An Analysis of Chronometric Cosmology

Maxwell H. Kaye

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Abstract

The standard model for cosmology, also known as Big Bang cosmology, is motivated by interpreting extra-galactic redshift as a result of the expansion of the Universe, which alters the frequency of traveling light similarly to the Doppler effect. In the Standard Model, space is infinite, flat, and expanding. From a small set of axioms about the symmetries of the universe and causality, we construct an alternative cosmology, proposed in the 1970s by Irving Ezra Segal known as chronometric cosmology. In Segal's universe $\overline{M}$, the spatial part is finite, closed, and curved in the shape of a 3-Sphere, and redshift appears naturally as a consequence of generalized stereographic projection from the curved 3-sphere to its flat tangent space, the Minkowski spacetime M. Locally, these two universes, Minkowski M and Segal's $\overline{M}$, are indistinguishable. However, when one considers phenomena at large scales, the predicted dependence between redshift and distance varies in the two models, and hence in principle they can be tested against astronomical data. We consider two tests for the relationship between redshift and distance: the redshift-luminosity relation, and the redshift-number count relation.

Contents

1 Introduction

With the development of new measurement technology, the field of cosmology has made incredible strides in the last century, as well as in the last few decades. Increased precision in telescopes and the ability to deploy them outside of the Earth's atmosphere has provided cosmologists with a wealth of data against which to test the validity of countless theories. At present, the scientific community has settled on a standard theory, known as the ΛCDM cosmological model, the standard model of cosmology, or more colloquially, the Big Bang Theory.

This theory is a familiar story to many. Once upon a time[1], there was a Big Bang. The universe was an extremely dense soup of energy, so dense in fact that matter itself could not form. There was a period of "cosmic inflation", in which space rapidly expanded. Eventually, expansion made the density of energy low enough to allow atoms to form. With the synthesis of matter, gravity started pulling the matter together into stars and other celestial objects, and the universe became more or less what it is today. Almost completely empty space, with galaxies here and there, each including billions of stars and planets.

How then, you might ask, did scientists come to know all of this? According to Carl Sagan's "Cosmic Calendar", if the Big Bang happened on January 1st, humans don't invent the alphabet until the last 10 seconds of December 31st [21]. Despite the challenge of these immense scales, there are a few key clues which allow us to deduce everything we know (or at least everything we think we know) about the universe.

First and foremost, there is redshift. Other than the light sources within or quite close to our own Milky Way, the light spectra from each galaxy, quasar, and super nova we see is redshifted[2]. Not only do we observe the redshift in 99% of observed galaxies, but there is a pattern: as we observe light sources that are farther and farther away, we tend to observe greater and greater redshifts. Astronomers have long interpreted this redshift as a result of a Doppler-like effect[3], and this interpretation has become the basis for the standard model of

[1] Perhaps one shouldn't say time, as time didn't even exist before the Big Bang. For that matter, there was no before the Big Bang. Already at the beginning of our story, there are some difficult conceptual paradoxes to rectify.

[2] Stars are about 3/4 hydrogen and about 1/4 helium, containing negligible amounts of other elements. Due to the laws of quantum mechanics, each element has a characteristic spectrum of wavelengths of light which it can emit. However, when we observe light from distant objects, the expected wavelengths are shifted to large values than we observe in the laboratory, or red-shifted.

[3] ie, since the universe is expanding, distant objects are moving away, and even farther away objects are moving away even faster, thus the light that reaches us from these galaxies has a stretched wavelength, much in the same way that as a car speeds away from your location on the highway, you hear hear the pitch decrease (wavelength of sound increases). We say Doppler-like effect because, not only are sources moving at different speeds with respect to the observer, but the expansion of space itself also contributes to the redshift.

cosmology.

There is another extremely important clue to the true nature of our universe, known as the *Cosmic Microwave Background*. We observe microwave-frequency light coming at us from all directions, according to a nearly uniform distribution that is very close to a black-body spectrum of $T \sim 2.7\ K$. Just as is the case of galactic redshift, the CMB is a phenomenon which any good cosmological model must explain. The standard model of cosmology predicts the CMB as a remnant from the photon decoupling stage of cosmic development. Up until about 400,000 years after the Big Bang, the universe was so dense with energy that atoms could not form, and the entire universe was filled with opaque plasma. This meant that photons were continuously scattered and absorbed by charges, and never had the chance to radiate through free space. Only when the universe expanded to the critical point where hydrogen atoms could form, were photons finally free to travel through space with a significant probability of not getting re-absorbed instantly. These photons, originating from the "surface of last scattering", are the standard explanation for the origin of the CMB.

The Big Bang Theory is perhaps the most famous theory in all of physics. Despite the fact that it is a theory reaching 13.8 billion years into the past, physicists and astronomers have been able to make extremely precise theoretical prediction (or perhaps a better word would would be "retro-dictions") of the first few moments of our universe.

However, if history is any indicator, we must approach every theory with a healthy skepticism. In the year 140, Ptolemy put forth the geocentric model, which put the earth at the center of the solar system, and was the prominent theory for centuries. In the Ptolemic model, all motion was modeled as uniform circular motion, an assumption that led to an overly complex theory. Ptolemy had to include "epicycles," where the orbiting objects themselves were moving in small circles around some central point. It wasn't until the time of Copernicus that people began to accept the heliocentric model of the universe. However, Copernicus continued to advocate for the uniform circular motion of orbits, and in fact also needed to introduce new epicycles into his model in order to match observation. About 100 years later, Kepler came along and introduced elliptical orbits, which completely removed the clunky and incorrect epicycles from the astronomical cannon. Time and again throughout the history of cosmology, physics, and science as a whole, theories become replaced by better, more elegant theories, as progressively better data, technology, and mathematics allow us to understand the universe more deeply.

Perhaps the Big Bang Theory is no different.

1.1 History of Modern Cosmology

In the early 20th century, Edwin Hubble was gathering and analyzing redshift data, when he stumbled upon an earth-shattering discovery: redshifts of distant light sources seemed to increase with distance. Meanwhile, Alexander Friedman and Georges Lemaitre independently showed that an expanding universe was a valid solution to Einstein's field equations[10][12]. Their work was then built upon by Robertson and Walker in the 1930s [20][32], and hence the FRLW metric was born. Loosely speaking, a metric is a description of how to measure the distance between two points. In physics, the spatial component of the metric (derived in section 3.1.2) is conventionally written

$$dl^2 = R^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right) \tag{1}$$

where $R = R(t)$ is a scale factor, which in the case of an expanding universe, depends on time. A more well defined mathematical description which encodes the same information would be the associated Riemannian metric tensor, which might be written in matrix form with respect to the same generalized spherical coordinates as

$$R^2\begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}.$$

The great mathematician Bernhard Riemann himself developed the theory of Riemmannian geometry out of Differential geometry in the 1800s, laying the groundwork for the study of spatial curvature. In fact, he even mentioned the possibility that space might be a 3-dimensional sphere in his famous 1854 lecture[19]:

> *That space is an unbounded three-fold manifoldness, is an assumption which is developed by every conception of the outer world; ... if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value. If we prolong all the geodesics starting in a given surface-element, we should obtain an unbounded surface of constant curvature, i.e., a surface which in a flat manifoldness of three dimensions would take the form of a sphere, and consequently be finite.*

Returning now to the 20th century, Edwin Hubble's research culminated in a 1935 paper that he co-authered with Richard Tolman[11]. Most physicists are familiar with the first section: the authors propose expansion and the Doppler effect as a possible reason for the observed redshift. However, it is less widely known that the second section investigates curvature as an alternative explanations for the redshift. The final sentence of the paper reads

> *It also seemed desirable to express and open-minded position as to the true cause of the nebular red-shift, and to point out the indications that spatial curvature may have a part in the explanation of existing nebular data.*

The scientific community has largely ignored this second section. Perhaps this is because this section is extremely vague, since neither Hubble nor Tolman had the mathematical tools to understand such curvature. Perhaps also humans are drawn to a story of creation, whether motivated by religion, or the fact that everything in our finite lives knows a beginning, and the Big Bang provides such a creation story.

The discovery of the cosmic microwave background by Robert Wilson and Arno Penzias in the 1960s [17], was seen by many as proof of the the Big Bang, as the CMB was predicted to exist by the standard model of cosmology as leftover radiation from the Big Bang. Everything seemed to be going well for the Big Bang Theory, and as the old saying goes, *if it ain't broke, don't fix it.* However, mathematical physicist Irving Segal (1918-1998) seemed to think it was broken. He was troubled by the fact that, in the ΛCDM model, the energy of radiation decreases due to expansion, or as Hermann Bondi calls it, radiation is lost to the "cosmic sink." Segal sought out to explore whether its is possible to restore conservation of energy through an alternative cosmological model.

From here our story departs form the mainstream to a theory which is not widely accepted by the scientific community. Chronometric cosmology, put forth by Irving Segal in his 1976 book[29], is an alternative cosmological model, which predicts the redshift of astronomical objects, as well as the cosmic microwave background, however its predictions differ slightly from that of the standard model. According to Segal's calculations, the values of the redshift data fit much better with the chronometric predictions than with the expansionary predictions[4].

Segal's universe is in many ways a revival of Einstein's closed universe. After formulating special and general relativity, Einstein went on to consider their cosmological implications. In 1917, he wrote a paper[7] positing a static universe of constant spatial curvature, which is consistent with his equations for an empty universe (and if matter is included, then local curvature arises, but the global shape can still be a sphere). However, Einstein later turned his back on this idea due to the redshift evidence for expansion presented by Hubble. It wasn't until 50 years later that Segal demonstrated that this very redshift data might be explained by spatial curvature.

[4]Segal died in 1998; his conclusions were based on the data available at the end of 20th century.

1.2 Organization of this Thesis

Segal devotes the second chapter of his book to the mathematical development of the theory, the third chapter to the physical development, and the remaining chapters to the experimental data. In this thesis, we mimic this structure, although with two key differences.

