# DECAY OF OSCILLATING UNIVERSES

A dissertation

submitted by

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# Abstract

It has been suggested by Ellis *et al* [8, 10] that the universe could be eternal in the past, without beginning. In their model, the "emergent universe" exists forever in the past, in an "eternal" phase before inflation begins. We will show that in general, such an "eternal" phase is not possible, because of an instability due to quantum tunneling. One candidate model, the "simple harmonic universe" has been shown by Graham *et al* [11] to be perturbatively stable; we find that it is unstable with respect to quantum tunneling. We also investigate the stability of a distinct oscillating model in loop quantum cosmology with respect to small perturbations and to quantum collapse. We find that the model has perturbatively stable and unstable solutions, with both types of solutions occupying significant regions of the parameter space. All solutions are unstable with respect to collapse by quantum tunneling to zero size. In addition, we investigate the effect of vacuum corrections, due to the trace anomaly and the Casimir effect, on the stability of an oscillating universe with respect to decay by tunneling to the singularity. We find that these corrections do not generally stabilize an oscillating universe. Finally, we determine the decay rate of the oscillating universe. Although the wave function of the universe lacks explicit time dependence in canonical quantum cosmology, time evolution may be present implicitly through the semiclassical superspace variables, which themselves depend on time in classical dynamics. Here, we apply this approach to the simple harmonic universe, by extending the model to include a massless, minimally coupled scalar field $\phi$ which has little effect on the dynamics but can play the role of a "clock".

# Acknowledgements

First, I would like to thank my advisor, Alex Vilenkin, for his guidance, support, and patience over the years. I am also grateful to my committee – Ken Olum, Larry Ford, Krzysztof Sliwa, and Jaume Garriga – for their careful reading of this thesis and many useful comments and suggestions. In addition, I would like to thank my parents, John and Sue Todhunter, for encouraging me in all my endeavors from the beginning.

This thesis is dedicated to Ayaan Mithani.

# Contents

# List of Figures

# Chapter 1

# Introduction

Did the universe have a beginning, or has it simply existed forever? This is perhaps one of the more compelling questions that cosmology may address. Originally, it was shown by the singularity theorems of Penrose and Hawking that an initial singularity is not avoidable [1]. These theorems relied on the assumption that certain energy conditions apply.

High precision measurements of cosmic microwave background (CMB) are consistent with an inflationary phase in the cosmic past [2]. During such a phase, a source of energy density with negative pressure forces the universe to expand very rapidly; the scale factor increases exponentially

$$a(t) \propto e^{Ht} \tag{1.1}$$

during this phase. By pushing large inhomogeneities outside of the horizon of the observable universe, inflation explains the large scale homogeneity and isotropy and almost complete flatness of the universe [3]. In addition, quantum fluctuations of fields during inflation result in density perturbations in agreement with those observed in the CMB.

While inflation may provide an explanation for observations in our universe, there is no definite answer to what preceded inflation. Inflation violates the strong energy condition, and the quantum fluctuations of fields during inflation violate the weak energy condition, so the singularity theorems of Penrose and Hawking do not apply. However, it was more recently shown that spacetimes which are on average expanding, $H_{avg} > 0$, cannot be complete in the past. The theorem of [4] does not rely on Einstein's equations or require any energy conditions to be satisfied; instead, it says that as long as the average expansion rate along a geodesic is positive, it must reach the past boundary of the expanding region in a finite proper time.

### Eternal Inflation

One possibility is that our universe could be a part of a larger eternally inflating universe, containing many distinct inflating bubbles which are disconnected from our own. It has been shown [5] that such models may inflate eternally, at least in the future. However, inflating space-times are expanding with $H_{avg} > 0$, and are therefore subject to the theorem of [4], and cannot be eternal in the past.

### Cyclic Models

Alternatively, our universe could be the re-birth of a universe in the past, that contracted from infinite size to a big crunch before re-expanding into our present universe – a "cyclic" model [6]. The second law of thermodynamics requires that entropy grows, so through each phase of contraction and expansion the total entropy of the universe must grow [7]. For infinite cycles to be possible (infinite cycles would represent a universe having no beginning, or end), entropy would grow without bound, signaling a "thermal death" of the universe. One way around this is keeping the entropy density $S/V$ finite; this would require the volume of the universe to increase as it cycles. Although there are periods of expansion and contraction, the overall growth of volume means that the cyclic universe has $H_{avg} > 0$, and the theorem of [4] applies. Therefore, cyclic universe models must have a beginning.

### The "Emergent" Universe

There is one way to avoid the singularity theorem of [4], which is requiring $H_{avg} = 0$. If the universe is in a phase satisfying $H_{avg} = 0$ before inflation, the initial singularity is avoided and there is no beginning of the universe. Such a scenario has been recently proposed by Ellis et al [8] (see also [9]); it is the aim of this thesis to investigate the plausibility of the emergent universe.

In order for the emergent universe to be possible, two main ingredients are needed. First, a phase which is truly "eternal" in the past must be possible. Second, the "eternal" phase must somehow reach an end and transition into the inflating universe. The emergent universe was first suggested by Ellis et al; they proposed a model including a scalar field $\phi$ with a flat potential $V(\phi \to -\infty) = \text{const.}$, but reaching a hill at $\phi \to 0$ [10]. This is

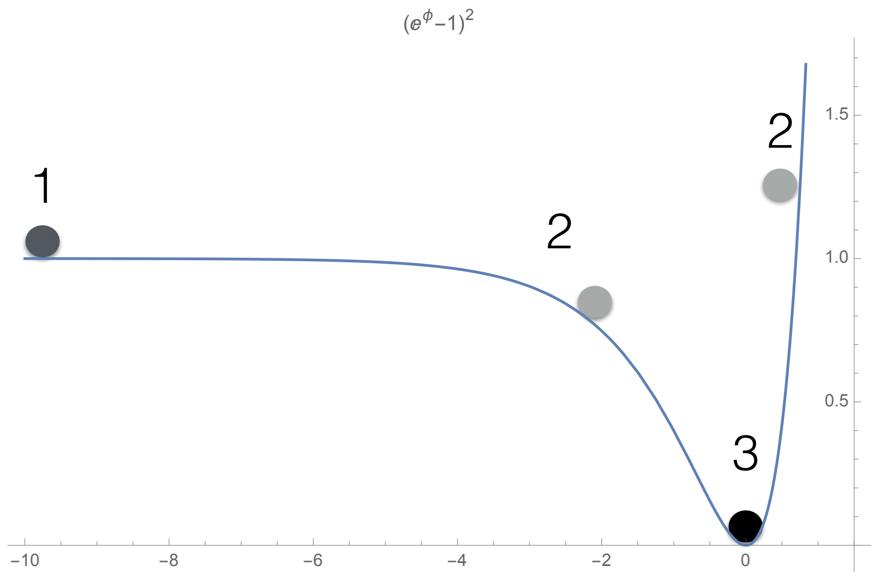



Figure 1.1: Schematic potential for the emergent universe scenario (see [10] for details). In the "eternal" phase, the scalar field rolls along a flat potential $V(\phi \to -\infty) = \text{const}$ (1) before reaching a hill in the potential, where $V(\phi) \propto (e^{B\phi} - 1)^2$, with $B \sim const$. Slow-roll inflation occurs (2), after which inflation ends and re-heating begins (3).

depicted in Fig. 1.1. The field rolls along the flat potential in the infinite past, but begins inflation when it reaches the hill. In this sense, a past-eternal phase is possible with inflation.

However, the first condition, that the eternal phase is possible to arrange, is not easily met. For example, the Einstein Static universe is well-known to be perturbatively unstable. Perturbations caused by quantum fluctuations of matter fields present in the universe can easily de-stabilize potential emergent universe models.

One model, the simple harmonic universe (SHU) was discussed by Graham et al [11], who showed that it is completely perturbatively stable, for some values of model parameters. We will also show, in Chapter 3, another oscillating model from loop quantum cosmology which is perturbatively stable for some values of model parameters. While we don't claim that this list of two examples is exhaustive, we wish to show that such perturbatively stable solutions are in fact possible, indicating that it is at least plausible that our universe could have begun in such a state. Evidence will show, however, that in fact it is not possible due to quantum instabilities.

In order for the emergent universe to be truly eternal, it must not succumb to instabilities of any sort. The main focus of this thesis is on the quantum mechanical instability of the emergent universe.

### Hamiltonian Framework of Quantum Cosmology

The goal of "quantum cosmology" is to apply quantum mechanics to the entirety of the universe. We understand the universe to be made up of spacetime and matter fields [1]; our particular observed universe has a particular combination of matter fields coexisting with a particular structure of spacetime. However, any object, regardless of how macroscopic, is always subject to the laws of quantum mechanics, and therefore the entire universe itself must be described by a wave function of some sort. The canonical Hamiltonian framework of quantum gravity, which we will use to determine an appropriate description of the quantum mechanical behavior of the classically oscillating emergent models, was originally pioneered by DeWitt [12].

The wave function of the universe is defined on a space of all possible three-geometries $h_{ij}$ and matter field $\{\phi\}$ configurations $\psi(h_{ih}, \{\phi\})$ (the "superspace"). This may be roughly interpreted as probability amplitudes for the universe to exist in a certain state as a function of these parameters. The wave function is the solution to the Wheeler-DeWitt equation

$$\mathcal{H}\psi(h_{ih}, \{\phi\}) = 0. \tag{1.2}$$

Here, $\mathcal{H}$ is the Hamiltonian operator. The wave function determines the quantum state of the universe; we observe only the classical universe.

### The role of time in quantum cosmology

In canonical quantum cosmology, the wave function of the universe is not dependent on time. However, temporal evolution may be present implicitly through the semiclassical superspace variables, which themselves depend on time in classical dynamics. In this chapter, we apply this approach to the Simple Harmonic Universe discussed in Chapter 2 of this thesis. By extending the model to include a massless, minimally coupled scalar field $\phi$ which has little effect on the dynamics but can play the role of a "clock," we determine the decay rate of the oscillating universe in Chapter 5.

We first present, in Chapter 2, the simplest model of a perturbatively stable oscillating model, SHU. We review the classical dynamics and then show that it is unstable with respect to quantum tunneling. We discuss a distinct model motivated by loop quantum

[1] Matter fields includes gauge fields, which are only fixed up to some gauge choice. Probability amplitude is independent of gauge choice, but it may be present as a phase in the wave function.

cosmology in Chapter 3. Like the SHU, it is perturbatively stable but unstable with respect to tunneling. Because of the LQC aspect of the model, its dynamics is significantly different than the SHU; it is therefore important to demonstrate separately the tunneling instability of this model. In Chapter 4, we address the effect on the tunneling probability of quantum corrections to the energy-momentum tensor.

We focus on the stability of the emergent universe phase, setting aside the explicit inclusion of the inflationary mechanism proposed by Ellis. The reason for this is straightforward: if it is impossible to construct a perfectly stable eternal phase, then the emergent universe scenario itself does not avoid some sort of beginning. However, it should be relatively straightforward to extend either of the models we analyze to include a scalar field in the prescribed potential (Fig. 1.1).

# Chapter 2

# The Simple Harmonic Universe

One oscillating model that satisfies the condition $H_{avg} = 0$, called the "Simple Harmonic Universe," was shown to be perturbatively stable by Graham et al [11]. In this model, oscillation is driven by the balance between negative cosmological constant ($\Lambda$), which turns around the expansion at large distances, and the presence of a material having equation of state

$$P = w\rho. \tag{2.1}$$

with

$$-1 < w < -\frac{1}{3}, \tag{2.2}$$

which causes the bounce. The energy density of the universe is then

$$\rho = \Lambda + \rho_0 a^{-3(1+w)} \tag{2.3}$$

with $\Lambda < 0$ and $\rho_0 > 0$. The SHU is closed, having positive spatial curvature $k = +1$, and the Friedmann evolution equation is

$$\dot{a}^2 + 1 = \frac{8\pi G}{3}\rho a^2. \tag{2.4}$$

The need for a material with equation of state in the range 2.2 to drive oscillations can be seen from the second Friedmann evolution equation, given by

$$\frac{\ddot{a}}{a} = -4\pi G\left(3p + \rho\right). \tag{2.5}$$

With the relation (2.1), a material for which $w < -1/3$ corresponds to $\ddot{a} > 0$, which means that the material is gravitationally repulsive. This causes the bounce; with positive curvature ($k = +1$), the bounce radius is non-zero.

The model is radially stable, provided that $w$ satisfies (2.2). For a perfect fluid source, the speed of sound $c_s$ can be found from $c_s^2 = dP/d\rho = w$. With the equation of state (2.2), this gives $c_s^2 < 0$, indicating instability with respect to short-wavelength compressional perturbations. Hence, it is important that the exotic matter source should not be a perfect fluid [11].[1] It could, for example, be an assembly of randomly oriented domain walls, which cannot be regarded as a perfect fluid, so instead $w = -2/3$ but the speed of sound $c_s^2 > 0$ [13]. For this choice of $w$, Eq. (2.3) takes the form

$$\rho = \Lambda + \rho_0 a^{-1}, \tag{2.6}$$

and the evolution equation (2.4) has a simple oscillatory solution

$$a = \omega^{-1}(\gamma - \sqrt{\gamma^2 - 1}\cos(\omega t)) \tag{2.7}$$

where

$$\omega = \sqrt{\frac{8\pi}{3} G|\Lambda|} \tag{2.8}$$

and

$$\gamma = \sqrt{\frac{2\pi G \rho_0^2}{3|\Lambda|}}. \tag{2.9}$$

A static universe solution is obtained from (2.7) by setting $\gamma = 1$; then $a = 1/\omega$. It has been shown in [11] that this solution is stable with respect to arbitrary small perturbations, including all scalar and tensor modes.

## 2.1 Collapse through tunneling

We consider a spherical universe with metric ansatz

$$ds^2 = dt^2 - a^2(t)\left(\frac{1}{1 - kr^2} dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right) \tag{2.10}$$

[1] A perfect fluid source could be acceptable if one allows an equation of state $P(\rho)$ more general than Eq. (2.1). All one needs is that $w = P/\rho$ satisfies Eq. (2.2) and $dP/d\rho > 0$.

with $k = +1$ for a closed universe. Matter content is described by Eq. (2.3); the scale factor $a(t)$ is the single dynamical degree of freedom. In classical theory, such a universe can be regarded as a dynamical system with a Hamiltonian

$$\mathcal{H} = -\frac{G}{3\pi a}\left(p_a^2 + U(a)\right), \tag{2.11}$$

where

$$p_a = -\frac{3\pi}{2G} a\dot{a} \tag{2.12}$$

is the momentum conjugate to $a$ and

$$U(a) = \left(\frac{3\pi}{2G}\right)^2 a^2 \left(1 - \frac{8\pi G}{3} a^2 \rho(a)\right). \tag{2.13}$$

The Hamiltonian constraint $\mathcal{H} = 0$ then yields the evolution equation (2.4). This can be thought of as expressing the fact that the total energy of a closed universe is zero.

In quantum theory, the universe is described by a wave function $\psi(a)$, the conjugate momentum $p_a$ becomes the differential operator $-id/da$ and the constraint is replaced by the Wheeler-DeWitt (WDW) equation [12] (for a review see, e.g., [14, 15, 16])

$$\mathcal{H}\psi = 0, \tag{2.14}$$

or

$$\left(-\frac{d^2}{da^2} - \frac{\beta}{a}\frac{d}{da} + U(a)\right)\psi(a) = 0. \tag{2.15}$$

Here, the parameter $\beta$ represents the ambiguity in the ordering of the non-commuting factors $a$ and $p_a$ in the Hamiltonian (2.11). Its value does not affect the wave function in the semiclassical regime $a \gg l_{Planck}$. In most of this thesis we set $\beta = 0$.

One might expect that for a simple harmonic universe the potential $U(a)$ should be of the same form as for a harmonic oscillator. This, however, is not the case: the motion in the potential (2.13) is simple harmonic only for a particular value of the energy, $\mathcal{H} = 0$. With $\rho(a)$ from (2.6), we have

$$U(a) = \left(\frac{3\pi}{2G}\right)^2 a^2 \left(1 - \frac{8\pi G}{3}(\rho_0 a + \Lambda a^2)\right). \tag{2.16}$$

It will be convenient to introduce a rescaled variable $x = \omega a$ with $\omega$ from Eq. (2.8). In terms of this variable the WDW equation takes the form

$$\left(-\frac{d^2}{dx^2}+U(x)\right)\psi(x)=0, \tag{2.17}$$

where

$$U(x)=\lambda^{-2}x^2(1-2\gamma x+x^2), \tag{2.18}$$

$\gamma$ is given by Eq. (2.9), and

$$\lambda=\frac{16G^2|\Lambda|}{9}. \tag{2.19}$$

The classically allowed range is defined by $U(x)\leq 0$. This range is non-empty when $\gamma\geq 1$. The shape of the potential in this case is illustrated in Fig. 2.1. In the classical solution, the radius of the universe oscillates forever between the values $x_+$ and $x_-$ where $U(x_\pm)=0$,

$$x_\pm=\gamma\pm\sqrt{\gamma^2-1}. \tag{2.20}$$

However, it is clear from the figure that quantum-mechanically the universe can tunnel through the barrier to a vanishing size at $x=0$. The WKB tunneling action is given by

$$S=\int_0^{x_-}\sqrt{U(x)}dx \tag{2.21}$$

and the corresponding tunneling probability can be estimated as

$$\mathcal{P}\sim e^{-2S}. \tag{2.22}$$

This can be interpreted as the probability of collapse through quantum tunneling as the universe bounces at radius $x=x_-$.

