NON-POSITIVE CURVATURE IN GROUPS

A dissertation

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Abstract

In this document we explore some of the relationships between different classes of groups important to geometric group theory and different properties they enjoy. These classes often coincide and the properties can have non-trivial intersection. Looking at these intersections, i.e. at groups having multiple properties of interest, we notice even stronger properties that come as a result. In particular we characterize the intersection of acylindrical hyperbolicity with CAT(0) and apply this classification to right angled Coxeter groups.

Some other classes we are particularly concerned with are those of the braid groups and automorphism groups. Specifically, in the second class, we will look at automorphisms of free groups, as well as automorphisms of the Universal Right Angled Coxeter groups, better known as the free product of a finite number of copies of $\mathbb{Z}_2$. Both these types of groups have had questions raised about them that we will address, in particular as to whether they are CAT(0).

In exploring the relationship between these classes, and the relatively new notion of Acylindrical Hyperbolicity, we get some interesting results pertaining to these questions. This comes from discovering that the braid groups (modulo a cyclic subgroup) and the automorphisms of our universal groups (modulo their inner elements) satisfy this notion of acylindrical hyperbolicity, which can be seen as a very generalized version of negative curvature.

In doing this, we will discover that, should either of these classes of groups prove to be CAT(0), then they must act on a specific kind of CAT(0) space, particularly a space which is 'rank one'. Because we know that sufficiently low index braid groups are indeed CAT(0), we explore such a space it acts on geometrically, and distinguish exactly such a 'rank one' element which we now know must exist.

We continue on to explore more about these acylindrical actions generally and their relationship to relative hyperbolicity. Both hyperbolicity and relative hyperbolicity exhibit a nice characterization of limit sets of appropriate actions in the form of boundary. We demonstrate why an analogous structure doesn't appear for acylindrically hyperbolic groups. In doing so, we explicitly construct two actions, both universal in the appropriate sense, that have markedly different end behavior.

This leads us towards a desire to classify the rigidity present in these classes, and how it weakens as we relax the hypotheses on the action of interest. In doing so, we prove a folklore result about relatively hyperbolic groups and neatly summarize the rigidity of these group actions.

To my family and friends, who kept me going through this endeavor. Most especially to my mother for her unconditional love, my father for his unconditional support, and my partner Caroline, for her love, devotion, and camaraderie

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My advisor Genevieve has for years been an astute advocate of my work, helping to push me in the right direction and avoid pitfalls. She has been accessible, patient, knowledgable, kind, and encouraging, which has given me the support I needed to persevere to reach this point. She helped me learn so much more than mathematics, but about cooperation, advocating for oneself, determination, and the importance of believe in in what one does. I can think of no better mentor than the one Genevieve has been to me.

Many other faculty at Tufts have been instrumental on my journey as well. Kim Ruane, the graduate director, was the staunchest supporter of graduate students in our department possible. Many, many times I watched her go to bat, not just for me, but for all of us. She always managed to win, too. It baffled me how Kim was able to do this, while also being a prolific researcher, amazing teacher, and caring mother of two. It wasn't until 3 years into my graduate study that I found out she doesn't sleep.

Fellow graduate students were also a font of support. Seth Rothschild and I were first year students together, who took all of our qualifying exams in the same month. This wasn't the original plan, but we talked each other into it when neither of us were willing to admit we couldn't do it. From here we followed step for step together, pushing each other to be better. Having somebody around who understood was invaluable to me and I wouldn't have traded his friendship for anything.

Charlie Cunningham was larger than life. Every new graduate student who hears a story about him will swear up and down that he's a myth. But he was real, and contributed his heart and soul to the Tufts math department during his years here.

Almost all of the social cohesion of our graduate community can be traced back to him. As a prospective graduate student visiting Tufts, the Organization of Graduate Students in Mathematics, which he co-founded, was one of the main selling points that made me say yes. The department would have been a much less entertaining place without his influence, and I think it benefits every PhD candidate that filters through Bromfield Pearson, and will for years to come.

I could go on and on about the people at Tufts who were instrumental in turning me into the mathematician, and person, I am today. Emily Stark, my academic older sister, was always ready to help discuss problems, as was Chris O'Donnell. Andy Eisenberg has forgotten more about LaTeXthan I'll ever know, and helped me learn it to an incredible degree. Jeff Carlson spent long nights in conversation with me about math, music, politics, and many other topics both interesting and banal. Ayla Sanchèz shared with me a passion for video games and metal music, as well as trivial groups. Michael 'Ben-Zvi' Ben-Zvi organized more events that I attended than I could count on dozens of hands. Peter Ohm, Jeremy Marcq, and Rob Kropholler made game night an unmissable spectacle every week (or multiple nights a week). My time preparing this work was one of the most exciting periods of my life, and I'd do it all again in a heartbeat.

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Introduction

Geometric group theory is all about doing *coarse* group theory. To a group, we assign a space called its Cayley graph, which is a metric object, and then consider these objects up to a coarse equivalence. In doing so, we ignore a lot of the fine structure of these algebraic objects. In particular, all finite groups are equivalent up to quasi-isometry. However, we are able to preserve just enough of the structure to be able to talk about the things that we care about.

We are mostly interested in notions of nonpositive curvature. We discuss the specific properties we care about in the background material, located in Chapter 2. One main result of this document is the classification of how two of these concepts intersect. We then apply this classification to groups of interest and explore some of the implications.

One of the key properties we care about, called *acylindrical hyperbolicity*, arose out of a study of hyperbolicity and relative hyperbolicity. Specifically, it is a loosening of the algebraic assumptions on the group, in order to discuss a larger class, while retaining some of the major machinery for working on large scale geometric structure. This idea came to prominence after many precursor ideas, such as weak proper discontinuity, and was explored in depth in a survey article by [26]. In it, Osin is able to go back and forth between group structure and the properties enjoyed by individual group elements. We use much of this structure to our advantage, linking group element structure in acylindrical hyperbolicity to that of CAT(0) groups. This work is done in Chapter 3, as well as an immediate application to right angled Coxeter groups.

We go on to discuss further implications with regards to groups of interest, namely the braid groups. The (possible) negative curvature properties of these groups have been a question interest in GGT for a long time, with a standing conjecture about whether they enjoy the CAT(0) property. We assert a restriction on the structure of the space the associated action must have, if this is the case. We also take a look at the case of the 4-strand braid group, which is known to be CAT(0) and explicitly demonstrate the structure which is proven to exist. This is contained in Chapter 4.

Chapter 5 is devoted to proving the acylindrical hyperbolicity of $\text{Out}(W_n)$, that is, the outer automorphism group of the universal right angled Coxeter group. We do this by relating it to $\text{Out}(F_n)$, a group which is discussed in the previous chapter as well. The outer automorphisms of the free group admit an action on a space quasi-isometric to a tree, which we adjust the structure on to allow an action by the group of interest. We continue to prove the hypotheses of the action that we desire.

Following this material, we shift gears slightly to discuss ideas of boundaries and limit sets. Hyperbolic and relatively hyperbolic groups admit a well defined boundary, however the same cannot be said for acylindrically hyperbolic groups. We explore the extent to which this fails, including a demonstration of non-homeomorphic limit sets for (universal) acylindrical actions on the same hyperbolic space. In an effort to codify the rigidity lost by passing to looser and looser nonpositive curvature conditions for groups, we prove some facts about quasi-isometries and relatively hyperbolic spaces. Chapter 6 contains these ideas and ends with a chart summarizing these results and known facts.

The end of the dissertation is shorter and ties up some loose ends. Chapter 7 includes a discussion of future work the author would like to pursue, including some questions that arise naturally from the material. Chapter 8 consists of appendices conducting some algebraic exercises to support the claims made in previous chapters, in order to not distract from the larger points with combinatorial group theory.

This document is intended to be legible to any mathematician with a background of some basic algebra and topology.

Chapter 2

Background

2.1 Hyperbolicity

2.1.1 Hyperbolicity and its Generalizations

To begin, we will be using a shorthand convention that is common in geometric group theory, which we record here

Definition 2.1.1. *An action $G \circlearrowright S$ of a group on a metric space is called* geometric *if it is cocompact, proper, and by isometries.*

The motivation for this convention is the fact that the Cayley graph for any group is quasi-isometric to a space it acts on geometrically. Hence the action 'records' the large scale geometry of the group.

The idea of a hyperbolic space is quite an old one. While it has many different formulations, one of those which is most common is what's called $\delta-$hyperbolicity.

Definition 2.1.2. *A geodesic metric space S is said to be $\delta-$hyperbolic for a $\delta \geq 0$ if, for any geodesic triangle Δ with sides $\Delta_A, \Delta_B, \Delta_C$, it is true that*

$$\Delta_A \subset N_\delta(\Delta_B \cup \Delta_C)$$

$$\Delta_B \subset N_\delta(\Delta_A \cup \Delta_C)$$

$$\Delta_C \subset N_\delta(\Delta_A \cup \Delta_B)$$

A space is simply called hyperbolic *if it is $\delta-$hyperbolic for some δ.*

Fact 2.1.3. *The metric space $\mathbb{H}^2$ is $\delta-$hyperbolic with $\delta = \ln(1+\sqrt{2})$.*

Fact 2.1.4. *A tree with the standard graph metric is a $0-$hyperbolic space*

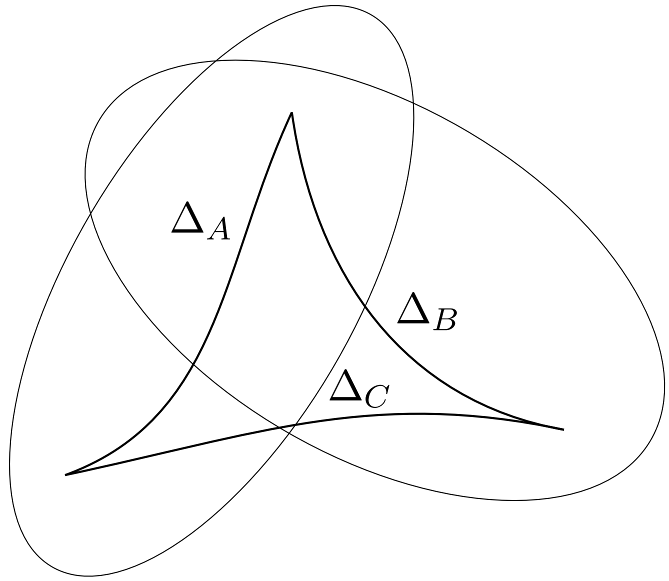


Figure 2.1: A $\delta-$thin triangle

Definition 2.1.5. *The* boundary *of a hyperbolic space S based at s_0, denoted $(\partial S, s_0)$ is defined to be equivalence classes of geodesic rays starting at s_0. Two rays are considered equivalent if their images in S have bounded Hausdorff distance from each other.*

$$\partial S := [\gamma]/\sim$$

$\gamma : [0,\infty), \gamma(0) = s_0, \gamma \sim \gamma' \iff \exists K \in \mathbb{R}, N_K(\gamma) \supset \gamma'.$

Fact 2.1.6. *The boundary, up to homeomorphism, is independent of basepoint. That is, for any $(\partial S, s_i)$, there are canonical maps which are bijective and continuous taking any to any other.*

Of course, we haven't specified what continuous means here, as we haven't given these boundaries topologies yet, however the homeomorphism respects the natural topology we would want it to have. As a result of this, we will normally supress the basepoint in notation.

It is worth noting here the flavors in which isometries of hyperbolic space come in.

Theorem 2.1.7. *[10] Let $g \in Isom(S)$ where S is $\delta-$hyperbolic. Then g is either:*

- Elliptic*: Orbits of points under g are bounded. In particular, this happens if g is finite order, and/or if g fixes a point in S.*
- Parabolic*: Powers of g fix exactly one point in ∂S.*
- Loxodromic*: Powers of g fix two points in ∂S.*

Oftentimes we will consider loxodromic elements as those which have axes that are quasi-isometric to copies of $\mathbb{R}$ in S. This can equivalently be taken as the definition of loxodromic elements.

Definition 2.1.8. *We call a finitely generated group* hyperbolic *if for some (and actually, any) finite generating set, the corresponding Cayley Graph is hyperbolic as a metric space*

Although later on in this document we will relax our requirements on generating sets, its very important we specify finite here. The reason for this is because if, for a discrete group, we chose a generating set that included every element of the group, the corresponding Cayley Graph would be an infinite complete graph and as such, have diameter 1, implying hyperbolicity. Therefore we would get that any group would be hyperbolic.

Fact 2.1.9. *Groups which are $\delta-$hyperbolic include:*

- *Virtually cyclic (including finite) groups*
- *Free groups*
- *Surface groups excluding those with low complexity*
- *Discrete, finitely generated subgroups of isometries of hyperbolic spaces*

We can also define a weaker form of hyperbolicity for groups that includes these and many other groups. In fact there is an analogous definition for spaces as well, but for our purposes this is all we care about.

We begin this exposition by defining a modification of the Cayley Graph

Definition 2.1.10. *Let G be a group with subgroup H. Define a graph $\hat{\Gamma}(G, H)$, labelled just $\hat{\Gamma}$ if the subgroup is clear from context, as follows. Let Γ be a Cayley graph for some chosen generating set. Then define $\hat{\Gamma}^{(0)} = \Gamma^{(0)} \sqcup v_{gH}$ where g ranges over a chosen set of coset representatives for H. Now let $\hat{\Gamma}$ inherit all the edges from Γ, and add in edges from each v_{gH} to each vertex representing an element of gH.*

Definition 2.1.11. *A group G is called* relatively hyperbolic *relative to a subgroup H if $\hat{\Gamma}$ is hyperbolic and it is* fine *as a graph, i.e. for every integer L, any edge belongs to finitely many cycles of simplicial length L.*

Without qualifier, relatively hyperbolic will refer to the stronger condition above. A group can also be relatively hyperbolic to a collection of subgroups, in the sense that the associated $\hat{\Gamma}(G, H_\lambda)$ coned-off Cayley graph has added vertices and edges for each coset of *each subgroup.*

2.1.2 Negatively Curved Directions

In this subsection we assume that S is an unbounded geodesic metric space, not necessarily hyperbolic. Let γ be a bi-infinite geodesic in S. We give some definitions of what it means for γ to represent a negatively curved direction, then explore some relationships between these concepts.

Definition 2.1.12. *The geodesic γ is called* Morse *if for any values $k, l > 0$, there exists a value $R > 0$ such that any (k, l)-quasigeodesic segment λ with endpoints on γ is contained inside $N_R(\gamma)$.*

Definition 2.1.13. *Let B be a metric ball which is disjoint from γ and π_γ be the closest point projection function onto γ. If for any such B, the diameter of $\pi_\gamma(B)$ is bounded above by a fixed value D, then γ is called* $(D-)$Contracting.

For a geometric action, we will often abuse notation, and call a group element Morse/contracting/rank one exactly when it's axes in the space are Morse/contracting/rank one. This is well defined because these notions are both invariant under quasi-isometry [29].

