7 = 21 different outcomes.

In mathematics, the term *permutations* refers to an arrangement of some number of objects, where the order in which the objects are arranged matters. That is, the same objects presented in a different order would be a different permutation of those objects. Within permutations, there are two initial cases. First, the case with n objects, where all n must be arranged. For example, if we have three different letters, how many ways can we arrange all three of them? The number of permutations for a set of n objects where all n are arranged is $n! = n*(n-1)*(n-2)*...*1$. Second, the case with n objects where some number less than n must be arranged. The number of permutations for r of the objects from a set of n objects (for $r \leq n$) is denoted $P(n,r) = n! / (n-r)!$. (We can see that the first case is really a simplification of the second, since if $n = r$, $P(n,r) = n! / (n-r)! = n! / 0! = n! / 1 = n!$.) For our example from above, given all 26 letters in the English language alphabet, asking how many three-letter passwords (with no repeating letters) can be formed, the number of permutations is $P(26,3) = 26! / (26-3)! = 26! / 23! = 26*25*24$.

Mathematically, multiplication principle and permutation problems are connected but distinct; however, the analysis presented here does not focus on the distinctions. In this analysis, the focus is the students' production of representations that differ from those that are expected (i.e., slots, formulae, tree diagrams, and lists) for both types of problems.

Existing work on learning and teaching combinatorics

In considering the developmental aspects of understanding combinatorics, Piaget and Inhelder (1975) suggest that children and adolescents' understanding progresses through stages that correlate with other, more general developmental stages. Specifically, they suggest that it is only as they reach the formal operations stage (12 to 13 years of age) that children are able to consider or enumerate a set of all possible outcomes. For instance, in creating permutations of small sets of distinct objects, Piaget and Inhelder found that pre-operational children (before seven years of age) have no system for creating different arrangements or for considering how many arrangements are possible. As they grow older and reach the concrete operations stage (between ages 7 and 11) they are able to create the different permutations more readily, but still do not use a consistent system to do so and often miss items or create the same item more than once. It is only in the third stage that students use a consistent system to create permutations or can make a conjecture about how many permutations were possible.

Understanding combinatorics may also be related to one's school or out of school experiences. For instance, Schliemann and Acioly (1989) interviewed bookies with different levels of formal schooling, including those with no formal schooling at all, who were accustomed to taking bets that involved the determination of the number of permutations of a fixed set of digits. While the bookies used tables listing the number of permutations for different scenarios during their work, the researchers interviewed them about permutations of colored chips and alphabetic characters, finding that some of the bookies connected this activity to the way that numeric digits are permuted in their work, while others did not make this connection and even rejected it when it was suggested. Relating the responses to the stages suggested by Piaget and Inhelder (1975) described above, they found that the level of schooling was positively and significantly related to the stage suggested by the response. In addition, while none of the bookies had formal instruction on probability, those with some formal schooling were more able to make logical probabilistic arguments.

Schliemann and Acioly's work confirms the types of reasoning about permutations seen by Piaget and Inhelder (1975). However, the progression through stages is shown to depend on factors other than development, such as schooling, work, and cultural factors. Even without the added element of the bookies' work, an individual's level of understanding of combinatorics may vary from one context or situation to another.

What children initially display about their understanding of permutations may also develop further through discussions in meaningful contexts, as suggested by Maher and Marino's (1996) work with students. Although their analysis is focused on children's justifications and proofs, rather than the mathematics of combinatorics, Maher and Martino show us young children engaged in simple problems of permutations. As part of a longitudinal study, students in fourth grade were asked to build all possible towers of blocks, given the height of the tower and two different colors of cubes to use in construction. Consistent with Piaget and Inhelder's (1975) theory regarding children in the concrete operations stage (between ages 7 and 11), students often did not have a foolproof system for organizing the possible permutations. However, with Maher and Martino's emphasis on students proving their answers to an interlocutor, over time some students felt the need to create organizational schemes. In doing so, students created either *patterns* of the colored towers, or *categories* of the towers. Patterns were organized visually and often led the students to count the same permutation more than once. For example, a student created all the towers of five cubes with exactly one blue cube and four orange cubes. The student then created all the towers with blue cubes on the bottom and orange cubes on the top, varying the number of blue cubes. However, both of these patterns included the tower with one blue cube on the bottom and four orange cubes on the top; as a result, this tower was counted more than once (Maher & Martino, 1996b). Categories, however, enabled students to prove that they had all possible permutations, as they were able to generate all the possibilities within a category. For example, one category could be thought of as "towers three cubes high with exactly two blue cubes", and students generated all three possibilities within this category. Aside from this increased organization in thinking about permutations, students also generated the beginnings of a recursive argument about the number of possible towers as a function of tower height, recognizing that the number doubled when the height was increased by one block. Their explanation of this suggests their reasoning is close to the classic permutation representation of a tree diagram, as they consider each existing tower with a height of *n-1* blocks to branch into two possibilities for the *n*th block. This example shows the richness and variety in combinatorial techniques, even for very simple problems.

The above studies speak to the challenges and to the role of experience and instruction in learning and understanding combinatorics. Combinatorics often receives short shrift in educational treatment and may be peripheral to the other mathematics taught within the same school year. The implication of this is that there is little instructional time to deal with the complexities and nuances in student understanding, as described above.

Combinatorics and Representations

Several established representations of combinatorics exist, and these are used both to find solutions to problems and as tools to give meaning and justification to both problem and solution. One possibility is a *list* of all the outcomes. This brute force method is effective for small sets. *Tree diagrams* are also commonly used. The *slot method* is another option, and, of course, there are established mathematical *formulae* for problems of this type.

Even seemingly simple problems in combinatorics may have multiple solution strategies and multiple ways to represent and consider what is happening in the situation. One way to think about the breadth of student knowledge is to look at the way in which they generate and use representations and their understanding of the connections between them. While multiple representations of combinatorics problems are often presented in student texts (e.g., Carter et al., 2010; Collins et al., 1997; Lappan et al., 2006; Rosen, 2003; Wheeler & Brawner, 2005), the links between the representations are not explicitly discussed, nor the advantages and disadvantages of each type considered. Wheeler and Brawner (2005), in their text for pre-service teachers, mention the wealth of representations in this mathematical area in their introductory notes, but do not incorporate discussion of this into the lessons and exercises that they propose.

Similarly, research and scholarly analysis has been carried out to examine student's production, interpretation, and use of different representations in other mathematical areas, such as that of algebraic functions. However, we do not have much information on the intersection between work on combinatorics and work on representations. As a result, the analysis presented here is exploratory. However, it is grounded in three aspects of existing research on representations in other areas of mathematics: (1) different representations of a mathematical object provide different affordances, (2) there is value in student-generated representations, and (3) representations can serve as tools for thinking and reasoning.

On the first point, Moschkovich, Schoenfeld, and Arcavi (1993) describe how different representations of a function make different features of that mathematical object more apparent. That is, not every representation of the same mathematical entity emphasizes the same aspects of the entity (Goldin, 1998). The same is likely true of different representations of a combinatorics problem, but we do not currently have a framework through which to consider these in a detailed manner. In order to explore how this view of representations connects to combinatorics, as part of the analysis below I will present the four canonical representations for each of the mathematical tasks described in this paper and compare them briefly.

On the second point, the analysis here takes the position that student-generated representations are a valuable part of their development of conceptual understandings (Brizuela, 2005; Goldin, 1998). In particular, this analysis will look at non-canonical representations that students generated in the classroom and in interviews. These representations are valuable to the students as they work on combinatorics problems, even as they both mimic and diverge from the canonical representations. This is similar to the value ascribed to both process and product in the students' invented graphs described by diSessa et al. (1991).