Semiclassical quantum tunneling in oscillating universe models has been studied by Dabrowski and Larsen [17, 18]. They considered a closed universe containing nonrelativistic matter (dust), a domain wall fluid with equation of state $w=-2/3$, and a negative cosmological constant. Due to the presence of dust, this model has another classically allowed range at small values of $a$. The WKB action (2.21) for tunneling between the two classically allowed regimes can then be expressed in terms of elliptic integrals. In the absence of dust, the model of [18] reduces to the simple harmonic universe, but the authors have not discussed this case.

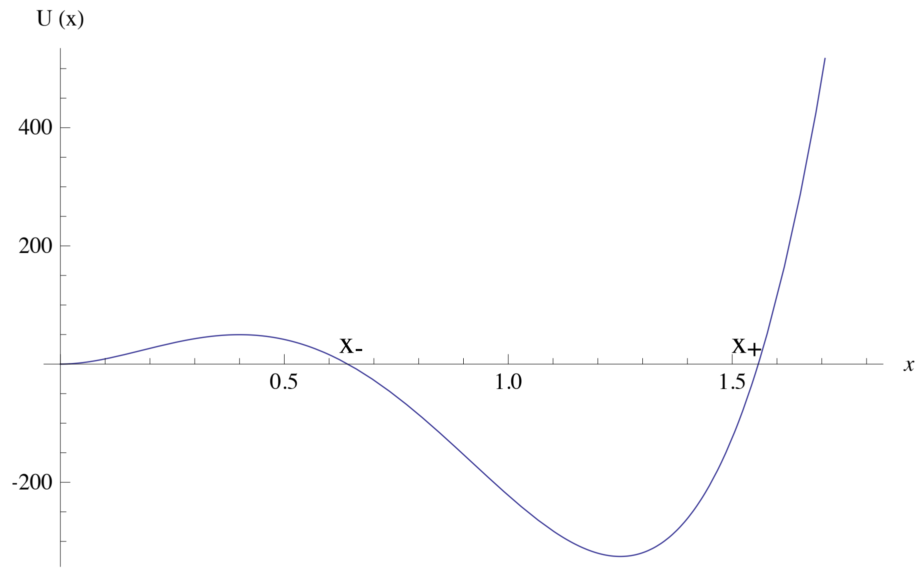



Figure 2.1: The WDW potential $U(x)$ for the parameter values $\lambda = 0.03$ and $\gamma = 1.1$.

For a simple harmonic universe, the integral in (2.21) can be expressed in terms of elementary functions,

$$S = \lambda^{-1}\left[\frac{\gamma^2}{2} + \frac{\gamma}{4}\left(\gamma^2 - 1\right)\ln\left(\frac{\gamma - 1}{\gamma + 1}\right) - \frac{1}{3}\right], \tag{2.23}$$

Since the tunneling probability (5.84) is nonzero, such a universe cannot survive forever.

For $\gamma = 1$, the classically allowed range reduces to a single point, and the WKB action is given by the simple formula

$$S_{\gamma=1} = \frac{1}{6\lambda}. \tag{2.24}$$

The classical solution in this case is a static universe with $x = 1$, and Eq. (5.84) can be interpreted as being proportional to the probability of quantum collapse per unit time.

## 2.2 Tunneling from nothing

We note that the tunneling between $x = x_-$ and $x = 0$ can also go in the opposite direction, in which case Eq. (5.84) with $S$ from (2.23) or (2.24) can be interpreted as describing spontaneous creation of an oscillating or static universe from nothing. The corresponding instanton can be found by solving the Euclideanized Friedmann equation,

$$\dot{x}^2 = \omega^2(x_+ - x)(x_- - x), \tag{2.25}$$

where the dot stands for differentiation with respect to the Euclidean time $\tau$. The solution can be expressed as

$$\omega\tau = \int_0^x \frac{dx'}{\sqrt{(x_+ - x')(x_- - x')}} = -2\ln\left(\frac{\sqrt{x_+ - x} + \sqrt{x_- - x}}{\sqrt{x_+} + \sqrt{x_-}}\right). \tag{2.26}$$

Solving this for $x$ as a function of $\tau$ we find

$$x(\tau) = \gamma - \frac{1}{2}(\gamma - 1)e^{\omega\tau} - \frac{1}{2}(\gamma + 1)e^{-\omega\tau}. \tag{2.27}$$

Introducing

$$\tau_0 = \omega^{-1}\ln\left(\frac{\gamma + 1}{\gamma - 1}\right), \tag{2.28}$$

Eq. (2.27) can be rewritten as

$$x(\tau) = \gamma - \sqrt{\gamma^2 - 1}\cosh[\omega(\tau - \tau_0/2)]. \tag{2.29}$$

Note that this is related to the Lorentzian solution (2.7) by a simple analytic continuation, as one might expect. The instanton solution (2.29) starts with $x = 0$ at $\tau = 0$, grows until it reaches a maximum value $x(\tau_0/2) = x_-$, and then returns to $x = 0$ at $\tau = \tau_0$. It is symmetric with respect to the point $\tau = \tau_0/2$.

The geometry of the instanton,

$$ds^2 = d\tau^2 + a^2(\tau)d\Omega_3^2, \tag{2.30}$$

is similar to a 4-dimensional ellipsoid. We note that

$$\dot{a}(0) = -\dot{a}(\tau_0) = 1, \tag{2.31}$$

which indicates the absence of conical singularities. In other words, the "poles" at $\tau = 0, \tau_0$ are rounded off.

For $\gamma = 1$ the instanton solution (2.27) simplifies to

$$x(\tau) = 1 - e^{-\omega\tau}. \tag{2.32}$$

It interpolates between $x = 0$ at $\tau = 0$ and $x = 1$ at $\tau \to \infty$. The geometry of this instanton is that of a cigar. It is rounded off at $a = 0$ and asymptotically approaches a static sphere at large $\tau$. The instanton action in this case is given by

$$|S_{inst}| = \frac{3\pi}{4G} \int_0^\infty d\tau a \left[\dot{a}^2 + (\omega a - 1)^2\right] = \frac{1}{6\lambda}. \tag{2.33}$$

Of course it is the same as in Eq. (2.24). Note that the action is finite, even though the instanton has an infinite 4-volume. We note also that the boundary term, which is proportional to the normal derivative of the boundary volume, vanishes for this instanton.

Even though there are no conical singularities, a closer examination shows that somewhat milder singularities are still present at the poles.[2] The scalar curvature for the metric (2.30) is

$$R = 6a^{-2}(1 - \dot{a}^2 - a\ddot{a}). \tag{2.34}$$

The first two terms in the parentheses cancel out at the poles, but in the last term $\ddot{a}(0) = -\gamma\omega \neq 0$, and thus $R \propto a^{-1} \propto \tau^{-1}$. This singularity is integrable, so the instanton action is finite.

It is possible that the curvature singularity can be removed by modifying Einstein's equations or the equation of state at small values of $a$. We could imagine, for example, that for a gas of domain walls the equation of state parameter gradually changes from $w = -2/3$ to $w = -1$ as we approach $a = 0$ (so the equation of state becomes that of the symmetric vacuum in the wall interiors). This would cure the singularity.

The situation here is somewhat similar to that with the Hawking-Turok (HT) instanton [19], which was proposed to describe quantum creation of open universes. Garriga has shown that this singular instanton can be regulated with a suitable matter source [20] and can also be obtained by dimensional reduction from a regular instanton in a higher-dimensional theory [21]. It is possible that the Euclidean solutions presented here can similarly be regarded as approximations to instantons of a more fundamental theory.

An important difference between HT and our instantons is that in the HT case the vicin-

[2] We are grateful to Jaume Garriga for pointing this out to us.

ity of the singular point makes a significant contribution to the action. For our instantons the contributions of singular points are negligible. This indicates that the instanton action and the tunneling probability are not sensitive to short-distance modifications of the theory.

## 2.3 The wave function

Having studied the semiclassical tunneling of the universe, we shall now examine solutions of the WDW equation (2.17) for the wave function of the universe $\psi(a)$. By analogy with a quantum harmonic oscillator, one might expect the wave function to oscillate in the classically allowed range and to decay exponentially in the two classically forbidden ranges on both sides of it. However, the situation we have here is rather different. In the case of an oscillator, we solve the Schrodinger equation

$$\frac{1}{2}\left(-\frac{d^2}{dx^2}+\omega^2 x^2\right)\psi(x)=E\psi(x) \tag{2.35}$$

with boundary conditions $\psi(x\to\pm\infty)=0$. Solutions exist only for certain values of the energy, $E=(n+\frac{1}{2})\omega$; this determines the energy spectrum of the oscillator.

Now, in our case the eigenvalue of the WDW operator is fixed: it is equal to zero. If we impose boundary conditions requiring, e.g., that $\psi(x\to\infty)=\psi(x=0)=0$, the system would be overdetermined and no solutions would exist, except for some special values of the parameters $\lambda$ and $\gamma$. For generic values of the parameters, we have the freedom to impose only a single boundary condition. A natural choice appears to be

$$\psi(x\to\infty)=0. \tag{2.36}$$

This fully specifies the solution. In the classically forbidden region $0\le x\le x_-$, the wave function is a superposition of exponentially growing and exponentially decreasing solutions. The solution that grows towards $a=0$ will dominate, unless the parameters of the model are fine tuned to suppress its contribution. Some numerical solutions to the WDW equation (2.17) are illustrated in Figs. 2.2 and 2.3.

The interpretation of these solutions is not completely clear, since we do not have a well established procedure for extracting probabilities from the wave function of the universe (see, e.g., Ref. [14] and references therein). But a nonzero value of $\psi(0)$ signals a non-

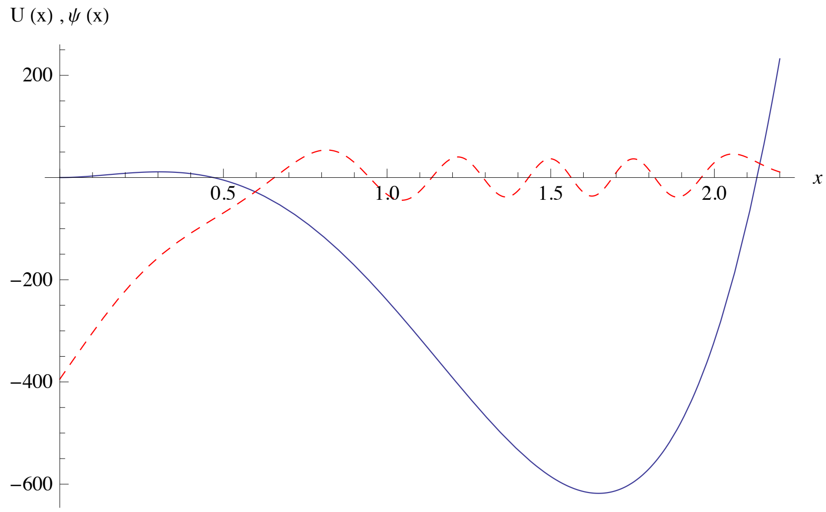



Figure 2.2: Solution of the WDW equation for the parameter values $\lambda = .05$ and $\gamma = 1.3$ (red dashed line). The WDW potential is also shown (blue line).

vanishing probability of collapse and appears to be inconsistent with the picture of an eternal oscillating or static universe.

Here, we assume that hitting the singularity at $x = 0$ is fatal for the universe. It is conceivable that wave functions similar to those in Fig. 2.3 could describe an eternal universe tunneling back and forth between a finite radius $a = \omega^{-1}$ and a Planck-size nugget. However, analysis of this possibility would require a full theory of quantum gravity and is beyond our present level of understanding. Our simple minisuperspace model certainly becomes inadequate at $a \sim l_{Planck}$.

## 2.4 Discussion

Is it possible to save the simple harmonic universe from quantum collapse? One possibility is to impose the boundary condition

$$\psi(0) = 0. \tag{2.37}$$

(This boundary condition was introduced in [12]; for a recent discussion see [22].) Together with the boundary condition at infinity (4.18), this will enforce a relation between the parameters of the model $\gamma$ and $\lambda$. As Figs. 2.2 and 2.3 illustrate, the value of $\psi(0)$ can be

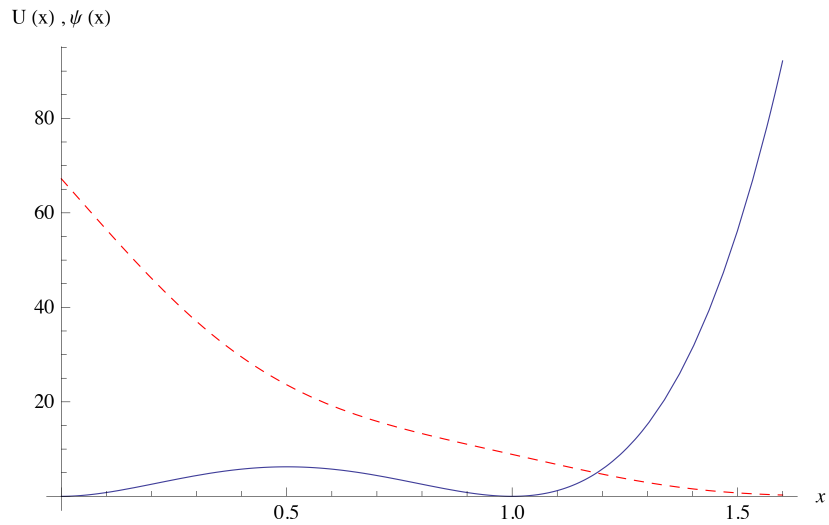



Figure 2.3: Solution of the WDW equation with $\lambda = 0.1$ and $\gamma = 1$ (red dashed line). The WDW potential is also shown (blue line). The classical solution in this case is a static universe at $x = 1$.

either positive or negative. This is determined by whether $\psi(x)$ is growing or decreasing near $x = x_-$, which is in turn determined by the number of oscillations $N$ of $\psi$ that fit into the classically allowed range $x_- < x < x_+$. Suppose for definiteness that we decrease $\lambda$ while keeping $\gamma$ fixed. This makes the potential well deeper, so $N$ monotonically increases and $\psi(0)$ oscillates between positive and negative values, making one oscillation as $N$ changes by $\Delta N \sim 1$. By continuity, $\psi(0)$ should go through zero twice per such oscillation. Values of $\lambda \ll 1$ correspond to the semiclassical regime, where $N \gg 1$ and the boundary condition (2.37) can be satisfied by a relatively small change in $\lambda$.

Thus, for each value of $\gamma > 1$ we expect an infinite set of values of $\lambda$ for which the condition (2.37) can be enforced. Fig. 2.4 shows the wave function for a universe with the parameters fine-tuned in this way. This approach appears to avoid the collapse, but the following argument indicates that it may not be possible to extend it beyond minisuperspace.

The WDW equation (2.14) can be interpreted as stating that the energy of a closed universe is equal to zero. Quantum states with different occupation numbers of matter particles have different energy of matter, but this energy is exactly compensated by the negative energy of gravity, so the total energy is zero. Then one expects that transitions between different states should be possible, as long as they have the same conserved quantum numbers. For example, there seems to be nothing to prevent spontaneous nucleation of

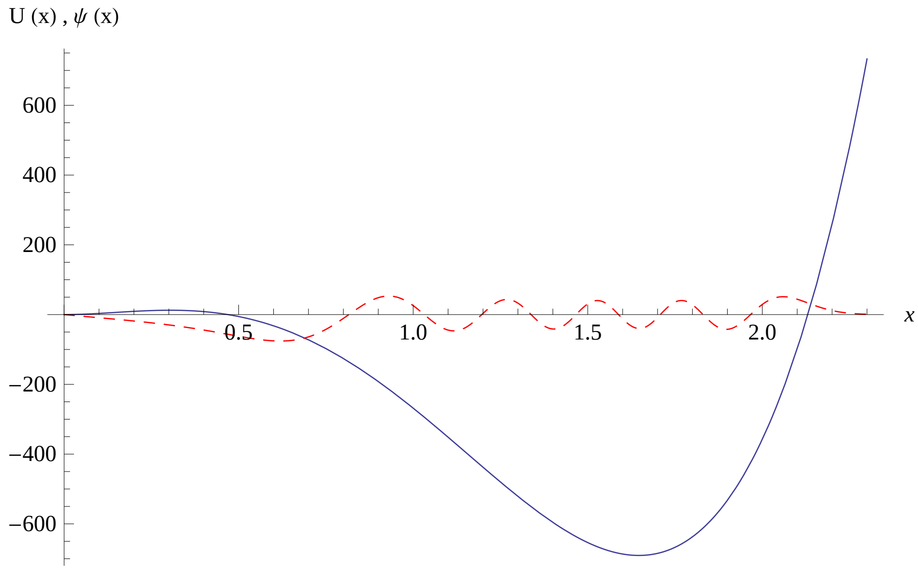



Figure 2.4: Solution of the WDW equation with the parameter values $\lambda = .0473$ and $\gamma = 1.3$, fine tuned so that $\psi(0) = 0$ (red dashed line). The WDW potential is also shown (blue line).

particle-antiparticle pairs. This seems to suggest that the universe will evolve to a state with large occupation numbers and high entropy. In terms of the wave function, we expect $\psi$ to be a superposition of states with different occupation numbers. The value of $\psi(0)$ cannot be fine-tuned for all of them, and we expect that quantum collapse cannot be avoided by fine tuning in more realistic models including a matter field.