The mathematical and physical development (sections 2 and 3) will hopefully be accessible to an advanced undergraduate. This means we will explain some concepts which Segal omitted because he thought them to be trivial, and many other points will be omitted, in an effort to concisely provide a picture of the theory. Section 2 is largely a summary of the Mathematical Development Section of Segal's book([29] pgs 22-49), as well as Daigneault's paper [6], and Segal's papers[27][23]. A key resource for section 3 is Sandage's *Observational Tests of World Models* [22].

The data analysis in section 4 will be based on modern data, which is another key difference between the present work and Segal's analysis. At the time Segal wrote his book, the highest measured redshift values were around 2.5, although now we have data from much farther away, with redshifts as large as 7 and 8. Modern data therefore provides an opportunity to truly put the theory to the test. The goal of this thesis is not to give conclusive evidence for the validity of either model, but rather to explore the physics and mathematics behind chronometric cosmology, and give some indication of how the models might be compared against observations.

2 Mathematical Development

2.1 Axioms and Definitions

In his book, Segal develops an axiomatic mathematical framework within which we can understand the cosmos[5]. Below are the axioms Segal puts forth, taken verbatim from pages 51-59 of his book[29].

- The Cosmos is a four-dimensional manifold
- The Cosmos is endowed with a notion of causality
- The Cosmos admits stationary observers
- Space is homogeneous and isotropic
- Any given timelike direction at a point p is tangential to the forward direction of some admissible observer
- Two different observers at the same point see the Cosmos in causally compatible ways, i.e., the transformation between their respective maps of the Cosmos should be causality-preserving

These axioms are all based on commonly held intuitions about the nature of the universe, causality, space, and time. In order to understand the specific meaning and implications of these axioms, we require a few definitions.

Definition 1 (Convex Cone)**.** *A convex cone is a subset* $\mathbf{C}$ *of a linear space* $\mathbf{L}$ *with the property that* $\mathbf{u}, \mathbf{v} \in \mathbf{C}$ *and* $a, b \geq 0 \Rightarrow a\mathbf{u} + b\mathbf{v} \in \mathbf{C}$.

Definition 2 (Infinitesimal Causal Orientation)**.** *An infinitesimal causal orientation in a manifold* M *is an assignment to each point* $p \in M$ *a nontrivial closed convex cone* $C(p)$ *in the tangent space space* T_pM*.*

The non-triviality requirement of the convex cones is to exclude non-relativistic notions of causality[6]. With these definitions, it becomes possible to define the notion of causality from the second axiom, by assigning to each point in space-time a convex cone, which exists in the tangent space of that point. All vectors within the cone are time-like directions, while those outside the cone are space-like (and those on the boundary are light-like, which are included in the future direction by the requirement that the cone is closed).

Definition 3 (Causally Oriented Manifold)**.** *A manifold* M *is causally oriented if it is endowed with an infinitesimal causal orientation such that the relation* $\prec: M \times M \to \{true, false\}$ *defined as* $p \prec q \iff q \in p + C(p)$ *is transitive. The relation* $\prec$ *is known as "precedence."*

[5] Throughout this thesis, the terms "cosmos," "universe," and "spacetime" are used interchangeably.

[6] If there were not a finite speed of causality (the speed of light), then the convex cone at any point p would trivially be the entirety of the tangent space at p.

On pages 22-23 of [29], Segal makes a clear distinction between infinitesimal causal orientations and finite global causal orientations, and the latter is not necessarily based on the former. Our definition of a causally oriented manifold is what Segal would refer to as a globally causally orientated manifold which has a precedence relation defined by the infinitesimal causal orientation.

Segal also adds in his discussion of the second axiom that not only must the Cosmos be causally oriented, but also no curve that always points into the future can be closed (ie time cannot wind back on itself). From this definition of causality, we can begin to discuss causal morphisms, which are essential for understanding the group structure of the Cosmos.

Definition 4 (Causal Isomorphism)**.** *A causal isomorphism between two causally oriented manifolds M and N is a map $F : M \to N$ that is a manifold isomorphism that carries one causal orientation into another.*

The third axiom is equivalent to the existence of a temporal group. Consider the set of all causal automorphisms on a manifold M which take any point p to any point q such that $p \prec q$. These are forward displacements in time. Similarly, one can define the set of all causal automorphisms on M which take p to q such that $q \prec p$ as backwards displacements in time. Hence axiom 3 postulates the existence of an identity element in the temporal translation group (consisting of all forward and backwards displacements), which is necessary for the temporal group to be considered a group.

Implicit in the statement of the fourth axiom is the ability to factor spacetime into its space and time components. For example, in the case of Minkowski space, $M = T \times S$ where $T = \mathbb{R}$ is time and $S = \mathbb{R}^3$ is space. With this factorization, it becomes possible to define the Lagrangian of the Cosmos as a function of space and time components:

$$\mathcal{L} = \mathcal{L}(x, \dot{x}, y, \dot{y}, z, \dot{z}, t).$$

Spatial homogeneity is equivalent to $\mathcal{L}$ carrying spatial translational invariance, and spatial isotropy is equivalent to $\mathcal{L}$ carrying spatial rotational invariance. Therefore by Noether's theorem [15], conservation of momentum and angular momentum follow directly from axiom 4.

Axioms 5 and 6 refer to the temporal symmetries of spacetime. Segal does not give much detail on axiom 5, and even omits it in some future papers [27], so we will not describe it in much detail here. The key point is that axiom 5 asserts the existence of admissible observers for all time-like directions, which along with axiom 3, define a one parameter group of temporal translations, and hence $\mathcal{L}$ carries temporal translation invariance. Segal refers to the implications of axiom 6 as "temporal isotropy." It asserts that two different observers at the same point in spacetime must make the same conclusions about the time precedence of different events. Temporal isotropy is equivalent to $\mathcal{L}$ carrying

Lorentz boost invariance.

So what do the axioms tell us about the mathematical structure of the universe? Well it must be four dimensional and be causally oriented, but on top of that there are 10 required symmetries:

- 3 space translation symmetries (spatial homogeneity)
- 1 time translation symmetry (temporal homogeneity)
- 3 space rotations symmetries (temporal isotropy)
- 3 Lorentz[7] boost symmetries (temporal isotropy)

which leads us to the conclusion that the isometry group of spacetime must include the Poincaré group. However, we can go a step further in determining the group structure of spacetime, by not only studying the isometry group, but also the causal group of a spacetime manifold.

Definition 5 (Causal Group)**.** *The causal group of M, a causally oriented manifold, is a Lie group that preserves an invariant causal orientation.*

It turns out that the causal group of interest is the conformal group. We will not give formal proof of this, but for motivation consider the fact that conformal transformations preserve angles in the space, which is tantamount to preserving the causal cones.

The conformal group of Minkowski space is $SO(4,2)$. The associated Lie Algebra $\mathfrak{so}(4,2)$ is composed of 15 generators: the 10 generators of Poincaré symmetries, as well as 4 generators of conformal inversions and 1 scale transformation. This is consistent with the well known fact that the dimension of $SO(p,q)$ is $1/2(p+q)(p+q-1)$, and hence $SO(4,2)$ is 15 dimensional. In the next section, we give an overview of the restrictions that the axioms and causal group structure put on candidate world manifolds.

2.2 Admissible World Manifolds

There are two possible manifolds which satisfy the above axioms for the cosmos. We will call these manifolds M and $\overline{M}$. Segal shows that these two manifolds are in fact the only possible spacetimes which satisfy the axioms.

$$M = \mathbb{R}^3 \times \mathbb{R} \qquad\qquad \overline{M} = \mathbb{S}^3 \times \mathbb{R}$$

[7] Technically, Lorentz invariance is specific to Minkowski space. More generally, temporal isotropy guarantees the equivalence of any two time-like directions, a sort of pseudo-lorentzian symmetry.

M is the familiar flat Minkowski spacetime, whereas $\overline{M}$ is referred to as the "universal cosmos". We will see that these two manifolds are closely related, and the very nature of this relationship is what determines the structure of chronometric cosmology.

In both the standard cosmological model and the chronometric model, the universe is a 4-dimensional manifold called spacetime, which consists of 1 temporal dimension and 3 spatial dimensions. In the Standard Model, the spacetime manifold M is isomorphic to $\mathbb{R}^4$. In this sense spacetime is flat[8]. According to the standard model of cosmology, spacetime is the 4-dimensional analogue of a plane. Also, according to this model, the universe is expanding.

In the chronometric model, the spacetime manifold $\overline{M}$ is isomorphic to $\mathbb{R} \times \mathbb{S}^3$, which can be thought of as the 4-dimensional analogue of a cylinder, where the principal direction on the surface with Normal Curvature of 0 (the straight direction) is the direction of cosmic time, and the other principal direction is space. Furthermore, there is a distinction between the cosmic time (the $\mathbb{R}$ in $\mathbb{R} \times \mathbb{S}^3$) and the local time (which is the time direction in $T_p\overline{M}$, the tangent space of $\overline{M}$ at any point $p \in \overline{M}$). This is why the theory is called "chronometric"; we can measure the difference between the two times. In contrast, since $\mathbb{R}^n$ is isomorphic to its own tangent space for every point $p \in \mathbb{R}^n$, there is no distinction between local time and global time in the Standard Model. Even in the chronometric model, it is not so easy to measure the difference between cosmic time and local time, because at small scales (on the order of a few thousand light years), the tangent space to $\overline{M}$ is a fantastic approximation for $\overline{M}$ itself. The "radius of the universe" is so large, that a distinction between the two times only becomes measurable at extremely large scales.

Furthermore, the two possible spacetimes M and $\overline{M}$ satisfying the axioms are related geometrically. Minkowski spacetime M is tangent to $\overline{M}$, or to any other Lorentzian manifold, at any point, and specifically at the point of observation. At the same time, M is causally embedded in $\overline{M}$ by a relativistic generalization of stereographic projection. In order to demonstrate this, we must first develop suitable representations for M and $\overline{M}$.

2.2.1 Preservation of Causality

While there are many possible manifolds that one might imagine being our universe, only M and $\overline{M}$ can be endowed with the necessary causal structure to satisfy the axioms.