On the third point, the representations described here, both canonical and student-generated, are useful as tools to find solutions, communicate information, and act as a means of expression for their creators. This point is best summarized by Nemirovsky, Tierney, and Wright (1998): "This philosophical tradition we are following argues that the meaning of symbols is to be found neither in the specific thoughts that they express nor in the objects to which they refer but in their use, that is, in the practices they serve" (p. 123). Their reference to a philosophical tradition draws on the work of Vygotsky (1978), who emphasized the idea of representations not as equivalent to concepts, but as a means (or tool) to work on concepts. That is, a creator's representations may change and develop as the creator changes his or her understanding of the concepts being represented.

In the analysis that follows, all four canonical and expected representations (list, tree diagram, slot, and formula) will be shown here for all three of the mathematical problems being discussed. Two of the problems that are the focus of the study here reported are taken from an interview with students and the third is taken from classroom observations; this will be described in more detail in the methodology section below.

Methodology

Summary

The data sources for this study were observations from two different classrooms, as well as individual interviews with 11 students from these classrooms. During interviews with students, participants were asked to work on combinatorics problems, and also to reflect upon them, explain their solutions, and evaluate alternative solutions. Classroom observations took place in the classes of two teachers, while the teacher was covering a topic within combinatorics.

During the observations and interviews, students generated two types of representations that were not explicitly presented or explored within the curriculum, the classroom instruction, or the anticipated responses – based on the canonical representations – to the interview questions. The two non-canonical representations will be referred to as "set-to-set lines" and "ordered lines" and will be described below. As a result of this finding from the interviews, all interview videos and all logs from classroom observations were reviewed for instances of the same types of non-canonical responses.

Participants

The researcher had the opportunity to meet several teachers while assisting with a professional development course. All teachers received a letter explaining this study and inviting them to participate, as well as clarifying that the study was independent of their participation in the professional development course.

Teachers were asked if they would be willing to allow classroom observations and to have student participants sought from within their classrooms.

Nine teachers consented to participate; of these, only two teachers both were willing to allow student participants and were planning to teach a combinatorics unit that included lessons on permutations during the following school year. Once these two teachers were identified, the administration of each of their schools was contacted with a letter explaining the project and asking for their participation. Both participating teachers teach at secondary schools in a large urban school district in the state of Massachusetts. While the teachers were in the same district, they were in separate secondary schools and they did not use the same instructional materials in their classrooms.

After the administrations of the schools had consented to participate, the families of all students of the two teachers received a consent letter and explanation of the study. The students were also asked for their assent. Fourteen students assented and had their parent or guardian consent as well. Ultimately, eleven of these students were available and were interviewed; seven students were from one class and four students were from the other.

Measures and Data Collection

Data was collected through classroom observations and individual student interviews. An overview of the data types is provided in Table 16, followed by a detailed description of each source. Classroom observations were completed in May and June 2010. After the classroom observations had taken place, student interviews were carried out in June 2010. Student interviews were conducted after instruction related to these topics was complete in each classroom. The time between the classroom instruction and the student interviews was between 5 and 18 days.

Table 16. *Data collected.*

Measure	Time of measurement	Mode
Classroom observations	May / June 2010	Written observation notes; collection of handouts and instructional materials
Student interviews	June 2010	Videotaped individual interview; subsequently logged and excerpts transcribed

Classroom observations. The first source of data was classroom observations. Classes were observed during the time that the teacher was providing instruction on combinatorics. In the secondary school curriculum in the school district in which this study took place, these topics are introduced in Algebra I at the end of the school year, in May or June. Each of the two participating teachers was teaching two sections of Algebra I, each attended by students 14 to 16 years old (i.e., freshmen to juniors in high school), at the time of data collection. The first teacher, who we will identify by the pseudonym Shana, taught related topics during five school days; classroom observations were carried out on all five days. Shana taught two sections of the same class; each section received the same 5 days of instruction and both sections were observed. The second teacher, Whitney (also a pseudonym), taught these topics during two school days, so the classroom observations were carried out on two days only. As with Shana, Whitney also taught two sections of the same class; each section received the same 2 days of instruction and both sections were observed. These details are summarized in Table 17.

Table 17. *Number of classes and students.*

Teacher name	Number of classes spent on material	Number of students interviewed
Shana	5 per section	7
Whitney	2 per section	4

During classroom observations, I took detailed notes, recording on paper as much as possible of the classroom activity. Additionally, I made note of all mathematical problems that were addressed during the class and collected blank versions of all handouts and paper assignments used. In addition, I used a checklist for quickly noting topics, concepts, and representations addressed during the class. A new checklist was used for each 15-minute interval. The checklist provided a structure to allow quick notation of the type of mathematical problem, the type of representations being used, and the type of classroom activity, such as teacher-led discussion, group work, or independent work. The section of the checklist related to this analysis is shown in Table 18.

Table 18. *Classroom observation checklist.*

Category	Item	Time

Representations	Formula
	List
	Tree
	Slot
	Explicit linking

Student interviews. All consenting and available students were interviewed individually. Eleven students were interviewed, seven of them from Shana's two classes and four from Whitney's two classes, as shown in Table 17. Each interview was videotaped. While interviews were flexible and open-ended, they had a goal of discussing problems about permutations. To do this, the students were given problems to solve (shown in Appendix A). After solving each problem, they were asked to explain what they had just done and tell me how they knew to do each step. They were then asked for a different way to solve the same problem and a different explanation. This process repeated for each mathematical problem. I attempted to elicit a full explanation for both correct and incorrect answers and strategies. The student interviews consisted of six combinatorics problems. The problems were selected to cover different cases of permutations and probability that are covered in secondary school curricula. The selection was judged to be both appropriate and adequate based on pilot interviews with teachers.

The combinatorics problems. The mathematical problems presented to the students during the interviews could all be solved using more than one solution method. Here, the term method refers specifically to an established or canonical technique of solving a combinatorics problem. In concert with this, they could use multiple types of canonical representations: a formula, a list, slots, and a tree diagram. Only two of the six interview problems are described in this analysis; these two were selected to illustrate the connections between both canonical and student-generated representations because the mathematical topic, permutations, is the same for both questions and the same as the material covered in the classroom observations. This is not to say that students could not use their own unconventional representations on the other four problems. In fact, this did occur in some interviews; however, they are not included in this analysis because the specific topics of the questions (probability and combinations) are different from the questions on permutations. Thus, we would not be able to contrast student choice of representations as clearly due to complications introduced by the varying problem structure.

In addition to the two interview problems, there is a third problem described in this analysis. The third problem is one that Shana's students completed during classroom observations; it is included as a supplementary example since multiple students in Shana's class were observed using the same type of non-canonical representations that were later seen in the interviews. This third question, posed in the classroom, was not designed for this study or suggested by the researcher, but rather was part of the teacher's classroom practice.

In this section, the expected solutions for the problems will be shown, and for each question I will summarize some notable points regarding how the representations relate to the problems. This exercise is undertaken to explore and explain the canonical representations, without regard to what the students actually did during their interviews or in the classroom; their work will be examined below in the results and discussion sections.

Interview Problem 1: "How many ways can you arrange: A, B, C, D?"

Formula: Using the formula for a permutation of k of n objects, denoted P(n,k), for k less than or equal to n. The formula is $P(n,k) = n! / (n-k)!$

In this case: $n = 4$, $k = 4$; $P(4,4) = 4! / (4-4)! = 4! = 24$

List: Listing all arrangements. This would take the form ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA. This yields 24 arrangements also.