Our analysis indicates that oscillating and static models of the universe, even though they may be perturbatively stable, are generically unstable with respect to quantum collapse. Here we focused on the simple harmonic universe with matter content described by Eq. (2.6), but we expect our conclusions to apply to a wider class of models. If we consider additional sources of energy density including strings, domain walls, dust, radiation, etc., the energy density may be represented as

$$\rho(a) = \Lambda + \frac{C_1}{a} + \frac{C_2}{a^2} + \frac{C_2}{a^3} + \frac{C_4}{a^4} + \ldots . \tag{2.38}$$

For positive values of $C_n$, the effect of this is that the potential $U(a)$ develops another classically allowed region at small $a$. So the tunneling will now be to that other region, but the qualitative conclusion about the quantum instability remains unchanged. Altering this conclusion would require rather drastic measures. For example, one could add a matter component $\rho_n(a) = C_n/a^n$ with $n \geq 6$ and $C_n < 0$. Then the height of the barrier becomes infinite at $a \to 0$ and the tunneling action is divergent. Note, however, that such a negative-energy matter component is likely to introduce quantum instabilities of its own. In addition, we consider the effect of quantum corrections such as Casimir energy and the trace anomaly in Chapter 5.

Additionally, one could investigate the quantum stability of braneworld, loop quantum cosmology, and other modified gravity inspired models.[3] We will discuss an oscillating model in loop quantum cosmology in the next Chapter.

[3]Some relevant discussion of quantum cosmology in Horava-Lifshitz gravity models can be found in Ref. [23].

# Chapter 3

# LQC oscillating model

Another class of oscillating models arise in loop quantum cosmology (LQC). LQC is a minisuperspace of loop quantum gravity, which is an approach to a quantized theory of gravity (this is analogous to canonical quantum cosmology, which is a minisuperspace of the standard Hamiltonian approach to quantum gravity). A key feature of LQC which motivates our interest here is that it generically has bouncing solutions, which allows for a class of oscillating models relevant to the emergent universe scenario. One spatially closed model ($k = +1$) was suggested by Mulryne et al [24] has been shown to have eternally oscillating solutions. Additionally, a scalar field potential can cause the spontaneous ignition of an inflationary epoch, preceded by eternal oscillation, giving a concrete example characterizing the emergent universe scenario.

Mulryne et al [24] have shown that such a model has eternally oscillating solutions with positive spatial curvature ($K = +1$), and that inflation may arise with an appropriate potential for the scalar field. An LQC model with flat spatial geometry ($K = 0$) and the energy density given by a massless scalar field $\phi$ and a negative cosmological constant was recently discussed by Mielczarek et al [25], who found a simple oscillating solution in this case. It should be noted that Loop Quantum Gravity, on which LQC is based, is still an incomplete theory; in this sense the foundations of LQC are not very reliable. However, LQC has now been studied in great detail and is an interesting (albeit restricted) theory in its own right (see Ref. [26] for an up to date review). It resolves the singularities of FRW models and provides a useful framework for investigating Planck-scale physics.

In this Chapter, we investigate the stability of the oscillating universe of Ref. [25] with respect to classical scalar field perturbations and to quantum tunneling.

## 3.1 Classical dynamics of oscillating LQC model

In LQC, the effective Friedmann equation is modified – as we will show below – such that the energy density $\rho$ is replaced by $\tilde{\rho}$, where

$$\tilde{\rho} = \rho \left(1 - \frac{\rho}{\rho_{max}}\right), \tag{3.1}$$

where $\rho_{max} = const$ is the maximum energy density the universe can reach; the universe bounces at this energy density. This is the source of one of the key differences for the purposes of tunneling between the LQC-inspired model and the SHU model.

Here, we consider an oscillating model in a flat spacetime described by the metric

$$ds^2 = N^2(t)dt^2 - a^2(t)dx_i dx^i. \tag{3.2}$$

Again, we choose $N = 1$ for the lapse function and $a(t)$ is the scale factor. The Cartesian coordinates are in the range

$$0 \leq x^i \leq L, \tag{3.3}$$

where $L$ represents the overall comoving size of the universe. The boundary at $L$ is identified with 0, forming a closed toroidal universe with finite total volume $V = a^3 L^3$.

The dynamics of this model in LQC are accurately described by the effective Hamiltonian [26][1]

$$\mathcal{H} = -\frac{3}{8\pi G\gamma^2}\frac{\cos^2(\ell\beta)}{\ell^2}V + \mathcal{H}_{matter}, \tag{3.4}$$

where,

$$\beta = 4\pi G\gamma p_V, \tag{3.5}$$

and $p_V$ is the canonical momentum conjugate to $V$. Here, $\gamma$ is the so-called Barbero-Immirzi parameter, and $\ell^2 = 4\pi\sqrt{3}\gamma G$. It is usually assumed that $\gamma \sim 1$; then $\ell$ is comparable to the Planck length.

We consider a model with energy density sourced by a massless scalar field $\phi$ and a

[1] The effective Hamiltonian commonly used in the LQC literature is given by (3.4) with $\cos(\ell\beta)$ replaced by $\sin(\ell\beta)$. These two forms of $\mathcal{H}$ are related by a simple change of variable $\beta \to \beta + \pi/2\ell$.

negative cosmological constant $\Lambda$, The corresponding matter Hamiltonian is

$$\mathcal{H}_{matter} = \frac{p_\phi^2}{2V} + \Lambda V \equiv \rho V, \tag{3.6}$$

where $p_\phi$ is the momentum conjugate to $\phi$, and we have defined the matter energy density $\rho$.

The classical equations of motion for the canonical variables are then

$$\begin{aligned} \dot{p}_\phi &= 0 && (3.7)\\ \dot{\phi} &= \frac{p_\phi}{V} && (3.8)\\ \dot{V} &= -\frac{3}{\gamma} V \frac{\cos(\ell\beta)}{\ell} \sin(\ell\beta) && (3.9)\\ \dot{\beta} &= -4\pi G\gamma \frac{p_\phi^2}{V^2}. && (3.10) \end{aligned}$$

where in the final relation we have used the Hamiltonian constraint

$$\mathcal{H} = 0. \tag{3.11}$$

Squaring the equation for $\dot{V}$ and combining with Eqs. (3.6) and (3.11), we find the modified Friedmann equation:

$$\frac{\dot{a}^2}{a^2} = \frac{\dot{V}^2}{9V^2} = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{max}}\right), \tag{3.12}$$

where

$$\rho_{max} = \frac{\sqrt{3}}{32\pi^2\gamma^3 G^2} \tag{3.13}$$

is the maximum energy density that can be attained in this model.[2] We assume that $|\Lambda| \lesssim \rho_{max}$. When the maximum energy density is reached, $\dot{V} = 0$ and the volume reaches its minimum.

The equations of motion for $\phi$ and $p_\phi$ in Eq. (3.7-3.8) result in the familiar equation for a homogeneous massless scalar field,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = 0 \tag{3.14}$$

[2]The same form of the modified Friedmann equation is obtained in the Shtanov-Sahni braneworld model with a timelike extra dimension [27, 28].

Using Eq. (3.6), the matter energy density can be expressed as

$$\rho = \Lambda + \frac{C}{V^2}, \tag{3.15}$$

where $C = p_\phi^2/2 = \text{const.}$

With this matter density, the Friedmann equation (4.9) has an oscillating solution [25]:

$$\begin{aligned} V(t) &= V_{min}(2\lambda)^{-1/2}\left(-\cos(\omega t) + 1 + 2\lambda\right)^{1/2}, & (3.16)\\ \omega &= \sqrt{96\pi G \rho_{max}\lambda\,(1+\lambda)}, & (3.17) \end{aligned}$$

where $\lambda = |\Lambda|/\rho_{max}$ and

$$V_{min} = \left(\frac{C}{\rho_{max} + |\Lambda|}\right)^{1/2} \equiv L^3 a_{min}^3. \tag{3.18}$$

The universe oscillates at frequency $\omega$ between minimum volume $V_{min}$, where $\rho = \rho_{max}$, and maximum $V_{max} = V_{min}\frac{1+\lambda}{\lambda}$, where $\rho = 0$. The value of the parameter $L$ depends on the normalization of the scale factor $a$. We shall choose it so that $a_{min} = 1$. Then $V_{min} = L^3$ with $L = (C/\rho_{max})^{1/6}(1+\lambda)^{-1/6}$.

We may also find a solution for the momentum $\beta$:

$$\beta(t) = \left(\frac{8\pi G\gamma^2\rho_{max}}{3}\right)^{1/2} \arctan\left(\sqrt{\frac{1+\lambda}{\lambda}}\tan\left(\frac{\omega t}{2}\right)\right). \tag{3.19}$$

In the limit $\Lambda \to 0$ the solution (3.16) goes into

$$V(t) = V_{min}^{(0)}\left(1 + 24\pi G\rho_{max}t^2\right)^{1/2}, \tag{3.20}$$

where $V_{min}^{(0)} = (C/\rho_{max})^{1/2}$. This solution, which describes a contracting, bouncing and re-expanding universe, has been discussed earlier by a number of authors [29].

In the following section, we study perturbative stability of the oscillating universe by considering space-dependent contributions from the scalar field $\phi(\mathbf{x}, t)$. The inhomogeneous components of the field to contribute perturbatively to the energy density; unbounded solutions to equations of motion for the inhomogeneous field perturbations correspond to de-stabilizing growth of the energy density.

## 3.2 Perturbative stability

Growth in mode functions of the scalar field corresponds to increased energy density of the scalar field. This must manifest by increased number density of particles having the corresponding energy level present in the universe. If perturbations of the scalar field modes grow without bound, this signals runaway particle production in the universe, and the geometry must correspondingly be destabilized.

The dynamics of perturbations in LQC has been studied by Agullo et al [30], with the conclusion that it is accurately given by the usual quantum field theory on the FRW background described by the solutions to the effective field equations.[3] In our case, the scalar field perturbations should then satisfy the Klein-Gordon equation

$$\Box\phi = \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\nabla^2\phi = 0 \tag{3.21}$$

in the oscillating background (3.16). Metric perturbations, describing gravitons, satisfy the same equation, so it will be sufficient to investigate the stability of the solutions of Eq. (3.21).

It will be convenient to introduce a new variable $\tau = \omega t$. The field $\phi$ can then be expanded into plane waves,

$$\phi(\mathbf{x}, \tau) = \sum_{\mathbf{k}} \phi_k(\tau) e^{i\mathbf{k}\mathbf{x}}, \tag{3.22}$$

where the mode functions $\phi_k$ satisfy

$$\phi''_k + 3\frac{a'}{a}\phi'_k + \frac{k^2}{\omega^2 a^2}\phi_k = 0. \tag{3.23}$$

Here, primes stand for derivatives with respect to $\tau$.

Periodic boundary conditions in the range (3.3) require that $k_i = 2\pi n_i/L$, where $n_i$ are integers. The eigenvalues of the Laplacian are then given by

$$k^2 = (2\pi/L)^2 n^2, \tag{3.24}$$

where $n^2 \equiv n_1^2 + n_2^2 + n_3^2$.

[3] It is shown in Ref. [31] that this prescription can become inaccurate in the regime where $C \ll 10^2\ell^6\rho_{max}$, so that the bounce is at a near-Planckian volume $V_{min} \ll 10\ell^3$. Here we shall assume that $C \gg 10^2\ell^6\rho_{max}$.

The scale factor corresponding to the solution (3.16) can be written as

$$a(\tau) = (2\lambda)^{-1/6} \left(-\cos\tau + 1 + 2\lambda\right)^{1/6}, \tag{3.25}$$

where we have normalized to $a_{min} = 1$. With this form of the scale factor, the mode equation (3.23) becomes

$$\phi''_k + \frac{\sin\tau}{2f(\tau)}\phi'_k + \frac{q^2}{f^{1/3}(\tau)}\phi_k = 0, \tag{3.26}$$

where

$$f(\tau) = -\cos\tau + 1 + 2\lambda \tag{3.27}$$

and

$$q^2 = \frac{(2\lambda)^{1/3}k^2}{\omega^2} = \frac{\pi}{12}\frac{n^2}{GL^2\rho_{max}(2\lambda)^{2/3}(1+\lambda)}. \tag{3.28}$$

Eq. (3.26) is a form of Hill's equation, which is a more general form of the familiar Mathieu equation. As with the Mathieu equation, we can find stable and unstable regions in the parameter space. The modes $\phi_k$ represent excitations of the field at a certain momentum $\mathbf{k}$, so growing solutions to the mode equation signal particle production in that state.

With a transformation to conformal time $d\eta = a^{-1}dt$ and defining $y(\eta) = a(\eta)\phi(\eta)$, Eq. (3.26) can be brought to the form

$$y'' + (k^2 + a''/a)y = 0, \tag{3.29}$$

where primes stand for derivatives with respect to $\eta$. One can then perform the standard stability analysis using Floquet theory [32]. We did not follow this path because we could not obtain an analytic solution for $a(\eta)$ and using a numerical solution would take an excessive amount of computer time. We therefore modified the Floquet method as described in Appendix B, so that it can be directly applied to Eq. (3.26). The resulting stability diagram in the parameter space of $q$ and $\lambda$ is shown in Fig. 3.1.

The parameter $\lambda$ in our model determines the ratio $a_{max}/a_{min}$. For $\lambda \sim 1$, we have $a_{max} \sim a_{min}$, so the scale factor oscillates about the value $a \sim 1$ with an amplitude also $\sim 1$. In this regime, the diagram exhibits the characteristic pattern of narrow parametric resonance, with instability confined to narrow bands. For $\lambda \ll 1$, $a_{max}/a_{min} \sim \lambda^{-1/3} \gg 1$, so the size of the universe changes by many orders of magnitude in the course of one

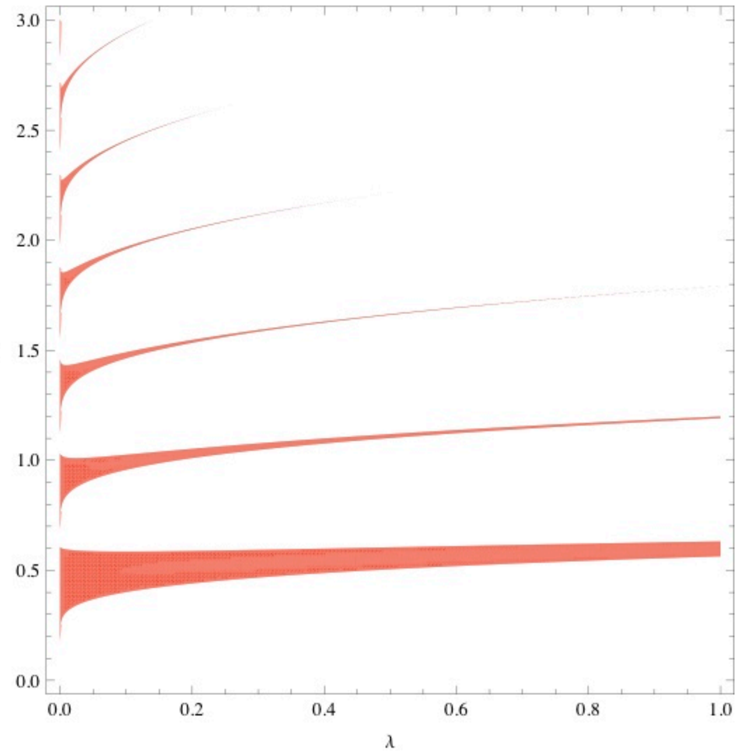



Figure 3.1: Stability diagram in the parameter space of $q$ and $\lambda$. Red bands are unstable.

oscillation. The instability bands become broader in this limit.

These results can be qualitatively understood as follows. In the absence of oscillations, the energy spectrum of a scalar field in a compact universe is discrete. If the oscillation amplitude is relatively small, the oscillations act as a periodic perturbation of frequency $\omega$, and particle production occurs only if $\omega$ is very close to one of the resonant frequencies of the modes. As the oscillation amplitude gets large, the perturbation effectively includes a wide range of frequencies, so a larger number of modes are affected.