2.1.3 Acylindrical Hyperbolicity

Here we will discuss the next extension of hyperbolicity, called acylindrical hyperbolicity. After introducing this property, we'll note how it is in fact a generalization of relative hyperbolicity as well.

Definition 2.1.14. *An metric space action $G \curvearrowright S$ is called* acylindrical *if for every $\epsilon > 0$ there exist $R(\epsilon), N(\epsilon) > 0$ such that for any two points $x, yinS$ such that $d(x, y) \geq R$, the set*

$$\{g \in G \mid d(x, g.x) \leq \epsilon, d(y, g.y) \leq \epsilon\}$$

has cardinality less than N.

We quickly note here that this property does not imply properness. It's easy to see that this still allows individual points to have infinite stabilizers.

Definition 2.1.15. *A group G is called* acylindrically hyperbolic *if it admits an acylindrical action on a hyperbolic space which is not elementary, that is, has a limit set inside the boundary of the space of cardinality strictly greater than 2.*

It's worth pointing out that this means we do not wish to consider groups which are finite or virtually cyclic as being acylindrically hyperbolic, despite the fact that they are (elementary) hyperbolic. These are the only hyperbolic groups we exclude, as any non-elementary hyperbolic group will satisfy this requirement by the natural geometric action on a Cayley Graph.

Fact 2.1.16. *For any acylindrical group action on a hyperbolic space, no elements act as parabolics. This means any individual action for a group element is either loxodromic or elliptic.*

Definition 2.1.17. *Let G be an acylindrically hyperbolic group. An element $g \in G$ is called a* generalized loxodromic *if there's an acylindrical action $G \circlearrowright S$ for S hyperbolic such that g acts as a loxodromic.* Note: *this action is not required a priori to be non-elementary, however it is easy to see that this will necessarily be the case.*

The status of being a generalized loxodromic is a *group theoretic* property, and while one qualifying action might have a particular element acting loxodromically, in another it may act elliptically. In fact, the existence of a generalized loxodromic can be taken to be an alternate definition of acylindrical hyperbolicity. We list a few more here after a necessary definition.

Definition 2.1.18. *A subgroup $H < G$ is called* hyperbolically embedded *if there exists a (possibly infinite) subset $X \subset G$ such that the (possibly nonproper) Cayley graph $\Gamma(G, X \sqcup H)$ is hyperbolic, and the metric space $(H, \hat{d})$ is proper, where $\hat{d}$ is the standard metric which disallows travel along edges in the complete subgraph* $(H, H) \subset \Gamma$ (paths taking edges in cosets of H are allowed).

The following equivalence and subsequent list comes to us from Osin [26]

Theorem 2.1.19. *[26] For a group G, the following are equivalent*

- *G is acylindrically hyperbolic.*
- *G contains a generalized loxodromic and is not virtually cyclic.*
- *There exists a (possibly infinite) generating set X such that $\Gamma(G, X)$ is hyperbolic, the natural action is acylindrical, and $|\partial\Gamma(G, X)| > 2$*
- *There exists an infinite, proper, hyperbolically embedded subgroup.*

Fact 2.1.20. *A list of some acylindrically hyperbolic groups:*

- *Mapping Class Groups (of sufficient complexity)*
- *Out(F_n)*
- *Indecomposable RAAGs*
- *3+ generator groups with 1 relator*
- *"most" 3-manifold groups*
- *Artin-Tits groups of spherical type (including braid groups modulo center) [11]*

2.2 CAT(0) Spaces

To set up a main theorem, we will begin by defining a new type of space which is instead of being negatively curved, is simply non-positively curved. Most information in this section, including definitions, will come from [10].

Definition 2.2.1. *Let X be a geodesic metric space and Δ a geodesic triangle in X. The* comparison triangle *for Δ is a geodesic triangle in $\mathbb{E}^2$ with the same side lengths.*

Definition 2.2.2. *A geodesic metric space X is called a* CAT(0) *space if for all geodesic triangles Δ, it is true that any $x, y \in \Delta$ satisfy*

$$d_X(x,y) \leq d_{\mathbb{E}^2}(\bar{x},\bar{y})$$

where $\bar{x}, \bar{y}$ are comparison points for x, y, that is, points on the corresponding side of the triangle at the same distance along the edge

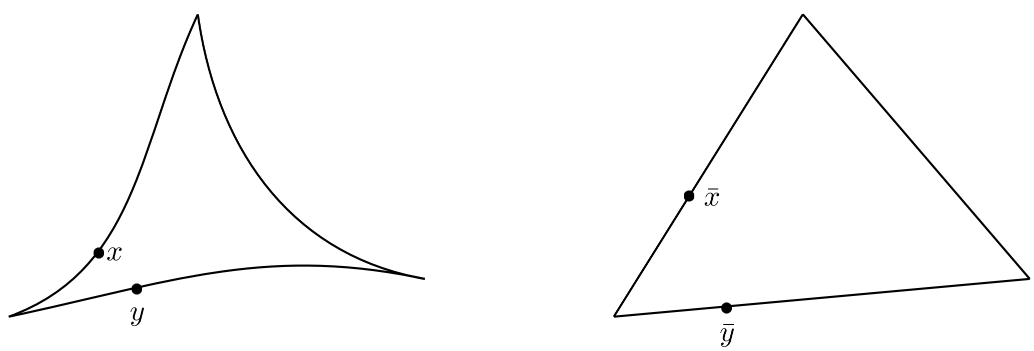


Figure 2.2: A Comparison Triangle in $\mathbb{E}^2$

We make immediate note of the fact that any CAT(0) space is necessarily simply connected.

To add to the types of axes we saw in our introduction to hyperbolicity, CAT(0) spaces have another type of geodesic we care about.

Definition 2.2.3. *The bi-infinite geodesic γ is called* rank one *if there is no isometric* *embedding of* $\mathbb{E}^2_{\geq 0}$ *such that* $\partial\mathbb{E}^2_{\geq 0} = \gamma$. *In this context, a copy of* $\mathbb{E}^2_{\geq 0}$ *will be called a* half flat.

Note that in this case this property is *not* a quasi-isometry invariant. Indeed any small perturbation of a half flat can change its isometry type. However, for groups which are CAT(0), we get a very useful equivalence. The following is proven using explicit constants in [13]

Theorem 2.2.4. *[13] Let* X *be a* CAT(0) *space and* γ *an infinite geodesic. Then* γ *is* $D-$*contracting for some* D *if and only if* γ *is* $M-$*Morse for some* M.

Recalling the definition of contracting, we see that a contracting geodesic is necessarily rank one. To see this, suppose it did bound a copy of $\mathbb{E}^2_{\geq 0}$. We can imagine taking larger and larger metric balls disjoint from our axis. In particular, for $n \in N$, imagine the ball of radius n centered about $(0, 2n)$, where our geodesic here is served by the y-axis. This will remain disjoint for all n, and will project to a segment of radius $2n$ on the geodesic, thus meaning it is not contracting for any D, as we can take n such that $2n > D$. Therefore if a CAT(0) group has a group element which is contracting, any CAT(0) space it acts on must be rank one.

2.2.1 Translation Length

We use some machinery developed in [10] to discuss types of CAT(0) isometries, which will look identical in nomenclature to hyperbolic isometries. In order to understand this trichotomy, we need one property of such isometries.

Definition 2.2.5. *Let* g *be an isometry of a metric space* X. *The* displacement function $d_g : X \to \mathbb{R}_{\geq 0}$ *is defined as*

$$d_g(x) = d(g.x, x).$$

In addition, to each isometry we assign a value called its translation length*, which* *is defined as*

$$|g| := inf\ \{d_g(x) | x \in X\}$$

It is worth noting that translation length is a conjugacy class invariant. Out of this falls a natural definition that will serve as the basis for classifying elements of $Isom(X)$.

Definition 2.2.6. *The set of $x \in X$ where an element of $Isom(X)$ attains its minimum is labelled $Min(g)$ and is called its "min set". An element is called* semi-simple *if its min set is nonempty, and a group action $G \circlearrowright X$ is said to act semi-simply if all elements satisfy this condition.*

We are now able to classify isometries. Note this will exactly coincide with the same terms for hyperbolic isometries.

Definition 2.2.7. *Let $g \in Isom(X)$. Then g is called:*

- Loxodromic *if $|g| > 0$.*
- Parabolic *if $|g| = 0$ and $Min(g)$ is empty.*
- Elliptic *if g has a fixed point (i.e. $|g| = 0$ and $Min(g)$ is nonempty).*

2.3 Automorphisms of Free Groups

2.3.1 Generating Aut(F_n)

For this section, fix a value of $n \geq 3$.

Definition 2.3.1. Aut(G) *is the* automorphism group *of a group G, under the operation of composition.*

We are interested in what Aut(F_n) looks like. We get an elegant classification theorem courtesy of Nielsen [25], almost a century ago.

Theorem 2.3.2. *[25] The group* $\mathrm{Aut}(F_n)$ *can be generated by elements of the following three forms:*

- *Permutations of Generators,* $x_i \mapsto x_{\pi(i)}$ *for some* $\pi \in \Sigma_n$
- *Inversions,* $x_1 \mapsto x_1^{-1}$
- *Transvections,* $x_1 \mapsto x_1 x_2$

Note that we only need to list the single inversion and transvection, as the other analogous elements can be achieved by composing with permutations. Furthermore, this is a finite generating set.

Next we note a certain type of automorphism that will be very important in the coming discussion.

Definition 2.3.3. *Let* $w \in F_n$*. Then we define an automorphism* ϕ_w *such that*

$$\phi_w(x_i) = w x_i w^{-1}$$

The subgroup of such automorphisms is called the inner automorphisms*, and is denoted* $\mathrm{Inn}(F_n)$*. In fact it is a normal subgroup, and we can define the quotient by it, which we do by calling*

$$\mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$$

the outer automorphisms.

These terms are defined for groups other than F_n as well, it just may not always be the case that $\mathrm{Inn}(G)$ is nontrivial, i.e. in the case of abelian groups. However it will always be normal, so $\mathrm{Out}(G)$ is well-defined.

2.3.2 The Free Factor Complex

Definition 2.3.4. *A* free factor *of F_n is a subgroup $H < F_n$, necessarily isomorphic to F_k for some k, such that there exists another subgroup G, such that*

$$F_n = H * G$$

Such a G is called the co-factor. A proper *free factor is one which is not the the whole group or trivial. Note immediately for a free factor, $k < n$.*

Definition 2.3.5. *Two free factors H_1, H_2 are considered in the same conjugacy class if there exists an element $g \in F_n$ such that*

$$H_1 = gH_2g^{-1}$$

Let $\mathcal{F} = \{[H] \mid [H]$ is a conjugacy class of a free factor of $F_n\}$. We put a poset structure on $\mathcal{F}$ by declaring $[H] < [G]$ if there exists representatives of these classes such that $H' \subset G'$.

Definition 2.3.6. *The* free factor complex *for a free group F_n with $n \geq 3$, which through abuse of notation we will also denote as $\mathcal{F}$, is the simplicial realization of $(\mathcal{F}, <)$.*

It is possible to define $\mathcal{F}$ for F_2, however we will not worry about that case here. To give a little bit of intuition about this complex, let us explore the case of F_3. In this setting, there are two kinds of vertices, those that represent subgroups isomorphic to $\mathbb{Z}$ or F_2. None of these vertices will be connected to members of their same class, so the one skeleton will be a bipartite graph. In fact, the one skeleton will be the entire complex for this case, and more generally this complex will have topological dimension $n - 2$ (hence why it is ill defined in the F_2 and $\mathbb{Z}$ case).

In $\mathcal{F}$ for higher rank, we can say something about the simplicial structure. Every simplex will be contained in some top level simplex, which will correspond to a choice

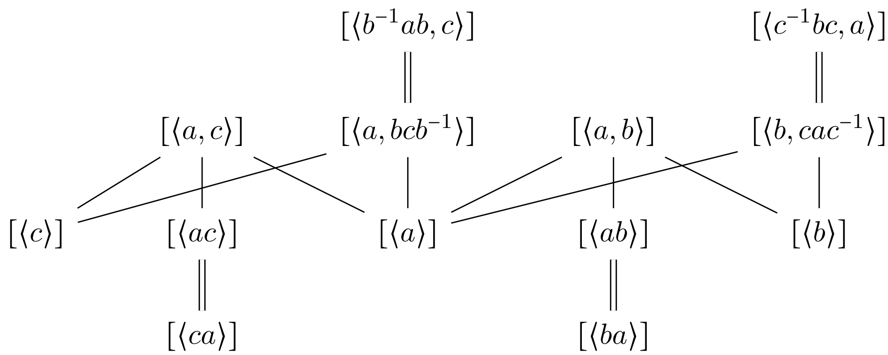


Figure 2.3: A Snapshot of $\mathcal{F}(F_3)$

of $n-1$ basis elements *in order*. Two top level simplices will share a face if one swaps the order of two adjacent choices of element.

The following theorem comes to us from [5].

Theorem 2.3.7. *[5] $\mathcal{F}$ with the standard simplicial metric is hyperbolic.*

Furthermore because finite-dimensional complexes are quasi-isometric to their one skeletons, $\mathcal{F}^{(1)}$ is also hyperbolic.

We can think of $\mathrm{Aut}(F_n)$ acting on $\mathcal{F}$ by acting on vertices in the natural way (applying the automorphism to the subgroup), and extending continuously to the higher dimension cells. Immediately we note that, because free factors are defined only up to conjugacy class, all elements of $\mathrm{Inn}(F_n)$ act trivially on $\mathcal{F}$. Therefore we can think of this as an action by $\mathrm{Out}(F_n)$. We note that this action is not a proper, cocompact action, meaning that $\mathrm{Out}(F_n)$ is not quasi-isometric to this space; in particular $\mathrm{Out}(F_n)$ is not itself hyperbolic.

Definition 2.3.8. *An automorphism of F_n is called an* elementary partial conjugation *if for some $i \neq j$, $\psi_{ij}(x_i) = x_j x_i x_j^{-1}$ and ψ_{ij} fixes all other generators.*

Fact 2.3.9. $\mathrm{Out}(F_n)$ *has flats.*

Proof. All ψ_{ij} are infinite order even after passing to the quotient. It is easy to show that $[\psi_{21}, \psi_{31}] = id$. □

Chapter 3

CAT(0) Groups

3.1 CAT(0) and Acylindrical Hyperbolicity

We would like to tie together the ideas of negative curvature in the form of acylindrical hyperbolicity and nonpositive curvature in the form of CAT(0). We do so with the following theorem. One direction of this was already proven in [27] and noted in [26]. We use the same idea to expand the result to a biconditional statement.