[8]Note that this is counter-intuitive. Many of us have been told that spacetime is curved, and this curvature is a manifestation of gravity. This is indeed true, but this curvature is local and on a smaller scale. Globally, Minkowski space is flat.

As given by definition 1, the light-cone at any point of Minkowski space is defined by a set of null-vectors ($\mathbf{x}$ such that $g(\mathbf{x}, \mathbf{x}) = 0$). Segal argues that a similar quadric Q in $SO(4,2)$ defines a natural Lorentz structure on M, which remains invariant under the action of $SO(4,2)$ (and hence is causally invariant).

In order to show that $\overline{M}$ also can admit such a structure, Segal remarks on page 39 of [29] that "any covering space of a conformal manifold is again a conformal manifold, in a unique way so that the defining covering local homeomorphism is locally conformal; and is conformally [globally] causal if the original manifold is such." The manifold $\mathbb{S}^1 \times \mathbb{S}^3$ is the (two-fold) covering space of M, while $\overline{M} = \mathbb{R}^1 \times \mathbb{S}^3$ is its universal cover. As a covering space of M, the space $\mathbb{S}^1 \times \mathbb{S}^3$ admits a local notion of causality, but it does not admit a global causal orientation (since there exist closed time-like curves, due to the fact that time is a circle). However, the space $\mathbb{R}^1 \times \mathbb{S}^3$, is the universal cover of Minkowski space M which is globally causal.

2.2.2 Coordinate Representations

Now that it has been established that M and $\overline{M}$ are the two candidate manifolds which obey the axioms, our task becomes the formulation of representations of both manifolds.

Any vector $(x_0, x_1, x_2, x_3) \in M$ (Minkowski space, the world manifold for the standard model, and also the tangent space to $\overline{M}$ at any point) can be uniquely represented by a Hermitian matrix, since Hermitian matrices are a 4 dimensional vector space, generated by the Pauli matrices along with the identity [9].

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So we can represent any point in M as

$$H = x_0\mathbb{I} + x_1\sigma_x + x_2\sigma_y + x_3\sigma_z = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \tag{2}$$

In a similar vein, since $\mathbb{S}^3$ is diffeomorphic to the matrix group $SU(2)$, we can represent any point in $\overline{M}$ as a pair

$$(t, V) \text{ for } V \in SU(2).$$

As we will see in the next section, the relationship between some point

$$x = (x_0, x_1, x_2, x_3) = H_x \in H(2) \cong M$$

and its corresponding point

$$\hat{x} = (t, V) \in \mathbb{R} \times SU(2) \cong \overline{M}$$

[9] Segal chooses for σ_y what is conventionally $-\sigma_y$, but the formulation is equivalent.

is the key to understanding chronometric cosmology. This is the relationship between observations of phenomena made by an observer at a point p (observations are made in the tangent space $M = T_p\overline{M}$), and the temporal evolution of phenomena in the $\overline{M}$ itself.

2.3 The Relationship Between M and $\overline{M}$

Using these matrix group representations of M and $\overline{M}$, Segal is able to analyze the relationship between observations made in M of events occurring in $\overline{M}$ through the following mapping (which is layed out clearly by Daigneualt[6]):

$$M \cong H(2) \quad \rightarrow \quad U(2) \quad \rightarrow \quad \mathbb{R} \times SU(2) \cong \overline{M}$$

2.3.1 Stereographic Projection

The mapping from $H(2)$ to $U(2)$ is given by the inverse of a generalized form of stereographic projection, known as the Cayley Tansform. Stereographic projection is a concept familiar to anyone who has taken a geography class. It is the reason Greenland appears far larger on maps than it it does on a globe. In essence, stereographic projection is the mapping of a flat space onto a curved space (specifically onto a sphere)[10]. The distortion of the curved surface $\mathbb{S}^2$ in a map created by stereographically projecting $\mathbb{S}^2$ onto $T_p\mathbb{R}^2$ at $p \in \mathbb{S}^2$ increases with the distance from p.

Geometrically speaking, 2-dimensional stereographic projection is the 1 to 1 + 1 mapping between a sphere and a plane. For any point p on the sphere, stereographic projection takes p to the point q on the plane tangent to the south pole of the sphere, such that the line through p and q goes through the north pole of the sphere. The north pole is referred to as the "point at infinity", and it is the reason we call the mapping 1 to 1 + 1: every point on the sphere is in 1 to 1 correspondence with a point on the plane, except for the point at infinity. Whether transforming a line to a circle, a plane to a sphere, or n-dimensional euclidean space to an n-dimension sphere, this process of streographic projection involves the inclusion of one additional point at infinity, and hence we refer to this process as "one point compactification," because we are taking a non-compact manifold and curving it in on itself, making it compact by connecting the boundary to this point at infinity.

A key step in the transformation between M and $\overline{M}$ is the compactification of $H(2)$ to $U(2)$. This can be seen as a generalization of inverse stereographic projection, where the 1 dimensional analogue of $H(2)$ is the imaginary axis, and the 1 dimensional analogue of $U(2)$ is the complex unit circle. Just as the lie

[10]In truth, we use the Mercator Projection for most maps, which is closely related to stereographic projection, except rather than mapping the sphere to a plane, it is mapped to a cylinder.

algebra of the complex unit circle is the the imaginary axis, the lie algebra of $U(2)$ is $iH(2)$ which is isomorphic to $H(2)$, and we can transform any $H \in H(2)$ to its corresponding $U \in U(2)$ via the "Cayley Transform", a generalization of sterographic projection, given by .

$$U = (I + iH/2)(I - iH/2)^{-1}.$$

The conformal group structure perserveres through this compactification of Minkowski space. One can compactify the M, O(3,1), by including it into the projective light cone (i.e. the space of all null lines through the origin) in 6-dimensional Euclidean space $\overline{M}$ with (4,2) signature. SO(4,2) naturally acts on this space.

2.3.2 Extracting $\overline{M}$ from $U(2)$

The next step in the mapping is a simple bijection between two representations of a 4-sphere. Any $U \in U(2)$ can be represented by a pair $t, V \in SU(2)$, where

$$exp(it)V = U.$$

This is because elements of $SU(2)$ have unit determinant by definition of the special linear group, while elements of $U(2)$ have determinants of complex magnitude 1.

However, as noted in section 2.1, causally oriented manifolds cannot admit closed time-like curves. Thus rather than allowing time to range over the whole complex unit circle, we exclude one point. $\mathbb{S}^1 \times \mathbb{S}^3$ becomes $\mathbb{R}^1 \times \mathbb{S}^3$, and $\overline{M}$ is obtained (through a causality preserving mapping) from M.

3 Physical Development

Science is meaningless without measurable and experimentally verifiable predictions. So after considering these two cosmological models, the natural next question becomes, how can we determine which universe we live in? The answer is light. Electromagnetic waves are essentially the only things that travel fast enough and far enough for us to compare curvature at such large scales. While there are many ways to use light to try and determine our universe's cosmology, this paper will focus on the redshift-luminosity relation.

3.1 Preliminaries

3.1.1 Definitions and Coordinate Conventions

Definitions:

Definition 6 (Redshift)**.** *The displacement of spectral lines toward longer wavelengths (the red end of the spectrum) in radiation from distant galaxies and celestial objects. This is interpreted as a Doppler shift that is proportional to the velocity of recession and thus to distance (Oxford Dictionary).*

The redshift z is defined as the fractional difference between emitted and observed wavelengths:

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}} \tag{3}$$

Definition 7 (Luminosity)**.** *The intrinsic brightness of a celestial object (as distinct from its apparent brightness diminished by distance) (Oxford Dictionary).*

Luminosity is measured in units of power (energy per time).

Definition 8 (Magnitude)**.** *Magnitude is an equivalent definition of luminosity used by astronomers, where $m \approx -2.5\log_{10}(L) + C$. Observed magnitude m, absolute magnitude M, observed luminosity L_{obs}, and absolute luminosity L, are related by the equation*

$$m - M = -2.5\log_{10}(\frac{L_{obs}}{L}) \tag{4}$$

There is an important distinction to be made between intrinsic (absolute) luminosity and observed luminosity. Consider a 60 watt light bulb in a large room. Anyone looking at the light from 10 feet away could try to calculate the luminosity of the light bulb by measuring the energy of the light that reaches them, and then realizing that the luminosity decreases according to a $1/r^2$ relation (which is described in depth in section 3.2). The value that the observer sees is known as *observed luminosity*, whereas the value 60 watts (800 lumens) is the light bulb's *absolute luminosity.*

However, now let us suppose that their is a smoke machine in the room going at full blast. The absolute magnitude calculation would not account for the dimming of the light due to the smoke, and hence if we ignore absorption/scattering effects (known to astronomers as *extinction*), we obtain a lower value than 60 watts. This is an important distinction, as when we look out on such vast distances into the cosmos, absorption due to cosmic dust and other matter can significantly affect the disparity between absolute and observed luminosity. Neither of the redshift-magnitude relations derived in section 3.2 account for absorption when relating observed luminosity to absolute luminosity.

<u>Coordinate Conventions:</u>
In the following physical derivations, we use a variety of spacetime coordinates, when referring to both local and global measurements. Below is a list of "global" coordinates. These are values associated with $\overline{M}$.

- $\tau \in [0, 2\pi)$ is the cosmic time, which we take to be unit-less. $\tau = ct/R$ where t is cosmic time, c is the speed of light, and R is the radius of the universe.
- l is the manifold distance between two points. In a 2-dimensional analogy, it would be the distance an ant walks as it travels along the surface of a sphere.
- ρ is mathematically equivalent to τ as a unit-less quantity, except we use it in cases when we describing a distance in the 3-sphere, rather than as cosmic time in $\overline{M}$. $\rho = l/R$, where R is the radius of the universe.

Now on to the "local" coordinates, which are associated with measurements made in the the M, the Minkowski space tangent to $\overline{M}$ at the point of observation.