Slot: Using one slot for each space in the arrangement, and placing in that slot the number of possible entries. Since each object appears only once, the number of possible entries is decremented with each slot, and the product of the possibilities yields the answer. In this case, $\underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 24$

Tree: From the root, or start, of the tree, the number of branches corresponds to the number of possibilities for the first item in the arrangement, and the item itself is marked at the end of the branch. From each of these nodes, the next set of branches corresponds to the number of possibilities for the second item in the arrangement, and so on. Here also, since each object appears only once, the number of branches decreases with each additional item. In this depiction, the tree is organized horizontally, with the choices moving from left to right, although other orientations are possible. For brevity, only the arrangements starting with "A" are fully drawn in *Figure 15*.

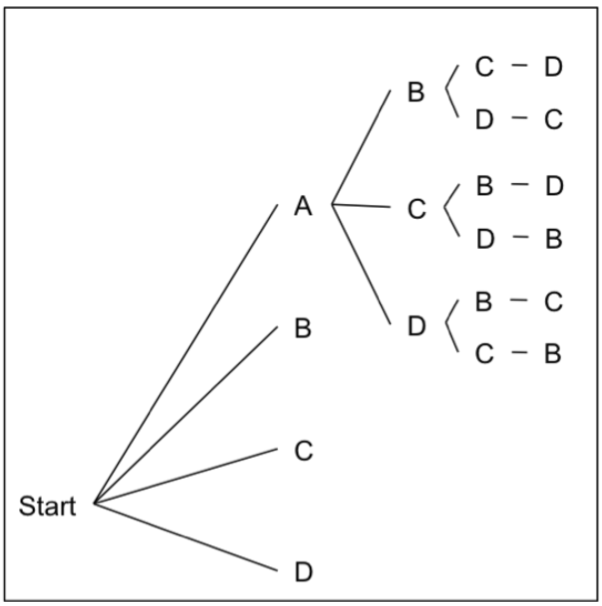


Figure 15. Tree diagram for interview problem 1.

Notable points: For this problem, since there are 24 possible arrangements, both the list and the tree may become tedious but are not beyond reason. However, they also give meaning and explicitly illustrate the final answer. The formula could also be employed, either with or without attending to the meaning of the structure of the formula and the sense of the final answer. The slot method seems to be the most likely choice for this problem, as it requires far less repetition than either the list or the tree, yet one does not have to recall the details of the formula. The user of the slot method has only to decide how many spaces there are in the arrangement, and then decide how many possibilities there are for the first space, how many for the second space, and so on. The user does have to decide to multiply the numbers in the slots, which implies either that one must recall this as a procedural requirement, or else understand the underlying reason for the multiplication. That is, the formula and the slots both obscure the details of the permutations and the rationale for the final answer, although to varying degrees. With the formula, no instances of the various permutations are visible and the reason why the formula enables one to find the correct numerical response is unclear as shown (although individuals who already understand the connections between these representations could potentially justify the formula to themselves or others). With the slots, the four slot positions do stand for the four positions in each possible permutation, so the representation is more closely connected to the sense of the problem. However, it is only the list and the tree diagram that explicitly show any of the permutations and that fully explain the final answer.

Interview Problem 2: "If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?"

Formula: Using the formula for a permutation of k of n objects, denoted $P(n,k)$, for k less than or equal to n. The formula is $P(n,k) = n! / (n-k)!$

In this case: $n = 10$, $k = 3$; $P(10,3) = 10! / (10-3)! = 10! / 7! = 10*9*8 = 720$

List: Listing all arrangements. If we refer to the students as numbers 1 through 10, this would take the form 1,2,3; 1,2,4; 1,2,5; 1,2,6; ... , 1,3,2; 1,3,4; 1,3,5; 1,3,6 .... This yields 720 arrangements also.

Slot: Using one slot for each space in the arrangement, and placing in that slot the number of possible entries. Since we are assuming that each person can only hold one club officer position at a given time, the number of possible entries is decremented with each slot, and the product of the possibilities yields the answer. In this case, $\underline{10} \times \underline{9} \times \underline{8} = 720$.

Tree: As described for Problem 1. Here also, since each person appears only once in a given arrangement, the number of branches decreases with each additional item. For brevity, only a few of the arrangements starting with "1" are fully drawn in the example displayed in *Figure 16*.

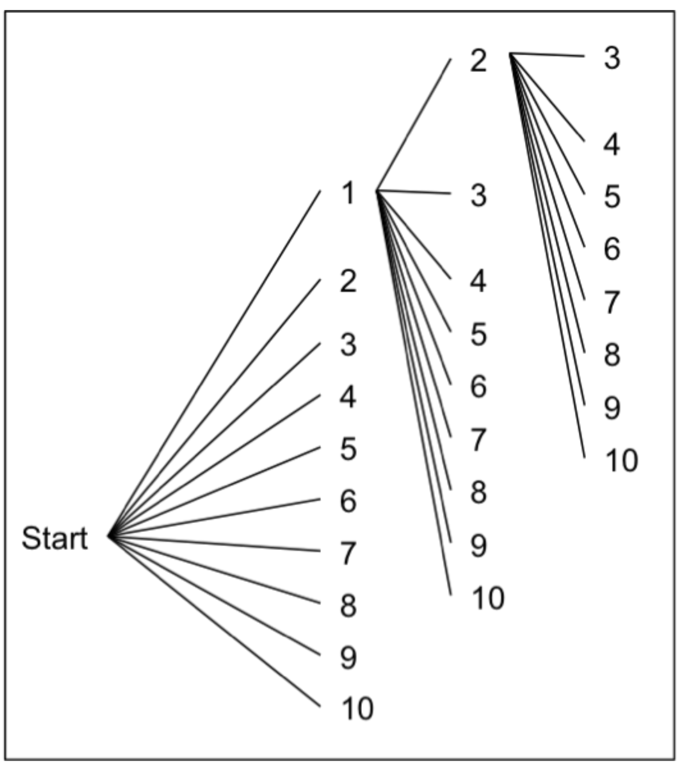


Figure 16. Tree diagram for interview problem 2.

Notable points: For this problem, since there are 720 possible arrangements, both the list and the tree are arduous beyond reason, despite the fact that they illustrate and make explicit the meaning of the final answer. As with Problem 1, the formula could be employed, either with or without attending to the meaning of the structure of the formula and the sense of the final answer. The slot method seems again to be the most likely choice for this problem, as it requires far less repetition than either the list or the tree, yet one does not have to recall the details of the formula. The user of the slot method has only to decide how many spaces there are in the arrangement, and then decide how many possibilities there are for the first space, how many for the second space, and so on. Since this question involves choosing officers of a club, the user needs to decide that each person is not used more than once in an arrangement, or else they would perhaps decide that each of the three slots has ten possibilities, yielding an (incorrect) answer of 1000. Additionally, the user of the slot method would need to decide to multiply the numbers in each slot, instead of adding them.

Classroom Problem: "How many different codes are possible if the first spot in the code is 1, 3, 5, 7, or 9, and the second spot in the code is either x or y?" (These were described to students as passwords or secret codes; they are referred to here simply as "codes".)

Formula: Using the multiplication principle (if an event occurs in *m* ways and another event occurs independently in *n* ways, then the two events can occur in $m*n$ ways), where we take the first event to be choosing the first part of the code and the second event to be choosing the second part of the code.

In this case: $m = 5$, $n = 2$; number of outcomes = 5*2 = 10.

List: Listing all codes. This would take the form 1x, 3x, 5x, 7x, 9x, 1y, 3y, 5y, 7y, 9y. This yields 10 outcomes also.

Slot: Using one slot for each space in the code, and placing in that slot the number of possible entries. The product of the possibilities yields the answer. In this case, $\underline{5}$ x $\underline{2}$ = 10.