As unstable mode functions oscillate, their amplitudes grow exponentially,

$$\phi_k(\tau) \propto e^{\alpha\tau}, \tag{3.30}$$

with the rate of growth $\alpha$ getting higher as $\lambda$ gets smaller and the instability band widens. The time evolution of the energy density,

$$\rho_k(\tau) = \frac{\omega^2}{2}\left({\phi'}_k^{\,2} + \frac{k^2}{\omega^2 a^2}{\phi_k}^2\right), \tag{3.31}$$

is shown in Fig. 3.2 for $q = 0.55$ and $\lambda = 0.5,\ 0.05,\ 0.01$. The corresponding growth rates

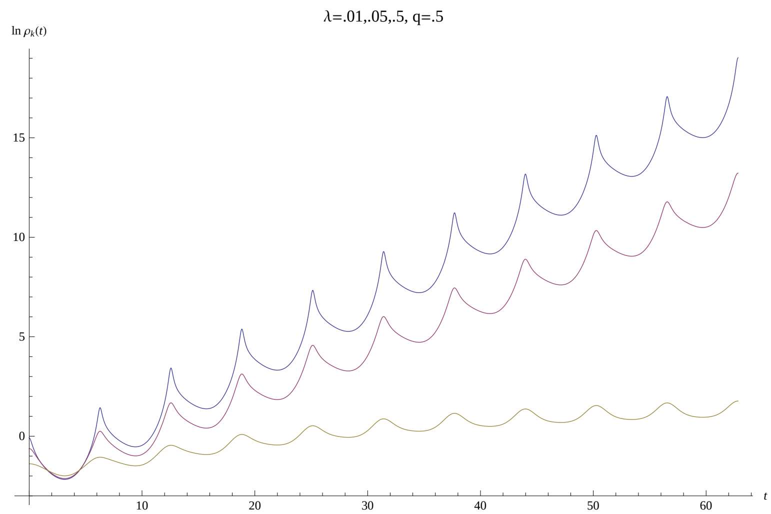



Figure 3.2: The growth of energy density in unstable modes with $q = 0.55$ and $\lambda = 0.5,\ 0.05,\ 0.01$. Lower curves on the graph correspond to larger values of $\lambda$.

are $\alpha = 0.09,\ 0.23,\ 0.31$, respectively.

The parameter $q$ defined in Eq. (3.28) depends on $\lambda$, as well as on $L$ and the mode number $n^2$. In Fig. 3.3 we show regions of instability for the independent parameters of the model, $\lambda$ and

$$\kappa = (GL^2\rho_{max})^{-1/2}. \tag{3.32}$$

Each unstable band of Fig. 3.1 now splits into an infinite number of bands, corresponding to different values of $n^2$. We have included in Fig. 3.3 only modes with $n^2 \leq 10$, with higher values of $n^2$ indicated by lighter shades of grey.

It is interesting to compare our results with those of Graham et al [11], who studied the stability of their simple harmonic universe model. In that model, the solution for $a(\tau)$ is inversely proportional to the oscillation frequency $\omega$, so $\omega$ drops out of the mode equation (3.26). This leaves a single model parameter $\tilde{\gamma} = 3|\Lambda|/2\pi G\rho_0^2$, where $\Lambda < 0$ is the cosmological constant and $\rho_0$ characterizes the contribution of matter with $w = -2/3$ to the total density. The role of this parameter is similar to that of $\lambda$ in our model. Graham et al find that for $\tilde{\gamma} \sim 1$, $a_{max}/a_{min} \sim 1$ and all modes are stable, while for $\tilde{\gamma} \ll 1$, $a_{max}/a_{min} \gg 1$ and there is a large number of unstable modes. (An oscillating solution exists only for $\tilde{\gamma} < 1$.) In contrast, our diagram in Fig. 3.3 shows a non-trivial pattern of stability and instability regions. In particular, we find wide ranges of $\kappa$ where the model is unstable when $\lambda \sim 1$ and where it is stable when $\lambda$ is very small.

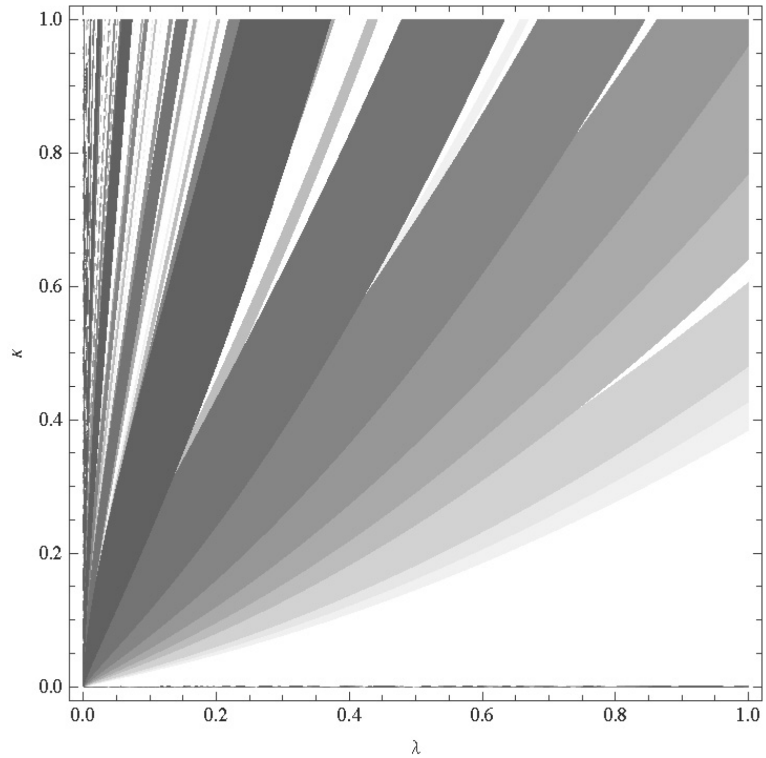



Figure 3.3: Stability diagram in the parameter space of $\kappa$ and $\lambda$, with instability regions corresponding to different values of $n^2$ marked by different shades of grey. Lighter shades correspond to larger values of $n^2$.

## 3.3 Tunneling to nothing

So far we have found that the oscillating LQC universe is stable with respect to inhomogeneous perturbations of the scalar field for certain values of the parameters. However, it is possible that the oscillating universe could tunnel from the bounce point to nothing, as is the case with the simple harmonic universe [33]. Even if the oscillating LQC universe is stable with respect to classical perturbations, it may tunnel quantum mechanically from the bounce point at $V = V_{min}$ to a state of zero volume. Here we estimate the semiclassical tunneling probability.

The application of standard semiclassical methods to this case is complicated by the fact that the field $\phi$ has a nonzero velocity at the bounce point, $\dot{\phi}(V_{min}) = p_\phi / V_{min}$ [see Eqs. (3.7-3.8)]. This problem, however, is not difficult to resolve if we note that the variables $V$ and $\phi$ are essentially decoupled from one another. The Hamiltonian (3.4),(3.6) is independent of $\phi$, so $p_\phi = (2C)^{1/2} = $ const, and the wave function can be factorized

$$\Psi(V,\phi) = e^{ip_\phi \phi}\psi(V). \tag{3.33}$$

The problem then reduces to a one-dimensional tunneling problem with a single dynamical variable $V$, which is described by the action

$$S = \int_i^f dt \left( \frac{1}{4\pi G\gamma} \beta \dot{V} - N\mathcal{H} \right), \tag{3.34}$$

where

$$\mathcal{H} = -\rho_{max} V \cos^2(\ell\beta) + \frac{C}{V} + \Lambda V. \tag{3.35}$$

The semiclassical tunneling probability is given by

$$\mathcal{P} \sim e^{-2S_E}, \tag{3.36}$$

where $S_E$ is the Euclidean action of the instanton solution to the Euclidean equations of motion. The instanton can be obtained from the Lorentzian solution (3.16),(5.7) by analytic continuation $t \to -i\tilde{t}$, $\beta \to i\tilde{\beta}$ to the classically forbidden range of $0 < V < V_{min}$. This gives

$$V(\tilde{t}) = L^3 (2\lambda)^{-1/2} \left( -\cosh(\omega \tilde{t}) + 1 + 2\lambda \right)^{1/2}, \tag{3.37}$$

$$\tilde{\beta}(\tilde{t}) = -\left( \frac{8\pi G\gamma^2 \rho_{max}}{3} \right)^{1/2} \tanh^{-1} \left[ \sqrt{\frac{1+\lambda}{\lambda}} \tanh\left( \frac{\omega \tilde{t}}{2} \right) \right]. \tag{3.38}$$

The classically forbidden range extends from $\tilde{t} = 0$, where $\tilde{\beta} = 0$, to $\tilde{t} = \tilde{t}_f$, where $V = 0$. ($\tilde{t}_f$ can be found from $\cosh(\omega \tilde{t}_f) = 1 + 2\lambda$.) Taking into account the Hamiltonian constraint $\mathcal{H} = 0$, the instanton action is given by

$$S_E = \frac{1}{4\pi G\gamma} \int_0^{\tilde{t}_f} d\tilde{t} \left| \tilde{\beta} \frac{dV}{d\tilde{t}} \right| = \frac{1}{4\pi G\gamma} \left| \int_0^{V_{min}} dV \tilde{\beta}(V) \right|. \tag{3.39}$$

The form of $\tilde{\beta}(V)$ may be determined from the constraint,

$$\tilde{\beta}(V) = \ell^{-1} \cosh^{-1} \left[ \rho_{max}^{-1/2} \sqrt{\frac{C}{V^2} - |\Lambda|} \right]. \tag{3.40}$$

Then, introducing a new variable $x = V/V_{min}$, the integral (4.32) can be rewritten as

$$S_E = \frac{V_{min}}{4\pi G\gamma\ell} \left| \int_0^1 dx \cosh^{-1} \left( \frac{1+\lambda}{x^2} - \lambda \right)^{1/2} \right|. \tag{3.41}$$

It can be expressed as an elliptic integral, but this expression is not particularly illuminating and we do not present it here. An interesting special case is the limit $\lambda \to 0$, when $V_{max} \gg V_{min}$. In this limit we obtain

$$\mathcal{P} \sim \exp\left( -\frac{\pi\sqrt{3}}{2} \frac{V_{min}}{\ell^3} \right), \tag{3.42}$$

where we have used

$$\int_0^1 dx \cosh^{-1} \frac{1}{x} = \frac{\pi}{2}. \tag{3.43}$$

The above semiclassical treatment indicates that when the universe reaches its minimum volume, there is a non-zero probability of tunneling to a singularity. The tunneling is strongly suppressed when the minimum volume at the bounce $V_{min}$ is much larger than the Planck volume.

We note finally that our conclusions here are somewhat different from those of Ashtekar et al [34], who studied quantum tunneling in the same LQC model with $\Lambda = 0$. They found that the Euclidean action in the classically forbidden region between $V = 0$ and $V = V_{min}$ is $S_E = 0$, suggesting that the tunneling to $V = 0$ is unsuppressed. On the other hand, numerical calculations in Ref. [34] indicate that the wave function is actually suppressed in this region,[4] and the authors interpret this as a breakdown of the semiclassical approximation. In our view, the semiclassical approximation is accurate under the usual conditions (roughly, $S_E \gg 1$). The reason for the discrepancy is that the Euclidean continuation in [34] was performed in the full action, including both $V$ and $\phi$ variables, while we considered a reduced action (3.34) with a single variable $V$. The latter appears to be the correct prescription in the presence of classical motion in $\phi$.[5]

[4]We also mention a related result by Craig [35] who showed analytically that eigenfunctions of the quantum evolution operator in LQC decay exponentially in the region between zero volume and the bounce.

[5]For a discussion of multi-dimensional tunneling in the presence of classical motion, see Ref. [36].

# Chapter 4

# Effect of Quantum Corrections on Stability of the Oscillating Universe

So far, we have considered two distinct oscillating models which may be perturbatively stable but which decay quantum mechanically. We may understand the reason for the decay in a rather straightforward manner: the effective potential at large values of scale factor $a$ grows infinitely, forcing the boundary condition $\Psi(a \to \infty) \to 0$ and resulting in non-zero probability for $\Psi$ elsewhere, including at $a \to 0$.

Graham *et al* suggested an interesting possibility that the SHU model can be stabilized by Casimir energy due to the zero-point fluctuations of quantum fields [42]. They assume that the Casimir energy density is given by

$$\rho_C(a) = -\frac{C}{a^4} \tag{4.1}$$

with $C > 0$. If $\rho_C(a)$ is added to the energy density of the SHU in Eq. (2.38), the potential $U(a)$ in (2.13) takes the form shown in Fig. 2.1. The minimum of the potential at $a = 0$ is lifted to a positive value, and Graham *et al* argued that this should prevent the tunneling from taking place.

In this chapter we examine in some more detail the effect of Casimir and other quantum corrections to the energy-momentum tensor on the tunneling of the universe. A Casimir

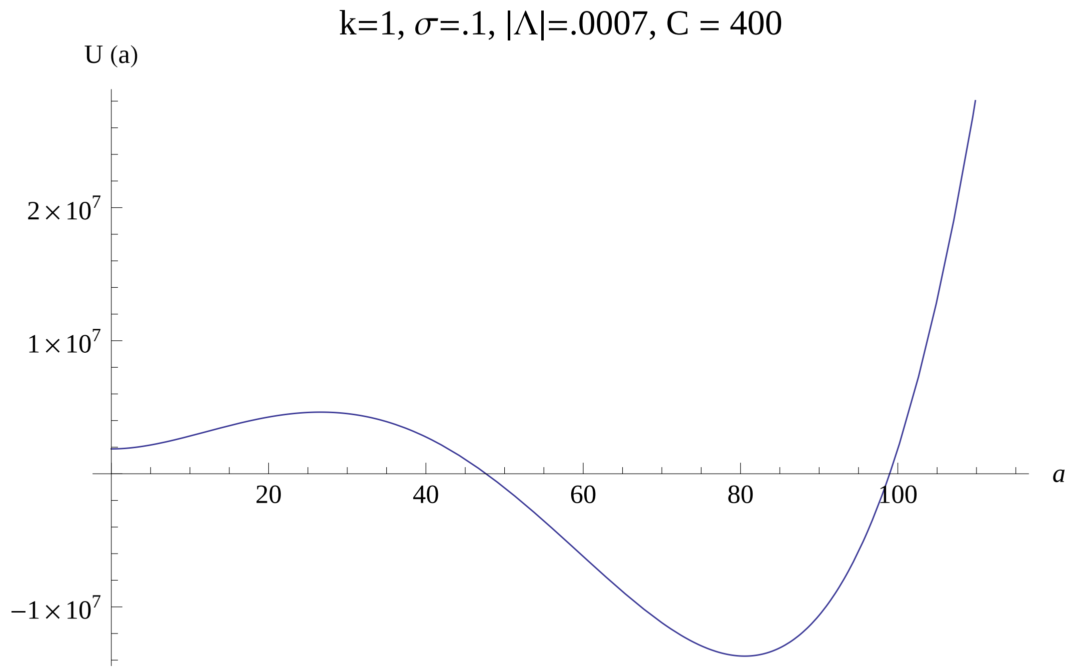


Figure 4.1: The effective potential $U(a)$ of the simple harmonic universe with Casimir energy.

energy density of the form (4.1) was derived for massless quantum fields in a static Einstein universe. In an oscillating universe like the SHU, there are additional contributions, depending on time derivatives of the scale factor. Even in the static limit, when $a_+ = a_-$, the tunneling is described by an instanton solution $a(\tau)$, which depends on the Euclidean time $\tau$, so the additional terms in $T_{\mu\nu}$ must be included. We shall see that these terms can have a significant effect on the tunneling.

Even when keeping only the Casimir energy is a good approximation, we argue that it does not generally prevent the universe from quantum decay. It may be possible, however, to stabilize the universe in some finely-tuned non-vacuum states.

## 4.1 Energy-momentum tensor

Calculation of the expectation value of the energy momentum tensor $\langle T^q_{\mu\nu} \rangle$ of quantum fields in a curved spacetime is a rather challenging task and can be done in a closed form only in a few simple cases (see [37, 38] for a review). The case most relevant to our considerations, which we shall adopt here, is that of free, massless, conformally coupled fields in a FRW universe. Then $\langle T^q_{\mu\nu}\rangle$ can be represented as

$$\langle T^q_{\mu\nu}\rangle = \alpha\ {}^{(1)}H_{\mu\nu} + \beta\ {}^{(3)}H_{\mu\nu} + T^{(C)}_{\mu\nu} + T^{(s)}_{\mu\nu}, \tag{4.2}$$

where $R_{\mu\nu}$ is the Ricci tensor and we have used the standard notation [37]

$$ {}^{(1)}H_{\mu\nu} = 2R_{;\mu;\nu} - 2g_{\mu\nu}R^{;\sigma}_{;\sigma} + 2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 \tag{4.3}$$

and

$$ {}^{(3)}H_{\mu\nu} = R^{\sigma}_{\mu}R_{\nu\sigma} - \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\sigma\tau}R^{\sigma\tau} + \frac{1}{4}g_{\mu\nu}R^2. \tag{4.4}$$

The coefficient $\beta$ in (4.2) is determined by the trace anomaly; it is given by

$$\beta = \frac{1}{1440\pi^2}(N_0 + 11N_{1/2} + 31N_1), \tag{4.5}$$

where $N_0$, $N_{1/2}$ and $N_1$ are the numbers of quantum fields of spin 0, 1/2 and 1, respectively. (Note that $N_{1/2}$ is the number of chiral spinors; a Dirac spinor is counted as two chiral spinors.) The tensor ${}^{(3)}H_{\mu\nu}$ is not identically conserved, but it is conserved in conformally flat spacetimes (and thus in FRW spacetimes). This tensor cannot be obtained by varying a local action. On the other hand, the tensor ${}^{(1)}H_{\mu\nu}$ can be obtained by varying an $R^2$ term in the action and is identically conserved. The coefficient $\alpha$ is affected by the $R^2$ counterterm that has to be added in order to cancel infinities in $T_{\mu\nu}$.By a suitable choice of the counterterm, this coefficient can be tuned to zero. We shall adopt this choice here, in order to simplify the discussion.