- x_0 is the local time, although we take it to be unit-less (so $x_0 = ct/R$, where t is the local time, c is the speed of light, and R is the radius of the universe). Similarly, x_1, x_2, and x_3 are local Euclidean coordinates of M.
- r is the local distance between two points. It can be thought of as the local coordinate of a coordinate chart for $\overline{M}$, as well as as a projection from the 3-sphere onto its tangent space. In a 2-dimensional analogy, imagine a sheet of glass balanced perfectly on the north pole of a sphere. The distance from the north pole to any other point p on the sphere, measured by r, would be the distance along the sheet of glass an ant must walk such that it is standing directly above the point p. $r = \sin(l/R) = \sin(\rho)$.

3.1.2 The Metric in $\mathbb{S}^3$

Both M and $\overline{M}$ are 4-dimensional manifolds, having 3 dimensions of space and one dimension of time. However, their spatial parts have different geometries, namely $\mathbb{R}^3$ and $\mathbb{S}^3$, and hence the distance between two points in each space is measured differently, using different metrics. In $\mathbb{R}^3$, the metric can be succinctly expressed as the familiar Pythagorean equation for the line element.

$$dl^2 = du_1^2 + du_2^2 + du_3^2$$

While it is possible to use the interior calculus on the manifold to derive an expression for the line element in $\mathbb{S}^3$, for simplicity we will do so by embedding $\mathbb{S}^3$ in $\mathbb{R}^4$.

In $\mathbb{R}^4$, the line element is given by

$$dl^2 = du_1^2 + du_2^2 + du_3^2 + du_4^2$$

We can view $\overline{M} \cong \mathbb{R} \times \mathbb{S}^3$ as the subset

$$\{x \in \mathbb{R}^5 \mid x = (u_0, u_1, u_2, u_3, u_4) \text{ and } u_1^2 + u_2^2 + u_3^2 + u_4^2 = R^2\}$$

Since we are concerned with spatial curvature, we can restrict our view to $\mathbb{S}^3 \in \mathbb{R}^4$.

Taking the differential of both sides of the equation for a sphere gives us

$$2u_1du_1 + 2u_2du_2 + 2u_3du_3 + 2u_4du_4 = 0$$

Hence we can replace du_4 in the equation for the line element with

$$du_4 = -(u_1du_1 + u_2du_2 + u_3du_3)/u_4 \qquad \text{where } u_4 = \sqrt{R^2 - u_1^2 - u_2^2 - u_3^2}$$

yielding

$$dl^2 = du_1^2 + du_2^2 + du_3^2 + \frac{(u_1du_1 + u_2du_2 + u_3du_3)^2}{R^2 - u_1^2 - u_2^2 - u_3^2}$$

Now that we have eliminated u_4, let us further simplify the equation using 3D spherical coordinates. Making the following substitutions:

$$u_1 = r' \sin\theta \cos\phi \qquad u_2 = r' \sin\theta \sin\phi \qquad u_3 = r' \cos\theta$$

One finds that the sum of the squares of the three differentials is

$$du_1^2 + du_2^2 + du_3^2 = dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2(\theta) d\phi^2$$

Furthermore,

$$u_1^2 + u_2^2 + u_3^2 = r'^2,$$

which is just the equation of a 2-sphere. Taking the differential of both sides (and dividing both sides by 2) yields

$$u_1du_1 + u_2du_2 + u_3du_3 = r'dr'$$

Thus the equation for the line element can be written

$$dl^2 = dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2(\theta) d\phi^2 + \frac{(r'dr')^2}{R^2 - r'^2}$$

Finally we substitute r' with $r = r'/R$, to put the equation in a more natural set of coordinates.

$$dl^2 = R^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 + \frac{r^2 dr^2}{1 - r^2})$$

Which when simplified gives us the metric given by equation 7 in Sandage[22], which was originally derived by Robertson[20] and Walker[32].

$$dl^2 = R^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \right) \tag{5}$$

where k is 1, since $\mathbb{S}^3$ has positive curvature ($k = -1$ corresponds to negative curvature, or hyperbolic space; $k = 0$ corresponds to no curvature, or flat space).

3.1.3 Derivation of the Two Times Formula

Here we derive the namesake of chronometric cosmology, a formula relating the cosmic time to the local time.

Theorem 1. *Our observed time, x_0, is given as a function of τ by the formula*

$$x_0 = 2\tan(\frac{\tau}{2}) \tag{6}$$

Proof. Let $X \in \overline{M}$ be a point in spacetime, and let H_x be the 2×2 Hermitian Matrix representing X. So

$$H_x = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}$$

Let U_x be the corresponding Unitary Matrix, via the Cayley Transform.

$$U = (I + iH/2)(I - iH/2)^{-1}$$

So we have

$$U_x = (I + iH_x/2)(I - iH_x/2)^{-1}$$

and

$$H_x = -2i(U_x - I)(U_x + I)^{-1}$$

Let $X(\tau)$ denote the same point in space as X, translated forward in time by τ. Thus

$$H_x(\tau) = -2i\frac{e^{i\tau}U_x - I}{e^{i\tau}U_x + I}$$

Now after inserting the expresion for U_x above, we get

$$H_x(\tau) = -2i\frac{e^{i\tau}(I + iH_x/2)(I - iH_x/2)^{-1} - I}{e^{i\tau}(I + iH_x/2)(I - iH_x/2)^{-1} + I}$$

Rewriting I as $(I - iH_x/2)(I - iH_x/2)^{-1}$ and canceling the factor $(I - iH_x/2)^{-1}$ yields

$$H_x(\tau) = -2i\frac{e^{i\tau}(I + iH_x/2) - (I - iH_x/2)}{e^{i\tau}(I + iH_x/2) + (I - iH_x/2)}$$

Grouping terms gives us

$$H_x(\tau) = -2i\frac{(e^{i\tau} - 1)I + (e^{i\tau} + 1)iH_x/2}{(e^{i\tau} + 1)I + (e^{i\tau} - 1)iH_x/2}$$

Using Euler's Formula and trigonometry, we can rewrite

$$e^{i\tau} - 1 = 2i\sin(\frac{\tau}{2})e^{i\tau}$$

$$e^{i\tau} + 1 = 2\cos(\frac{\tau}{2})e^{i\tau}$$

So our equation becomes

$$H_x(\tau) = -2i\frac{-2i\sin(\frac{\tau}{2})e^{i\tau}I + 2\cos(\frac{\tau}{2})e^{i\tau}iH_x/2}{2\cos(\frac{t}{2})e^{i\tau}I - 2i\sin(\frac{\tau}{2})e^{i\tau}iH_x/2}$$

After canceling out the $e^{i\tau}$ terms and simplifying we get

$$H_x(\tau) = 2\frac{2\sin(\frac{\tau}{2})I + \cos(\frac{\tau}{2})H_x}{2\cos(\frac{\tau}{2})I - \sin(\frac{\tau}{2})H_x}$$

Without loss of generality, assume $H_x = 0$, by setting it as the origin of our coordinate system. So we now have the following formula for the temporal translation of a point X, written as a 2×2 Hermitian matrix:

$$H_x(\tau) = 2\tan(\frac{\tau}{2})I$$

Finally extracting the local time from the definition of H_x (equation 2), we find that

$$x_0 = 2\tan(\frac{\tau}{2}) \tag{7}$$

which gives us a relationship between the local time x_0 and the cosmic time τ, the "Two Times Formula".

□

3.1.4 Observation of Light Waves in $\overline{M}$

When working in the standard model, the tangent space to $\mathbb{R}^4$ is again $\mathbb{R}^4$, so there is no physical distinction between vectors in the tangent space to M, and points in M itself. The same cannot be said of $\overline{M}$ however. We are forced to decide whether the world we observe exists in the tangent space to the spacetime manifold, or in the spacetime manifold itself. This distinction is subtle, but has wide reaching implications to both our observation of redshift, and the principle of conservation of energy.

In any introductory course to quantum mechanics, students learn that the energy operator is given by the Hamiltonian: $-i\hbar\frac{\partial}{\partial t}$, where $\frac{\partial}{\partial t}$ is a derivative with respect to time. We are now forced to ask ourselves, should this be a derivative with respect to cosmic time τ, or local time x_0? At its heart, the Hamiltonian is the generator of the one-parameter group of time translations, so it is reasonable to suppose that the true Hamiltonian involves a derivative with respect to τ, and the corresponding frequency operator would then be $i^{-1}\frac{\partial}{\partial \tau}$.

The assumption that our observations of the universe are made in the tangent space (associated with the Minkowski time x_0), whereas "free temporal evolution is as given on the global curved manifold" (Segal 20), not only determines a redshift-time relation unique to $\overline{M}$, but also has the profound effect of preserving the conservation of energy at the cosmic scale. This originates from fact that cosmic time and local time are locally indistinguishable, but if our observations are made in the tangent space, then we would observe distortion effects from generalized stereographic projection (as described in section 2).

3.1.5 Derivation of Redshift-Time Relation

The redshift-time relation is a direct consequence of the two times relation, due to this interpretation of the frequency operator.

Theorem 2. *The redshift z of a light ray at time τ, which was emitted at time $\tau = 0$, is given by the formula:*

$$z = \tan^2(\frac{\tau}{2}) \tag{8}$$

Proof. Since our observations are made in the tangent space to the comsos, whereas frequency is given by the derivative with respect to cosmic time (Section 3.1.4), differentiating local time with respect to cosmic time gives the ratio of observed frequency to emitted frequency.

$$\frac{\nu_{emit}}{\nu_{obs}} = \frac{dx_0}{d\tau} = \sec^2(\frac{\tau}{2}).$$

From this formula we can derive the redshift-time relation. By definition 3,

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}}$$

it follows that

$$z + 1 = \lambda_{obs}/\lambda_{emit} = (\frac{c}{\nu_{obs}})/(\frac{c}{\nu_{emit}}) = \frac{\nu_{emit}}{\nu_{obs}} = \sec^2(\frac{\tau}{2})$$

and hence the wavelength is magnified by a factor of

$$z + 1 = \sec^2 \frac{\tau}{2}.$$

Thus by trigonometric substitution we arrive at equation 8, the redshift time relation:

$$z = \tan^2 \frac{\tau}{2} \tag{8}$$

□

3.2 Redshift-Magnitude Relations

3.2.1 Derivation of Chronometric Redshift-Magnitude Relation

The Redshift-Luminosity relation is critical for contrasting the predictions of the standard model with Segal's model, as Luminosity is perhaps the best distance proxy available to us (other than, of course, redshift), for extremely distant objects.