Tree: As described for Problem 1. Here, since each spot in the code has a different (non-overlapping) set of possibilities, each level of the tree looks quite different. The full tree is shown in *Figure 17*.

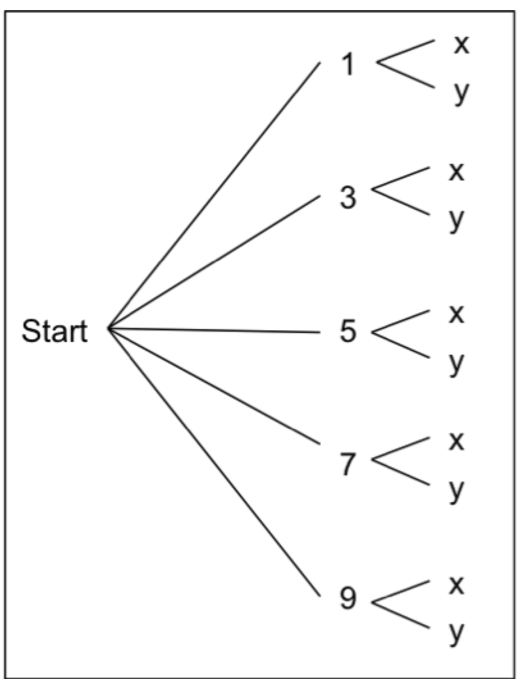


Figure 17. Tree diagram for classroom problem.

Notable points: Since this question has only 10 possible outcomes, any of the representations above are manageable. In particular, the list and tree diagram are much shorter to construct than for the two interview problems. The tree diagram is notably different for this problem, since the choices for each spot in the code are from separate sets. This means that (1) the same items do not appear at both levels of the tree, and (2) the user has to decide how many branches to make by considering the sets separately, not by determining the initial set size and then decrementing it to account for a reduced number of possibilities, as was done in the interview problems. Any of the representations shown is a viable choice.

Data Analysis

Each student interview was logged and then coded according to an *a priori* scheme that included flagging which external representations were used; the possibilities were the same as those shown above: formula, list, slot, and tree diagram. Careful examination of student written work and interview video resulted in several examples of non-canonical representations. Excerpts from the videos were transcribed verbatim in order to provide evidence for the claims made below.

Logs from classroom observations were reviewed to look for instances of the same type of student-generated representations. These data sources provide additional instances of the student representations. They are used to enrich the analysis.

Results

The representations students used in interviews are shown below, in Table 19. Note that there were 11 students interviewed, but some students may have used more than one type of representation. Note also that this table does not include comment on whether or not they arrived at the correct answer with a particular representation, just how often each was used. Two novel, student-generated representations were used; they are nicknamed "ordered lines" and "set-to-set lines". Both of these will be described below.

Table 19. *Student representation use in interviews.*

Problem	Formula	List	Tree	Slot	Ordered lines	Set-to-set lines
#1	-	9	3	4	2	-
#2	-	3	2	6	-	1

Among the canonical representations, there are two points to note. First, a list was more commonly used on the first problem (a problem with 24 possible outcomes) than on the second (with 720 possible outcomes). This was to be expected, as noted above, as it is less arduous to list 24 possibilities than 720. However, it is worth noting that students may or may not have had a sense of the size of the final answer when deciding whether or not to generate a list. Another possible explanation for the disparity between the numbers of students using a list on problems 1 and 2 is that Problem 1 (which is shown in Appendix A in the same format students received it during the interview) both named the objects to be arranged (A, B, C, D) and also represented them physically. This provided students with a naming convention and a form of shorthand to begin to generate a list. For Problem 2, there was no naming convention suggested, such as using the numbers 1-10 to stand for the 10 members of the after school club from among whom the president, vice-president, and treasurer would be selected.

The second point regarding the canonical representations is that the tree diagram was the least popular choice for both problems. The tree diagram, like a list of possibilities, can be time consuming and unwieldy, yet a list was used often, by 9 out of 11 participants on Problem 1, while the tree diagram was neglected. Pilot work on similar interview questions showed that the same was true for teachers solving permutation problems (Caddle, 2010). It is not completely clear why a list is frequently used, but the tree is not, and the information from this study is not sufficient to indicate that this would be the case for a larger population of students and teachers. It is possible that the tree diagram is simply less convenient to write out, as it can be hard to know, for instance, how to space the elements on the first branches to allow enough room for the later branches. During classroom instruction, both teachers used all four canonical representations: formula, list, tree, and slot. Neither teacher used non-canonical written representations. It is worth noting that the lists and tree diagrams were used as introductory techniques and that students were familiar with them already.

Ordered Lines

The first non-canonical representation described here is ordered lines. The name "ordered lines" is meant to indicate that in this representation, lines connect, in order, the elements being arranged for each instance of the permutation. That is, as seen in *Figure 18*, lines are drawn to connect, in order, the elements of the permutation CDAB. This figure shows just the ordered line for one possible permutation of the four objects. Each additional permutation demands a new ordered line, as with the line for CADB shown in *Figure 19*.

As shown in Table 19, two students, José and Chris, used the representation "ordered lines" with Problem 1. Note that these two students were not only from different classrooms, with José a student of Shana and Chris a student of Whitney, but also from different schools. The use of the ordered lines representation occurred when each student attempted to find the number of arrangements of four objects (A, B, C, D). Each student began by choosing one object to be first in an arrangement, and then drew a line from there to the second object, then to the third, then to the fourth. For example, Chris first generated the representation recreated in *Figure 18* (note that he did not draw arrowheads; the arrowheads are included to indicate that he first drew a line from C to D, then from D to A, and finally from A to B). He described his process as he made the lines: "First I looked at it [the problem] and took this as one way [drawing the lines shown in *Figure 18*]. It's already like that for me [meaning arranged left to right]. So that gave me one." He then drew the lines shown in *Figure 19*, saying "Then I just skip one, and go back and do that." In this second statement, he refers to starting again at C, but then skipping D to go to A, then going back to D after.

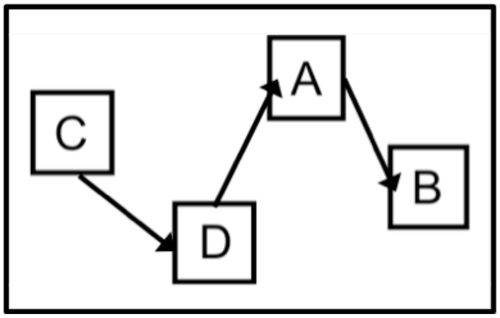


Figure 18. Chris' first step creating ordered lines for Problem 1.

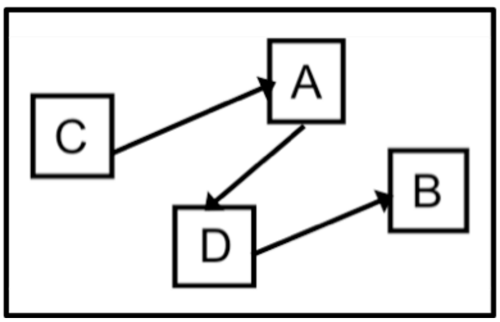


Figure 19. Chris' second step creating ordered lines for Problem 1.

As Chris continued to make his ordered lines, he superimposed sets of lines on top of each other, eventually creating the representation shown in *Figure 20*. This is his description of the process:

> "And then keep going. And usually I just do it from there. I started from this [pointing to C], and see which one of these [of the remaining letters] could go first, after this one [C]. Then I go to the next letter, or they might be numbers or something. So I go to here [pointing to A] and start with that, and then this would go next [C], or this [D], or this [B]. […] Once you find out all the ways C can go, it's basically almost the same because you just start it from here [pointing to A] and go backwards."