Theorem 3.1.1. *Let G be a group, which is not virtually cyclic, acting geometrically on a* CAT(0) *space X. Then G is acylindrically hyperbolic if and only if it contains an element g which acts as a rank one isometry on X. Furthermore, the set of generalized loxodromics is precisely the set of rank-one elements.*

Proof. ($\Leftarrow$) This follows from Theorem 5.4 in [6], where it is stated that a geodesic axis for an isometry in a CAT(0) space is contracting exactly when it fails to bound a half-flat, meaning rank-one elements are contracting. Next, contracting elements satisfy a property labelled *weakly contracting*, shown in [27]. Sisto goes on to prove in Theorem 1.6 that this property is strong enough to show that this element is contained in a virtually cyclic subgroup, labelled $E(g)$, which is hyperbolically embedded in the group. This is one of four equivalent conditions for being a generalized loxodromic, listed in Theorem 1.4 of [26].

($\Rightarrow$) If G is acylindrically hyperbolic, then it contains at least one generalized loxodromic. This is because we can take the action on a hyperbolic space guaranteed by the definition of acylindrical hyperbolicity, and knowing that it is devoid of parabolic elements, invoke the non-elementary condition on the action to verify at least one element must act as a loxodromic. Call this element g. We know by a result of Sisto that this element is Morse in G [28]. An equivalence in the setting of CAT(0) groups, proved in [13], says that a (quasi-)geodesic axis in a CAT(0) space is contracting if and only if it is Morse and if and only if it is rank-one. The geometric nature of our action, which says our space is quasi-isometric to our group, guarantees that because our group element is Morse, its axes are as well. Therefore our element g has axes which are rank-one, i.e., it acts as a rank-one isometry. □

This equivalence allows us to restate the Rank Rigidity Conjecture for CAT(0) groups, originally posited by Ballman and Buyalo.

Conjecture 3.1.2. Rank Rigidity Conjecture *[2]*
Let X be a locally compact geodesically complete CAT(0) *space and G a discrete group acting geometrically on X. If X is irreducible, then either*

- *X is a Euclidean building or higher rank symmetric space*

  or

- *G is acylindrically hyperbolic.*

3.2 An Application to right angled Coxeter Groups

In this section we use Theorem 3.1.1 to prove a trichotomy for a popular class of groups in geometric group theory.

Definition 3.2.1. *Given a finite, simple graph Γ, the* right angled Coxeter Group, *abbreviated* RACG, *associated to Γ, labelled $W(\Gamma)$, is the group generated by $V(\Gamma)$ such that each generator has order 2, and two generators commute exactly when they share an edge*

$$W(\Gamma) \cong \langle v_1, v_2, v_3, v_4 | v_i^2, [v_1, v_3], [v_3, v_4] \rangle$$

In his seminal work on Coxeter groups, [15], Michael Davis constructs a complex that RACGs act on geometrically.

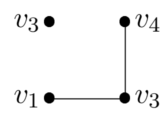


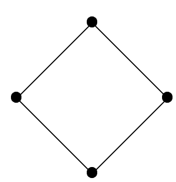
Figure 3.1: RACG from a given defining graph

Definition 3.2.2. *Let $W(\Gamma)$ be a right angled Coxeter group. Define the following cube complex Σ as follows. $\Sigma^{(1)}$ is the Cayley complex of $W(\Gamma)$, using the standard generating set of vertices. Then, for every edge in Γ between vertices v_1, v_2, add in squares along the cycle $(g, v_1.g, v_2v_1.g, v_2.g)$. Finally, for each $k-$clique in Γ, equivariantly add in a $k-$cube along the appropriate 1-skeleton in $\Sigma^{(1)}$. The complete complex is called the* Davis complex *and denoted $\Sigma(W(\Gamma))$.*

Theorem 3.2.3. *[15] For any RACG, the associated Davis complex is* CAT(0) *under the standard piecewise Euclidean metric.*

Further work into right angled Coxeter groups was done by Gabor Moussong in his thesis. He proves a necessary and sufficient condition for these groups to be hyperbolic.

Theorem 3.2.4. *[24] A right angled Coxeter group is hyperbolic if and only if its defining graph contains no squares. That is, there is no subgraph of the form:*



The following comes to us from Caprace and Fujiwara in [12]. In their paper, they use the term 'parabolic subgroup' to refer to any conjugate of a subgroup of the form $W(\Gamma')$, where Γ' $\subset \Gamma$ is an induced subgraph, meaning that if you include two vertices, you include any edge between them. Note the following result takes place in a general Coxeter group (not necessarily right angled).

Theorem 3.2.5. *[12, Proposition 4.5], Proposition 4.5 Let $\gamma \in W(\Gamma)$. Then either γ is rank-one or γ belongs to a parabolic subgroup P such that P is finite (1), or P splits as a direct product $P = P_1 \times P_2$ where each P_i is infinite (2), or P splits as $P = K \times P_{aff}$ where K is finite and P_{aff} is an affine parabolic group of rank ≥ 3 (3).*

As we use parabolic in a different context, we will change this notation. An 'induced subgroup' will refer to the type of subgroups defined in [12]. This can also be thought of as any conjugate of a subgroup induced by taking some subset of the standard generating set.

We see that because we will restrict our attention to right angled Coxeter groups, condition (3) becomes more precise. In particular, any affine parabolic group must split as a direct product of copies of $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Necessarily, then, this subgraph is a nontrivial join, so we may consider, for our purposes, the third condition a subcase of the second. We will call belonging to a finite induced subgroup criterion (1) and belonging to an induced subgroup that splits as above as criterion (2).

From these pieces, we can obtain the following trichotomy about RACG's. We define the term 'nontrivial join' to be a join between two graphs where neither graph is full (a clique). By convention, the empty graph and the graph of a single vertex are cliques.

Theorem 3.2.6. *Let $W(\Gamma)$ be a right angled Coxeter group. Then exactly one of the following is true:*

- *$W(\Gamma)$ is acylindrically hyperbolic.*
- *$W(\Gamma)$ is elementary hyperbolic.*
- *Γ is a nontrivial join.*

Proof. Suppose Γ is a finite simplicial graph such that $W(\Gamma)$ is nonelementary and not acylindrically hyperbolic. We know by 3.1.1 then that there are no rank one elements. We will use the proposition above then to prove that the graph Γ must be a nontrivial join.

The first thing we note is that in insisting the group is not elementary hyperbolic, the graph is not a clique on its vertices. This is because the RACG for a clique is a finite group. Furthermore, it is not the join of a two vertex set with a clique, because this associated group is virtually cyclic, with the two vertex set contributing an infinite dihedral group, and the clique being a direct factor with a finite group.

Now suppose all group elements met criterion (1). Then the defining graph must be a clique, otherwise we could take the product of two elements in separate complete subgraphs. Therefore our group is finite, and hence elementary hyperbolic, which is a contradiction.

Therefore atleast one group element meets criterion (2), as we are assuming the group is not acylindrically hyperbolic and thus has no rank one group elements. Importantly about group elements of the form (2) is that some representative of their conjugacy class belongs to an induced subgroup which is generated by a subset of the standard generators, where the corresponding induced subgraph Γ' is a non-trivial join. Specifically the generators corresponding to vertices of one (infinite, non-clique) factor must each be connected to every generator of the other factor.

Now, suppose all group elements meet either criterion (1) or (2). Let $S = \{\Gamma_1 \ldots \Gamma_n\}$ where the Γ_i is any induced subgraph which is a nontrivial join and not a full subgraph. If no such subgraphs existed, we must be in the previous case. Notice this union must necessarily give the whole graph by the assumption that all elements lie in a a subgroup of the form (1) or (2). This is because, for any vertex, either it is connected to every other, or it is not. If it is not, the product of generators corresponding to those vertices is a group element of the form (2), so those vertices belong to a nontrivial join, so that join is included in the set. If it *is* connected to every other vertex, because the graph is assumed not to be a clique, it is connected to two vertices which do not share an edge.

Therefore, the product of those two generators are of the form (2), so belong to a nontrivial join. We can include our vertex into the graph of that join, on either side, because it connects to every vertex.

Next, we will omit from the set S any element which is a strict subset of another element. We do not then change the property that

$$\bigcup_i \Gamma_i = \Gamma$$

Our goal will be to reduce the set to a single element to show that Γ itself is in the set, and therefore a nontrivial join. Let

$$\prod_i v_i \in W(\Gamma_1)$$

be the infinite order element obtained by taking the product of all standard generators corresponding to vertices in Γ_1 and

$$\prod_j w_j \in W(\Gamma_2)$$

be analogous. These are infinite order because we assume these subgraphs are not full. We now consider their disjoint product,

$$g := \prod_i v_i \prod_{j, w_j \neq v_k \forall k} w_j \quad \in W(\Gamma_1 \cup \Gamma_2).$$

By assumption, this either lies in a finite induced subgroup or a subgroup of form (2). If the former were true, either the whole graph is a clique, which is a contradiction, or the clique is a subgraph of another join, in which case we omit it from our list, and start again.

So we may assume the case (2) for the element g, which means that this element arises from a subgraph which is a nontrivial join, call it $\Gamma_{1,2}$. Because we chose our group elements to involve all generators from these subgraphs, it is necessarily true that $\Gamma_i \subset \Gamma_{1,2}$ for $i = 1, 2$. We may now replace in our list $\{\Gamma_i\}$ the elements Γ_1 and Γ_2 with $\Gamma_{1,2}$, thereby strictly reducing the cardinality of this finite set. We repeat this process as needed until the set contains a single nontrivial join, implying that the full graph Γ itself was a nontrivial join. $\square$

Chapter 4

Braid Groups and $\mathrm{Out}(F_n)$

4.1 Braid Group Actions

Braid groups are an important example of groups that are intermediate between hyperbolic and flat. They are not hyperbolic (nor even relatively hyperbolic); indeed they have a number of flats. However, they do have free subgroups and many properties shared by groups which are non-positively curved.

Definition 4.1.1. *The* braid group on n strands*, denoted B_n, is the set of arrangements of n strings, fixed at top and bottom, up to homotopy. The group action is concatenating these strings, bottom of one to the top of the next.*

An example group element is available in figure 4.2. A finite group presentation for B_n was first discovered by Artin [1].

$$B_n \cong \langle \sigma_1 \ldots \sigma_{n-1} \mid [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| \geq 2, \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \quad \forall k \rangle$$

The following is obtained by Bowditch in [7] by noting that $\overline{\mathrm{B_n}} \coloneqq \mathrm{B_n}/\mathrm{Z}(\mathrm{B_n})$ represents the mapping class group of a punctured surface.

Theorem 4.1.2. *[7] Let $n \geq 4$. The group $\overline{\mathrm{B_n}} \coloneqq \mathrm{B_n}/\mathrm{Z}(\mathrm{B_n})$ is acylindrically hyperbolic.*

A stronger statement holds. A result from [11] shows that all Artin–Tits groups of spherical type, otherwise known as *generalized braid groups*, are acylindrically hyperbolic after modding out their center.

Now the results 3.1.1 and 4.1.2 give us the following:

Theorem 4.1.3. *Let B_n be the braid group with $n \geq 4$, and suppose that X is a* CAT(0) *space on which B_n acts geometrically. Then Y is a rank-one* CAT(0) *space in a natural splitting of $X = Y \times \mathbb{R}$. In particular, B_n does not act on a Euclidean building.*

Proof. Because the action is assumed to be geometric, all isometries are semi-simple i.e. either elliptic or loxodromic. Our central element is infinite order, and so it must act loxodromically, meaning it has positive translation length. We know from [8] that the min set of an element with positive translation length breaks down as a metric product, $Min(g) = Y \times \mathbb{R}$, where g fixes the Y factor pointwise.

Finally, we see that any group element that commutes with our infinite order g also respects this product structure. But we are considering an element in the center of the group, so the whole group commutes and thus respects this structure. Therefore $Min(Z(B_n)) = X = Y \times \mathbb{R}$, and $\overline{\mathrm{B_n}}$ acts geometrically on the factor Y. Because $\overline{\mathrm{B_n}}$ is acylindrically hyperbolic by [11], this tells us that it acts with a rank-one isometry, i.e. Y is rank-one. □

This holds for all Artin–Tits group of spherical type. By combining Theorem 3.1.1 with a result of Calvez and Wiest [11], we get that any CAT(0) space acted on geometrically by Artin–Tits group of spherical type must be of rank-one.

To illustrate this theorem, we give an example from an explicit CAT(0) complex. In [9], Brady constructs a 2-complex that $B_4/Z(B_4)$ acts on made up of equilateral triangles which is CAT(0) using the standard piecewise Euclidean metric. This is obtained by 'projecting' down the infinite cylic factor corresponding to

$$Z(B_4) = \langle (\sigma_1\sigma_2\sigma_3)^4 \rangle \cong \mathbb{Z}.$$

The link of any vertex in this projected complex looks as in Figure 4.1.

Importantly, we note that the top right and bottom left vertices of this link

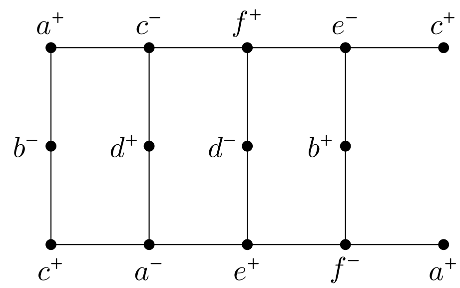


Figure 4.1: An Arbitrary Link

are identified, as well as the top left and bottom right. We recognize this as the 1-skeleton of a CW-complex homeomorphic to the Möbius strip. Because this complex is CAT(0), this link has a standard CAT(1) metric on it [10]. The corresponding triangles are equilateral, so this metric assigns each edge in this link a length of $\frac{\pi}{3}$. We now examine the vertices labelled d^- and b^-, specifically noting that

$$d_{lk}(b^-, d^-) = \frac{4\pi}{3} > \pi$$

Because this angle is larger than π, that means that the path obtained in the space obtained by concatenating the paths from x_0 to those points in the link is a local geodesic. We note this path is also a portion of an axis for the group element bd^{-1}. (Note: one of these elements must have an inverse, as both vertices have a minus, which means that group element takes us *towards* x_0.)

This space enjoys the property that all vertices have isometric links, so if we look at the link of the vertex $b.x_0$, we see the path the axis of bd^{-1} takes is in through b^- and out through d^-, which also have distance greater than π. This implies our axis, which we will label $\gamma := \gamma_{bd^{-1}}$, is also a local geodesic at this vertex. Combining this with the fact that edges of triangles in our space are local geodesics because the metric is CAT(0) and locally Euclidean, this tells us that γ is a geodesic axis for the action of the group element bd^{-1}.

Indeed this axis is rank-one, as [2] proves the Rank Rigidity Conjecture in the case where the dimension of the complex is 2. Because X is two dimensional, if γ

bounded a space isometric to $\mathbb{E}^2_+$, then any vertices of links it went through would have to have diameter at most π, as the link of that vertex would contain faces forming a half-disk portion of the copy of $\mathbb{E}^2_+$. As this axis is rank-one, we know that our group element is a generalized loxodromic, by Theorem 4.1.3.