$T^{(C)}_{\mu\nu}$ is an additional Casimir contribution, which arises if the space has nontrivial topological identifications. One example is a spatially flat universe with a toroidal topology, like we discussed in Chapter 3. (See in particular Eq. (3.2). There, we had $0 \le x_i \le L$.) Then

$$T^{(C)}_{\mu\nu} = -\frac{C}{a^4}\mathrm{diag}(1, 1/3, 1/3, 1/3) \tag{4.6}$$

where this tensor is traceless and covariantly conserved: $T^{(C)\nu}_{\nu} = 0$, $T^{(C)\nu}_{\mu;\nu} = 0$. The coefficient $C$ for a real scalar field is $C_0 = 0.8375$. In general, for non-interacting conformal fields,

$$C = (N_b - N_f)C_0, \tag{4.7}$$

where $N_b = N_0 + 2N_1$ and $N_f = 2N_{1/2}$ are the numbers of bosonic and fermionic spin degrees of freedom.[1] A flat toroidal universe will be discussed in detail in Sec. 4.2. Note that for a spherical ($k = +1$) universe, there are no additional Casimir terms: the Casimir contribution is already included in ${}^{(3)}H_{\mu\nu}$. The reason is that a $k = 1$ universe is conformally related to a static Einstein universe, which has a vanishing trace anomaly.

The last term in (4.2) depends on the choice of quantum state; it can be thought of as 'radiation', representing particle excitations of the fields. For conformal fields, there is a natural choice of vacuum state – the conformal vacuum, whose mode functions are obtained by a conformal transformation from those of a static universe with $a = \text{const}$. In this state $T^{(s)}_{\mu\nu} = 0$. (Note that for conformal fields the time dependence of the scale factor does not give rise to particle production.) For now, we shall assume that all fields are in their conformal vacuum states; a more general case will be considered in Sec. 4.3.

To investigate the effect of quantum corrections on the SHU, we shall use the semiclassical Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -T^{SHU}_{\mu\nu} - \langle T^q_{\mu\nu} \rangle, \tag{4.8}$$

where $T^{SHU}_{\mu\nu}$ is the classical energy-momentum tensor of the SHU. Here we have used Planck units to set $8\pi G = 1$. The corresponding Friedmann equation is

$$\frac{\dot{a}^2 + k}{a^2} = \beta \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 + \frac{1}{3} \left( \Lambda + \frac{\sigma}{a} \right), \tag{4.9}$$

where we have set $\alpha = 0$ and $T^{(C)}_{\mu\nu} = 0$ (assuming spherical topology).

As we already mentioned, the trace anomaly term in this case includes the effect of Casimir energy (proportional to $a^{-4}$ in the Friedmann equation). However, we will see that this energy is positive, so its effect is to lower the potential, creating another classically allowed region near $a = 0$.

We can find the classical turning points by setting $\dot{a} = 0$ in the Friedmann equation (4.9), and finding solutions for

$$1 - \frac{1}{3} \left( \sigma a - \Lambda a^2 \right) = \beta a^{-2}. \tag{4.10}$$

With a suitable choice of the parameters, the classical turning points of the SHU (see Eq. (2.20) in Chapter 2) remain approximately unchanged. (The condition for that is

[1] Note that $T^{(C)}_{\mu\nu}$ vanishes in supersymmetric models, where $N_b = N_f$.

$\beta|\Lambda|^3/\sigma^4 \ll 1$.) Then it is easy to see that there must be an additional turning point at small $a$. With $\beta\sigma^2 \ll 1$, this turning point is at

$$a_* \approx \beta^{1/2}. \tag{4.11}$$

If the number of quantum fields appearing in Eq. (4.5) is sufficiently large, $N \gg 10^3$, so that $\beta \gg 1$, then $a_*$ is large in Planck units. There is then a classically allowed region between $a_*$ and 0, and clearly the oscillating universe can tunnel to this region.

Graham *et al* have pointed out that a negative Casimir energy for a $k = +1$ SHU may be possible to arrange by considering non-conformal fields. The case of a massless non-conformally coupled scalar field was studied in Ref. [39], with the conclusion that the Casimir energy density is given by Eq. (4.1), where the sign of the coefficient $C$ depends on the coupling $\xi$ of the field to the curvature. It is negative for conformal coupling, $\xi = 1/6$, but is positive for the minimal coupling, $\xi = 0$. Unfortunately the calculation of $\langle T_{\mu\nu}\rangle$ in Ref. [39] was performed only for a static Einstein universe, which is not sufficient for the analysis of tunneling. In the next section, we shall therefore introduce a version of the SHU model which allows a negative Casimir energy for conformally coupled fields, so that Eq. (4.2) can still be used.

## 4.2 A flat oscillating model

We consider a spatially flat ($k = 0$) universe,

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2, \tag{4.12}$$

which is compactified on a torus: the coordinates $x^i$ ($i = 1, 2, 3$) take values in the range $0 \le x^i \le 1$ and the surfaces $x^i = 0$ and $x^i = 1$ are identified. (This is similar to the flat oscillating model discussed in Chapter 3, Eq. (3.2), where in that case $0 \le x_i \le L$ with $L$ corresponding to the comoving size of the universe.) In this case the expectation value of $T_{\mu\nu}$ has an additional Casimir contribution $T^{(C)}_{\mu\nu}$, given by Eqs. (4.6),(4.7). With $N_b > N_f$, we have $C > 0$ and the Casimir energy is negative. The $k = 0$ Friedmann equation including corrections to Einstein's equations, the trace anomaly and Casimir contributions, is

$$\frac{\dot{a}^2}{a^2} = \beta\left(\frac{\dot{a}^2}{a^2}\right)^2 + \frac{1}{3}\left(\Lambda + \frac{\sigma}{a} - \frac{C}{a^4}\right). \tag{4.13}$$

The turning points are the solutions to the Friedmann equation when $\dot{a} = 0$:

$$\Lambda a^4 + \sigma a^3 - C = 0. \tag{4.14}$$

Though we cannot find analytic solutions, there are two classical turning points for $C < \frac{27\sigma^4}{256|\Lambda|^3}$. For $\sigma^4/|\Lambda|^3 \gg C$, the turning points are approximately given by

$$a_+ \approx \sigma/|\Lambda|, \quad a_- \approx (C/\sigma)^{1/3}. \tag{4.15}$$

Following Graham *et al* [42], we shall first consider the regime where the trace anomaly term can be neglected. (The effect of this term will be discussed in the next section.) The Friedmann equation then corresponds to the Hamiltonian constraint $\mathcal{H} = 0$ with $\mathcal{H}$ from Eq. (2.11) and $\Omega(k=0) = 1$, so the potential is given by

$$U(a) = 12\left(-\sigma a^3 - \Lambda a^4 + C\right). \tag{4.16}$$

The Casimir contribution to the potential is a positive constant, so $U(a)$ remains positive all the way from $a_-$ to $a = 0$ and has the general form illustrated in Fig. 4.1. Graham *et al* argued that in this case the universe should be stable with respect to quantum decay.

To examine this claim, we consider the wave function of the universe, which can be found by solving the Wheeler-DeWitt (WDW) equation[2]

$$\left(\frac{d^2}{da^2} - U(a)\right)\psi(a) = 0. \tag{4.17}$$

In our model, the probability for the universe to have infinite size should vanish; hence we have to require that

$$\psi(a \to \infty) = 0. \tag{4.18}$$

In order to exclude the singular state at $a = 0$, one would also want to require $\psi(0) = 0$ [12]. However, the general solution of Eq. (5.10) depends on only two arbitrary constants. One

[2]Here we assume the simplest ordering of the operators $a$ and $p_a$ in the Hamiltonian (2.11).

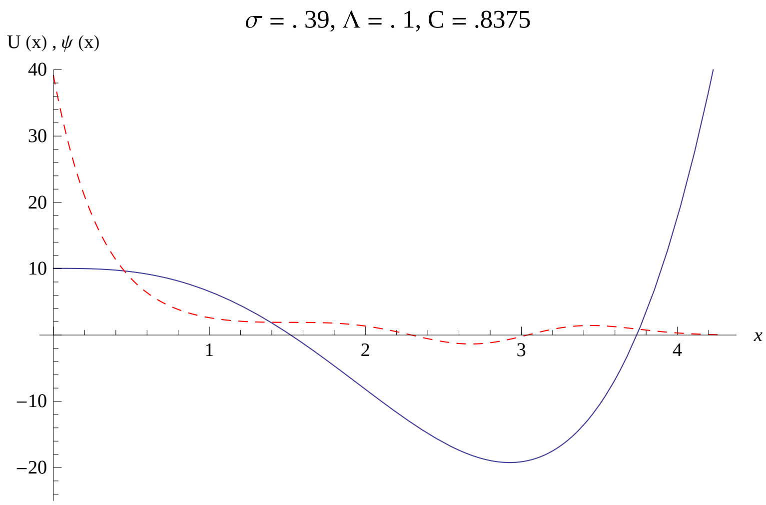



Figure 4.2: Numerical solution (dashed red line) to the WDW equation with $\sigma = .39$, $\Lambda = -.1$ and $C = .8375$. The potential is plotted with a solid blue line.

of them is used to fix the overall normalization of $\psi$ and the other to enforce the boundary condition (4.18). Thus there is no freedom left to enforce a boundary condition at $a = 0$ [33].

The WKB solutions for $\psi$ in the classically forbidden range $0 < a < a_-$ have the form

$$\psi_\pm(a) \propto (U(a))^{-1/4} e^{\pm W(a)}, \tag{4.19}$$

where

$$W(a) = \int_a^{a_-} \sqrt{U(a)} da. \tag{4.20}$$

The wave function will generally include a superposition of $\psi_+$ and $\psi_-$. Since $\psi_+$ grows exponentially with decreasing $a$, we expect $\psi$ to be large at $a = 0$. This behavior is illustrated in a numerical solution shown in Fig. 4.2.

The fact that $\psi(0)$ is large indicates that the UV physics at $a \lesssim 1$ cannot be ignored. Without a UV-complete theory of quantum gravity, we cannot tell what the effect of this physics will be, but we shall try to consider some possible alternatives. If reaching $a \lesssim 1$ means disappearance of semiclassical spacetime, then a stationary state with $\psi(0) \neq 0$ simply cannot exist. To illustrate this point, let us consider a hypothetical world where the electron wave function in a hydrogen atom is such that the probability density for the electron to be at the location of the proton is nonzero. Suppose further that the proton, electron, neutron and neutrino masses satisfy $m_p + m_e > m_n + m_\nu$, so that proton and electron can always scatter into neutron and neutrino,

$$e^-p^+ \to n\nu. \tag{4.21}$$

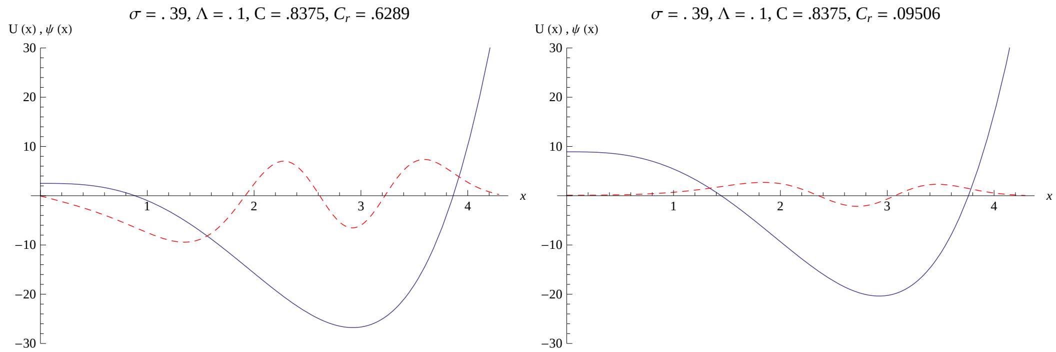
Under these assumptions, it is clear that the atom would have a nonzero decay rate. We can imagine that the interaction between electrons and protons in this world is such that the electron 'orbit' is separated from the center of the atom by a potential barrier. There may or may not be a small classically allowed range near the center. Our conclusion applies in either case: a stationary state of the atom is not possible if the probability density for the electron at the center is nonzero. In this example, the reaction (4.21) represents the UV physics, which is not included in the Schrodinger equation for the hydrogen atom, just like the disintegration of the classical spacetime at $a = 0$ is not reflected in the WDW equation.

An alternative scenario is that the UV-complete theory will resolve the singularity at $a = 0$, replacing it with a non-singular Planck-size nugget. The universe may then tunnel back and forth between the classical oscillating regime and the nugget, resulting in a stationary quantum state. The WDW equation should be accurate in the semiclassical regime at $a \gg 1$; hence our conclusion regarding the growth of the wave function towards small values of $a$ should still apply. This indicates that the most probable states of the universe will be in the Planck regime at $a \sim 1$.

## 4.3 Stable oscillating universes

Even though the boundary condition $\psi(0) = 0$ cannot be generally enforced, it may be satisfied for some special values of the parameters $\Lambda$, $\sigma$ and $C$. These parameters are assumed to be fixed, but the effective value of $C$ can be changed if we allow more general states of the quantum fields contributing to the Casimir energy. We assumed so far that these fields are all in conformal vacuum states, so that $T^{(s)}_{\mu\nu} = 0$ in Eq. (4.2). Suppose now that some particle excitations on top of the vacuum are also present. For massless particles, the expectation value of $T_{\mu\nu}$ will have the form of the radiation energy-momentum tensor,

$$T^{(s)}_{\mu\nu} = \frac{C_r}{a^4}\mathrm{diag}(1, 1/3, 1/3, 1/3) \tag{4.22}$$




(a) $C_r = .6289$ (b) $C_r = .09506$

Figure 4.3: Numerical solutions of the WDW equation (dashed red line) with parameters $\sigma = .39$, $\Lambda = -.1$ and $C = .8375$, and two values of $C_r$ such that $\psi(0) = 0$. The potential is plotted with a solid blue line.

with $C_r > 0$. This has the same form as the Casimir term (4.6). Hence, including particle excitations has the effect of replacing the parameter $C$ with an effective value

$$C_{eff} = C - C_r. \tag{4.23}$$

Increasing $C_r$ (decreasing $C_{eff}$) widens the potential well and increases the number of nodes of the wave function in the classically allowed region, causing $\psi(0)$ to oscillate between positive and negative values and pass through zero at special values of $C_{eff}$. We can then tune the particle content in such a way that $C_{eff}$ takes one of the values that yield $\psi(0) = 0$. For a given set of parameters $\Lambda$, $\sigma$, and $C$, there is a finite number of such nonsingular solutions, given by the difference in the number of nodes at $C_r = C$ and at $C_r = 0$.

As an illustration, we show in Fig. 4.3 the non-singular solution for the parameter values $\sigma = .39$, $\Lambda = -.1$ and $C = .8375$ (the same as in Fig. 4.2). There are only two such solutions in this case, corresponding to the values $C_r = .6289, .09506$.

## 4.4 The effect of the trace anomaly

Let us now consider the effect of the trace anomaly term (proportional to $\beta$) in Eq. (4.13). The effective Lagrangian corresponding to this equation is [40, 41]

$$\mathcal{L} = -3a\dot{a}^2 + \beta\frac{\dot{a}^4}{a} - a^3\rho(a), \tag{4.24}$$

where

$$\rho(a) = \Lambda + \frac{\sigma}{a} - \frac{C}{a^4}. \tag{4.25}$$

Wheeler-DeWitt quantization of this model is problematic, because the momentum $p_a = \partial\mathcal{L}/\partial\dot{a}$ depends nontrivially on $\dot{a}$, and as a result the Hamiltonian operator involves fractional powers of differential operators. We shall therefore take an alternative route and analyze tunneling in terms of solutions to Euclideanized Friedmann equation.