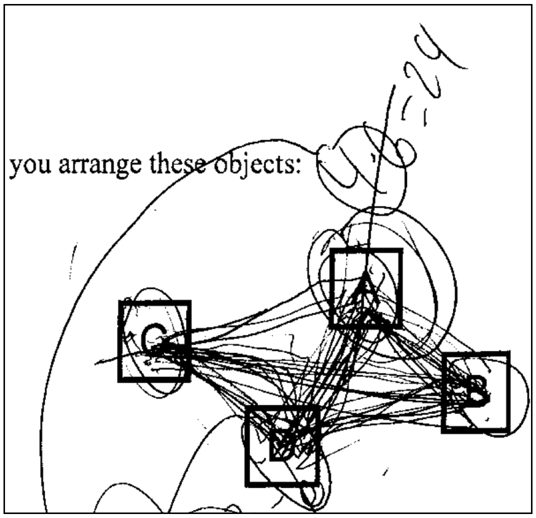


Figure 20. Chris' work on problem 1.

In his description of the process, Chris stated that he could “start it from here”, meaning that once he used his ordered lines representation to find all the arrangements starting with C, he could proceed with finding all the arrangements starting with A. However, he was actually able to use his findings of the number of arrangements starting with C to determine the total number of arrangements:

Interviewer: So how many ways can you go, starting with C, do you know?

Chris: [redraws lines, correctly denoting six possibilities starting with C.] It’s a total of eight, I believe.

Interviewer: Eight? Can you say which ones they are?

Chris: [retracing lines again] CDAB, CDBA. So that’s two. CABD, CADB, that’s four. Did I say CA? So it’s six… CBAD, CBDA.

Interviewer: So there’s six ways you can do it starting with C? How many ways do you think there would be to do it starting with A?

Chris: It’s basically the same process. Since I got my first result, since it’s six that one letter can go, I’d multiply that, since there’s four

> letters, I'd multiply how many letters there are times how many possible ways there are, so it's four times six gives you 24.

Another student, José, also used ordered lines, as shown in *Figure 21*. It is worthwhile to note that this student first gave an answer of 16, saying he multiplied the number of objects (i.e., 4 letters) by itself. He then offered that you could also go "like that – there's the letter right here [D], so you can go C, then A, then B." He constructed several ordered lines while naming each corresponding permutation, but he stated that he would still reach 16; he did not complete his ordered lines to get to a final answer. José used this representation to show the interviewer a different way of finding an answer, but it was not what he initially used to solve the problem. His use of ordered lines shows us that he understood what individual permutations would look like, but since he felt confident in his answer of 16, he did not continue with this work. We can't know, with the information available, whether or not he would be able to get a correct answer of 24 if he continued with this representation.

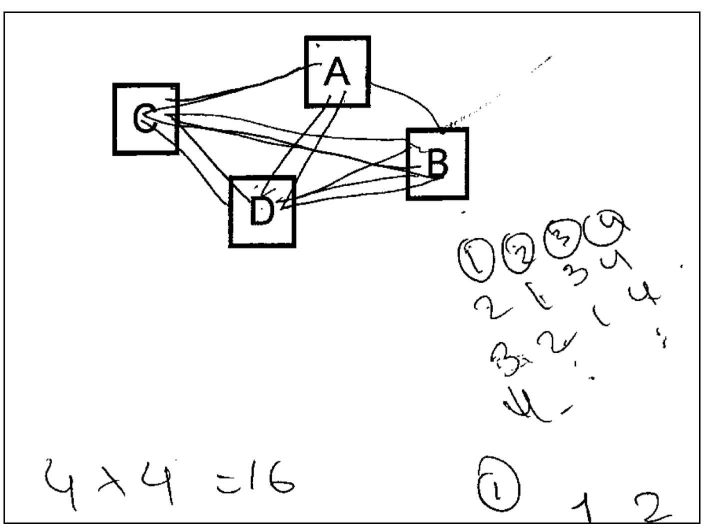


Figure 21. José's work on Problem 1.

The two students who used the ordered lines representation clearly show that they understand (a) the concept of permutation as an ordered arrangement of a set of objects, (b) that they are being asked to find all possible arrangements, and (c) that they can use a connecting line to indicate a particular arrangement. One note is that the finished image of the ordered lines representation, as seen in *Figure 20* and *Figure 21*, does not allow an observer who comes along later to easily determine the answer. Rather, it is the *process* of generating the representation that allowed students to discuss and respond to the problem. Taking this into consideration, we can see that the ordered lines could be used in concert with a more traditional representation, such as a list. The generation of ordered lines could enable a student to build a list. In fact, this technique could supply significant support for a list representation, as a common problem with student lists is that they are not well ordered and possibilities are missed or counted more than once. The representation generated in this analysis by just two students could provide support for many more.

It is also worth considering the way in which Chris used ordered lines to count the permutations starting with C and then extended the pattern to the total number of permutations. In this way, he only needed to solve a sub-problem. As described by Hadar and Hadass (1981) and later recommended by Watson (1996), many problems in combinatorics can be broken into sub-problems if the whole is too complex. While the whole problem here is not as complex as those described by the studies referenced, it certainly can be considered using separate cases, as Chris does when he looks just at the permutations starting with C. This not only breaks the problem into smaller pieces, but allows students to notice patterns and hopefully to begin to glean generalizations from them, as Chris does here. The ordered lines representation was a tool for Chris to think about this topic. Also note that other students may be visualizing something similar to ordered lines already when generating lists, even though they do not explicitly draw them. On Problem 1 in the interviews, two additional students (besides Chris and José) were observed moving their pencil in a pattern similar to that of someone drawing ordered lines, although they did not actually make any marks on their papers.

Set-to-Set Lines

The second non-canonical representation is set-to-set lines. The name "set-to-set lines" was chosen because the representation involves showing more than one set of items to be arranged and then drawing lines from an item in one set to an item in another. As with the ordered lines, a new line is drawn for each arrangement. However, it differs from the ordered lines representation in that the options for each position in the arrangement are listed as separate sets. In the case of ordered lines, only one set of objects was shown. In the example shown here, in *Figure 22*, the student generates three sets and each set, consisting of the numbers 1 through 10, is presented in a column.

During the interview, one student, Matthew, from Shana's classroom, used the "set-to-set lines" representation. This occurred on problem 2, "If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?" Matthew's work is shown in *Figure 22*.

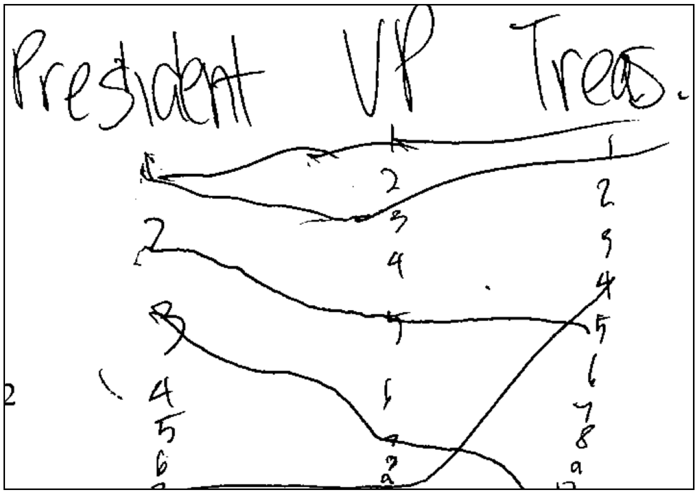


Figure 22. Matthew's work on Problem 2.