So what does it look like? If we translate bd^{-1} into our standard generating set, noting that σ_1, σ_3 commute, we get the element

$$\sigma_2\sigma_1\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}$$

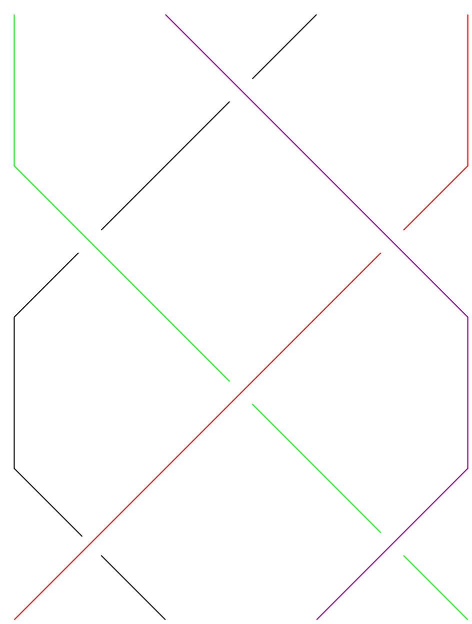


Figure 4.2: The element bd^{-1}

We note a few properties about this braid immediately. First, it involves every strand. This is a requirement because any element that did not act on a puncture of the sphere would be a reducible element of the mapping class group, and hence not a pseudo-Anosov element, which are exactly the class of generalized loxodromics.

Secondly we see that this element counts each generator zero times with signed multiplicity. This means that the abelianization of this element is trivial, i.e. that it is in the kernel of the map $ab : B_4 \to B_4^{ab} \cong \mathbb{Z}$. It is worth noting that this map is defined by $ab(\sigma_i) = 1$ for all i, and so $ab((\prod \sigma_i)^n) = 3n$. Thus it must not belong to the center $Z(B_4)$, so it indeed represents a non-trivial element of $B_4/Z(B_4)$.

4.2 $\mathrm{Out}(F_n)$

Lemma 4.2.1. *Let $H < G$ and suppose there exists $h \in H$ which has infinite order, such that there exists $g \in C_G(h)$ and $g \notin H$. Then H does not hyperbolically embed into G.*

Proof. While this result follows as an immediate corollary of [26], 2.11, a direct proof is enlightening as to the behavior of hyperbolically embedded subgroups. The following construction will take place in $\Gamma(G, H \sqcup X)$. Recall that the definition a hyperbolically embedded group required that the relative metric on $(H, \hat{d})$ be proper, i.e. that metric balls be compact. While this metric, denote it d_h, is prohibited from taking paths in the complete subgraph $(H, H) \subset \Gamma$ it *may travel along edges in the cosets of H*. We then use our assumption to show that a metric ball around our vertex h is not compact. In particular, consider $B_3(h)$. We see that for any $n \in \mathbb{Z}$, we can take the following path:

$$h \to gh \to h^n gh \to g^{-1} h^n gh = h^{n+1}$$

This path is length three, because there is an edge connecting g and $h^n g$ in the coset gH. Because we assume h is infinite order, this gives us infinitely many elements contained in $B_3(h)$

□

Given that we know $\overline{\mathrm{B_n}}$ and $\mathrm{Out}(F_n)$ are both acylindrically hyperolic groups, we might expect the embedding respects the acylindrically hyperbolic structure. In particular, this would mean that generalized loxodromics of the braid group modulo center would map to generalized loxodromics of $\mathrm{Out}(F_n)$. However, we find in Theorem 4.2.3 that this is not true.

Before stating the theorem, we recall what type of elements are generalized loxodromics in $\mathrm{Out}(F_n)$. These include what are called *fully irreducible* automorphisms, which are maps such that no power of them fix any conjugacy classes of a free factor. Braid groups do not produce such elements.

Lemma 4.2.2. *Let* $\iota : \overline{\mathrm{B_n}} \hookrightarrow \mathrm{Out}(F_n)$*. Then* $\iota(\overline{\mathrm{B_n}})$ *contains no fully irreducible automorphisms.*

Proof. First we note *how* this injection works. Identify F_n with $\pi_1(D_n)$, i.e. as the fundamental group of the n−punctured disk. Let $\sigma \in B_n$ be a braid. We can view σ as a unit cube in 3-space where every horizontal slice is exactly a copy of D_n, and the image of the punctures vary continuously as we move from top to bottom, finally assuming that in the bottom the punctures are in their original place. This interpretation of σ can be seen as follows:

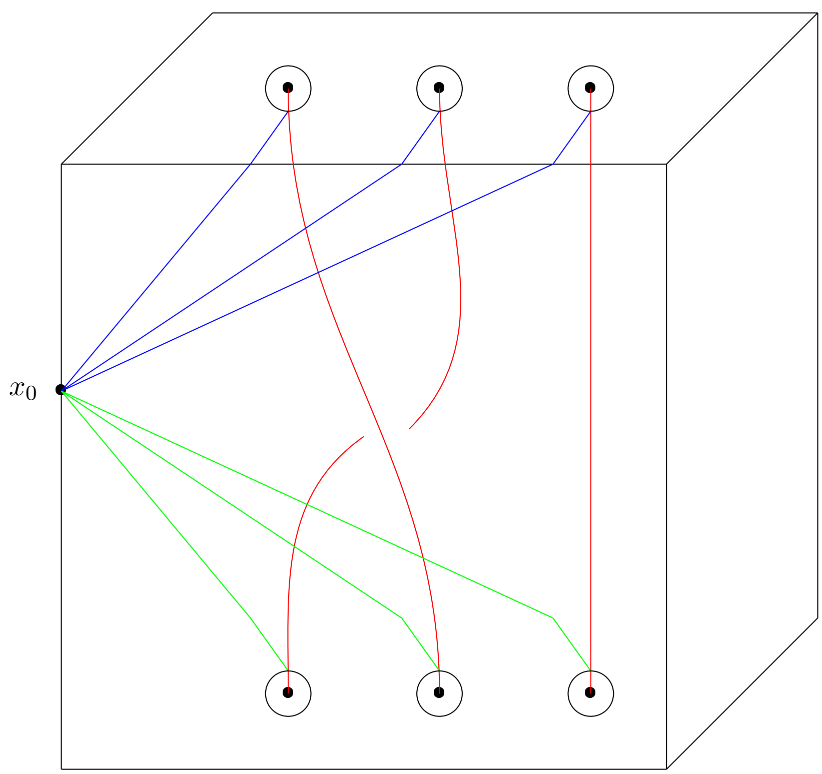


Figure 4.3: Braid Group Action on $\mathrm{Aut}(F_n)$

Note that any element of the braid group then induces a permutation on the punctures. There is a map, thus, $\mathrm{B_n} \twoheadrightarrow \Sigma_n$. The kernel of this map is the pure braid group $\mathrm{P_n}$. Thus we get that $\mathrm{P_n}$ has finite index in $\mathrm{B_n}$ and furthermore any braid has some finite power which is an element of $\mathrm{P_n}$.

To recover from σ an automorphism of F_n, we note that $\sigma : D_n \to D_n$ acts as a homeomorphism, preserving the punctures. Therefore it induces an isomorphism on fundamental groups σ_*, meaning we have a map

$$\iota' : \mathrm{B_n} \to \mathrm{Aut}(F_n)$$

By taking the quotient of the image of this map modulo inner automorphisms, we recover $\bar{\iota} : \mathrm{B_n} \to \mathrm{Out}(F_n)$. A calculation that we perform in the appendix, Claim A, shows that the kernel of this map contains $Z(B_n)$, verifying that $\bar{\iota} : \overline{\mathrm{B_n}} \hookrightarrow \mathrm{Out}(F_n)$ is well defined.

Next let's consider the image of the pure braid group in $\mathrm{Out}(F_n)$. If $\tau \in \mathrm{P_n}$ then this means it fixes all punctures pointwise in the homeomorphism, not simply setwise. Graphically what this represents is sending a loop around a given puncture to a loop composed of a path γ, the loop around that same puncture, and γ^{-1}. This tells us that, if we pick $\{x_i\}$ as our generating set for F_n, where each x_i represents a loop around each puncture with some predetermined basepoint, then

$$\tau_*(x_i) = w_i x_i w_i^{-1}$$

So it fixes the conjugacy class of each generator. In particular, this tells us that the image of the pure braid group is composed entirely of *reducible* automorphisms. Because we know any braid has a power in $\mathrm{P_n}$, that means that no braids induce fully irreducible automorphisms.

An entirely combinatorial proof of this fact is located in the appendix as Claim B.

□

4.2.1 Failure of Hyperbolic Embedding

Theorem 4.2.3. *The natural injection* $\iota : \overline{\mathrm{B_n}} \hookrightarrow \mathrm{Out}(F_n)$ *is not a hyperbolic embedding for* $n \geq 4$.

Proof. Let $\phi \in \mathrm{Out}(F_n)$ be defined as follows:

$$\phi(x_j) = \begin{cases} x_4 & \text{j} = 3 \\ x_3 & \text{j} = 4 \\ x_j & \text{else} \end{cases}$$

Recall the element $\sigma_1 \in \overline{\mathrm{B_n}}$, abusing notation by considering it as the image of the same element in $\mathrm{B_n}$. When considered as an element of $\mathrm{Out}(F_n)$, $\iota(\sigma_1)$ has the following effect on generators:

$$\iota(\sigma_1)(x_j) = \begin{cases} x_1 x_2 x_1^{-1} & \text{j} = 1 \\ x_1 & \text{j} = 2 \\ x_j & \text{else} \end{cases}$$

First we are tasked with showing that ϕ is not in $\iota(\overline{\mathrm{B_n}})$. The first thing we know is that any automorphism which appears as the image of a braid must fix the word $x_1 x_2 \ldots x_n$. This fact is available in [23]. This is shown by noting that

$$\phi(x_1 \ldots x_n) = x_1 x_2 x_4 x_3 \ldots x_n$$

so it does not fix the product of the generators, a necessary (and in fact sufficient) condition to be in the image. Note this also implies further that
$\Sigma_n \cap \iota(\mathrm{B_n}) = e$.

Next we want to invoke Lemma 4.2.1 by finding that $\phi \in C_G(\iota(\sigma_1))$. Indeed $[\phi, \iota(\sigma_1)] = 1$. To see this, evaluate the commutator on an arbitrary x_i. Note $\phi = \phi^{-1}$

- $i \neq 1, 2, 3, 4.$

  Then each of these autmorphisms fixes x_i, therefore $[\phi, \iota(\sigma_1)](x_i) = x_i$

- $i = 1$

$$
\begin{aligned}
[\phi, \iota(\sigma_1)](x_1) &= \phi\iota(\sigma_1)\phi^{-1}\iota(\sigma_1^{-1})(x_1) \\
&= \phi\iota(\sigma_1)\phi^{-1}(x_2) \\
&= \phi\iota(\sigma_1)(x_2) \\
&= \phi(x_1) \\
&= x_1
\end{aligned}
$$

- $i = 2$

$$
\begin{aligned}
[\phi, \iota(\sigma_1)](x_2) &= \phi\iota(\sigma_1)\phi^{-1}\iota(\sigma_1^{-1})(x_2) \\
&= \phi\iota(\sigma_1)\phi^{-1}(x_2^{-1}x_1x_2) \\
&= \phi\iota(\sigma_1)(x_2^{-1}x_1x_2) \\
&= \phi(x_2) \\
&= x_2
\end{aligned}
$$

- $i = 3$

$$
\begin{aligned}
[\phi, \iota(\sigma_1)](x_3) &= \phi\iota(\sigma_1)\phi^{-1}\iota(\sigma_1^{-1})(x_3) \\
&= \phi\iota(\sigma_1)\phi^{-1}(x_3) \\
&= \phi\iota(\sigma_1)(x_4) \\
&= \phi(x_4) \\
&= x_3
\end{aligned}
$$

- $i = 4$

$$
[\phi, \iota(\sigma_1)](x_4) = \phi\iota(\sigma_1)\phi^{-1}\iota(\sigma_1^{-1})(x_4)
$$

$$= \phi\iota(\sigma_1)\phi^{-1}(x_4)$$

$$= \phi\iota(\sigma_1)(x_3)$$

$$= \phi(x_3)$$

$$= x_4$$

Therefore the centralizer of $\iota(\sigma_1)$ contains ϕ, which is infinite order. Thus the hypotheses for Lemma 4.2.1 are met. $\square$

Chapter 5

Outer Automorphisms of Universal RACGs

5.1 $\mathrm{Out}(W_n)$

Assume here that $n \geq 4$.

We will use the conventions in this section that

$$F_n = \langle x_i | \quad \rangle, \quad W_n = \langle w_i \mid w_i^2 \rangle$$

As $w_i^{-1} = w_i$ we will suppress inverse notation when working in W_n.

We will make use of the following result by Gutierrez, Piggot, and Ruane, so it is helpful to list it here.

Theorem 5.1.1. *[22]*

$$\mathrm{Aut}(W_n) = \mathrm{Aut}^0(W_n) \rtimes \Sigma_n = (W_n \rtimes \mathrm{Out}^0(W_n)) \rtimes \Sigma_n$$

where the W_n factor is the whole of $\mathrm{Inn}(W_n)$, $\mathrm{Aut}^0(W_n)$ *is generated by partial conjugations,* $\mathrm{Out}^0(W_n)$ *is* Aut^0 *the quotient of inner automorphisms, and Σ_n is the full symmetric group on n letters, corresponding to permuting the generators.*

The following two results can be found in [18]

Lemma 5.1.2. *Consider the subgroup $W_n > G := \langle w_1 w_i \mid i \in \{2, \dots n\} \rangle$. Then $G \cong F_{n-1}$.*

We will identify the generators for F_{n-1} with words in W_n by

$$x_i := w_1 w_{i+1} \text{ for } 1 \leq i \leq n-1$$

This gives a natural map $i : \mathrm{Aut}(W_n) \hookrightarrow Aut(F_{n-1})$ as the subgroup G is characteristic, so an automorphism of W_n give an automorphism of $G \cong F_{n-1}$. In fact, elements in the subgroup G as above are sometimes referred to as the 'even subgroup'. Because all cancellation happens in pairs, it is well defined to speak of words of even length (including the empty word).

Using this characterization, it is easy to see that this even subgroup must be characteristic. Given our decomposition of $\mathrm{Aut}(W_n)$, we see generators come in the form of graph automorphisms and partial conjugations. In the former case, all w_i are sent to words of length one, and in the latter, w_i are sent to either words of length one or three. In either case, after possible cancellation in pairs, words of even length will remain of even length.

Lemma 5.1.3. *This map is injective.*

We include a proof of this here for completeness.

Proof. Let $\phi : W_n \to W_n$ be an automorphism that fixes pointwise the set $\{w_1 w_i\}$. Our goal is then to show that it must be the case that

$$\phi = id_{\mathrm{Aut}(W_n)}$$

We do this by considering ϕ on each generator. If ϕ fixes each generator, it is necessarily the identity.