With the replacement $t \to -i\tau$, the Friedmann equation (4.13) takes the form

$$\frac{a'^2}{a^2} + \beta\frac{a'^4}{a^4} + \frac{1}{3}\rho(a) = 0, \tag{4.26}$$

where primes stand for derivatives with respect to the Euclidean time $\tau$. Solving this for $a'$, we have

$$a'^2 = \frac{a^2}{2\beta}\left(-1 \pm \sqrt{1 - \frac{4\beta}{3}\rho(a)}\right). \tag{4.27}$$

In the classically forbidden range $\rho(a) < 0$, so we have to choose the positive sign of the square root. Then it is easy to check that for $|\rho(a)| \ll \beta^{-1}$ the trace anomaly term is unimportant and Eq. (4.27) reduces to the usual (Euclidean) Friedmann equation,

$$a'^2 = -\frac{1}{3}\rho(a). \tag{4.28}$$

At small $a$, $\rho(a)$ is dominated by the Casimir term, $\rho_C(a) = -C/a^4$, and the condition $|\rho(a)| \ll \beta^{-1}$ gives

$$a \gg (\beta C)^{1/4} \equiv a_q. \tag{4.29}$$

For generic numbers of quantum fields, $N_0 \sim N_{1/2} \sim N_1 \sim N$, $a_q \sim 0.2N^{1/2}$, so we can have $a_q \gg 1$ if $N$ is sufficiently large. We note also that with sufficiently small $\sigma$ Eq. (4.15) gives $a_- \gg a_q$. Thus, with a suitable choice of parameters the scale factor ranges $1 \ll a \lesssim a_q$ and $a_q \ll a \lesssim a_-$ should both allow semiclassical treatment.

In the range $a \ll a_q$, where the trace anomaly term is important, Eq. (4.27) takes the form

$$a'^2 \approx (C/3\beta)^{1/2}, \tag{4.30}$$

with the solution

$$a(\tau) \approx (C/3\beta)^{1/4}\tau. \tag{4.31}$$

The Euclidean action evaluated in this range is

$$S_E(\tau) = \int_{\tau}^{\tau_-} d\tau \left( 3aa'^2 + \frac{\beta a'^4}{a} - a^3 \rho(a) \right), \tag{4.32}$$

where $\tau_- \sim C^{1/12} \beta^{1/4} \sigma^{-1/3}$ corresponds to $a = a_-$. We see immediately that with $a(\tau)$ from (4.31) the integral in (4.32) diverges at $\tau \to 0$. Expressing $\tau$ in terms of $a$, we have

$$S_E(a \to 0) \approx Q \ln \frac{a_-}{a}, \tag{4.33}$$

where

$$Q = (3\beta C^3)^{1/4} \sim N. \tag{4.34}$$

This suggests that the wave function at small $a$ grows as a large negative power of $a$,

$$\psi(a) \propto e^{S_E(a)} \propto a^{-Q}. \tag{4.35}$$

The growth may or may not be terminated by quantum gravity effects at $a \sim 1$. In either case, normalizable wave functions with $\psi(a \to 0) = 0$ may exist for some special states of the quantum fields, as discussed in Sec. 4.3.

## 4.5 Summary and discussion

We used a simple minisuperspace model to analyze the effect of Casimir and trace anomaly corrections on the quantum decay of classically stable oscillating universe models. We found that these corrections can significantly modify the wave function of the universe $\psi(a)$ at small values of the scale factor $a$ and may even cause it to diverge at $a \to 0$. However, the vacuum corrections do not generally stabilize an oscillating universe. The reason is simple: the wave function of the universe must satisfy the boundary condition $\psi(a \to \infty) = 0$. This leaves no freedom to impose a boundary condition at $a = 0$, and as a result the wave function generally grows towards small $a$. In this regard the situation is the same as for the simple harmonic universe model without vacuum corrections [33].

We found also that the wave function can be tuned to zero at $a = 0$ for non-vacuum states of the quantum fields, corresponding to certain fine-tuned amounts of 'radiation' in the universe. Such states may correspond to absolutely stable, stationary quantum states of the universe.

This possibility, however, should be regarded with caution. Our treatment here has been based on the semiclassical gravity approximation, Eq. (4.8), in which the quantity $C_r$ characterizing the amount of radiation in Eq. (4.22) is a continuous parameter. This allowed us to tune this parameter to enforce $\psi(a = 0) = 0$. It is not clear that such tunable parameters will exist in the full theory of quantum gravity. One might expect that, on the contrary, $C_r$ could be quantized. For example, if we simply add quantum conformal free fields to a compact minisuperspace FRW model, the WDW equation will separate and the fields will contribute a radiation term with a discrete spectrum of $C_r$, which is not likely to overlap with the set of values required for $\psi(0) = 0$. This treatment, however, would not account for the Casimir and trace anomaly contributions and for the possibility of having the quantum fields in a superposition of occupation number eigenstates. A definitive resolution of these issues will require a better understanding of the quantum theory of gravity.

# Chapter 5

# Decay Rate of Simple Harmonic Universe

In Chapters 2, we showed that the simple harmonic universe has a non-zero tunneling probability, signaling that it cannot be eternal. However, in quantum cosmology, the wave function of the universe is independent of time.

In the semiclassical regime, the decay rate can be expressed as

$$\Gamma = \mathcal{A} e^{-2|S_E|}, \tag{5.1}$$

where $S_E$ is the under-barrier Euclidean action. The action $S_E$ has been calculated in Refs. [18, 33] for some simple FRW models. Here, we would like to go beyond that and also calculate the pre-exponential factor $\mathcal{A}$, at least in the framework of the FRW models under consideration.

The problem we have to address is that the rate $\Gamma$ is the decay probability per unit time, and the time variable is conspicuously absent in the formalism of quantum cosmology. Any time evolution should then be understood implicitly, in terms of the canonical variables themselves. We adopt this approach here and use it to calculate the decay rate in the simple harmonic universe, extended to include a clock. Here, as before, we focus on the case where $w = -2/3$ for calculation simplicity, though the situation is qualitatively similar for other values of $w$.

The SHU model includes a single dynamical variable – the scale factor $a$. In the case of an oscillating universe, the scale factor evolution is not monotonic; hence it cannot serve as a time variable. We therefore introduce a second minisuperspace variable – a homogeneous, massless, minimally coupled scalar field $\phi$, which will play the role of a clock. We shall assume that the contribution of this field to the total energy density of the SHU is negligible, so that its presence has little effect on the dynamics of oscillations and does not alter the stability analysis of [42].

We now modify the SHU model by adding a homogeneous, massless, minimally coupled scalar field $\varphi(t)$. The corresponding Hamiltonian constraint is

$$\mathcal{H} = -\frac{G}{3\pi a}\left(p_a^2 - \frac{3}{4\pi G a^2}p_\varphi^2 + \tilde{U}(a)\right) = 0, \tag{5.2}$$

where

$$p_\varphi = 2\pi^2 a^3 \dot{\varphi} \tag{5.3}$$

is the momentum conjugate to $\varphi$. The momentum $p_\varphi$ is a constant of motion, $\dot{p}_\varphi = 0$. Without loss of generality, we shall assume that $p_\varphi > 0$. Then it follows from Eq. (5.3) that the scalar field $\varphi$ increases monotonically; hence it can be used as a time variable.

In quantum cosmology, we make the replacement $p_a \to -i\partial/\partial a$ and $p_\varphi \to -i\partial/\partial\varphi$, and the Hamiltonian constraint $\mathcal{H} = 0$ becomes the Wheeler DeWitt (WDW) equation

$$\left[-a\frac{\partial}{\partial a}a\frac{\partial}{\partial a} + a^2\tilde{U}(a) + \frac{3}{4\pi G}\frac{\partial^2}{\partial\varphi^2}\right]\Psi(a,\varphi) = 0. \tag{5.4}$$

Here, we have adopted the ordering of the non-commuting factors $a$ and $\partial/\partial a$ proposed in [43], for which the differential operator in Eq. (5.4) becomes a covariant Laplacian.

With the change of variables $\alpha = \ln(\omega\gamma a)$, $\phi = (4\pi G/3)^{1/2}\varphi$, the WDW equation becomes

$$\left[-\frac{\partial^2}{\partial\alpha^2} + U(\alpha) + \frac{\partial^2}{\partial\phi^2}\right]\Psi(\alpha,\phi) = 0, \tag{5.5}$$

where the potential $a^2\tilde{U}(a) = U(\alpha)$ is

$$U(\alpha) = \beta^{-2}e^{4\alpha}\left(1 - 2e^{\alpha} + \gamma^{-2}e^{2\alpha}\right), \tag{5.6}$$

where

$$\beta = \left(\frac{2G}{3\pi}\right)\omega^2\gamma^2 = \frac{32\pi}{27}G^3\sigma^2 \ll 1. \tag{5.7}$$

The last inequality follows from the assumption that both parameters $|\Lambda|$ and $\sigma$ are small in Planck units:

$$|\Lambda| \ll G^{-2} \qquad ; \qquad \sigma \ll G^{-3/2}. \tag{5.8}$$

The WDW equation (5.5) separates, and the general solution can be expressed as a superposition of terms of the form

$$\Psi(\alpha, \phi) = e^{ip\phi} f_p(\alpha), \tag{5.9}$$

where the separation parameter $p$ is the eigenvalue of the momentum $p_\phi$ and the function $f_p(\alpha)$ satisfies the equation

$$\left[ -\frac{\partial^2}{\partial \alpha^2} + U_p(\alpha) \right] f_p(\alpha) = 0 \tag{5.10}$$

with

$$U_p(\alpha) = U(\alpha) - p^2. \tag{5.11}$$

The effective potential $U_p(\alpha)$ is plotted in Fig. 5.1.

We see that inclusion of a scalar field has the effect of decreasing the potential $U(\alpha)$ by a constant term, $-p^2$. We assume that this term is small compared to the characteristic scale of the potential, that is,

$$p \ll \beta^{-1}. \tag{5.12}$$

This term, however, does have an effect near the turning points

$$\alpha_\pm = \ln \left( \gamma^2 \pm \gamma^2 \sqrt{1 - \gamma^{-2}} \right) \tag{5.13}$$

of the unperturbed potential, $U(\alpha_\pm) = 0$. The turning points in the presence of a scalar field,

$$\alpha_1 = \alpha_- - \delta\alpha_1 \tag{5.14}$$

$$\alpha_2 = \alpha_+ + \delta\alpha_2 \tag{5.15}$$

can be found by solving $U_p(\alpha) = 0$. To the lowest order in $p^2$, we have

$$\delta\alpha_1 \simeq \frac{p^2}{|U'(\alpha_-)|} \tag{5.16}$$

$$\delta\alpha_2 \simeq \frac{p^2}{U'(\alpha_+)}. \tag{5.17}$$

The derivatives of the potential appearing in Eqs. (5.16),(5.17) are

$$U'(\alpha_-) = -\frac{1}{\beta^2}\left(-1+\sqrt{1-\gamma^{-2}}\right)^4 \gamma^8 \left(1+\left(-1+\sqrt{1-\gamma^{-2}}\right)\gamma^2\right), \tag{5.18}$$

$$U'(\alpha_+) = \frac{1}{\beta^2}\left(1+\sqrt{1-\gamma^{-2}}\right)^4 \gamma^8 \left(-1+\left(1+\sqrt{1-\gamma^{-2}}\right)\gamma^2\right), \tag{5.19}$$

or, by order of magnitude,

$$U'(\alpha_-) \sim -\beta^{-2}, \quad U'(\alpha_+) \sim \beta^{-2}\gamma^{10}. \tag{5.20}$$

Since $\gamma \gtrsim 1$, we can write

$$\delta\alpha_\pm \lesssim \beta^2 p^2 \ll 1. \tag{5.21}$$

This implies that the relative displacement of the turning points is small, $\delta a_\pm / a_\pm \sim \delta\alpha_\pm \ll 1$.

Apart from shifting the turning points $\alpha_\pm$, the scalar field also modifies the character of the potential at small $a$ ($\alpha \to -\infty$), introducing another classically allowed region (region I in Fig. 5.1). The potential at $\alpha \to -\infty$ can be approximated as $U_p(\alpha) \sim \beta^{-2} e^{4\alpha} - p^2$, so the boundary of this region is approximately

$$\alpha_0 \simeq \frac{1}{2}\ln(\beta p). \tag{5.22}$$

In order to justify semiclassical treatment, we shall require that the corresponding scale factor is large in Planck units,

$$a_0 = \left(\frac{2Gp}{3\pi}\right)^{1/2} \gg G^{1/2}, \tag{5.23}$$

which implies $p \gg 1$.

The two classically allowed regions are separated by a barrier (region II) extending between the turning points $\alpha_0$ and $\alpha_1$. The situation is therefore analogous to a particle

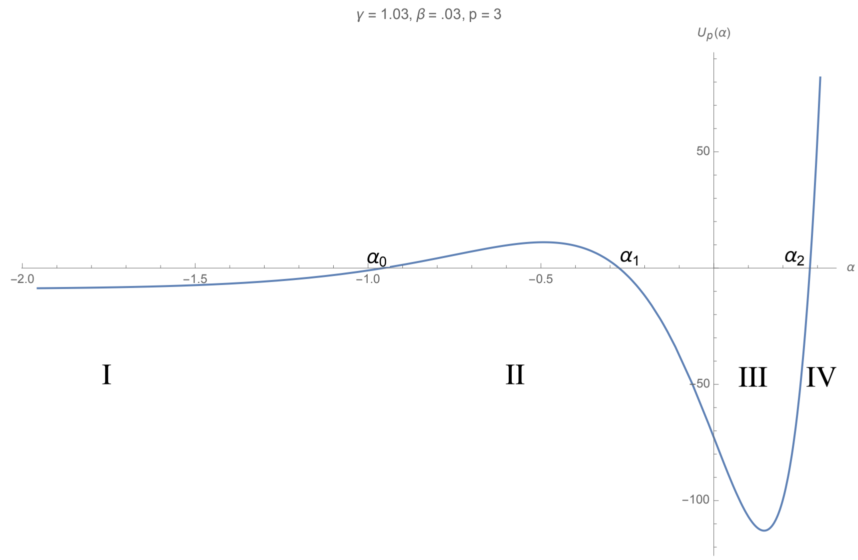



Figure 5.1: Effective potential $U_p(\alpha)$, with $\gamma = 1.03$, $\beta = .03$, and $p = 3$. The parameter values were chosen such that the qualitative features of the potential are clear; our semiclassical analysis relies on $1 \ll p \ll \beta^{-1/2}$ and $\gamma - 1 \gg \beta$ (see Eqs. (5.66)-(5.68)).

in a metastable state: the particle is localized in an approximate energy eigenstate, but the corresponding energy eigenvalue takes on an imaginary part, indicating a non-vanishing decay rate. We shall see that the wave function in our model exhibits a very similar behavior.

## 5.1 Semiclassical solutions

Solutions to the WDW equation are specified by imposing boundary conditions appropriate to the problem. Here, we first require that the wave function must decay under the infinite barrier to the right of $\alpha_2$, $f_p(\alpha) \to 0$ as $\alpha \to \infty$ (this is the same boundary condition chosen in earlier Chapters). In addition, we require that the solution be outgoing (left-moving) in the region $\alpha < \alpha_0$. Physically, this means that once the oscillating universe tunnels through the barrier, it collapses to the singularity at $a = 0$ ($\alpha \to -\infty$). In other words, the singularity is a point of no return: the probability of getting back from $a = 0$ to $a = a_0$ is zero.

Sufficiently far from the turning points, we can determine the solutions using the semiclassical approximation:

$$f_p(\alpha) \simeq \frac{C_1 e^{-i\pi/4}}{[-U_p(\alpha)]^{1/4}} e^{+i\int_{\alpha_*}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'} + \frac{C_2 e^{i\pi/4}}{[-U_p(\alpha)]^{1/4}} e^{-i\int_{\alpha_*}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'}. \tag{5.24}$$

for left (+) and right (−) moving waves in the classically allowed region, $U_p(\alpha) < 0$, and

$$f_p(\alpha) \simeq \frac{D_1}{(U_p(\alpha))^{1/4}} e^{+\int_{\alpha_*}^{\alpha} \sqrt{U_p(\alpha')} d\alpha'} + \frac{D_2}{(U_p(\alpha))^{1/4}} e^{-\int_{\alpha_*}^{\alpha} \sqrt{U_p(\alpha')} d\alpha'}. \tag{5.25}$$

for growing (+) and decaying (−) solutions under the barrier, $U_p(\alpha) > 0$.

Near a turning point, $\alpha = \alpha_*$, where[1]

$$\text{Re } U_p(\alpha_*) = 0, \tag{5.26}$$

the semiclassical approximation breaks down. In such regions we use the standard technique [44] and approximate the potential by a linear function,

$$U(\alpha_* + \delta\alpha) \simeq U(\alpha_*) + U'(\alpha_*)\delta\alpha = U'(\alpha_*)(\alpha - \alpha_*). \tag{5.27}$$

Setting $z = (U'(\alpha_*))^{1/3}(\alpha - \alpha_*)$, the approximate WDW equation near a turning point is

$$\left( \frac{\partial^2}{\partial z^2} - z \right) \Psi(z) = 0. \tag{5.28}$$

The solution is a linear combination of Airy functions $Ai(z)$ and $Bi(z)$, having the asymptotic ($z \to \infty$) forms

$$\begin{aligned} Ai(z) &\sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3} z^{3/2}} && (5.29) \\ Bi(z) &\sim \frac{1}{\sqrt{\pi}} z^{-1/4} e^{\frac{2}{3} z^{3/2}} && (5.30) \\ Ai(-z) &\sim \frac{1}{\sqrt{\pi}} z^{-1/4} \sin\left[\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right] && (5.31) \\ Bi(-z) &\sim \frac{1}{\sqrt{\pi}} z^{-1/4} \cos\left[\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right]. && (5.32) \end{aligned}$$

On the other hand, for the linearized potential,

$$\begin{aligned} \int_{\alpha_*}^{\alpha} [U(\alpha)]^{1/2} d\alpha \simeq [U'(\alpha_*)]^{1/2} \int_{\alpha_*}^{\alpha} (\alpha - \alpha_*)^{1/2} d\alpha &= \frac{2[U'(\alpha_*)]^{1/2}}{3} (\alpha - \alpha_*)^{3/2} && (5.33) \\ &= \frac{2}{3} z^{3/2}. && (5.34) \end{aligned}$$

[1]We have to use the real part, since the parameter $p$ is generally complex, and thus the potential $U_p(\alpha)$ is also complex. We shall see, however, that the imaginary part of $p$ is exponentially suppressed. Hence we shall disregard it everywhere, except for the calculation of the decay rate.