It is important to note a few elements of Matthew's response. First, he initially found the correct answer (720) by using a slot representation, labeling the slots as president, vice president, and treasurer, and multiplying 10 times 9 times 8. The interviewer then asked if he could find the answer a different way. While working on the representation shown in *Figure 22*, he said that there would be too many lines to draw out; in fact, that there would be 720 of them. This is similar to the work José did, as described above, where he did not use this representation as a means to find a final answer, but rather as a way to support and justify his initial answer. Second, he repeatedly referred to his set-to-set lines representation as a "tree". This suggests that he remembered the name for the canonical tree diagram representation, but not how to construct one. Third, when Matthew was describing how he solved the problem using slots, he was able to explain why there were 10 possibilities for the first slot, yet only 9 for the second and 8 for the third. But when drawing his set-to-set lines and naming possibilities for the interviewer, he named invalid arrangements "first person, first person, first person", and "first person, second person, first person". This is shown in *Figure*

22, as the lines connecting 1-1-1, 1-2-1, and 2-5-5. While this could reflect an unintended interpretation of the problem in which the same person can hold more than one office, it is not consistent with the explanation Matthew gave initially. This suggests that in this case, the set-to-set lines representation is not helping him with the problem.

One reason for the apparent unhelpfulness of the set-to-set lines may become apparent if we look at the classroom problem described above ("How many different codes are possible if the first spot in the code is 1, 3, 5, 7, or 9, and the second spot in the code is either x or y?"). This problem was presented in Shana's classroom as a small group activity. During classroom observations, several students in different groups generated a set-to-set lines representation for the problem; *Figure 23* shows what this would look like if executed correctly for this problem. While similar to a tree diagram, this appeared in Shana's class before she discussed tree diagrams, although some students may have seen traditional tree diagrams in previous years in school. The set-to-set lines representation was not discussed or presented in Shana's class, but it is possible that the students had seen it or used it earlier in their schooling. However, without knowledge of their backgrounds, we cannot say whether this representation was ever presented or endorsed in their classrooms. Regardless, the fact that it was used without prompting by several students suggests that they found it to be a useful and economical way to find an answer for this type of problem.

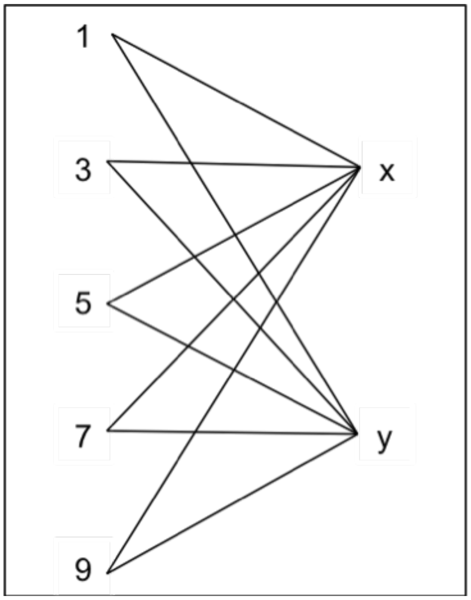


Figure 23. An example of a set-to-set lines representation.

The correct use of the set-to-set lines for the problem presented in Shana's class does include connecting every item in the first column to every item in the second column in order to obtain all possibilities. So it seems that Matthew may have been trying to demonstrate this in his response to the interview question. In doing so, he did not recognize that the columns he made during the interview were three instances of the same set of objects, while the columns in the classroom problem were two distinct sets of objects; this difference necessitates a change in the use of the set-to-set lines representations. We might also note that the large number of arrangements that was the correct answer in the interview problem might make a full and complete use of the set-to-set lines representation untenable as a means to find the answer. However, it was a reasonable way for Matthew to attempt to demonstrate the meaning of the problem and could even have been used to work out sub-problems, as we saw Chris do with the ordered lines.

Discussion

The data and analysis presented here leads to three main conclusions. First, students can and do generate their own non-canonical representations in combinatorics, and these representations are useful to them. Second, students may be more likely to use non-canonical representations when the problem presentation supports this. Third, recognizing and incorporating these representations in the curriculum and the classroom could be a useful tool for students.

Regarding the first point, the few instances we see of ordered lines and set-to-set lines in the interviews and in the classroom observations are sufficient to show that these exist and that they may be useful to students when solving combinatorics problems. The student-generated representations were useful both in finding answers and in giving explanations and information to the interviewer. In the example of Chris using ordered lines to find that there were 24 permutations of four objects and the example of students in Shana's classroom using set-to-set lines to find the number of possible codes, the students were able to find correct solutions. With Chris' ordered lines, he was able to correctly execute a sub-problem and then extrapolate that result to find a fully correct answer. In the case of José and Matthew's interview questions, they did not find correct solutions using the non-canonical representations. However, that does not mean they were not useful. They both turned to these representations when asked to support or explain their initial answers. They were able to use the representations to give meaning to their work and articulate their reasoning. Articulation of ideas is a valuable mathematical activity, even when the reasoning is incorrect. In addition to the mathematical value to the students, the non-canonical representations are potentially valuable to teachers. They reveal the student thought. This could allow teachers to make decisions about their response to the students based on the details of the difficulties.

Second, while we do not have enough information to show quantitatively when and why students use particular representations, these cases suggest that the way we present our combinatorics problems has a notable influence on the student's choice of representations, and perhaps on their ability to successfully solve the problem. Why was the set-to-set lines representation used several times by different students in the classroom but only by one student in the interview? Why was it seen frequently in Shana's classroom and never in Whitney's classroom? Why did two students use ordered lines on the first interview problem but none used it on the second interview problem?

The instances that we have available to us suggest that the way in which the problem statement is represented is a primary factor in student choice of representations. When the items to be combined or arranged were presented to the students as objects or words that were all marked on paper for them, as was the case in the first interview problem and the classroom problem, students made use of this presentation to create one of these non-canonical representations. We can see two cases where different problem presentations – those in problems 1 and 2, for instance – seem to be connected to the kinds of non-canonical representations students use.

The first case is the use of ordered lines on the first interview problem but not the second. In the first interview problem, the items to be arranged were not only given the designations of A, B, C, and D, but they were also depicted as letters inside squares (as shown in Appendix A). This reduced the work for a student to use ordered lines. They also did not need to name the objects, as they would have if the question had asked simply, "How many ways can you arrange four different objects?" They did not need to draw or write anything to show the objects; the blocks were already shown. In contrast, in order to be able to generate ordered lines for the second interview question, they would have needed to choose a way to name as well as to represent 10 distinguishable items. While someone proficient in combinatorics could do this step quickly, it would require a student to have the intent to create these; this is not an obvious step. Note also that ordered lines did not appear in student work in either classroom; however, in the classrooms observed there were no permutation problems where the objects to be arranged were explicitly presented through drawings or icons.

The second case of differing problem presentations is related to the way in which the combinatorics problem was presented in Shana's classroom, but not in Whitney's. In the problem in Shana's classroom, the items that could make up the code were presented to students in two columns, just as shown in *Figure 23*. In order to construct the set-to-set representation, students needed to create only the lines; they did not have to name or denote the items and they did not have to arrange them in columns. In contrast, students in Whitney's classroom worked on problems that were similar mathematically, but the items were not arranged in columns. As with the difference between the presentation of the two interview problems, when the problem in Shana's classroom was presented with each set in its own column, students did not need to complete this first step (making the items explicit) in order to produce a set-to-set lines representation. The only instance of a set-to-set lines representation in the interviews was with Matthew. In his case, he explicitly represented his own set of objects on which to make this representation.