- Suppose $\phi(w_1) = z$. Assume z is fully reduced. We know that $z \neq id$, as this is an automorphism. Because ϕ fixes $w_1 w_i$, it must be the case that for all i, $\phi(w_i) = z^{-1} w_1 w_i$. We invoke the fact that $\mathrm{Aut}(W_n)$ is generated only by conjugations and permutations, following from the decomposition above, to observe that this word we map to must begin and end with the same letter after being reduced. If z^{-1} does not end with the letter w_1 (meaning z starts with the letter w_1), then z^{-1} must start with the letter w_i *for each* i. This is clearly a contradiction.

Therefore we assume z starts with the letter w_1. The last letter of z will be the first letter of $z^{-1}w_1w_i$, which again because $\mathrm{Aut}(W_n)$ has no transvections, must be either w_i or empty. Because the former is impossible for all i simultaneously, it must be that it is empty. This tells us that $z = w_1$, so ϕ fixes w_1.

- Now suppose $\phi(w_i) = z_i \neq w_i$ for some $i \neq 1$. Then because $\phi(w_1w_i) = w_1w_i$, we get that $\phi(w_1) = w_1w_iz_i^{-1}$. This must be true for every i, meaning that for all $i \neq 1$, $\phi(w_i) = z_i = z_j = \phi(w_j)$. This contradicts our assumption that ϕ is an automorphism, because the image of the generators is no longer a generating set.

□

Due to the injectivity of this map, we will label it

$$\iota : \mathrm{Aut}(W_n) \hookrightarrow \mathrm{Aut}(F_{n-1})$$

Because our goal is to say something about $\mathrm{Out}(W_n)$, we look at what happens to elements of $\mathrm{Inn}(W_n)$. While it is not quite true that they map into $\mathrm{Out}(F_{n-1})$, we find this is *almost* the case.

Lemma 5.1.4. *Let $r \in \mathrm{Aut}(F_n)$ be defined by $r(x_i) = x_i^{-1}$ for all i. Then*

$$i(\mathrm{Inn}(W_n)) \subset \mathrm{Inn}(F_{n-1}) \rtimes \{r\}$$

Proof. Recall $x_i := w_1w_j$. We write $\mathcal{C}_{x_i}$ for conjugation by x_i in $\mathrm{Aut}(F_{n-1})$.

We will consider the effect that an inner automorphism of W_n will have on

$\{w_1 w_i\}$, recalling that this set of automorphisms is generated by conjugation of all generators by another set generator. These will come in two flavors, denoting an inner conjugation by $\mathcal{C}_j$ for conjugation of all generating elements by w_j

- $\mathcal{C}_1(w_1 w_j) = w_j w_1 = (w_1 w_j)^{-1}$. Therefore $\mathcal{C}_1 = r$.
- $\mathcal{C}_i, i \neq 1$

$$\begin{aligned}
\mathcal{C}_i(w_1 w_j) &= w_i(w_1 w_j)w_i \\
&= w_i w_1(w_1 w_1 w_j w_1)w_1 w_i \\
&= w_i w_1(w_j w_1)w_1 w_i \\
&= w_i w_1(w_1 w_j)^{-1} w_1 w_i \\
&= (w_1 w_i)^{-1}(w_1 w_j)^{-1}(w_i w_1)^{-1} \\
&= r \circ \mathcal{C}_{x_1}(x_j)
\end{aligned}$$

□

Finally this gives us the following fact

$$\bar{\iota}(\mathrm{Out}(W_n)) \subset \mathrm{Aut}(F_{n-1})/(\mathrm{Inn}(F_{n-1}) \rtimes \{r\}) \subset \mathrm{Out}(F_{n-1})/\langle\langle R \rangle\rangle$$

where $R = \langle r | r^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

This relationship is summarized in the following diagram:

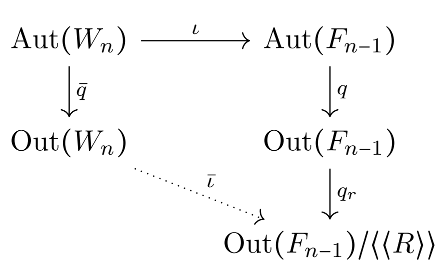


Figure 5.1: Diagrammatic Relations of the Groups

Denote by $\mathcal{F}_R \subset \mathcal{F}$ the subspace which is obtained by including all vertices which correspond to subgroups that are fixed by the involution r, and then including any simplices all of whose vertices lie in $\mathcal{F}_R^{(0)}$. The goal will be to create an action of $\bar{\iota}(\mathrm{Out}(W_n)) \circlearrowright \mathcal{F}_R$. We first need to know just a little bit more about the image of $\mathrm{Out}(W_n)$. While the following seems technical, it will be a key observation allowing us to define an action.

Lemma 5.1.5. *For any $\phi \in \mathrm{Aut}(W_n)$, and r the automorphism as above,*

$$[r, \iota(\phi)] \subset \mathrm{Inn}(F_{n-1})$$

Proof. We break this into cases, depending on what kind of automorphism ϕ is. We need only consider the case where ϕ is a generator, because the inner automorphisms are normal. To demonstrate this for a normal subgroup $H \triangleleft G$, and an element $r \in G$, assume that $a, b \in G$ are such that $[r, a] \in H, [r, b] \in H$, we'd like to show $[r, ab] \in H$. Then

$$\begin{array}{ll}
[r, ab] \in H & \\
\Updownarrow & \\
rabrb^{-1}a^{-1} \in H & \\
\Updownarrow & brb^{-1}r = k' \in H \\
rark'a^{-1} \in H & \\
\Updownarrow & \text{conjugate by } a^{-1}\text{, H is normal} \\
a^{-1}rark' \in H & \\
\Updownarrow & \\
a^{-1}rar \in H & \\
\Updownarrow & \text{conjugate by } a \\
rara^{-1} \in H &
\end{array}$$

and we note the last line is true by hypothesis. Now, let's look at the cases:

- ϕ is a graph automorphism (i.e. permutes the generators), so $\phi \in \Sigma_n$. This subgroup is generated by transpositions of w_i. We further break this into subcases.

- $\phi = (ij), i \neq 1 \neq j$. Then it is easy to see $\iota(\phi)$ permutes x_{i-1}, x_{j-1} and that this map commutes with inverting every generator.
- $\phi = (1i)$. In this case, $\phi(w_1 w_j) = w_i w_j = w_i w_1 w_1 w_j$ for $i \neq j$ and $\phi(w_1 w_i) = w_i w_1$. Then $\iota(\phi)(x_j) = x_{i-1}^{-1} x_j$ for $j \neq i-1$ and $\iota(\phi)(x_{i-1}) = x_{i-1}^{-1}$. More computation reveals that $[\iota(\phi), r]$ is conjugation by x_{i-1}^{-1}, and so an inner automorphism.

- ϕ is a partial conjugation. Once more, we are relegated to subcases
  - Neither the acting letter nor the acted-on letter is w_1. Then for $i \neq 1 \neq j$, $\phi(w_j) = w_i w_j w_i$ and ϕ fixes all other generators. Then $\iota(\phi)$ fixes all generators of F_{n-1} except x_{j-1}, and $\iota(\phi)(x_{j-1}) = x_{i-1} x_{j-1}^{-1} x_{i-1}$. Note quickly that ϕ is order two in the domain, so $\iota(\phi)$ is also order two, this is borne out by performing the calculation on the right hand side. Clearly $[r, \iota(\phi)](x_i) = x_i$ for $i \neq j-1$.

$$
\begin{aligned}
\iota(\phi) \circ r \circ \iota(\phi) \circ r(x_{j-1}) &= \iota(\phi) \circ r \circ \iota(\phi)(x_{j-1}^{-1}) \\
&= \iota(\phi) \circ r(x_{i-1}^{-1} x_{j-1} x_{i-1}^{-1}) \\
&= \iota(\phi)(x_{i-1} x_{j-1}^{-1} x_{i-1}) \\
&= x_{j-1}
\end{aligned}
$$

  - The acting letter is w_1. Then call the acted on letter w_i, so that $\phi(w_i) = w_1 w_i w_1$ and fixes other generators. In fact, in this case, $\iota(\phi)$ inverts x_{i-1} and fixes the other free generators. This automorphism commutes with inverting all generators.
  - The acted on letter is w_1, so $\phi(w_1) = w_i w_1 w_i$. Quick calculations show that $\iota(\phi)(x_j) = x_{i-1}^{-2} x_j$. Then

$$
\iota(\phi) \circ r \circ \iota(\phi) \circ r(x_j) = \iota(\phi) \circ r \circ \iota(\phi)(x_j^{-1})
$$

$$
\begin{aligned}
&= \iota(\phi) \circ r(x_j^{-1} x_{i-1}^{2}) \\
&= \iota(\phi)(x_j x_{i-1}^{-2}) \\
&= x_{i-1}^{-2} x_j x_{i-1}^{2} \\
&= x_j^{x_{i-1}^{-2}}
\end{aligned}
$$

□

This allows us to make the observation that

$$R \lhd \iota \circ q(\mathrm{Aut}(W_n)) < \mathrm{Out}(F_{n-1})$$

or in other words

$$\langle\langle R \rangle\rangle \cap \iota \circ q(\mathrm{Aut}(W_n)) = R$$

where $\langle\langle R \rangle\rangle$ is the normal closure of R.

More to the point, this allows us to replace Figure 5.1 with the following:

$$
\begin{array}{ccccc}
\mathrm{Aut}(W_n) & \xrightarrow{\iota} & im(\iota) & \hookrightarrow & \mathrm{Aut}(F_{n-1}) \\
\downarrow{\scriptstyle \bar{q}} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
\mathrm{Out}(W_n) & & im(\iota \circ q) & \hookrightarrow & \mathrm{Out}(F_{n-1}) \\
& \overset{\bar{\iota}}{\searrow} & \downarrow{\scriptstyle q_r} & & \\
& & im(\iota \circ q)/R & &
\end{array}
$$

Figure 5.2: Involution Normality in the Image

It is shown in [4] that $\mathrm{Out}(F_n)$ is acylindrically hyperbolic. This is proven by way of its action on the free factor complex, although it is unknown if this action is itself acylindrical. Hyperbolicity of this complex is demonstrated in [5]. In this action *fully irreducible* elements of $\mathrm{Out}(F_n)$ act with Weak Proper Discontinuity (WPD), which tells us that they are generalized loxodromics (in an action on a different space). Theorem **H** in [4] constructs a new action on a space quasi-isometric to a tree, which we will denote by $\mathcal{Q}$, that satisfies the conditions required by acylindrical hyperbolicity, in which these same fully irreducible elements act loxodromically. The fact that this action is acylindrical is stated in the discussion after Theorem **I** [4]. Furthermore we are guaranteed, again by [4] that all fully irreducible group elements act loxodromically in this action on $\mathcal{Q}$.

Theorem 5.1.6. $\mathrm{Out}(W_n)$ *is acylindrically hyperbolic, for* $n \geq 4$.

Proof. We will abuse notation throughout this proof, letting r represent both the automorphism in $\mathrm{Aut}(F_{n-1})$ and its image under the map q.

The first thing we will do is make a slight modification to $\mathcal{Q}$. Unlike in uniquely geodesic spaces such as CAT(0) spaces, fixed point sets in arbitrary hyperbolic spaces aren't as nice as we like, so we will add in a little extra structure. Let δ represent the constant of hyperbolicity for $\mathcal{Q}$

Define $\hat{\mathcal{Q}} := \mathcal{Q} \cup E$, where E consists of combinatorial edges of length 4δ between any two points which are at distance at most 4δ in $\mathcal{Q}$. We note that these two spaces are quasi-isometric by noting that $\mathcal{Q}$ embeds into $\hat{\mathcal{Q}}$ in the natural way such that distances are not changed, and the embedding is 2δ quasi-onto. The group $\mathrm{Out}(F_{n-1})$ will act on $\hat{\mathcal{Q}}$ in the natural way on the embedded copy $\mathcal{Q}$ and permute the edges in E according to their endpoints.

Label $\hat{\mathcal{Q}}_R$ the fixed point set of $R = \langle r \rangle$ acting on $\hat{\mathcal{Q}}$.

Now define an action $\mathrm{Out}(W_n) \circlearrowright \hat{\mathcal{Q}}_R$. We start by noting there is a natural action of $M := \iota \circ q(\mathrm{Aut}(W_n)) \circlearrowright \hat{\mathcal{Q}}$, because it is a subgroup of $\mathrm{Out}(F_{n-1})$. Now to say what an element of $\psi \in \mathrm{Out}(W_n)$ does, we look at its image, $\bar{\iota}(\psi) \in q_r(M)$. Using the structure of Figure 5.2, $\bar{\iota}(\psi) = gR$ for some element $g \in \mathrm{Out}(F_{n-1})$. For any $f \in \hat{\mathcal{Q}}_R$, we can define

$$\bar{\iota}(\psi).f = g.f.$$

This is well-defined, because no matter which element of R we pick (i.e., either id or r), they both have the same effect on f, (i.e., $r.f = f$ by definition of $\hat{\mathcal{Q}}_R$.)

Finally we claim the image of $\mathrm{Out}(W_n)$ leaves $\hat{\mathcal{Q}}_R$ invariant set-wise. Let $\mathcal{C} \in im(\iota \circ q)$ which encompasses any element coming from $\mathrm{Out}(W_n)$, and let $f \in \hat{\mathcal{Q}}_R$. We know from 5.1.5 that $[r, \mathcal{C}] = 1$ in $\mathrm{Out}(F_{n-1})$. Then

$$
\begin{aligned}
r\,\mathcal{C}\,.f &= \mathcal{C}\,r.f \\
&= \mathcal{C}(r.f) \\
&= \mathcal{C}\,.f.
\end{aligned}
$$

So the point f is moved to under $\mathcal{C}$ is indeed fixed by r, as $r\,\mathcal{C}\,.f = \mathcal{C}\,.f$.

In order to show that this action on $\hat{\mathcal{Q}}_R$ satisfies acylindrical hyperbolicity, we must show three things:

1. $\hat{\mathcal{Q}}_R$ is hyperbolic.

2. This action satisfies acylindricity.

3. This action is non-elementary.

For the first task we recall that hyperbolicity is a quasi-isometry invariant, so we know that $\hat{\mathcal{Q}}$ is hyperbolic. We claim that $\hat{\mathcal{Q}}_R$ is quasi-convex in $\hat{\mathcal{Q}}$, making it also hyperbolic by [10] H.1.9. To show quasi-convexity, let f_0, f_1 be in $\hat{\mathcal{Q}}_R$ such that a geodesic between them leaves $\hat{\mathcal{Q}}_R$. If such points don't exist, our subspace is directly convex. Otherwise, label x_0, x_1 the points (possibly the same as f_i) such that the chosen geodesic first leaves then re-enters $\hat{\mathcal{Q}}_R$. Let $\lambda = [x_0, x_1]$, which by assumption only intersects $\hat{\mathcal{Q}}_R$ in the endpoints. If we take $r.\lambda$, we obtain another, distinct geodesic between x_0, x_1. Take any point x_i on this segment. By the closeness of geodesics with the same endpoints in a hyperbolic space, the distance between x_i and $r.x_i$ is bounded by 4δ [10]. Therefore there is a combinatorial edge between them of length 4δ. Because r is order 2, it acts by inversion on this edge, and therefore fixes its midpoint. This means that every point on this geodesic is within distance at most 2δ of a fixed point. So this (and any) geodesic between points in $\hat{\mathcal{Q}}_R$ lies in a 2δ neighborhood of $\hat{\mathcal{Q}}_R$. This means $\hat{\mathcal{Q}}_R$ is 2δ quasiconvex and therefore hyperbolic.