Theorem 3. *The redshift magnitude relation in the universal cosmos is given by the formula:*

$$m = 2.5\log(z) - 2.5(2 - \alpha)\log(1 + z) + C \tag{9}$$

Proof. Let us start with equation redshift time relation [29](p 79).

$$z = \tan^2(\frac{\tau}{2})$$

We can choose to represent the universal spacetime manifold $\overline{M}$ in natural units, thus the the temporal redshift relation becomes a spatial redshift relation, in terms of a distance ρ (which is scaled by $1/R$, so $\rho \in (0, 2\pi)$):

$$z = \tan^2(\frac{\rho}{2}) \tag{10}$$

This equation thus represents the redshift in the light that travels from some luminous object at distance ρ. Let us now calculate the luminosity of the light, using our knowledge of the geometry of the universal cosmos.

The observed luminosity L_{obs} at distance ρ is equal to the absolute luminosity of the source, diminished by the 2 factors:

- Surface area of a sphere with radius ρ
- Redshift of radiation after traversing the distance ρ

Luminosity is diminished by a factor of the surface area of a 2-sphere centered at the source, as the luminosity must be spread from a point to an area surrounding the point. Imagine a shell surrounding the luminous source. An observer on the shell would measure only the energy from the source that reached their section of the shell, which is why it is diminished by this factor.

The luminosity is also diminished by the redshift of the radiation due to traveling the distance ρ. In a static universe, this factor is $\frac{1}{1+z}$. This follows from the definition of redshift:

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}}$$

So the general form of the relationship between the observed luminosity, L_{obs}, and absolute luminosity, L, for a static universe is given by:

$$L_{obs} = \frac{1}{A}\frac{1}{1+z} * L$$

The area of a 2-sphere of radius r in Euclidean space is $4\pi r^2$, although since $\overline{M}$ is spherical, the area of a sphere of radius r (the manifold distance, with our luminous object at the center), is $4\pi \sin^2(\frac{r}{R})$, or in natural units (with the radius ρ), $4\pi \sin^2(\rho)$. Thus, the observed luminosity is given by the equation:

$$L_{obs} = \frac{1}{4\pi \sin^2(\rho)}\frac{1}{1+z} * L$$

The absolute luminosity, L, can be obtained by integrating the spectral function (a distribution of probabilities of different energies) over all frequencies.

$$L = \int f(\nu)d\nu$$

In practice however, we can only measure certain frequencies, so the integral becomes definite, of the form:

$$\int_{\nu 1}^{\nu 2} f(\nu)d\nu$$

In fact, the frequencies that we can measure are relatively high (corresponding to low wavelengths), compared to the frequencies giving maxima for the black body spectrum. Thus we can assume $f(\nu) = 1/\nu^{\alpha}$, because the frequency of the black body function grows exponentially for small wavelengths, as shown in figure 3, so it fits a decaying exponential in frequency.

[11] https://chem.libretexts.org/Courses/Pacific_Union_College/Quantum_Chemistry/

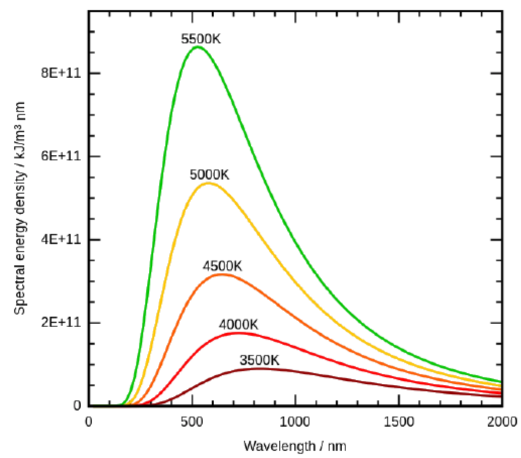

Figure 1: Black Body Spectrum[11]

The integral of the spectral function evaluates to:

$$L = (1+z)^{1-\alpha}$$

Putting it all together, we have

$$L_{obs} = \frac{1}{4\pi \sin^2(\rho)} \frac{1}{1+z} (1+z)^{1-\alpha} = \frac{(1+z)^{-\alpha}}{4\pi \sin^2(\rho)}$$

Now we want to write ρ as a function of z, so we can invert equation 10 to find

$$\rho = 2\arctan(\sqrt{z})$$

So the formula for observed luminosity becomes:

$$L_{obs} = \frac{(1+z)^{-\alpha}}{4\pi \sin^2(2\arctan\sqrt{z})} = \frac{(1+z)^{-\alpha}}{4\pi \frac{4z}{(z+1)^2}} \propto \frac{(1+z)^{2-\alpha}}{z}$$

We can convert luminosity to magnitude via the equation

$$m - M = -2.5\log(L_{obs}/L)$$

where m is the apparent magnitude and M is the absolute magnitude of the source. Hence we obtain the chronometric m-z relation:

$$m = 2.5\log z - 2.5(2-\alpha)\log(1+z) + C$$

where the absolute magnitude was absorbed into the constant C[12]. □

[12]We are implicitly making the assumption that all sources are "standard candles", ie that they all have the same intrinsic luminosity. This assumption is unfortunately indispensable, although not realistic.

3.2.2 The Standard Redshift-Luminosity Relation

Theorem 4. *The redshift magnitude relation in the standard model is given by the formula:*

$$m = M + 5\log q_0^{-2}[zq_0 + (q_0 - 1)(-1 + (2q_0 z + 1)^{1/2})] + C \tag{11}$$

Proof. Consider a sphere of radius r centered at a light source, where r is the distance to the earth. Since space is Euclidean in the standard model, the surface area of this sphere is

$$A = 4\pi r^2,$$

and hence the apparent bolometric flux f_b in terms of the absolute flux F_b is

$$f_b = \frac{F_b}{4\pi r^2} * \phi(z),$$

where ϕ is a function that accounts for the depletion in energy due to redshift.

Now let us determine ϕ, i.e. how the flux is affected by the redshift. In a static (non-expanding) universe, since each photon is redshifted by z, the observed energy from each photon is (using definition 3):

$$E_{obs} = h\frac{1}{\lambda_{obs}} = h\frac{1}{\lambda_{emit}(1+z)} = \frac{E_{emit}}{1+z}$$

Hence the flux is diminished by a factor of $1 + z$ simply by the definition of redshift.

However, since the Big Bang Universe is expanding, the increase in path length associated with the redshift comes with an associated decrease in energy density, yielding an additional diminishing of the the flux by $1 + z$. Hence $\phi = \frac{1}{(1+z)^2}$, and we have

$$f_b = \frac{F_b}{4\pi r^2(1+z)^2}.$$

The reader should note that while this equation for flux is diminished by two factors of $(1+z)$, the equation for Luminosity developed in the previous section is diminished by only a single factor, as this second source of energy loss due to expansion is not present. Now converting this equation into magnitude via 4 gives us

$$m - M = -2.5\log(f_b/F_b) = -2.5\log(\frac{1}{4\pi r^2(1+z)^2}).$$

Now just as in the derivation of the chronometric m-z relation, we must replace r with an expression in terms of z, using an analogous equation to equation 10.

Such an equation was derived by Mattig [13], who showed that the Friedman equation

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2\ddot{R}}{R} = -\frac{kc^2}{R^2}$$

is solved by

$$R_0 r = \frac{c}{H_0 q_0^2 (1+z)} \left(z q_0 + (q_0 - 1)(-1 + (2q_0 z + 1)^{1/2}) \right) \tag{12}$$

Plugging this redshift distance relation into the expression for magnitude yields

$$m = M + 5\log q_0^{-2}[zq_0 + (q_0 - 1)(-1 + (2q_0 z + 1)^{1/2})] + 5\log\frac{c}{H_0} + 2.5\log 4\pi$$

which, if we collect the constants into the symbol C, yields the equation 11:

$$m = M + 5\log q_0^{-2}[zq_0 + (q_0 - 1)(-1 + (2q_0 z + 1)^{1/2})] + C$$

□

3.3 Redshift-Number Count Relations

Luminosity is not the only distance proxy we can use to experimentally determine the nature of spacetime. By plotting redshift on one axis and number of bodies at that redshift on another, we can can compare chronometric cosmology with Big Bang cosmology. As distance from earth increases, we expect to find more astronomical objects at that distance from earth, by virtue of the cosmological principle (the homogeneity and isotropy of spacetime implies that number of bodies is linearly related to volume). However, whether the universe is Euclidean and expanding, or curved and static, will present different relationships between these two quantities.