In summary, José, Chris, and the students in Shana's class were able to build on a representation of the objects that was given to them, either by their teacher or the interviewer. While Matthew constructed his own objects, it seemed helpful to the other students to have the objects already available. The drawing or listing, on paper, of the items in the problem provides a scaffold that gives students an entry point into the problem. This is aligned with findings from Brizuela and Alvarado (2010) that young children are more able to solve complex addition problems when they have notational tools available to them. The study presented here suggests that older students working on combinatorics problems need these supports as well.

The third conclusion is that this indicates we may be missing a valuable instructional tool to help students understand the meaning of combinatorics problems. While we may eventually want students to extract the meaning of problems from words alone and to construct their representations from scratch, an early introduction to combinatorics should include providing students with names or notation for the set of objects they are using, upon which they can construct. By providing this scaffolding for students, we can enable them to produce their own representations that, in turn, they can use as tools to explain their understanding.

In addition, while the idea of embracing and working with student-generated representations is not new in mathematics education (e.g. Brizuela, 2005; diSessa et al., 1991; Nemirovsky et al., 1998), the form that these might take in combinatorics has not been explored in research or in curricular materials. This study begins the work of identifying these and assessing their utility and their connections to canonical representations. The next steps are to seek out other types of student representations and then to consider connections between these. As shown by Moschkovich et al. (1993) for representations of functions, some representations of combinatorics may make features of the mathematical situation more prominent or less prominent. It may also occur that the use of one representation leads naturally to the adoption of another. The perspective of multiple representations, which has enriched the examination of learning about numbers and of learning about algebra, can similarly enrich the examination of teaching and learning combinatorics.

References

Brizuela, B.M. (2005). Young children's notation for fractions. *Educational Studies in Mathematics, 62*(3), 281-305.

Brizuela, B.M. & Alvarado, M. (2010). First graders' work on additive problems with the use of different notational tools. *Revista IRICE Nueva Época, 21*, 37-43.

Caddle, M.C. (2010). *Analyzing solution methods for combinatorics problems: Differences across problems and across teachers.* Dissertation qualifying paper. Tufts University. December 2010.

Carter, J.A., Cuevas, G.J., Day, R., Malloy, C., Holliday, B., & Luchin, B. (2010). *Algebra 1*. Glencoe / McGraw-Hill, 2010.

Collins, W., Cuevas, G.J., Foster, A.G., Gordon, B., Moore-Harris, B., Rath, J., Swart, D., & Winters, L.J. (1997). *Algebra 2*. Glencoe / McGraw-Hill, 1997.

diSessa, A. A., Hammer, D., Sherin, B., Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *Journal of Mathematical Behavior, 10*, 117-160.

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior, 17*(2), 137-165.

Hadar, N. & Hadass, R. (1981). The road to solving a combinatorial problem is strewn with pitfalls. *Educational Studies in Mathematics, 12*(4), 435-443.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (2006). *Connected Mathematics 2, Grades 6, 7, 8*. Prentice Hall, 2006.

Maher, C.A. & Martino, A.M. (1996). Young children invent methods of proof: The gang of four. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning* (pp. 431-447). Hillsdale, NJ: Lawrence Erlbaum.

Maher, C.A. & Martino, A.M. (1996b). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27(*2), 194-214.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69-100). Hillsdale, NJ: Lawrence Erlbaum Associates/Taylor & Francis.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119-172.

Pérez Echeverría, M. P. & Scheuer, N. (2009). External representations as learning tools: An introduction. In C. Andersen, N. Scheuer, M. P. Pérez Echeverría, E. V. Teubal (Eds.), *Representational systems and practices as learning tools* (pp. 1-17). Boston, MA: Sense Publishers.

Piaget, J. & Inhelder, B. (1975). *The origin of the idea of chance in children*. New York: W. W. Norton & Company Inc.

Rosen, K. (2003). *Discrete Mathematics and Its Applications*. Boston, MA: McGraw Hill.

Schliemann, A.D., & Acioly, N.M. (1989). Mathematical knowledge developed at work: The contribution of schooling. *Cognition and Instruction, 6*(3), 185-221.

Vygotsky, L.S. (1978). *Mind in Society.* Cambridge, MA: Harvard University Press.

Watson, R. (1996). Students' combinatorial strategies. *Teaching Mathematics and its Applications, 15*(1), 27-32.

Wheeler, E., & Brawner, J. (2005). *Discrete Mathematics for Teachers*. Boston, MA: Houghton Mifflin Company.

Appendix A: Interview Questions

Question 1
How many ways can you arrange these objects:

A
C
B
D

Question 2
If there are 10 students in an after-school club, how many ways can the club select a president, vice-president, and treasurer?

Question 4a
You have 6 marbles in a bag. Four marbles are blue and two marbles are yellow. If you choose one marble without looking, what is the probability that the marble you pick is yellow?

Question 4b
If you choose one marble without looking, and then you choose a second marble without looking, what is the probability that they are both yellow?

Question 5
There are 4 students staying after school:

Angela
Byron
Carlos
David

How many ways can you choose 2 students to clean the board, 1 student to sharpen pencils, and 1 student to organize papers?

Question 6
You have 10 cards, numbered 1 through 10. If you draw 2 cards, what is the probability that the sum of the numbers on the cards is even?

General Conclusions

As described in the introductory section of this dissertation, the data from the study were used in four separate analyses. Each analysis formed a standalone paper with its own introduction, background literature, methodology, analysis, results, and conclusions. Each can be read and understood individually, with or without having read the dissertation introduction, the full methodology, or this concluding section. As a result, while some general conclusions are discussed below, any reader interested in specific results and recommendations should read the conclusions within one or more individual analyses in order to obtain detailed information.

The four analyses described above rely on and connect to different areas of research in mathematics education. The first two papers consider the complex characteristics of teacher knowledge and the qualities of instruction. In doing so, they aim to become part of the active and ongoing conversations about how to prepare, evaluate, and support math teachers. As mentioned above, two tenets foundational to this study are that (a) teachers of mathematics need a special kind of professional knowledge and it is important to understand the form of this knowledge, and (b) it is more useful to know how teacher attributes or interventions affect the teachers' students, rather than just how they appear at the teacher level. The first two papers seek to elaborate on these tenets by categorizing teachers' professional knowledge in a new way and then by connecting classroom instruction to student interviews.

The third and fourth papers focus on elements of student thinking specific to combinatorics. Issues surrounding student representations and overgeneralizations in mathematics have been the subjects of a great deal of rich research already. However, these papers provide additional examples to indicate that there is still a great deal we do not know about these matters. In addition, both of these papers provide connections between the issues of student cognition and the context of the classroom. Both papers point to how instruction can support student thinking, but also how it can affect thinking in ways that are surprising. The specific findings could ultimately be useful to teachers who are working with students on combinatorics.

A notable element of this dissertation as a whole is that the data yielded a great deal of information in distinct areas of mathematics education research. While the initial goal of the data collection most closely reflected the analysis in the first paper, the resulting materials provided a rich, and sometimes unexpected, view of teacher thought, classroom instruction, and student thought. By dividing this work into four independent analyses, the most interesting aspects of the data were made available to be analyzed and shared.

This richness speaks to the value of a methodology, such as this one, that gathers information from a variety of sources. By including teacher interviews, student interviews, and classroom observations, we can consider mathematical thinking and instruction in its full complexity. At the same time, the separate analyses enable examination of particular aspects of the larger system. The implication for small studies in educational research, including student-initiated studies, is that varying data sources is a beneficial technique for producing a well rounded picture of the situation. While a small study may seek to look at student thought and thus focus on student interviews, for example, that same study could be deepened by looking for other ways to elaborate on or support the interview data. The size of the study may limit the amount of data or the number of cases, but researchers should not hesitate to draw on multiple resources to enrich their work.