For acylindricity we begin by letting $R(\epsilon), N(\epsilon)$ be constants depending on ϵ that demonstrate the acylindricity of the action $\mathrm{Out}(F_{n-1}) \circlearrowleft \mathcal{Q}$. These same constants will work to demonstrate acylindricity of $im(\iota \circ q)$ because the relevant set of elements will be a subset of the one we consider in the supergroup. Our claim is that these same constants will once again work for $im(\iota \circ q)/R$. We proceed by contradiction. Let $\epsilon > 0$, and $R(\epsilon)$ as above. Then suppose

$$|\{\phi \in im(\iota \circ q)/R | d(x, \phi.x) \leq \varepsilon, d(y, \phi.y) \leq \varepsilon\}| \geq N.$$

Now consider the set of pre-images $\{q_r^{-1}(\phi)\}$ of these elements. Because q_r is surjective, this set has *no fewer* elements than the original. Furthermore, because our quotient is by R, which acts trivially on $\mathcal{Q}_R$, these elements also have the same induced action. Therefore, this violates the assumption that there are fewer than $N(\epsilon)$ elements in $im(\iota \circ q)$ that satisfy this property. Finally, adding these combinatorial edges to $\mathcal{Q}_R$ does not change the property of acylindrical hyperbolicity; it slightly modifies the constants. This is because it does not change the distance of points in $\mathcal{Q}$, and elements moving the new combinatorial edges close to themselves must bring those endpoints, which belong to $\mathcal{Q}$, close to themselves. Specifically, for $x, y \in \hat{\mathcal{Q}}$ with distance $d(x, y) \geq R$, the set

$$\{g \in G | d(x, g.x) \leq \epsilon, d(y, g.y) \leq \epsilon\}$$

is contained in the set, for $x, y \in \mathcal{Q}$ with $d(x, y) \geq R(\epsilon + 2\delta)$,

$$\{g \in G | d(x, g.x) \leq \epsilon + 2\delta, d(y, g.y) \leq \epsilon + 2\delta\}$$

which is finite by assumption.

Finally, we are tasked with showing this action is non elementary. In order to demonstrate that the limit set contains strictly more than two points, we will show that there are two elements, neither a power of the other, in the image of $\mathrm{Aut}(W_n)$ that act as loxodromics on $\hat{\mathcal{Q}}$. Due to quasi-convexity, any element which acts as a loxodromic on $\mathcal{Q}$ and fixes $\hat{\mathcal{Q}}_R$ set-wise will also act as a loxodromic on $\hat{\mathcal{Q}}_R$, so establishing a loxodromic action on $\mathcal{Q}$ is sufficient. To find these elements, we recall that $\mathcal{Q}$ is designed such that any elements acting loxodromically and with WPD on the free factor complex also act as such on $\mathcal{Q}$. These automorphisms are called *irreducible with irreducible powers*, often abbreviated *iwip*. Sufficient conditions to show that a given automorphism is an iwip can be pretty intricate, so we will relegate the finding of these elements in our image to Claim C in the appendix.

□

Again, we use result 3.1.1 to obtain the following corollary.

Corollary 5.1.7. *Suppose* $\mathrm{Out}(W_n)$ *acts geometrically on* X *a* $\mathrm{CAT}(0)$*space. Then* X *contains a rank one geodesic. In particular,* $\mathrm{Out}(W_n)$ *cannot act geometrically on a Euclidean Building or higher rank symmetric space.*

Chapter 6

Limit Sets of Universal Actions

6.1 Motivation

The goal of this chapter is to make a statement about the limit set of acylindrical actions which are universal, in a sense we will define below. In order to state why we should expect such a statement, we will remark on some existing structure for hyperbolic and relatively hyperbolic groups.

Theorem 6.1.1. *[19] Let X_0 and X_1 be hyperbolic spaces which are quasi-isometric. Then ∂X_i is well defined and*

$$\partial X_0 \cong \partial X_1$$

The following statement is commonly known as the Švarc-Milnor lemma, a proof of which can be found in [16].

Theorem 6.1.2. *[16] Let G be a group which acts geometrically on a proper geodesic metric space X. Then G is finitely generated, and X is quasi-isometric to any Cayley graph for G with a finite generating set. This implies in particular a QI-equivalence between any two spaces G acts on geometrically.*

The combination of these theorems allows us to talk about, for a hyperbolic group G, its boundary ∂G, without confusion. It should be noted that there are many possible metrics the boundary can be equipped with, but any natural metric will induce the same topology.

The following definitions are available in [20].

Definition 6.1.3. *For a connected, locally finite metric graph Γ with edge lengths*

1, the associated combinatorial horoball $\mathcal{H}(\Gamma)$ *is a graph with vertex set*

$$V(\mathcal{H}(\Gamma)) := \Gamma^0 \times \mathbb{N}$$

where the points (v,n) *and* $(v,n+1)$ *are connected by edges of length 1, and each level* $\Gamma^0 \times n$ *has an edge of length 1 between them if their distance in* Γ *was less than or equal to* 2^n*.*

Fact 6.1.4. *This space, for any such graph, is hyperbolic, and has a boundary consisting of a single point.*

Definition 6.1.5. *Let* G *be a group and* $\mathcal{H}$ *a finite collection of finitely generated subgroups. If* Γ *is a Cayley graph for* G*, then we construct a new space, called the* cusped space*, by attaching combinatorial horoballs to each coset of each element of* $\mathcal{H}$*. If this new space is hyperbolic, we say* $(G,\mathcal{H})$ *is* relatively hyperbolic*. The elements of* $\mathcal{H}$ *are called* peripheral subgroups.

This construction allows us to define a relative boundary of a relatively hyperbolic group $\partial(G,\mathcal{H})$, which is the boundary of the constructed cusped space. Bowditch, in [7], verifies for us that this boundary is unique up to homemorphism for any of the hyperbolic spaces on which G acts in this way. Additionally, we are assured that our various definitions of relative hyperbolicity, including those in the introduction, agree.

This leads us to wonder, because acylindrically hyperbolic groups are a generalization of relatively hyperbolic groups, can we define some sort of canonical notion for the boundary of an acylindrically hyperbolic group?

6.2 Universal Actions

We see initially that without additional hypotheses, these groups fail to have a well defined boundary. This is because there can be many hyperbolic, not coarsely-equivalent spaces they act on acylindrically, which can have distinct boundaries.

What we can ask about, however, is the limit set for these actions within the boundary. Here we will show that even given these assumptions, these limit sets fail to be homeomorphically rigid.

Finally, one last definition:

Definition 6.2.1. *For an acylindrically hyperbolic group, an action $G \circlearrowright X$ is called universal if it is acylindrical, X is hyperbolic, and all generalized loxodromics act as loxodromics.*

The next natural question to ask is whether all AH groups admit such actions. The answer is no, as proven recently by Carolyn Abbott, who gives a finitely generated (but not finitely presented group) which cannot have a universal acylindrical action. However, we will see that even when restricting our attention to this natural class of actions, the limit set will still fail to be well-defined. Recent work has been done on this by Abbott and Osin in a forthcoming paper, who assume more hypotheses than those below, and get some invariance of this structure. Their specified action is both universal and cobounded, which brings one closer to geometric actions, which are cocompact.

6.3 Non-Uniqueness of Limit Sets

Lemma 6.3.1. *Suppose a group action $G \circlearrowright X$ is acylindrical. Then if $H < G$, the natural action $H \circlearrowright X$ is also acylindrical. In particular, if $G \circlearrowright X$ is an acylindrically hyperbolic action, and $H \circlearrowright X$ is non-elementary, then it is also an acylindrically hyperbolic action.*

Note this does not mean acylindrical hyperbolicity passes automatically to subgroups. A subgroup can (and often does) have a strictly smaller limit set, so that condition needs to be checked as well. In particular, any finite or virtually cyclic subgroup of an AH group is not acylindrically hyperbolic.

Proof. Let $R(\epsilon), N(\epsilon)$ be the constants demonstrating acylindricity for $G \circlearrowright X$. Fix a value ϵ and choose two points x, y with $d(x,y) \geq R(\epsilon)$. Then because

$$\{g \in H | d(x, g.x) \leq \epsilon, d(y, g.y) \leq \epsilon\} \subset \{g \in G | d(x, g.x) \leq \epsilon, d(y, g.y) \leq \epsilon\}$$

we get also that

$$\#\left\{g \in H \left| \begin{matrix} d(x, g.x) \leq \epsilon \\ d(y, g.y) \leq \epsilon \end{matrix} \right.\right\} \leq \#\left\{g \in G \left| \begin{matrix} d(x, g.x) \leq \epsilon \\ d(y, g.y) \leq \epsilon \end{matrix} \right.\right\} \leq N(\epsilon).$$

This demonstrates acylindricity for the sub-action using the same constants.

By assumption the space X is hyperbolic so if the sub-action is non-elementary, we obtain that H is acylindrically hyperbolic. □

The following result is noted and used in [26]. We provide a proof here for the purpose of completeness.

Lemma 6.3.2. *[26] If a group action is geometric then it is also acylindrical.*

Proof. Suppose $G \circlearrowright X$ geometrically. Let $K \subset X$ be a compact fundamental domain for this action. Set $d = diam(K)$. We note by cocompactness that for any $x, y \in X$, there exists a group element $h \in G$ such that

$$d(x, h.y) \leq d.$$

We make one more claim, that is due to the action being by isometries. We claim that for all $\epsilon > 0, y \in X$

$$\{g \in G | d(y, g.y) \leq \epsilon\} = \{g \in G | d(h.y, g.(h.y) \leq \epsilon\}.$$

Now, for $\epsilon > 0$, pick $R(\epsilon) > d$. For any two points x, y, we can choose g such that $d(x, g.y) \leq d$, i.e. that both $x, g.y$ belong to the same translate of K. Without loss of generality, assume this translate is K itself. Then the set

$$\{g \in G | d(x, g.x) \leq \epsilon, d(y, g.y) \leq \epsilon\}$$

is exactly equal to the set

$$\{g \in G | d(x, g.x) \leq \epsilon, d(h.y, g.(h.y)) \leq \epsilon\}.$$

This set is a subset of the set of elements which translate K to a tile at distance a maximum of ϵ away, which is bounded because the group action is proper. This bound is a function of ϵ, so let this bound serve as $N(\epsilon)$ □

Lemma 6.3.3. *Let Γ be a Fuchsian group acting geometrically on $\mathbb{H}^2$. Then for the natural isometric embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{H}^3$, the induced action of $\Gamma \circlearrowright \mathbb{H}^3$ that comes from the inclusion $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$ is acylindrical.*

Proof. Label by X the original embedded copy of $\mathbb{H}^2$. Let $x \in X$ be a point in this subspace. Then x belongs to some fundamental domain K of the action $\Gamma \circlearrowright X$.

For any $\epsilon > 0$, let $N(\epsilon)$ be the (necessarily finite) number of translates of K that intersect $\mathcal{N}_\epsilon(K)$. That is to say $N(\epsilon) = |S|$ where

$$S = \{g \in \Gamma \mid gK \cap \mathcal{N}_\epsilon(K) \neq \varnothing\}$$

For any $g \in \Gamma$, all of which are loxodromic, because X is the only totally geodesic copy of $\mathbb{H}^2$ that Γ acts on geometrically, the geodesic axis lies entirely within X. What this tells us is that for any point $z \in \mathbb{H}^3$, there exists a point $x \in X$, such that

$$d(z, g.z) \geq d(x, g.x).$$

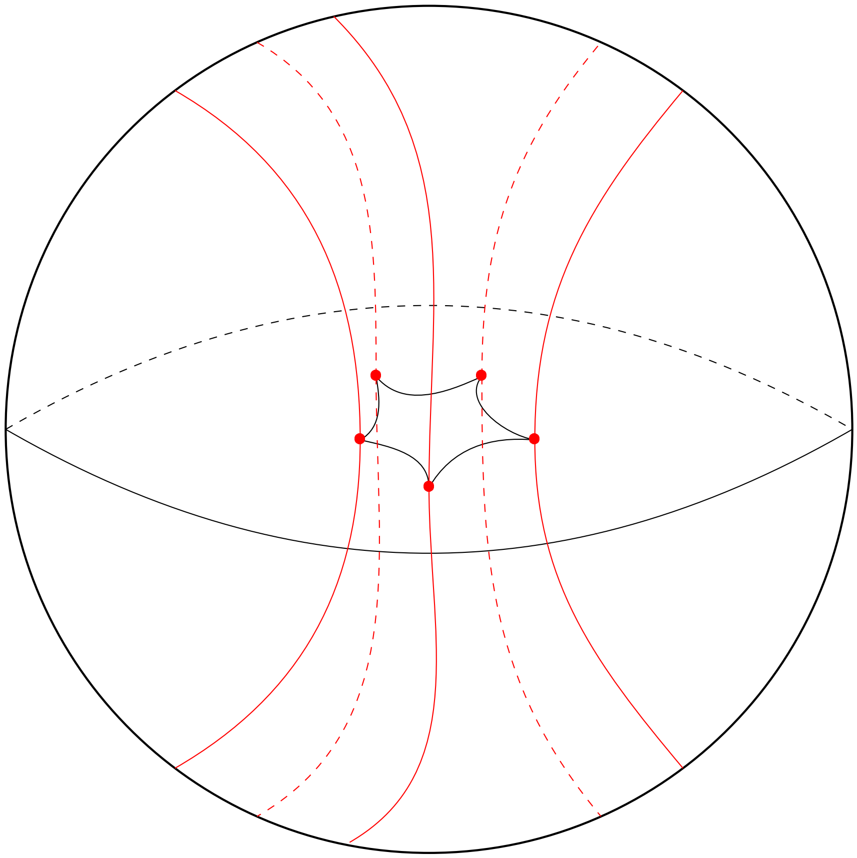


Figure 6.1: The convex hull of these geodesics serve as a fundamental domain

From this we can determine that

$$\{g \in \Gamma | d(z, g.z) \leq \epsilon\} \subset \{g \in \Gamma | d(x, g.x) \leq \epsilon\}.$$

The size of this right-hand set is bounded by $N(\epsilon)$, which thus also bounds the size of the left-hand set. Because for any points $w, z \in \mathbb{H}^3$

$$\{g \in \Gamma | d(z, g.z) \leq \epsilon, d(w, g.w) \leq \epsilon\} \subset \{g \in \Gamma | d(z, g.z) \leq \epsilon\}$$

we get that the action is acylindrical with constants $N(\epsilon)$ as chosen before, and any value of $R(\epsilon) > 0$. □

Theorem 6.3.4. *There exist acylindrically hyperbolic groups G, which admit two* *different universal actions* $G \circlearrowright X$*, such that in the representations*

$$\rho_1 : G \to \text{Isom}(X),\ \ \rho_2 : G \to \text{Isom}(X)$$

the limit sets $\Lambda_1(G)$ *and* $\Lambda_2(G)$ *are not homeomorphic.*

Proof. The space in question will be $\mathbb{H}^3$, and the group a closed surface group.