We thus see that the linearized approximation near the turning points, with Airy function solutions, matches onto the WKB solutions away from the turning points. We determine solutions in all regions by imposing the boundary conditions in regions I and IV, and match semiclassical solutions across the turning points $\alpha_0$, $\alpha_1$ and $\alpha_2$..

We first apply the boundary condition in region IV, to the right of $\alpha_2$. There, the solution consists only of a decaying mode, $f_p(\alpha \to \infty) \to 0$:

$$f_p^{IV}(\alpha) = \frac{A}{2(U_p(\alpha))^{1/4}} e^{-\int_{\alpha_2}^{\alpha} \sqrt{U_p(\alpha')} d\alpha'}, \tag{5.35}$$

where $A = \text{const}$. With the asymptotic form of the Airy functions, this fixes the coefficients across $\alpha_2$ in region $III$:

$$f_p^{III}(\alpha) = \frac{e^{-i\pi/4} A}{2[-U_p(\alpha)]^{1/4}} \left( e^{i\int_{\alpha}^{\alpha_2} \sqrt{-U_p(\alpha')} d\alpha'} + i e^{-i\int_{\alpha}^{\alpha_2} \sqrt{-U_p(\alpha')} d\alpha'} \right). \tag{5.36}$$

The second boundary condition – that the solution be outgoing in the region $\alpha < \alpha_0$ – means that the solution must take the form

$$f_p^{I}(\alpha) = \frac{e^{-i\pi/4}}{[-U_p(\alpha)]^{1/4}} B e^{-i\int_{\alpha}^{\alpha_0} \sqrt{-U_p(\alpha')} d\alpha'}, \tag{5.37}$$

where $B$ is a constant coefficient.[2] We can now use the same method as above to match the solutions across the turning point $\alpha_0$ and fix the coefficients of the wave function in region II:

$$f_p^{II}(\alpha) = \frac{-iB}{2[U_p(\alpha)]^{1/4}} e^{-\int_{\alpha_0}^{\alpha} \sqrt{U_p(\alpha')} d\alpha'} + \frac{B}{[U_p(\alpha)]^{1/4}} e^{\int_{\alpha_0}^{\alpha} \sqrt{U_p(\alpha')} d\alpha'}. \tag{5.38}$$

We now have expressions for solutions $f_p^I$ and $f_p^{II}$ in terms of coefficient $B$, and solutions $f_p^{III}$ and $f_p^{IV}$ in terms of coefficient $A$; we must now reconcile solutions everywhere in terms of a single coefficient. The general solution to the left of $\alpha_1$ is

$$f_p^{II}(\alpha) = \frac{A'}{2[U_p(\alpha)]^{1/4}} e^{-\int_{\alpha}^{\alpha_1} \sqrt{U_p(\alpha')} d\alpha'} + \frac{B'}{[U_p(\alpha)]^{1/4}} e^{\int_{\alpha}^{\alpha_1} \sqrt{U_p(\alpha')} d\alpha'}. \tag{5.39}$$

Matching this across $\alpha_1$ with the aid of the linearized approximation, we find the form of

[2]Note that two linearly independent solutions in the limit $\alpha \to -\infty$ are $f_p(\alpha) \propto \exp(\pm ip\alpha)$, so the outgoing mode can be unambiguously identified.

the solution to the right of $\alpha_1$,

$$f_p^{III}(\alpha) = \frac{e^{-i\pi/4}}{2[-U_p(\alpha)]^{1/4}}\left((A' + iB')e^{i\int_{\alpha_1}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'} + (iA' + B')e^{-i\int_{\alpha_1}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'}\right). \tag{5.40}$$

The coefficients $A'$ and $B'$ can now be determined by noting that the solution $f_p^{II}(\alpha)$ in Eq. (5.39) must match the solution in Eq. (5.38) determined with the boundary conditions. Defining

$$K = \int_{\alpha_0}^{\alpha_1}\sqrt{U_p(\alpha')}d\alpha', \tag{5.41}$$

we find the relations

$$A' = 2Be^{K} \tag{5.42}$$

$$B' = \frac{-iB}{2}e^{-K}. \tag{5.43}$$

Then the solution in region $III$ is

$$f_p^{III}(\alpha) = \frac{e^{-i\pi/4}B}{2[-U_p(\alpha)]^{1/4}}\left(\left(2e^K + \frac{e^{-K}}{2}\right)e^{i\int_{\alpha_1}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'} + i\left(2e^K - \frac{e^{-K}}{2}\right)e^{-i\int_{\alpha_1}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'}\right). \tag{5.44}$$

Similarly, we require that the solutions $f_p^{III}(\alpha)$ from Eq. (5.36) and Eq. (5.44) agree:

$$B\left(\left(2e^K + \tfrac{e^{-K}}{2}\right)e^{i\int_{\alpha_1}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'} + i\left(2e^K - \tfrac{e^{-K}}{2}\right)e^{-i\int_{\alpha_1}^{\alpha}\sqrt{-U_p(\alpha')}d\alpha'}\right) \tag{5.45}$$

$$= A\left(e^{i\int_{\alpha}^{\alpha_2}\sqrt{-U_p(\alpha')}d\alpha'} + ie^{-i\int_{\alpha}^{\alpha_2}\sqrt{-U_p(\alpha')}d\alpha'}\right). \tag{5.46}$$

Defining

$$J = \int_{\alpha_1}^{\alpha_2}\sqrt{-U_p(\alpha')}d\alpha', \tag{5.47}$$

the relations

$$A = iB\left(2e^K - \frac{e^{-K}}{2}\right)e^{-iJ} \tag{5.48}$$

$$A = -iB\left(2e^K + \frac{e^{-K}}{2}\right)e^{iJ} \tag{5.49}$$

must be simultaneously satisfied.

In order for a solution to exist, we must require

$$\frac{1-\frac{e^{-2K}}{4}}{1+\frac{e^{-2K}}{4}} = -e^{2iJ} \tag{5.50}$$

or

$$J = \left(n+\frac{1}{2}\right)\pi - \frac{i}{2}\ln\left(\frac{1-\frac{e^{-2K}}{4}}{1+\frac{e^{-2K}}{4}}\right), \tag{5.51}$$

where $n$ is an integer. For the semiclassical approximation to be justified, we have to require that

$$\mathrm{Re}J \gg 1, \quad \mathrm{Re}K \gg 1. \tag{5.52}$$

Then, expanding the logarithm up to the leading order in powers of $e^{-2K}$, we have the approximate relation

$$J \simeq \left(n+\frac{1}{2}\right)\pi + \frac{i}{4}e^{-2K}. \tag{5.53}$$

This condition will later be used to determine the momentum eigenvalue $p$ and the decay rate $\Gamma$.

## 5.2 Evaluation of $J$ and $K$

In the semiclassical regime, we shall assume that the contribution to $J$ from $p$ may be treated as a perturbation. We shall determine this contribution to the leading order in $p$ (or, more precisely, in $\beta p$). Representing $J$ as

$$\begin{aligned} J &= \int_{\alpha_1}^{\alpha_2} \sqrt{-U(\alpha)+p^2}d\alpha && (5.54) \\ &\simeq \int_{\alpha_-}^{\alpha_+} \sqrt{-U(\alpha)}d\alpha + \int_{\alpha_-}^{\alpha_+} \frac{p^2}{2\sqrt{-U(\alpha)}}d\alpha && (5.55) \\ &\quad + \left(\int_{\alpha_1}^{\alpha_-} \sqrt{-U(\alpha)}d\alpha + \int_{\alpha_+}^{\alpha_2} \sqrt{-U(\alpha)}d\alpha\right) \\ &\equiv J_0 + J_1 + J_2, && (5.56) \end{aligned}$$

we then evaluate $J_0$, $J_1$, and $J_2$ analytically:

$$
\begin{aligned}
J_0 &= \int_{\alpha_-}^{\alpha_+} \sqrt{-U(\alpha)}d\alpha = \frac{\pi}{8\beta}\gamma^3(\gamma^2-1)(5\gamma^2-1), && (5.57)\\
J_1 &= \frac{p^2}{2}\int_{\alpha_-}^{\alpha_+} \frac{1}{\sqrt{-U(\alpha)}}d\alpha = \frac{\pi\beta p^2}{2}, && (5.58)\\
J_2 &\simeq \int_{\alpha_1}^{\alpha_-} \sqrt{-U(\alpha)}d\alpha + \int_{\alpha_+}^{\alpha_2} \sqrt{-U(\alpha)}d\alpha && (5.59)\\
&\simeq \frac{2}{3}\left(\sqrt{-U'(\alpha_-)}\delta\alpha_1^{3/2} + \sqrt{U'(\alpha_+)}\delta\alpha_2^{3/2}\right) && (5.60)\\
&\simeq \frac{2}{3}\left(\frac{p^3}{|U'(\alpha_-)|} + \frac{p^3}{U'(\alpha_+)}\right) \sim \beta^2 p^3. && (5.61)
\end{aligned}
$$

Here, in the calculation of $J_2$ we have expanded $U(\alpha)$ near $\alpha_\pm$ and used Eqs. (5.16), (5.17) and (5.20). The contribution from the correction to the turning points, $J_2$, is small compared to $J_1$; hence we can write

$$J \simeq J_0 + \frac{\pi\beta p^2}{2}. \tag{5.62}$$

With $J_0$ from Eq. (5.57), we note that it follows from the first condition in (5.52) that $\gamma$ should not be very close to 1,

$$\gamma - 1 \gg \beta. \tag{5.63}$$

To evaluate $K$, we again expand perturbatively in p, $K = K_0 + \delta K$, where

$$K_0 = \int_{-\infty}^{\alpha_-} \sqrt{U(\alpha)}d\alpha = \frac{1}{24\beta}\left[15\gamma^6 - 13\gamma^4 + \frac{3}{2}\left(\gamma^3 - 6\gamma^5 + 5\gamma^7\right)\ln\left(\frac{\gamma-1}{\gamma+1}\right)\right]. \tag{5.64}$$

and $\delta K \sim \beta p^2$ includes all corrections due to $p$. For small values of $\gamma$, the term in brackets is $\sim 1/2$ (see Fig. 5.2). In the limit of large values of $\gamma$, the term in brackets simplifies to

$$\frac{48}{105}\left(1 + \frac{69}{16\gamma^2}\right), \tag{5.65}$$

which is also approximately $1/2$ for $\gamma \gg 1$. Since $\beta \ll 1$, this means that $K_0 \gg 1$ for any value of $\gamma$. For example, with $\gamma = 1.05$, and $\beta = .0001$, $K_0 \simeq 549$. The value of $K_0$ is largely unaffected by increasing the value of $\gamma$ (see Fig. 5.2), and decreasing $\beta$ has the effect of increasing $K_0$. Even though $\delta K \ll K_0$, we cannot generally neglect $\delta K$. Since $p \gg 1$ is required for our semiclassical analysis, we can have $\delta K > 1$ even if $\beta p \ll 1$. Neglecting $\delta K$ in Eq. (5.53) is justified only if $\beta p^2 \ll 1$. To simplify further analysis, we shall assume this condition to be satisfied.

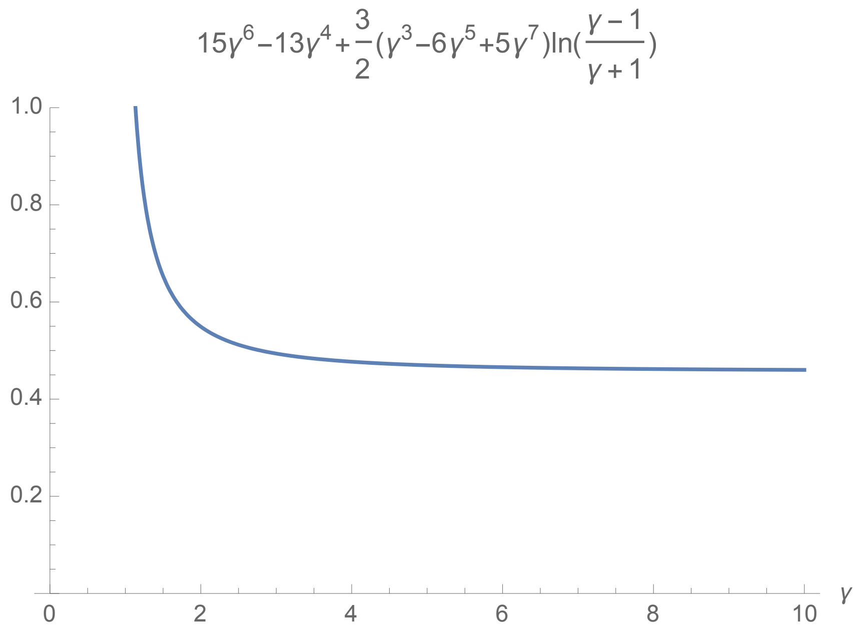



Figure 5.2: Plot of the behavior of the term in brackets, proportional to $K_0$, for values of $\gamma$ near 1.

Thus, the complete list of our assumptions is

$$\beta \ll 1, \tag{5.66}$$

corresponding to the fact that $|\Lambda|$ and $\sigma$ are small in Planck units;

$$1 \ll |p| \ll \beta^{-1/2}, \tag{5.67}$$

which is the regime where the effect of the scalar field on the turning points is small (Eq. (5.21)), and which keeps the turning point $a_0$ large in Planck units, $a_0 \gg G^{1/2}$;

$$\gamma - 1 \gg \beta, \tag{5.68}$$

corresponds to the requirement that $\gamma$ is not too close to 1 for the semiclassical analysis to be appropriate.

## 5.3 The decay rate

We shall now use Eqs. (5.53) and (5.62) to determine the momentum eigenvalue $p$. We first assign to $p$ real and imaginary parts,

$$p = p' + ip'', \tag{5.69}$$

with $p'$ and $p''$ real. We shall assume that $p'' \ll p'$; this will be justified below. (Note that we also neglected the effect of $p''$ on the classical turning points in earlier sections.)

Substituting $J$ from (5.62) in (5.53), using (5.69) and neglecting ${p''}^2$ compared to ${p'}^2$, we obtain two relations:

$$J_0 + \frac{\pi\beta p'^2}{2} = \left(n + \frac{1}{2}\right)\pi \tag{5.70}$$

$$\pi\beta p' p'' = \frac{1}{4}e^{-2K_0}. \tag{5.71}$$

With $J_0$ from Eq. (5.57), the first of these relations becomes

$$\gamma^3(\gamma^2 - 1)(5\gamma^2 - 1) = 4\beta[(2n+1) - \beta p'^2] \approx 4\beta(2n+1). \tag{5.72}$$

Disregarding the small correction introduced by the "clock", as we did in the last step, this is a quantization condition on the parameters of the model $\beta$ and $\gamma$. Note that if $\gamma$ is not very close to 1, the left hand side of (5.72) is $\mathcal{O}(1)$, and since $\beta \ll 1$, we must have $n \gg 1$. The spectrum of the parameters is then nearly continuous, as one would expect in the semiclassical regime.

The value of $p'$ is largely arbitrary, as long as it satisfies $1 \ll p' \ll \beta^{-1/2}$. Once $p'$ is selected, the imaginary part $p''$ is determined by Eq. (5.71). And since $p' \gg 1$ and $\beta \ll 1$ it is easy to see from (5.71) that $p'' \ll p'$.