3.3.1 Derivation of z-N(z) in the Standard Model

Again following the derivation from Sandage [22], we start from an expression for the interval dl^2 between two points in space.

$$dl^2 = R^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)] \tag{1}$$

where R(t) is a scale factor with units of distance (in the expansionary model it increases as a function of time) and $k \in \{-1, 0, 1\}$ is spatial curvature. For the Euclidean case, we take k to be 0. Now let us find the volume of a sphere of radius r from the origin in M. The volume can be found by integrating the corresponding volume form[13], which gives us

[13] A natural form is given by $dV = |g|dx^1 \wedge dx^2 \wedge ... \wedge dx^n$, where $|g|$ is the absolute value of determinant of the matrix of the metric tensor (equation 1), and $dx^1 = dr, dx^2 = d\theta, dx^3 = d\phi$.

$$V = \int_0^{Rr} \frac{(Rr)^2 dr}{\sqrt{1-kr^2}} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi$$

Since $k = 0$ in the Euclidean case, we get

$$V(r) = \frac{4\pi R^3 r^3}{3}$$

Now in order to write V in terms of z, we will need to write r as a function of z, which is given by equation 12

$$R_0 r = \frac{c}{H_0 q_0^2 (1+z)} \left(z q_0 + (q_0 - 1)(-1 + (2q_0 z + 1)^{1/2}) \right) \tag{12}$$

Setting q_0 to $1/2$, as we do throughout this preliminary analysis, we find that

$$r \propto 1 - \frac{1}{\sqrt{z+1}}$$

giving us the volume, and in turn number count, as a function of redshift (normalized to ignore constants).

$$V(z) = \frac{4}{3}\pi(1 - \frac{1}{\sqrt{1+z}})^3 \tag{13}$$

3.3.2 Derivation of z-N(z) in the chronometric Model

Following a similar procedure as in the previous section, we use the metric given by equation 1, this time with $k = 1$, to find the volume of a ball of radius r embedded in a 3-sphere of radius R

$$V = 2\pi(\arcsin r - r\sqrt{1-r^2})$$

Now we will use the inverse of the redshift distance relation, given by equation 10,

$$\rho = 2\arctan\sqrt{z}$$

along with the fact that the coordinate r is given by $\sin(\rho)$ to get

$$r = \sin(2\arctan\sqrt{z}).$$

Plugging this into the volume equation, we get

$$\begin{aligned} V &= 2\pi \left(2\arctan\sqrt{z} - \sin(2\arctan\sqrt{z})\sqrt{1 - (\sin(2\arctan\sqrt{z}))^2} \right) \\ &= 2\pi \left(2\arctan\sqrt{z} - \sin(2\arctan\sqrt{z})\cos(2\arctan\sqrt{z}) \right) \\ &= 4\pi \left(\arctan\sqrt{z} - (1/4)\sin(4\arctan\sqrt{z}) \right). \end{aligned}$$

Finally, after normalizing the constant term of 4π so the total volume is 1, we arrive at the chronometric volume as a function of redshift.

$$V(z) = \arctan(\sqrt{z}) - (1/4)\sin[4\arctan(\sqrt{z})] \tag{14}$$

4 Analysis

There are three important questions to consider when analyzing any physical theory:

- Is the theory mathematically consistent?
- Does the theory make measurable predictions?
- Are our observations consistent with the theoretical predictions?

The answer to the first two questions seems to be a resounding yes. While there are two published critiques of the mathematical rigor of chronometric cosmology, discussed in section 4.3, their arguments seem to have been effectively dismantled in Segal's responses. The answer to the third question however remains to be seen. This section gives a first attempt at answering it, although no decisive conclusions can be drawn from this approximate, preliminary analysis.

4.1 Redshift Data Analysis

With the two different redshift distance relations, it becomes possible to experimentally determine which cosmology we live in, using telescope data. Segal did this using the Noonan data set[16], as well as others, but that was almost half a century ago. Since then, technology has progressed significantly, and with new technology has come more accurate data, particularly for more distant sources. Figure 2 displays data from NASA's NED catalogue [2], and plots along with it the m-z predictions of the two models:

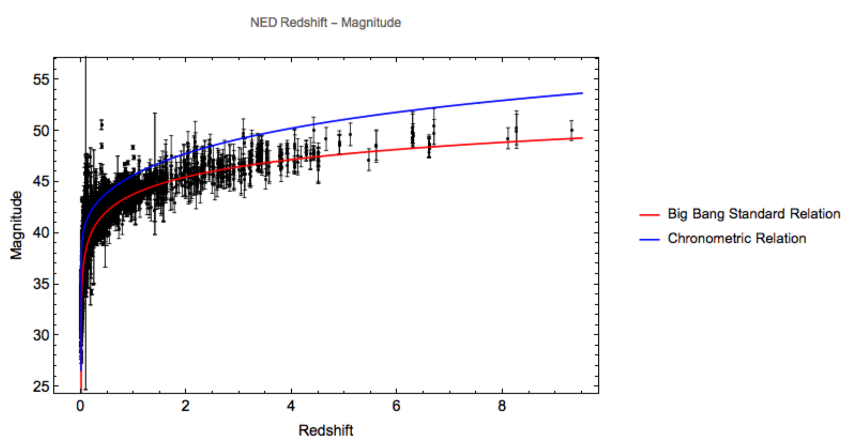

Figure 2: NED m-z data and fits

The data was taken directly from the NED data set [2], and the only bodies which were excluded were those that did not have error values for magnitude, or if the error value was 0. This was done so that the fits could account for the errors using mathematica's NonlinearModelFit function. In the case of the Big Bang m-z relation, equation 11 was used, with the constant C being fit to the data, and the q_0[14] set to $\frac{1}{2}$. The chronometric m-z relation is given by equation 9, and fit to the constants C and α. For this fit, $\alpha \approx 5.08$.

While it is somewhat useful to use data sets like the NED catalogue, which take surveys of various parts of the sky and combine them into one large data set, there is a high risk of a sampling bias due to the breadth of the survey. Figure 3 displays the same fits applied to the Hubble Deep Field data set[18], which contains data from all bodies observable light sources in a small patch of the sky, collected by the Hubble Space Telescope. Figure 3 displays only the magnitude data from the F160W filter. For both deep field data sets, we take the filter corresponding to the greatest magnitude values (and hence the least bright stars), in order to include the faint objects deeper in space. For the Hubble chronometric fit, $\alpha \approx 1.80$.

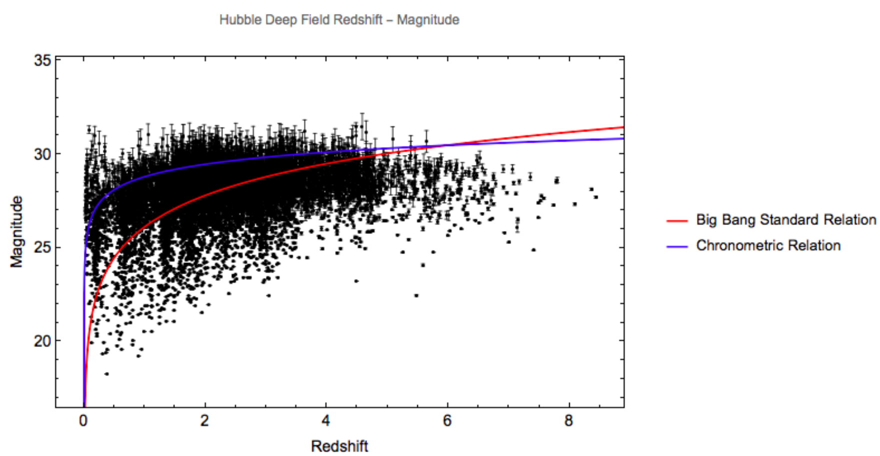

Figure 3: Hubble Deep Field m-z data and fits

Figure 2 was created using the Astro-Deep Field data set[14], a similar data set which includes data from 6 deep fields, collected using the Hubble Space Telescope, the Spitzer Space Telescope, and the VLT Hawk-I ground facility. The magnitudes displayed were collected using the MAG_B435 filter, and $\alpha \approx 1.00$. This means there was no spectral function dependence on the data, perhaps

[14]The current standard model paradigm posits that the expansion is currently accelerating (with q_0 being negative), and was decelerating in the distant past[30]. However a value of $q_0 = \frac{1}{2}$ is the simple case of a flat universe of constant acceleration, which is sufficient for the purpose of these plots, which is merely to investigate the approximate consistency of both models with the data.

because there are negligible absorption effects for this data set.

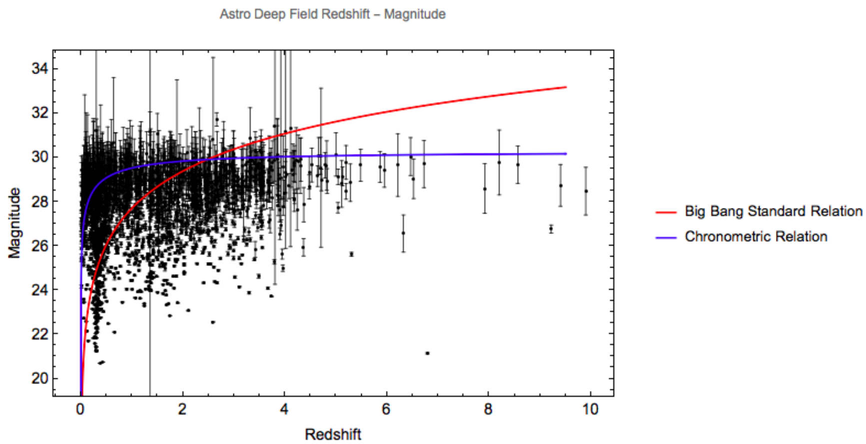

Figure 4: Astro Deep Field m-z data and fits

The final data set we consider is from the Super Nova Cosmology Project [1]. Type 1a supernova data has the advantage of being more or less a standard candle, as we know to significant precision how bright they are. Much like the NED data set however, absorption plays a greater role in this data set, due to the nature of a total sky survey.

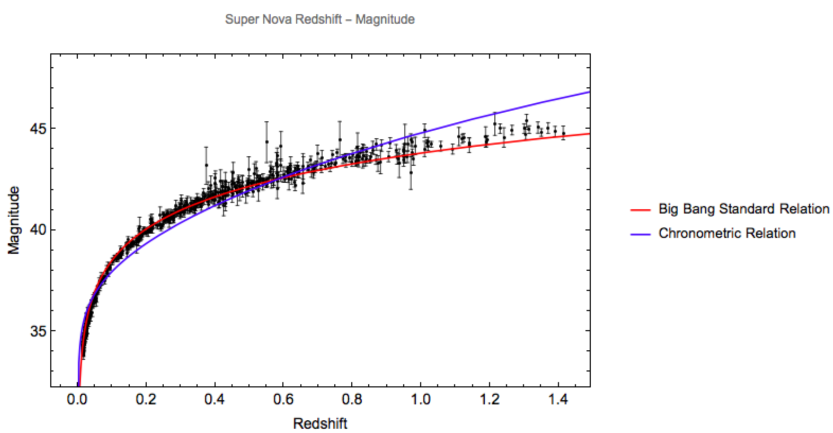

Figure 5: Supernova Cosmology Project m-z data and fits

The value for α in the chronometric fit in figure 5 is about 5.32. While α is not directly a measure of absorption, it is notable that the two data sets which one would expect to be more skewed by absorption have higher α values. This makes sense, as α corresponds to the rate of decay of the black body spectrum emitted by sources, and greater absorption might correspond to greater decay.