The following argument will work for the fundamental group of any closed surface of genus ≥ 2. However to be explicit, we will consider $G = \pi_1(\Sigma_2)$. A presentation for G is

$$G \cong \langle a, b, c, d \mid [a, b][c, d] = 1 \rangle.$$

Now, consider the action $G \circlearrowright \mathbb{H}^2$. This action that of deck transformations, recognizing $\mathbb{H}^2$ as the universal cover, $\widetilde{\Sigma_2}$. Because the quotient of this space is a closed manifold, the action is geometric, meaning it is acylindrical. Furthermore, it has full limit set; that is to say $\partial G = \partial \mathbb{H}^2 \cong S^1$. Finally, every element in this group action acts as a loxodromic, meaning it is a universal action.

By the lemma above, this action extends to an acylindrical action on $\mathbb{H}^3$, that has limit set $\Lambda(G) \cong S^1$, with all elements continuing to act loxodromically.

Now we want to exhibit another universal action by this group on $\mathbb{H}^3$ with distinct limit set.

Let ϕ be a Pseudo-Anosov element of $MCG(\Sigma_2)$. We can construct a hyperbolic 3-manifold, the geometry of which is given to us by [30], by taking the space $\Sigma_2 \times [0, 1]$, and identifiying $\Sigma_2 \times \{0\}$ with $\phi(\Sigma_2) \times \{1\}$. Denote this manifold by M. We get a decomposition of $\pi_1(M) = \pi_1(\Sigma_2) \rtimes_{\phi^*} \mathbb{Z}$, where $\phi^* \in Aut(\pi_1(\Sigma_2))$ is induced by ϕ.

Again because the quotient is a closed manifold, the natural covering space action $\pi_1(M) \circlearrowright \mathbb{H}^3$ is geometric, and therefore acylindrical. Also by the geometric nature of the action, we get $\partial \pi_1(M) = \partial \mathbb{H}^3 \cong S^2$.

We use a fact proved by Thurston in [31], Corollary 8.1.3 in Chapter 8, to assert that in fact the $\pi_1(\Sigma_2)$ has the same limit set as the entire group, by normality. Specifically, $\Lambda(\pi_1(\Sigma_2)) = \partial\mathbb{H}^3 \cong S^2$.

Now we need to know that all elements act as loxodromics. Because the action is geometric (and acylindrical), none will act as parabolics. Therfore, we need to rule out the possibility of elements acting elliptically. However, because $\mathbb{H}^3$ is CAT(0), we know that any element acting elliptically will have a fixed point on the interior of $\mathbb{H}^3$. We note that all elements of $\pi_1(M) = \pi_1(\Sigma_2) \rtimes_{\phi^*} \mathbb{Z}$ are infinite order. This implies that none can act elliptically. If $g \in \pi_1(M)$ was elliptic, then for some $k \in \mathbb{Z}$, g^k would fix a point, as would g^{mk} for all $m \in \mathbb{Z}$. This is an infinite number of elements fixing a point, which violates the properness assumption of our action.

Therefore, of the induced action $G = \pi_1(\Sigma_2) \circlearrowright \mathbb{H}^3$ has the following properties:

- It is acylindrical
- The space is hyperbolic
- It has limit set S^2
- All elements act as loxodromics

Therefore this is a universal action with a distinct (homeomorphism type) of limit set for the group G. □

6.4 Contrast with Relatively Hyperbolic Groups

We defined the notion of a relatively hyperbolic group in the background. However, in the current literature there are many distinct definitions of relative hyperbolicity, all of which coincide. We want to present one more classification which will be useful. First we must develop the notion of a convergence group action, which is explored in [7]. It is helpful in the following definition to imagine M as being the boundary of a hyperbolic space.

Definition 6.4.1. *A group action $G \circlearrowright M$, for M compact, is called a* convergence action *if the induced action on the set of distinct triples*

$$\{(m_1, m_2, m_3) | m_i \in M, m_i \neq m_j \text{ for } i \neq j\}$$

is properly discontinuous.

To this end, we want actions which are a certain kind of convergence action.

Definition 6.4.2. *A convergence action is called* geometrically finite *if every $m \in M$ is such that one of the following is true:*

- $\# stab_G(m) < \infty$.
- *There exist a sequence $g_i, i \in \mathbb{N}$ of group elements, and points $a, b \in M$ such that $g_i m \to a$ and $g_i m' \to b$ for all $m' \neq m$.*

We call a group acting on a hyperbolic space a *geometrically finite* action, if its induced action on the boundary of that space is a geometrically finite convergence action. Finally, we get the following:

Theorem 6.4.3. *[7] A pair $(G, \mathcal{H})$ is relatively hyperbolic if G admits a geometrically finite action on a proper, hyperbolic space X such that the set $\mathcal{H}$ consists of exactly the maximal parabolic subgroups and each of these are finitely generated.*

The goal of this section is to prove the following assertion:

Theorem 6.4.4. *Any relatively hyperbolic group with infinite peripheral subgroups acts as in [7] on hyperbolic spaces that are not equivariantly quasi-isometric.*

We start with a lemma, mentioned in [21] but not proven, which does most of the heavy lifting for the theorem.

Lemma 6.4.5. *Let $f(x) := 2^x$ and $g(x) := 2^{2^x}$. Then for any finitely generated infinite group H, the combinatorial horoballs $\mathcal{C}_f(H)$ and $\mathcal{C}_g(H)$ are not quasi-isometric.*

Furthermore, there are no values $N, M \in \mathbb{R}_{\geq 0}$ such that these spaces truncated above the N and M levels respectively are QI.

We note proving the later case implies the first statement, allowing $M = N = 0$.

Proof. Here we might be tempted to apply the idea, explained in [10], that the growth of balls in a graph is a quasi-isometry invariant, because these two graphs by design have different growth rates. However, this statement is made specifically for graphs which arise as Cayley graphs of finitely generated groups. Implicitly in this formulation, we are using the assumption that our graph has uniformly bounded valence, which regrettably is not true for these combinatorial horoballs. We must therefore do a little more work.

The first thing we observe about these spaces is that they are δ−hyperbolic for some δ, by [21] Corollary 2.28, and the boundaries are a single point. In the geodesic ray definition of the boundary, these points are the equivalence class of rays that point straight 'upwards', consisting entirely of vertical edges. Therefore any (c, c) quasi-isometry $\phi : \mathcal{C}_f(H)_N \to \mathcal{C}_g(H)_M$, which acts by homeomorphism on the boundaries of hyperbolic spaces, must take these geodesic rays to ones in the equivalence class of the boundary element on the right. In other words, the images of these rays must stay a bounded distance from a ray that enters into every level of the cusp $\mathcal{C}_g(H)$, meaning that it too, intersects every 'level set'.

A quick shorthand we will use will be $d_f := d_{\mathcal{C}_f(H)}$ and $d_g := d_{\mathcal{C}_g(H)}$.

Consider two rays, $\lambda_0 := \{x_0\} \times [N, \infty)$ and $\lambda_1 := \{x_1\} \times [N, \infty)$ such that $d_f(x_0, x_1) > R(c)$, where $R(c)$ is a constant that will be specified later. Our sub-graphs are infinite diameter, so this is guaranteed to be possible . Define two functions $h_1, h_2 : [N, \infty) \to [M, \infty)$ such that $h_i(k)$ is the height of the point

$\phi(\{x_i\} \times \{k\})$. Assuming the QI constant of ϕ to be $c \in \mathbb{N}$ then it follows that

$$d_{\mathcal{C}_g}(\phi(x_i) \times \{k\}, \phi(x_i) \times \{k+1\}) < 2c$$

as the distance of the preimage of these points is precisely 1. This tells us then that

$$h_i(k) < M + 2ck$$

even if the image takes the largest possible 'step' of $2c$ for each increase of 1 on the left.

To break QI, we will find points which are relatively far apart on the left, and force them to get uncomfortably close on the right. These points will be along the rays λ_i. First we note, because geodesics in a combinatorial horoball are paths that move up in the horoball, across a level set, and back down by [20] Lemma 3.10, that the distance must be at least as follows.

$$d_f := d_f(\lambda_0(n), \lambda_1(n)) \geq \min_{n \leq k \leq R} \{\lceil log_{2^k}(R) \rceil + 2(k-n)\}$$

Actually k is bounded above by $\max\{R, n\}$, but we can assume $R \geq n$ for whatever points we look for to break quasi-isometry. The reason k is only that large is because if $k = R$, then at that level the two points are connected by a single edge, as points closer than $2^k > R$ will be connected. Therefore, going any further up would not gain us anything.

Next we consider the distance of the image of these points. Because ϕ is a quasi-isometry, and we assumed the λ_i were geodesic rays, the images are quasi-geodesics. This means they are $B-$boundedly far from geodesics. The geodesic representatives in the equivalence class are those rays which point directly upwards, similar to λ_i. That means, in any 'level set' n, the distance between $\phi(\lambda_0)$ and $\phi(\lambda_1)$ (when

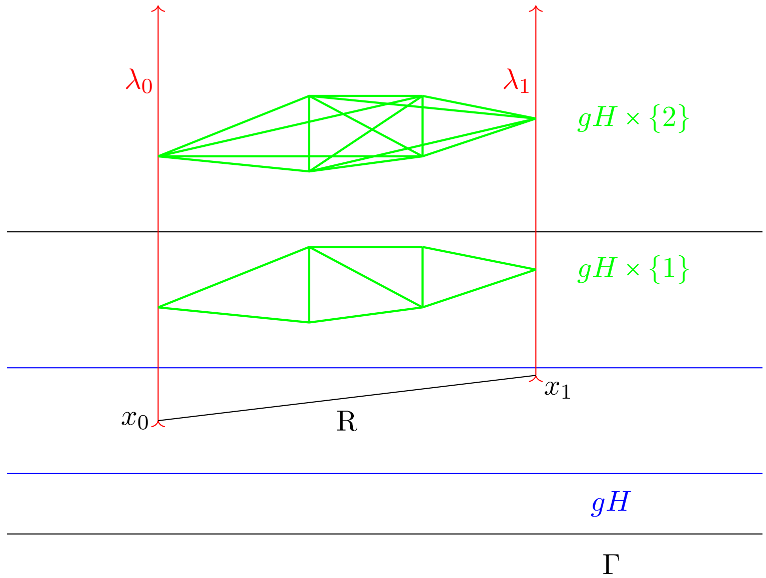


Figure 6.2: Combinatorial Horoball

projected down to Γ) is bounded above by $R + 2B$, where B is the distance that a (c, c)-quasigeodesic is in X from a true geodesic. Specifically, this tells us that

$$d_g := d_g(\phi(\lambda_0(n)), \phi(\lambda_1(n))) \leq \lceil log_{2^{2^n}}(R + 2B) \rceil + 2$$

.

We are assuming ϕ is a $c - QI$, so then it must be true for any values of R, n that

$$\frac{d_f}{c} - c < d_g$$

We now endeavor to slog through some algebra

$$\begin{aligned} \frac{d_f}{c} - c < d_g &\implies \frac{1}{c} \min_{n \leq k \leq R} \{\lceil log_{2^k}(R) \rceil + 2(k - n)\} - c < \lceil log_{2^{2^n}}(R + 2B) \rceil + 2 \\ &\implies \frac{1}{c} \min_{n \leq k \leq R} \{\lceil log_{2^k}(R) \rceil + 2(k - n)\} - c - 2 < \lceil log_{2^{2^n}}(R + 2B) \rceil \end{aligned}$$

We now choose values of R, n, which can be taken as large as desired, to fulfill the following condition. Set

$$D := log_{2^{2^n}}(R + 2B).$$

Choose R, n such that $D = D(c)$ is such that $\lceil D \rceil > \max(c, 2)$. Now we can divide both sides by $\lceil log_{2^{2^n}}(R + 2B) \rceil$.

$$\frac{1}{c}\frac{\min_{n \le k \le R}\{\lceil log_{2^k}(R) \rceil + 2(k-n)\}}{\lceil log_{2^{2^n}}(R + 2B) \rceil} - 2 < 1$$
$$\frac{\min_{n \le k \le R}\{\lceil log_{2^k}(R) \rceil + 2(k-n)\}}{\lceil log_{2^{2^n}}(R + 2B) \rceil} < c$$
$$\min_{n \le k \le R}\{\frac{\lceil log_{2^k}(R) \rceil + 2(k-n)}{\lceil log_{2^{2^n}}(R + 2B) \rceil}\} < 3c$$

Using log rules, we get

$$\log_{2^{2^n}}(R + 2B) = \frac{1}{2^n \ln(2)} \ln(R + 2B)$$

and

$$log_{2^k}(R) = \frac{1}{k \ln(2)} \ln(R)$$

This changes our previous expression into

$$\min_{n \le k \le R}\{\frac{\lceil \frac{1}{k \ln(2)} \ln(R) \rceil + 2(k-n)}{\lceil \frac{1}{2^n \ln(2)} \ln(R + 2B) \rceil}\} < 3c$$

To make this expression easier to work with, we note we can make this left side larger without loss of generality, because we are trying to show they are bounded above. To do this, we remove the ceiling function from the denominator, and remove the ceiling function but add one to the top. To this end, we get

$$\min_{n \le k \le R}\{\frac{\frac{1}{k \ln(2)} \ln(R) + 2(k-n) + 1}{\frac{1}{2^n \ln(2)} \ln(R + 2B)}\} < 3c$$

$$\min_{n\leq k\leq R}\{\frac{2^n}{k}\ln(-2B)+\frac{2k-2n+1}{\frac{1}{2^n\ln(2)}\ln(R+2B)}\}<3c$$

$$\min_{n\leq k\leq R}\{\frac{2^n}{k}\ln(-2B)+\frac{2^n\ln(2)(2(k-n)+1)}{\ln(R+2B)}\}<3c$$

We notice both terms of this expression are positive, so in order to bound this above by $3c$, which is independent of R, n, it must be true that both terms individually are bound above by $3c$. Let's start with the first term.

$$\frac{2^n}{k}\ln(-2B)<3c$$

We note that $\ln(-2B)$ is constant as well, so this is equivalent to stating that the fuction $\frac{2^n}{k}$ is bounded above, i.e. that $\frac{2^n}{k}<M$. This tells us that if we choose sufficiently large values of n, that the difference $k-n$ is unbounded.