Complete References

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure, and context. *Educational Studies in Mathematics, 15*(2), 129-147.

Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics, 12*(4), 399-420.

Brizuela, B.M. & Alvarado, M. (2010). First graders' work on additive problems with the use of different notational tools. *Revista IRICE Nueva Época, 21*, 37-43.

Caddle, M.C. (2010). *Analyzing solution methods for combinatorics problems: Differences across problems and across teachers.* Dissertation qualifying paper. Tufts University. December 2010.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Carpenter, T.P., Hiebert, J., Moser, J.M. (1981). Problem Structure and First-Grade Children's Initial Solution Processes for Simple Addition and Subtraction Problems. *Journal for Research in Mathematics Education, 12*, 27-39.

Carpenter, T.P. & Moser, J.M. (1982). The development of addition and subtraction problem-solving skills. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and Subtraction: A Cognitive Perspective* (pp. 9-24). Hillsdale, NJ: Lawrence Erlbaum.

Carpenter, T.P., & Moser, J.M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15,* 179-202.

Carter, J.A., Cuevas, G.J., Day, R., Malloy, C., Holliday, B., & Luchin, B. (2010). *Algebra 1*. Glencoe / McGraw-Hill, 2010.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., and Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3-29.

Collins, W., Cuevas, G.J., Foster, A.G., Gordon, B., Moore-Harris, B., Rath, J., Swart, D., & Winters, L.J. (1997). *Algebra 2*. Glencoe / McGraw-Hill, 1997.

diSessa, A. A., Hammer, D., Sherin, B., Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *Journal of Mathematical Behavior, 10*, 117-160.

Duke, N.K. & Beck, S.W. (1999). Education should consider alternative formats for the dissertation. *Educational Researcher, 28*(3), 31-36.

Gess-Newsome, J. (1999). Pedagogical content knowledge: An introduction and orientation. In J. Gess-Newsome & N.G. Lederman (Eds.), *Examining*

Pedagogical Content Knowledge: The Construct and its Implications for Science Education (pp. 3-17). Norwell, MA: Kluwer Academic Publishers.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.

Glencoe McGraw-Hill (2010). Algebra 2 Online Learning Center. Retrieved from http://glencoe.mcgraw-hill.com/sites/0078884829/

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior, 17*(2), 137-165.

Hadar, N. & Hadass, R. (1981). The road to solving a combinatorial problem is strewn with pitfalls. *Educational Studies in Mathematics, 12*(4), 435-443.

Hadfield, O. D., Littleton, C. E., Steiner, R. L., & Woods, E. S. (1998). Predictors of preservice elementary teacher effectiveness in the micro-teaching of mathematics lessons. *Journal of Instructional Psychology, 25*(1), 34-47.

Hill, H.C., Ball, D.L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351.

Hill, H.C., Ball, D.L, & Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H.C., Blunk, M.L., Charalambous, C.Y., Lewis, J.M., Phelps, G.C., Sleep, L., & Ball, D.L. (2008b). Mathematical Knowledge for Teaching and the

Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Dean, C., & Goffney, I. M. (2007). Assessing elemental and structural validity: Data from teachers, non-teachers, and mathematicians. *Measurement: Interdisciplinary Research and Perspectives*, *5*(2), 81–92.

Hill, H.C., Kapitula, L. Umland, K.U. (2011). A validity argument approach to evaluating teacher value-added scores. *American Educational Research Journal, 48*(3), 794-831.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H.C., Schilling, S.G., & Ball, D.L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Hill, H.C., Umland, K.U., Litke, E., & Kapitula, L. (under review). Teacher quality and quality teaching: Examining the relationship of a teacher assessment to practice. *American Journal of Education.*

Huinker, D., & Madison, S. K. (1997). Preparing efficacious elementary teachers in science and mathematics: The influence of methods courses. *Journal of Science Teacher Education, 8*(2), 107-126.

Karmiloff-Smith, A., & Inhelder, B. (1974). "If you want to get ahead, get a theory". *Cognition, 3*(3), 195-212.

Krathwohl, D.R. (1994). A slice of advice. *Educational Researcher, 23*(1), 29-32, 42.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (2006). *Connected Mathematics 2, Grades 6, 7, 8*. Prentice Hall, 2006.

Lowery, N. V. (2002). Construction of teacher knowledge in context: Preparing elementary teachers to teach mathematics and science. *School Science and Mathematics, 102*(2), 68-83.

Lubinski, C. A., & Otto, A. D. (2004). Preparing K-8 preservice teachers in a content course for standards-based mathematics pedagogy. *School Science and Mathematics, 104*(7), 336.

Maher, C.A. & Martino, A.M. (1996). Young children invent methods of proof: The gang of four. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning* (pp. 431-447). Hillsdale, NJ: Lawrence Erlbaum.

Maher, C.A. & Martino, A.M. (1996b). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27(*2), 194-214.

Massachusetts Department of Education (2000). *Massachusetts mathematics curriculum framework*. November 2000.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, T. P. Carpenter (Eds.), *Integrating research on the graphical representation of* *functions* (pp. 69-100). Hillsdale, NJ: Lawrence Erlbaum Associates/Taylor & Francis.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119-172.

Pérez Echeverría, M. P. & Scheuer, N. (2009). External representations as learning tools: An introduction. In C. Andersen, N. Scheuer, M. P. Pérez Echeverría, E. V. Teubal (Eds.), *Representational systems and practices as learning tools* (pp. 1-17). Boston, MA: Sense Publishers.

Piaget, J. (1976). *The child's conception of the world*. (J. Tomlinson & A. Tomlinson, Trans.). Savage, MD: Littlefield Adams. (Original work published 1926.)

Piaget, J. & Inhelder, B. (1975). *The origin of the idea of chance in children*. New York: W. W. Norton & Company Inc.

Philipp, R. A., Thanheiser, E., & Clement, L. (2002). The role of children's mathematical thinking experience in the preparation of prospective elementary school teachers. *International Journal of Educational Research, 37*(2), 195-210.

Rosen, K. (2003). *Discrete Mathematics and Its Applications*. Boston, MA: McGraw Hill.

Saxe, G.B., Gearhart, M., & Nasir, N.S. (2001). Enhancing students' understanding of mathematics: A study of three contrasting approaches to professional support. *Journal of Mathematics Teacher Education, 4*, 55-79.

Schliemann, A.D., & Acioly, N.M. (1989). Mathematical knowledge developed at work: The contribution of schooling. *Cognition and Instruction, 6*(3), 185-221.

Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction, 20*(2), 119-150.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Simon, M.A., & Schifter, D. (1993). Toward a constructivist perspective: The impact of a mathematics inservice program on students. *Educational Studies in Mathematics, 25*, 331-340.

Strauss, S. (1993). Teachers' pedagogical content knowledge about children's minds and learning: Implications for teacher education. *Educational Psychologist, 28*(3), 279-290.

Szydlik, J. E. (2000). Photographs and committees: Activities that help students discover permutations and combinations. *Mathematics Teacher, 93*(2), 93-99.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Tirosh, D., & Graeber, A.O. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics, 20*(1), 79-96.Brizuela, B.M. (2005). Young children's notation for fractions. *Educational Studies in Mathematics, 62*(3), 281-305.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning* (pp. 219-239). Hillsdale, NJ: Lawrence Erlbaum.

Vygotsky, L.S. (1978). *Mind in Society.* Cambridge, MA: Harvard University Press.

Watson, R. (1996). Students' combinatorial strategies. *Teaching Mathematics and Its Applications, 15*(1), 27-32.

Wheeler, E., & Brawner, J. (2005). *Discrete Mathematics for Teachers*. Boston, MA: Houghton Mifflin Company.