These redshift data from these same 4 data sets is plotted in figures 6, 7, 8, and 9 versus the number of objects found below that redshift, normalized by the length of the data set (so the N(<z) values range from 0 to 1). The theoretical plots are the volume-redshift relations derived in section 3.3.

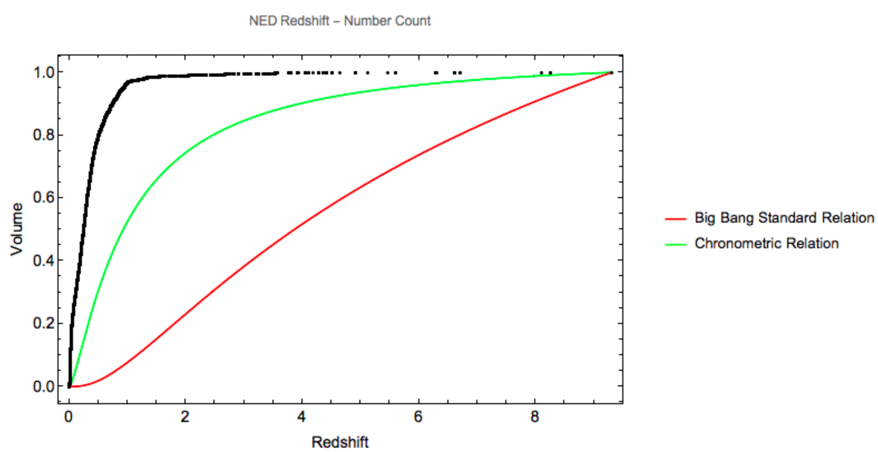

Figure 6: Ned z-N(<z) data and fits

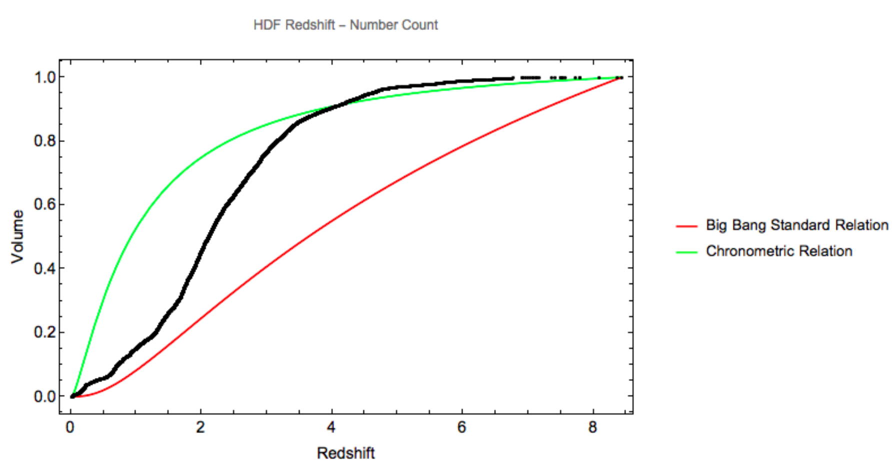

Figure 7: Hubble Deep Field z-N(<z) data and fits

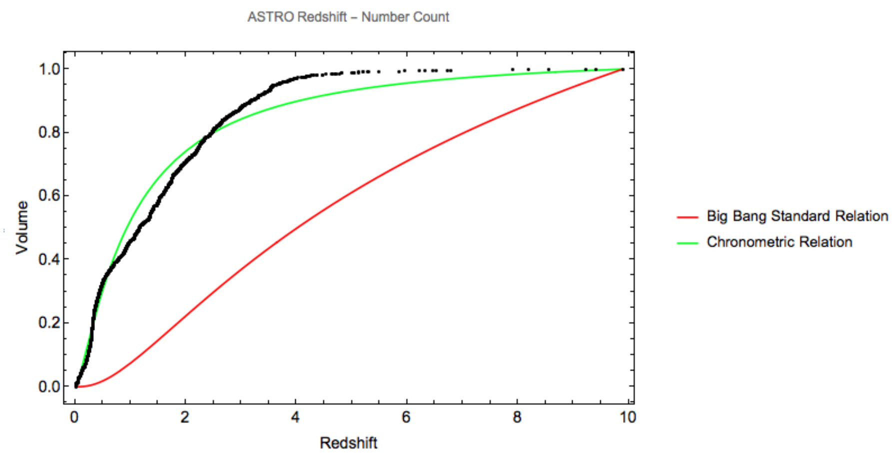

Figure 8: Astro Deep Field z-N(<z) data and fits

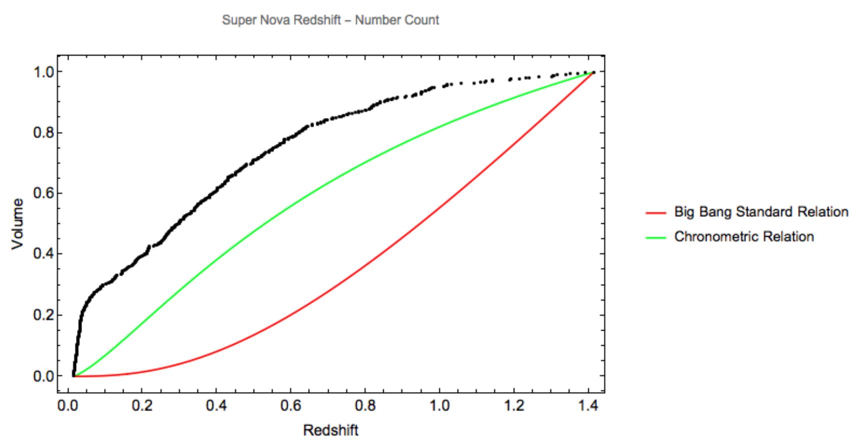

Figure 9: Super Nova Cosmology Project z-N(<z) data and fits

The data seems to fit the chronometric relation slightly better than the Big Bang relation (particularly at higher redshifts, which is what is of the most relevance to the discrepancy between the theoretical predictions), however this data is far from conclusive. Uncertainties in the conversion between observed and absolute magnitude can arise from a multitude of sources, including absorption, luminosity evolution, and the so called "K-correction" when converting the flux through a filter to bolometric flux.

4.2 Sources of Uncertainty

While magnitude and number count are two of the best distance proxies available to us for determining the veracity of any given cosmological theory, they are not without their limitations. In our comparisons of the two models with the data, we don't apply any additional absorption and K corrections, beyond what may have been done in the data analysis which produced the publicly available catalogues which we looked at. Our comparisons should be taken simply as an indication that both models seem to describe the data quite well.

4.2.1 Absorption

First of all, absorption of light by interstellar media (i.e. dust) presents an obvious problem to the $m - z$ relation, as the object's apparent magnitude appears fainter than can be expected. This problem is a strong motivator for the use of deep fields like Hubble and Astro Deep, but the effect of absorption is certainly always present, and its contribution is quite difficult to account for. While it is possible to estimate the matter density in our nearby region, and then use the principle of homogeneity to extrapolate this throughout the universe, it is extremely difficult to make this measurement precise. In his book on Cosmology[5], Bondi notes that

> *The bulk of this enormous amount of radiation arrives from very distant parts, in fact, half from regions so distant that the light has only a* 50% *chance of arriving without having been absorbed by other stars on the journey (21).*

Not only does absorption affect $m - z$ predictions, by way of both magnitude depreciation and alteration of the spectral index (as is clearly reflected in the high values of α fitted to the NED and Super Nova Cosmology Project survey catalogues), but it can also affect whether the light reaches us at all and in turn cause uncertainty in $N(< z) - z$ predictions.

4.2.2 Filters and the K-Correction

Another practical problem is posed by the necessity of magnitude filters and K-corrections. The redshift of a given source's spectrum to lower frequencies affects the apparent magnitude, since detectors can only measure magnitude in a fixed frequency range. The source rest-frame wavelength λ_0 will appear to the observer "stretched" by redshift to $\lambda = \lambda_0(1 + z)$, while the detector bandwidth $\delta\lambda$ is fixed. Hence modern catalogues must apply a "K-correction" when converting the measured fluxes for different frequency bands into magnitudes, but the K corrections vary for different fields where measurements are made. Modern catalogues give measured fluxes for different frequency bands, which helps by giving more information, but K corrections will vary from one deep-field in which measurements are made, and are difficult to estimate.

Furthermore, objects which are redshifted out of the frequency range of the filter would not be included in the data, and hence would affect the number count. It is clear that these effects are relevant for larger redhsifts, so there is some finite range of luminosities and redshift which can be sensibly compared with the data. at the very small, effects of local motion/lack of curvature make the data less useful, whereas at very large redshifts we might not even be observing the sources.

4.2.3 Luminosity Evolution

Another problem arises when one considers the possibility of Luminosity Evolution. While Segal postulates temporal homogeneity (on the large scale, the universe looks the same at different points in time), the same is not necessarily true in the standard model. In an evolving universe, the mean age of galaxies or other sources decreases with increasing redshift. It is difficult for us to know for sure, as we look farther into the past, whether on average the younger sources might have a different brightness.

With all of these uncertainties in mind, the plots from section 4.1 should be taken simply as an indication that the two models behave comparably well, and further investigation is merited.