Now we look at the second term. Note

$$\frac{2^n\ln(2)}{\ln(R+2B)}=\frac{1}{D(c)}$$

This tells us it must be the case that

$$2(k-n)+1<3cD(c).$$

The right side is a constant dependent on c. However, the restrictions on the first term already told us that $k-n$ must be unbounded as we let n get large. Thus we get a contradiction, and ϕ cannot be a $c-QI$ for any value of c. □

Proof. of 6.4.4

Let (G,H) be a relatively hyperbolic pair, such that H is infinite. We construct two spaces, X_1, X_2 as follows. X_i will be a copy of $\Gamma(G)$, the Cayley graph, with combinatorial horoballs glued on to all cosets of H. In X_1, allow the scaling function to be 2^n, and in X_2 allow the scaling function to be 2^{2^n}. Again by [21], we note that this resultant space is hyperbolic for both cases. Remark 2.23 in this paper was the inspiration for this result, where the authors conjecture this result.

Consider a potential equivariant $c-$quasi-isometry ϕ between these spaces. Because it is equivariant, it must induce a quasi-isometry on the quotient spaces. This means that ends are taken to ends, as it is a homeomorphism on the boundary. Coarsely, then, it must map the images of horoballs in the quotient to images of horoballs on the right hand space. Moving upstairs, the original quasi-isometry must coarsely take horoballs to horoballs. By our lemma, this is not possible for a quasi-isometry, for the given scaling functions. □

6.4.1 The Big Picture

We summarize now in the following table the existing knowledge on the limit set rigidity and quasi-isometric rigidity for classes of groups of interest

Group Property	Action Type on Hyperbolic X	Limit Set	QI type
Hyperbolic	Geometric	Yes	Yes
Relatively Hyperbolic	Geometrically Finite	Yes	No (H.)
Acylindrically Hyperbolic	Universal	No (H.)	No

Table 6.1: Quasi-isometric and Limit Set Rigidity of Hyperbolicity Generalizations

Chapter 7

Further Questions and Future Work

7.1 Questions

Much of this work leads naturally to other questions. We pose a few of them here.

Question 7.1.1. *Does* Out(W_n) *also act on a (necessarily modified) free factor complex? Is it hyperbolic? How does the lack of transvections reduce the size of this space?*

Question 7.1.2. *What are some implications of this restated version of the Rank Rigidity Conjecture? Can we now solve this question for a wider class of groups? Rank Rigidity is known for* CAT(0) *cube complexes and 2-dimensional* CAT(0) *spaces.*

Question 7.1.3. *Under what conditions is the class of A.H. groups closed under semi-direct products? Amalgamated products? Clearly the answer is not "any conditions" for either of these.*

Question 7.1.4. *What is a homeomorphism type of a limit set of a universal action for* $\overline{\mathrm{B_n}}$*? How does it change when embedded into* Out(F_n)*?*

Specifically, we can ask what a limit set for a largest, in the sense of Osin and Abbot, action looks like.

Question 7.1.5. *What is the divergence of* Out(W_n)*? Does it being acylindrically hyperbolic help us?*

Question 7.1.6. *We know how* CAT(0) *groups intersect acylindrically hyperbolic groups. What about systolic groups? Although closely related to* CAT(0) *groups, neither property implies the other.*

7.2 Other Continuation

Hyperbolic and relatively hyperbolic are both properties that can be spoken of for groups as well as spaces. We note, however, that acylindrical hyperbolicity only makes sense for groups. There is a notion that is somewhat in parallel which is designed with spaces in mind. This property is called being 'hierarchically hyperbolic', and is discussed in [3]. Though the definition is quite intimidating, an exceptionally readable introduction is available in the blog of Alessandro Sisto.

Naturally, one calls a hierarchically hyperbolic group one which acts in the correct way on an HHS. It turns out that a hierarchically hyperbolic group also admits an acylindrical action on a hyperbolic space [3], so this new property is in fact intimately related to acylindrically hyperbolic groups. Note however, this action is not guaranteed to be non-elementary nor the space even infinite diameter.

Based on this information, we may wonder what relationships the ideas in this document have to the new world of the HHS. Do the limit set results tell us anything about the boundary of an HHS? Can we recover some kind of hierarchical structure of a group from an acylindrical action on an arbitrary hyperbolic space (i.e. not one that arises from an HHS)?

Chapter 8

Appendix

8.1 Claim A

Claim A. *The natural map*

$$\bar{\iota} : \bar{B_n} \to \mathrm{Out}(F_n)$$

is well defined

Proof. We will show this by demonstrating that the kernel of the map $\iota : B_n \to \mathrm{Out}(F_n)$, $\iota = \pi \circ \iota'$ contains $Z(B_n)$, the center of the group, recalling that $\bar{B_n} := B_n/Z(B_n)$ and $\pi : \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$ is the quotient map. Label by β the generator of $Z(B_n)$, $\beta = (\sigma_1 \ldots \sigma_{n-1})^n$.

If $\alpha \in Z(B_n)$, then $\alpha = \beta^k$ for some $k \in \mathbb{Z}$, because the center is infinite cyclic. Therefore we only need to show $\iota(\beta) = id$. We demonstrate its effect on an arbitrary generator x_i $1 \le i \le n$.

$$\begin{aligned}
\iota(\beta)(x_i) &= \iota((\sigma_1 \ldots \sigma_{n-1})^n)(x_i) \\
&= \iota((\sigma_1 \ldots \sigma_{n-1}))^n(x_i) \\
&= \iota((\sigma_1 \ldots \sigma_{n-1}))^{n-1}(x_{i+1}) \\
&\vdots \\
&= \iota((\sigma_1 \ldots \sigma_{n-1}))^{i}(x_n) \\
&= \iota((\sigma_1 \ldots \sigma_{n-1}))^{i-1}(x_1) \\
&\vdots \\
&= x_i
\end{aligned}$$

This tells us that the map is well defined, i.e. that we can take as our domain the group $\bar{B_n}$.

□

8.2 Claim B

Claim B. *Any element of the braid group has a power which induces an automorphism of F_n which is a product of partial conjugations (i.e. one that is reducible).*

Proof. For historic reasons, we will label the map $\alpha : B_n \to Aut(F_n)$. This map is defined as follows:

$$\alpha_{\sigma_i}(x_m) = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } m = i \\ x_i & \text{if } m = i+1 \\ x_m & \text{else} \end{cases}$$

The braid group has a naturally defined subgroup known as the *pure braid group*, which consists of those elements which project to the trivial element of the symmetric group. In other words, all strands end up in the same position, after winding. Because this represents the kernel of a surjective map onto Σ_n, this subgroup is finite index in B_n.

A common generating set for P_n, also due to Arin [1], is that consisting of elements $A_{i,j}$ for all pairs of $i, j \in \{1, \dots n\}$, where

$$A_{i,j} = \sigma_{j-1} \sigma j - 2 \dots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}$$

We will show that the map α takes all pure braid elements to reducible automorphisms.

m < i

$$\alpha_{A_{ij}}(x_m) = \mathbf{x_m}$$

m=i

$$\begin{aligned}
\alpha_{A_{ij}}(x_m) &= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}^{-1}}(x_m)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2}(x_m)\ [x_m \text{ is fixed by } \sigma_i \text{ for } i > m]\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i}(x_m x_{m+1} x_m^{-1})\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}}(x_m x_{m+1}\mathbf{x_m} x_{m+1}^{-1} x_m^{-1})\ [\text{note this is } x_m^{x_m x_{m+1}}]\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+2}}(x_m(x_{m+1}x_{m+2}x_{m+1}^{-1})\mathbf{x_m}(x_{m+1}x_{m+2}x_{m+1}^{-1})^{-1}x_m^{-1})\\
&= x_m(x_{m+1}\ldots x_{j-1}x_j x_{j-1}^{-1}\ldots x_{m+1}^{-1})\mathbf{x_m}\\
&\quad (x_{m+1}\ldots x_{j-1}x_j x_{j-1}^{-1}\ldots x_{m+1}^{-1})^{-1}x_m^{-1}\\
&= \mathbf{x_m}^{x_m(x_{m+1}\ldots x_{j-1}x_j x_{j-1}^{-1}\ldots x_{m+1}^{-1})}
\end{aligned}$$

m = i+1 = j

$$\begin{aligned}
\alpha_{A_{ij}}(x_m) &= \sigma_i^2(x_m)\\
&= \sigma_i(x_i)\\
&= x_i x_m x_i^{-1} = \mathbf{x_m}^{x_i}
\end{aligned}$$

i < m < j-1

$$\begin{aligned}
\alpha_{A_{ij}}(x_m) &= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}^{-1}}(x_m)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_m^{-1}}(x_m)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{m-1}^{-1}}(x_{m+1})\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_m}(x_{m+1}) \text{ m+1 is fixed by indices more than one less}\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{m+1}}(x_m)\\
&= \mathbf{x_m}
\end{aligned}$$

m=j

$$
\begin{aligned}
\alpha_{A_{ij}}(x_j) &= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}^{-1}}(x_j)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}}(x_j^{-1}x_{j-1}x_j)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-3}^{-1}}(x_j^{-1}x_{j-1}^{-1}x_{j-2}x_{j-1}x_j)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i^2}(x_j^{-1}\ldots x_{i+1}\ldots x_j)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}\sigma_i}(x_j^{-1}\ldots x_{i+2}^{-1}x_ix_{i+2}\ldots x_j)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}}(x_j^{-1}\ldots x_{i+3}^{-1}\underline{x_{i+2}^{-1}x_ix_{i+1}x_i^{-1}x_{i+2}}x_{i+3}\ldots x_j)\\
&= \alpha_{\sigma_{j-1}\ldots\sigma_{i+1}}(x_j^{-1}\ldots x_{i+3}^{-1}\underline{x_{i+1}^{-1}x_ix_{i+1}x_{i+2}x_{i+1}^{-1}x_i^{-1}x_{i+1}}x_{i+3}\ldots x_j)\\
&\quad\vdots\\
&= \alpha_{\sigma_{j-1}}(x_j^{-1}x_{j-2}^{-1}\ldots x_{i+1}^{-1}x_ix_{i+1}\ldots x_{j-1}x_{j-2}^{-1}\ldots x_i^{-1}x_{i+1}\ldots x_j\\
&= x_{j-1}^{-1}x_{j-2}^{-1}\ldots x_{i+1}^{-1}x_ix_{i+1}\ldots x_{j-1}x_jx_{j-1}^{-1}x_{j-2}^{-1}\ldots x_i^{-1}x_{i+1}\ldots x_{j-1}\\
&= \mathbf{x_j}^{x_{j-1}^{-1}\ldots x_{i+1}^{-1}x_i\ldots x_{j-1}}
\end{aligned}
$$

j < m

$$\alpha_{A_{ij}}(x_m) = \mathbf{x_m}$$

To summarize this, we list the effect $\alpha_{A_{ij}}$ has on x_m. We do this by spelling out the *conjugating word*, where e represents the identity.

Generator	Conjugating Word
$m < i$	e
$m = i$	$x_i \ldots x_{j-1}x_jx_{j-1}^{-1}\ldots x_{i+1}^{-1}$
$i < m < j$	e
$m = j$	$x_{j-1}^{-1}\ldots x_{i+1}^{-1}x_i\ldots x_{j-1}$
$j < m$	e

Table 8.1: Effect of pure braid generators on free generators

□

8.3 Claim C

We need to show that the desired action is non-elementary by finding two elements which will act as loxodromics. In order to do so we make use of a criterion for irreducibility (with irreducible powers) first discovered by Gersten and Stallings in [17].

Lemma 8.3.1. *[17] Let $\mathcal{O}$ be an outer automorphism of F_n and let $M_{\mathcal{O}}$ be the associated matrix which transforms the standard basis elements of $\mathbb{Z}^n = F_n^{ab}$. Then if the characteristic polynomial for $M_{\mathcal{O}}$ is irreducible over $\mathbb{Q}$ and the matrix itself is primitive (nonnegative entries, and some positive power has all postive entries), the automorphism is irreducible with irreducble powers (iwip).*

The way we use this lemma is inspired by examples stated later in [17]. In particular, an automorphism with the following matrix meets the above criteria.

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \ddots & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 4 & 4 & \dots & 4 \end{bmatrix}$$

Elementary matrix operations verify that, assuming A to be an n by n matrix,

$$char(A) = x^n - 4x^{n-1} - 4x^{n-2} \dots - 4x - 1$$

which has no rational roots, and that A^k has all positive entries.

To find a second element, we use the following, similar matrix.

$$B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \ddots & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 8 & 8 & \dots & 8 \end{bmatrix}$$

Again,

$$char(B) = x^n - 8x^{n-1} - 8x^{n-2} \dots - 8x - 1$$

which has no rational roots, and B^n contains only positive entries.

A quick calculation of $ABA^{-1}B^{-1}$ verifies that these matrices do not commute, indicating that any automorphisms inducing these matrices also must not commute. Furthermore, using matrix operations we can verify that these matrices will not share a common power, indicating that the corresponding automorphisms do not share a common power.

Claim C. *The image of the map*

$$\iota : \text{Out}(W_n) \hookrightarrow \text{Out}(F_{n-1})/\langle\langle R \rangle\rangle$$

contains atleast two elements which factor through outer automorphisms that are iwip.

Proof. The goal will be to find two outer automorphisms whose induced matrices are A and B as above.

We make note of three kinds of automorphisms availabale in our image. First, we have the standard permutation group Σ_{n-1} that rotates around the generators at will (but not w_1). Second, we have partial conjugations by letters other than w_1. An example of this is $\mathcal{C}_{i,k}(w_i) = w_k w_i w_k$, which when mapped to $\text{Out}(F_{n-1})$ looks as follows:

$$\iota(\mathcal{C}_{i,k}(x_{i-1})) = x_{k-1}x_{i-1}^{-1}x_{k-1}$$

Finally, any partial conjugation by w_1 has the effect of changing the sign of any copy of the free group generator conjugated by. For example, if we compose $\mathcal{C}_{i,1}$ with the above, we recover

$$\iota(\mathcal{C}_{i,1} \circ \mathcal{C}_{i,k}(x_{i-1})) = x_{k-1}x_{i-1}x_{k-1}$$

Using pieces of this form, as well as symmetric permutations, we can find automorphisms in the image as follows. Let $\phi_A \in \mathrm{Out}(F_n)$ be as follows:

$$\phi_A(x_i) = x_{i+1}, \quad i \neq n-1, \quad \phi_A(x_n - 1) = x_{n-1}^2 x_{n-2}^2 \dots x_1 \dots x_{n-2}^2 x_{n-1}^2$$

We see here that the associated matrix is A as above. Analagously, we define ϕ_B as

$$\phi_B(x_i) = x_{i+1}, \quad i \neq n-1, \quad \phi_B(x_n) = x_{n-1}^4 x_{n-2}^4 \dots x_1 \dots x_{n-2}^4 x_{n-1}^4$$

Now, we would like to show that these two automorphisms do not share both limit points (indicating that together, they make up strictly more than 2 limit points for the action). We note that if they did share endpoints, then $\phi_B \in E(\phi_A)$ in the sense, by Lemma 6.5 in [14]. This is a contradiction, because ϕ_B is infinite order and does not share a power with ϕ_A, yet $E(\phi_A)$ must be virtually cyclic. □

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