With a complex momentum (5.69), the WDW wave function (5.9) has the form

$$\Psi(\alpha, \phi) = e^{ip'\phi - p''\phi} f_p(\alpha). \tag{5.73}$$

The corresponding probability distribution can be found in terms of the Klein-Gordon current [12, 45]. Up to a normalization constant, it is given by

$$\mathcal{J} = \frac{i}{2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*), \tag{5.74}$$

In our minisuperspace model, the current has two components,

$$\mathcal{J}^\alpha = \frac{i}{2}(\Psi^*\partial_\alpha\Psi - \Psi\partial_\alpha\Psi^*), \tag{5.75}$$
$$\mathcal{J}^\phi = -\frac{i}{2}(\Psi^*\partial_\phi\Psi - \Psi\partial_\phi\Psi^*), \tag{5.76}$$

and satisfies the continuity equation

$$\partial_\alpha\mathcal{J}^\alpha + \partial_\phi\mathcal{J}^\phi = 0. \tag{5.77}$$

With a proper normalization, the component

$$\mathcal{J}^\phi = p'|f_p(\alpha)|^2 e^{-2p''\phi} \tag{5.78}$$

can be interpreted as the probability density for $\alpha$ at a given "time" $\phi$,

$$d\mathcal{P} \propto \mathcal{J}^\phi(\alpha,\phi)d\alpha. \tag{5.79}$$

To express the decay rate in terms of the proper time $t$, we find the amount $\Delta\phi$ by which the field $\phi$ increases during one period of oscillation, $\tau \approx 2\pi/\omega$. Using the classical equation of motion[3] for $\phi$,

$$\dot{\phi} = \frac{2G}{3\pi}\frac{p}{a^3} \tag{5.80}$$

and ignoring, as before, the contribution to the turning points from $p$, we have

$$\Delta\phi = \frac{2Gp}{3\pi}\int_\tau dt\frac{1}{a(t)^3} \simeq \frac{4Gp'}{3\pi}\int_{a_-}^{a_+} da\frac{1}{\dot{a}a(t)^3}. \tag{5.81}$$

Expressing $\dot{a}$ from Eq. (4.9) we evaluate the integral:

$$\Delta\phi = 2p'\int_{\alpha_-}^{\alpha_+} d\alpha\frac{1}{\sqrt{-U(\alpha)}} = 2\pi\beta p'. \tag{5.82}$$

[3]Note that this is different from Eq. (5.3) because of the rescalings $\phi = (4\pi G/3)^{1/2}\varphi$ and $p_\phi = (3/4\pi G)^{1/2}p_\varphi$.

We now relate the field $\phi$ to the number of oscillations,

$$N = \frac{\phi}{\Delta\phi} = \frac{\phi}{2\pi\beta p'}, \tag{5.83}$$

so the probability in Eq. (5.84) becomes

$$\mathcal{J}^{\phi} \propto \; e^{-4\pi\beta p' p'' N} |f_{p'}(\alpha)|^2. \tag{5.84}$$

Finally, using Eq. (5.71), we obtain

$$\mathcal{J}^{\phi} \propto \exp\left(e^{-2K_0} N\right). \tag{5.85}$$

We see that the probability for the universe to remain in the oscillating state decreases by a factor of $e$ in $N = e^{2K_0}$ oscillations. The characteristic lifetime of a simple harmonic universe is thus

$$T = \frac{2\pi}{\omega} e^{2K_0}, \tag{5.86}$$

with $K_0$ given by Eq. (5.64).

## 5.4 The tunneling probability

In the semiclassical picture, we can think of the SHU as undergoing classical oscillations between the turning points $a_-$ and $a_+$, with some probability of tunneling through the barrier every time it hits the point $a_-$. We shall now calculate this tunneling probability and relate it to the tunneling rate that we found in the preceding section.

We shall focus on the case of small $\Lambda$, when $\gamma \gg 1$ and the turning points are approximately given by

$$a_- \approx \frac{1}{2\gamma\omega} = \frac{3}{8\pi G\sigma}, \tag{5.87}$$

$$a_+ \approx \frac{2\gamma}{\omega} = \frac{\sigma}{|\Lambda|}. \tag{5.88}$$

The turning points are then widely separated, $a_+/a_- \approx 4\gamma^2 \gg 1$, and the form of the barrier between $a_-$ and $a = 0$ is essentially independent of $\Lambda$. In this regime, we expect the tunneling probability to be nearly the same as for a $\Lambda = 0$ universe undergoing a single bounce at $a = a_-$. Then the probability for SHU to remain in the oscillating phase after $N$ oscillations is

$$\mathcal{P}_N \approx (1-Q)^N \approx e^{-QN}, \tag{5.89}$$

where $Q \ll 1$ is the tunneling probability for a $\Lambda = 0$ universe.

In order to calculate $Q$, we find the semiclassical WDW wave function as we did in Sec. III, except now we only have regions I, II and III to consider. We do not need a time variable in this case, but the scalar field still plays a useful role of introducing a classically allowed region near the singularity. This allows us to impose an outgoing boundary condition at $\alpha \to -\infty$, but we assume as before that the presence of the scalar field has little effect on the dynamics.

By the same argument as in Sec. III, the wave function at large values of $\alpha$ has the form of (5.9) with $f_p(\alpha)$ given by Eq. (5.44). No boundary condition is imposed at $\alpha \to +\infty$, so we do not have any quantization condition in this case, and the momentum eigenvalue $p$ can be set to be real. The wave function (5.44) describes an ensemble of contracting universes, which bounce at $\alpha = \alpha_-$ and re-expand. The expanding component has a smaller coefficient, accounting for the fact that some universes have been lost to tunneling decay. The probability to avoid decay is given by the ratio of the probability fluxes for the two components in Eq. (5.44),

$$1 - Q = \frac{\mathcal{J}^\alpha(\rightarrow)}{\mathcal{J}^\alpha(\leftarrow)} = \left(\frac{4 - e^{-2K}}{4 + e^{-2K}}\right)^2 \approx 1 - e^{-2K}, \tag{5.90}$$

where left and right arrows correspond to contracting and expanding branches, respectively. (Note that at large $\alpha$ both terms in (5.44) are very rapidly oscillating, so any interference effects between the two terms become completely negligible.) Thus, we have

$$Q \approx e^{-2K}. \tag{5.91}$$

The tunneling exponent $K$ can be found from Eq. (5.64). In the limit of large $\gamma$ it gives

$$K_0 \approx \frac{2}{105\beta}\left(1 - \frac{69}{16\gamma^2}\right), \tag{5.92}$$

where the second term can be dropped in the limit of $\gamma \to \infty$.

Substituting $Q$ from (5.91) in Eq. (5.89), we recover Eq. (5.85), as expected.

# Chapter 6

# Conclusions

The primary goal of this thesis was to determine the plausibility of the emergent universe scenario. We focused on two oscillating models which have been shown to be perturbatively stable, and investigated the stability with respect to quantum tunneling, using the canonical approach to quantum cosmology.

In Chapter 1, we presented the basic Simple Harmonic Universe scenario, and showed that generally it collapses to nothing, having non-zero solutions for the wave function at zero size, $\psi(a = 0) \neq 0$. The main reason for this was that, at large values of the scale factor $a$, the potential barrier is infinite, meaning that we have to impose as the single boundary condition $\psi(a \to \infty) \to 0$; it is not possible to simultaneously require $\psi(0) = 0$. We considered the general effect of other sources of energy density on the potential at small $a$, with the conclusion that there is not a source that would produce a stabilizing effect on the SHU. It was suggested by Graham *et al* that one exception could be a negative Casimir energy, which has the effect of producing a positive but finite potential barrier at small values of $a$. We examined the impact on the tunneling instability of negative Casimir energy and other quantum corrections in Chapter 4, and found that they generally do not stabilize the emergent universe agains quantum tunneling. Additionally, we considered in Chapter 3 a distinct oscillating model in loop quantum cosmology, and found that that model also has a nonzero tunneling probability.

In Chapter 5, we implemented DeWitt's prescription to describe time evolution in quantum cosmology in terms of semiclassical superspace variables, which can be used to define a "clock". We applied this approach to the calculation of the tunneling decay rate of a simple harmonic universe. The role of a clock in our model was played by a homogeneous, massless, minimally coupled scalar field $\phi$. The classical evolution of $\phi$ is monotonic, and thus it is a good time variable.

We found the WKB wave functions $\Psi(a, \phi)$, which are eigenstates of the momentum $p_\phi$ conjugate to $\phi$, and matched these wave functions across the turning points, where the WKB approximation breaks down. We imposed a boundary condition at $a \to \infty$ requiring that $\Psi$ vanishes in that limit and an outgoing boundary condition at $a = 0$. The latter condition means that collapse to $a = 0$ is irreversible, so collapsing universes do not bounce back from the singularity. These two boundary conditions determine the wave function completely and in addition provide two constraints on the parameters of the system and on the momentum eigenvalue $p_\phi$. We showed how these constraints can be used to calculate the decay rate.

We also considered the case of a vanishing cosmological constant $\Lambda$, when the universe experiences a single bounce off the barrier and found the tunneling probability through the barrier using the conserved Klein-Gordon-type current. The resulting probability agrees with our calculation of the decay rate in the limit of small $\Lambda$.

It would be interesting to extend our analysis to a static universe, which has $\gamma = 1$ and $a_+ = a_- = \omega^{-1}$. In this case, the classically allowed region III reduces to a single point, and the method of a linear approximation for the potential $U(a)$ around the turning points that we used in Sec. III cannot be applied. However, one can instead use a quadratic approximation $U(a) \propto (a - a_*)^2$ around the point $a_* = \omega^{-1}$. The wave function in that range can be expressed in terms of harmonic oscillator functions, which will then have to be matched to the WKB wave functions away from $a_*$. Alternatively, it should be possible to find the solution numerically.

In summary, we have shown that generally it is not possible to construct an eternal universe which is completely stable. While there are at least two models which are perturbatively stable, at least for much of the parameter space, they decay quantum mechanically. This indicates that an "emergent universe" scenario cannot be truly eternal in the past, and instead must have had some sort of beginning. Two other scenarios – the cyclic model and eternal inflation – address the question of the state of the universe before inflation, but both of these have expanding spacetimes, $H_{avg>0}$, and therefore cannot be past-eternal. Together with the fact that the emergent universe is not eternal in the past, either, we conclude that there are currently no viable models for a universe with no beginning.

# Appendix A

# FRW minisuperspace

As it is defined in the superspace, the full wave function seems to be of little practical value. However, we can ask questions about the wave function in a restricted number of degrees of freedom: a "minisuperspace." Since we observe the universe in a classical state – corresponding to a peak of the wave-function – we consider only a small, relevant subset of the possible spatial geometries, and determine the wave function solutions in that minisuperspace. Most importantly, the inflationary universe is an approximately deSitter spacetime, corresponding to a spherically symmetric, homogeneous spacetime having classical dynamics which are sufficiently described by the FRW scale factor $a(t)$. Then we can reasonably restrict the the dimension of superspace to a single minisuperspace variable $a(t)$, suppressing the rest, and determine solutions $\psi(a)$.

In this thesis, we investigate static and oscillating classical universes which homogeneous and isotropic – their classical dynamics are well described by the FRW solutions. So long as the solutions are perturbatively stable, we can restrict our attention to only wave function solutions along the FRW scale factor $a$. In this Appendix, we present the procedure to determine the WDW equation in the FRW minisuperspace, with some effective potential determined by the matter fields.

We take the FRW metric ansatz

$$ds^2 = -N^2(t)dt^2 + a(t)^2 d\mathbf{x^2}. \tag{A.1}$$

Here $a(t)$ is the scale factor, and the spatial part of the metric is

$$d\mathbf{x}^2 = \frac{1}{1-kr^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{A.2}$$

with $k > 0$ corresponding to positive spatial curvature, $k < 0$ to negative curvature, and $k = 0$ no curvature. The lapse function $N(t)$ is an arbitrary function describing the spacing of time-like slices of the space-time; dynamics are completely independent of the choice of $N(t)$.

In a closed FRW space ($k = +1$), the scalar curvature is

$$\mathcal{R} = \frac{6}{a^2N^3}\left(N^3 - a\dot{a}\dot{N} + N\left(\dot{a}^2 + a\ddot{a}\right)\right). \tag{A.3}$$

The Einstein-Hilbert action

$$\mathcal{S} = \int d^4x\sqrt{|g|}\left(\frac{1}{16\pi G}\mathcal{R} - \mathcal{L}_{matter}\right) \tag{A.4}$$

becomes

$$\mathcal{S} = 2\pi^2\int dt\left(\frac{3}{8\pi G}\left(Na - \frac{a\dot{a}^2}{N}\right) - Na^3\mathcal{L}_{matter}\right). \tag{A.5}$$

Here, we have already performed an integration by parts to get the second term, and have integrated over the volume of the three-sphere,

$$\int dx^3\sqrt{h} = 2\pi^2. \tag{A.6}$$

The conjugate momentum of the scale factor $a$ is defined in the usual way:

$$p_a = \frac{\partial\mathcal{L}}{\partial\dot{a}} = -\frac{3\pi}{2G}\frac{a\dot{a}}{N}. \tag{A.7}$$

and similarly

$$p_N = \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0. \tag{A.8}$$

In this minisuperspace, then,

$$\mathcal{L} = N\mathcal{H} - p_a\dot{a}. \tag{A.9}$$

Variation with respect to $N$ results in the Hamiltonian constraint

$$\mathcal{H} = 0. \tag{A.10}$$

With the choice of $N = 1$, this corresponds to the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho. \tag{A.11}$$

# Appendix B

# Stability criterion

In this Appendix we shall derive a stability criterion for solutions to the equation

$$\ddot{\phi}_k + 3\frac{\dot{a}}{a}\dot{\phi}_k + \frac{k^2}{a^2}\phi_k = 0, \tag{B.1}$$

where the scale factor $a(t)$ is a periodic function with period $T$.

According to Floquet's theorem, Eq. (B.1) admits solutions of the form

$$\phi(t) = e^{i\alpha t}p(t), \tag{B.2}$$

where $p(t)$ is a periodic function with period $T$, and $\alpha$ is a constant defined by the boundary conditions. It is apparent that when $\alpha$ is real, the solutions are oscillatory and therefore stable, whereas when $\alpha$ is complex, the solutions grow or decay and are unstable.

Note that if $\phi(t)$ is a solution of (B.1), then $\phi(t+T)$ is also a solution. With the ansatz (B.2), we have

$$\phi(t+T) = \zeta\phi(t), \tag{B.3}$$

where $\zeta = e^{i\alpha T}$ is a constant.

To derive the stability criterion, we first define two solutions, $\phi_1(t)$ and $\phi_2(t)$, by the initial conditions

$$\begin{aligned} \phi_1(0) &= 1 \qquad & \phi_2(0) &= 0 \\ \dot{\phi}_1(0) &= 0 \qquad & \dot{\phi}_2(0) &= 1. \end{aligned} \tag{B.4}$$

Any solution can be written as a linear combination of these two solutions:

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t). \tag{B.5}$$

In particular, the solutions $\phi_1(t+T)$ and $\phi_2(t+T)$ are

$$\begin{aligned} \phi_1(t+T) &= \phi_1(T)\phi_1(t) + \phi_1'(T)\phi_2(t) \\ \phi_2(t+T) &= \phi_2(T)\phi_1(t) + \phi_2'(T)\phi_2(t). \end{aligned} \tag{B.6}$$

For any solution of the form (B.3), we obtain, using Eqs. (B.5), (B.6), the following set of linear equations for $c_1$ and $c_2$:

$$(\phi_1(T) - \zeta)c_1 + \phi_2(T)c_2 = 0 \tag{B.7}$$

$$\phi_1'(T)c_1 + (\phi_2'(T) - \zeta)c_2 = 0 \tag{B.8}$$

Nonzero solutions to this set of equations exist when

$$\begin{vmatrix} (\phi_1(T) - \zeta) & \phi_2(T) \\ \phi_1'(T) & (\phi_2'(T) - \zeta) \end{vmatrix} = \zeta^2 - (\phi_1(T) + \phi_2'(T))\zeta + W(\phi_1(T), \phi_2(T)) = 0 \tag{B.9}$$

where $W(t) = W(\phi_1, (t), \phi_2(t))$ is the Wronskian. From Eq. (B.1), $W(t)a^3(t) =$ const, so that $W(T) = W(0)$. With the boundary conditions in Eq. (B.4), we have $W(0) = 1$, so that Eq. (B.9) becomes

$$\zeta^2 - (\phi_1(T) + \phi_2'(T))\zeta + 1 = 0. \tag{B.10}$$

The product of the two roots of this equation is equal to 1; hence the roots can be represented as $\zeta_{1,2} = \exp(\pm i\alpha T)$. The sum of the roots is

$$\zeta_1 + \zeta_2 = 2\cos(\alpha T) = \phi_1(T) + \phi_2'(T) \equiv b. \tag{B.11}$$

If $b > 2$, then the roots are real, while if $b < 2$, the roots are complex and conjugate to one another.

As we indicated above, the solutions are stable for real $\alpha$ and unstable for complex $\alpha$.

This corresponds to the condition

$$|\phi_1(T) + \phi_2'(T)| < 2 \tag{B.12}$$

for stable solutions, and

$$|\phi_1(T) + \phi_2'(T)| > 2 \tag{B.13}$$

for unstable solutions.[1] In order to check the mode stability in our model, we integrate Eq. (B.1) with $a(t)$ from Eq. (3.25), using the initial conditions in (B.4). Then we check whether the solutions at $t = T$ satisfy (B.12) or (B.13).

The stability diagram in Fig. 3.3 was produced by sampling the parameter space $0 < \lambda, \kappa < 1$ at logarithmic intervals, so there are many more points at small values of the parameters than at $\lambda, \kappa \sim 1$. This ensures that we have high resolution in regions where it is required.

[1]The case where $|\phi_1(T) + \phi_2'(T)| = 2$, corresponding to the boundary between the stable and unstable regions, is further analyzed in [32].

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