4.3 Relevant Literature

4.3.1 Chronometric Cosmology and Mathematical Astronomy

The seminal work on chronometric cosmology is Segal's 1976 book, *Mathematical Cosmology and Extragalactic Astronomy*[29], although he also published a number of papers on the subject in the PNAS journal, including the 1992 paper [28] with an updtated redshift-magnitude relation. Another excellent source on the topic of chronometric cosmology is Daignault's lecture[6], which, in 44 pages, summarizes the key points in Segal's book. In terms of geometrical cosmology, a foundational text is Sandage's 1988 paper, *Observational Tests of World Models*[22].

Irving Segal was a renowned mathematician. He did extensive work on laying the theoretical groundwork of C* algebras, with the Gelfand-Nainmark-Segal construction to his name. He coauthored papers with giants like John von Neumman, and did work with Albert Einstein. He was the recipient of three Guggenheim Fellowships, was elected into the national academy of Sciences in 1973, and received the Humboldt Award in 1981[4]. Despite his many accomplishments and clear mastery of his field, his work on chronometric cosmology (arguably his seminal work outside of pure mathematics) was largely ignored by the scientific community in the field of cosmology and astrophysics.

There were two responses to Segal's work, by the authors Wormald [33] and Fairchild [8]. Both papers claimed to have found a flaw in Segal's chronometric model. Segal responded to both papers with a refutation[24][25], although neither author published a response.

4.3.2 Fairchild's Critique

Fairchild's argument [8] is as follows: By the correspondence principle (that quantum mechanics should reduce to classical mechanics in the classical limit) the same redshift distance relation should be obtained from classical mechaincs. However, Fairchild finds a discrepancy of a factor of two in equation 10 through a classical derivation of the the formula. Fairchild suggests that the equation should be

$$z = \tan^2 \frac{d}{R}$$

rather than

$$z = \tan^2 \frac{d}{2R}.$$

However, Fairchild's argument rests on localization of the wave function, either near emission or near observation. Segal argues that "neither assumption is in fact compatible with Maxwell's equations, due to the diffusion of energy in the course of propagation"[25].

4.3.3 Wormald's Critique

According to Wormald, Segal's calculation of the theoretical redshift is inconsistent with certain results from geometrical optics. Wormald raises the following two objections to Segal's theory:

> "(i) Segal's concept of measurement of a distant event is not in accord with observational practice; moreover allowing his concept of measurement for nearby events there can be no observed red shift in the Segal universe.
> (ii) Segal's postulate that unitime is not directly observable is inconsistent with his postulate that unitime is the fundamental physical time, for provided we have massive particles in our universe we may construct a geometrical clock which provides a direct measurement of unitime along the world line of any freely falling observer" (401).

Wormald argues that, if we consider the source of any radiation as a device that "ticks", we can assume it does so with a period Δ (this is a unitime period, and thus consistent with global energy conservation). It thus becomes a straightforward calculation of redshift (since wavelength is directly proportional to period),

$$1 + z = \frac{\text{(observed T distant)}}{\text{(observed T nearby)}}$$

Now consider 4 events E_1, the emission of a tick, E_2 the subsequent emission of a tick, and O_1 and O_2, the respective observations of these ticks farther away. Wormald then claims that, since light travels along null geodesics of the universal cosmos (along the boundaries of the infinitesimal causal cones), the interval E_1E_2 is the same as the interval O_1O_2. This argument is then elaborated by examining Segal's derivation of the two times relation (pgs 83-84 of [29]), and then claiming that Segal relates two equations which are true only on different world lines, which hence cannot be related.
Segal responded to Wormald's critique, pointing out that

> *The times of arrival of successive ticks at a source are observationally quite unrelated to the direct wavelength measurements from which the red-shift phenomenon derives. In both chronometric and Doppler theories, the red shift is modeled as a single-photon phenomenon, whereas ticks involve a succession of photons.*[24]

Segal asserts the rigor of his original proof, and clarifies that the "unobservability of the universal energy [is a] purely microscopic phenomena."

Not only did Segal publish responses refuting both criticisms, but even after there was no follow-up discussion in the literature, Segal presented yet another derivation of his redshift formula in a later PNAS paper [23], where he does not employ as much mathematics as in the original one. This paper formed the basis for Daigneault's [6] derivation of the relation.

4.4 Further Discussion

4.4.1 The Cosmic Microwave Background

Proponents of the standard model cite the CMB as decisive evidence of the Big Bang theory, as it is interpreted as left over radiation from the Big Bang (or rather radiation from the end of the era of last scattering, when the density of the universe became low enough that the hydrogen atoms could form, around 400,000 years after the Big Bang). Segal poses an alternative explanation for the background radiation, within the schema of chronometric cosmology.

Consider the redshift distance equation,

$$z = \tan^2(\rho/2). \tag{10}$$

A close look at this equation reveals that as ρ approaches π, z approaches infinity. Since ρ is measured in natural units, a distance of $\rho = \pi$ corresponds precisely to halfway around the $\mathbb{S}^3$ component of $\overline{M}$, the antipodal point of the universe from the observer. Thus any radiation from such far flung regions of the universe would, according to Segal, reach a state of complete redshift.

But what about light which travels more than one half circuit around the sphere? Just as equation 10 oscillates with period 2π, so too would the redshift oscillate, however Segal's claim is that after just one half-circuit and "complete redshifting," the radiation will appear to be non-local. Segal argues that "By conservation of energy and maximization of entropy, this radiation would have a black body spectrum, as is consistent with observations of the microwave background, which is thereby theoretically predicted"([29] pg 21). This background radiation would theoretically reach an equilibrium state, at the point where the total energy emitted by black body radiation of matter is equal to the total energy absorbed.

Now that the existence of CMB has been theoretically predicted, the task becomes calculating its theoretical temperature, and comparing with observations. Unfortunately, absorption (see section 4.2) makes an accurate temperature difficult to predict precisely. Segal succinctly lays out the problem in his 1983 paper on the topic of the Cosmic Background Radiation[26].

> *Ideally of course the state of the CBR would be derived from that of the total system consisting of the CBR together with all matter in the universe, which on the usual statistical basis may be postulated to have a density matrix of the form* $D = e^{-bH}/tre^{-bH}$*, where* H *is the total energy operator:* $H = H_{CBR} + H_{matter} + H_{interaction} + H_{other}$*;* H_{other} *includes the energies of neutrinos, the x-ray background, and their interactions, as well as possible unknown components. Unfortunately the specification of all the components of* H *other than* H_{CBR} *would at best be highly speculative, in addition to which ultraviolet divergences in the treatment of interactions that are currently unresolved would foreclose the possibility of reliable calculations, even if analytic expressions for the components of* H *were given.*

While it is clear that chronometric cosmology does predict the existence of the CMB, or at least provides a mechanism that explains its existence, more work is needed to determine if it predicts an equilibrium state at the correct temperature, and if its fluctuations can be theoretically explained as well.

Additionally, it is worth noting that there has been recent evidence from the CMB which seems to be in tension with the standard model. In 2018, the Planck space observatory published CMB data [3] which was inconsistent with our previous understanding of the CMB power spectra. In a paper published in *Nature* in 2019 [31], authors Valentino, Melchiorri and Silk suggest that a closed universe with positive curvature might account for this discrepancy.

4.4.2 Matter Distribution

Many proponents of the standard model also cite the distribution of matter as evidence for the theory. We observe that out of all baryonic matter, about 75%

is hydrogen, about 25% is Helium 4, and less than 1% is comprised of other elements. This distribution is consistent with what one would expect from the predicted ratio of protons to neutrons during era of Big Bang nucleosynthesis. While Segal does not go into a great deal of detail on chronometric cosmology's explanation for this result, he does mention that it is possible to obtain the same ratios by the same mechanism of nucleosynthesis from regions of opaque plasma. The idea is that such regions are bound to exist in a universe of infinite age, due to fluctuations in matter distribution.

4.4.3 Future Work

Future work includes a precise determination of the chronometric prediction for the temperature of the CMB. Recent measurements[9] of the Cosmic Microwave Background have determined its temperature to be $2.7260 \pm 0.0013K$. Once the temperature is calculated to a reasonable degree of precise from equilibrium considerations, we will try to account for the observed fluctuations in the CMB as well. In order for future investigation of chronometric cosmology to be warranted, the model should explain the details of small fluctuations observed in the CMB spectrum. Additionally, a mechanism should be provided for how matter may be replenished in a static Universe to satisfy the axioms of spacial and temporal homogeneity. While Segal briefly touches on these topics, the precision of our modern measurements require significant development of these theoretical predictions.

5 Conclusion

For hundreds of years, humans have looked up at the stars and wondered at their nature. And for hundreds of years, we have been missing pieces of the puzzle. Could chronometric cosmology be the next step towards a deeper understanding of the universe? Will Segal's name some day be uttered in the same breath as the likes of Copernicus and Kepler, Newton and Einstein? Or is his theory nothing more than a beautiful piece of mathematics, without relevance to our physical world?

It can be derived from the 6 axioms (section 2) that there are only two admissible manifolds which can constitute our universe: either ([29] pg 58). M requires the the existence of dark energy, a period of cosmic inflation, and involves additional parameters (q_0), to describe the rate of expansion. $\overline{M}$ requires none of these, and additionally has the aesthetic advantages of preserving the conservation of energy, and being the universal cover of M. However, while these points might urge a mathematician or philosopher to prefer $\overline{M}$, the principled scientist must be swayed by observational evidence alone. In the words of the great astronomer Hermann Bondi[5],

> *Which one of [these] possible 'universes' is the one actually realized [then] becomes a secondary question, the answer to which must at present be found observationally, since there are no theoretical grounds for preferring any one of the possible models to any others.*

We have shown that by using both luminosity and number count as proxies for distance, the distinct redshift-distance relations can be contrasted between the two admissible world models. Irving Segal spent much of his life in an effort to do just this, and while the confident rhetoric of his works seems to suggest his theory is correct, the rest of the scientific community is quite confident to the contrary. While the veracity of the theory remains to be seen after this preliminary analysis, the elegance of the mathematics, the verifiability of the physics, and the promise of the data begs the earnest scientist for further consideration.

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