Tensors – 2 (eigenvalue problem)

Eigenvalue Problem

Transformation $\underline{T}$: $\underset{\sim}{u} \rightarrow \underline{T} \cdot \underline{u}$

Question: Are there any particular vectors, $\underset{\sim}{x}$, such that

$\underset{\sim}{T} \cdot \underset{\sim}{x}$ is parallel to $\underset{\sim}{x}$? (may stretch, but no rotation)

Such $\underset{\sim}{x}$ are called eigenvectors of tensor $\underset{\sim}{T}$

$$\underline{T} \cdot \underset{\sim}{x} = \underbrace{\lambda}_{\substack{\text{eigenvalue} \\ \text{(stretch ratio)}}} \overbrace{\underset{\sim}{x}}^{\text{eigenvector}}$$

equivalent to: $(\underset{\sim}{T} - \lambda \underset{\sim}{I}) \cdot \underset{\sim}{x} = 0$

or $(T_{ij} - \lambda \delta_{ij}) x_j = 0$ for each $i = 1,2,3$

This is a system of 3 linear algebraic eq. for unknowns $x_1 x_2$

$$\left.\begin{aligned} (T_{11} - \lambda) x_1 + T_{12} x_2 + T_{13} x_3 &= 0 \\ T_{21} x_1 + (T_{22} - \lambda) x_2 + T_{23} x_3 &= 0 \\ T_{31} x_1 + T_{32} x_2 + (T_{33} - \lambda) x_3 &= 0 \end{aligned}\right\}$$

trivial solution $x_1 = x_2 = x_3 = 0$

provides no information

Any other solutions? — Only if Det $= 0$

$$\text{Det} \left\| \begin{matrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{matrix} \right\| = 0$$

- cubic equation for λ.

Suppose λ_0 is a root. Then for $\lambda = \lambda_0$, det $= 0$ and there is

a non-trivial solution $(x_1^0, x_2^0, x_3^0) = \underset{\sim}{x}^0$

$$\underset{\sim}{T} \cdot \underset{\sim}{x}^0 = \lambda_0 \underset{\sim}{x}^0$$

Cubic eq-n has 3 roots

If $a+ib$ is a root, then: $a-ib$ also a root

$\Rightarrow$ either 1 or 3 real roots

For a symmetric matrix T_{ij} : all 3 roots are real (without proof)

Procedure of solution:

- find roots of the cubic eq
- for each root λ : substitute it into the system of 3 eq:

$$\left.\begin{array}{l} (T_{11}-\lambda)\,x_1 + T_{12}\,x_2 + T_{13}\,x_3 = 0 \\ \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \end{array}\right\}$$

and find eigenvector (x_1, x_2, x_3)

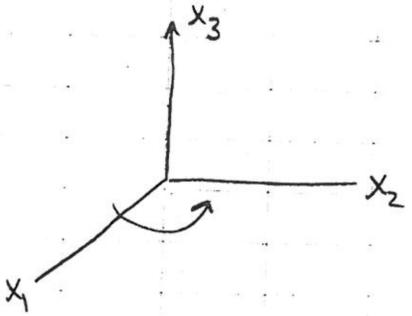
- Will have 3 pairs (solutions of the eigenvalue problem)

$(\lambda_1, \underset{\sim}{x}^{(1)})$ $\leftarrow$ $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)})$

$(\lambda_2, \underset{\sim}{x}^{(2)})$

$(\lambda_3, \underset{\sim}{x}^{(3)})$

Example: $90°$-rotation about X_3 axis



as found earlier: $T_{ij} = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$\text{Det} \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda^2 + 1) = 0$$

One real root (non-symmetric T! 3 real roots not guaranteed)

Eigenvector corresponding to $\lambda = 1$: from the system

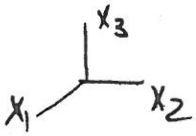
$$\left.\begin{aligned} (T_{11} - \lambda)x_1 + T_{12}x_2 + T_{13}x_3 &= 0 \\ T_{21}x_1 + (T_{22} - \lambda)x_2 + T_{23}x_3 &= 0 \\ T_{31}x_1 + T_{32}x_2 + (T_{33} - \lambda)x_3 &= 0 \end{aligned}\right. \Rightarrow \left.\begin{aligned} -x_1 - x_2 &= 0 \\ x_1 - x_2 &= 0 \\ 0 &= 0 \end{aligned}\right\}$$

$\Rightarrow$ $\boxed{X_3 - \text{axis}}$ – geometrically obvious

In this case, the eigenvalue problem could be solved just by observation.

Note: eigenvector is determined to within length. This corresponds to the nature of the problem: we are looking for direction in space, any vector along it determines direction.
Convention: eigenvectors are assumed to be of unit length ($\mathbf{e}_3$) for convenience & standartization

Example: Mirror reflection about $x_1 x_2$ plane



$$T_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

symmetric, 3 real roots

Eq-n for eigenvalues: $(1-\lambda)^2(1+\lambda) = 0$

$\lambda_1 = \lambda_2 = 1$ (double root)

$\lambda_3 = -1$

Eigenvectors:

For $\lambda_1 = \lambda_2 = 1$: $x_3 = 0$, $x_1 x_2$ - arbitrary
- any direction in $x_1 x_2$ plane

For $\lambda_3 = -1$: x_3 - axis

Example : two successive rotations

(expect : $\lambda = 1$ - no stretch)

Ⓐ 90° about x_3 countercloc

Ⓑ α-angle about clockwise

As found earlier:

$$T_{ij} = \begin{vmatrix} 0 & -\cos\alpha & -\sin\alpha \\ 1 & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha \end{vmatrix}$$

not symmetric

cubic eq-n (eigenvalue problem)

$$\det \begin{vmatrix} -\lambda & -\cos\alpha & -\sin\alpha \\ 1 & -\lambda & 0 \\ 0 & -\sin\alpha & \cos\alpha - \lambda \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda\cos\alpha - 1 - \lambda - \lambda^2) = 0$$

$\lambda = 1$ - the only real root - check

(as expected) (matrix not symmetric)

Eigenvector:
(axis of one equiv. rotation)

$$\left.\begin{aligned} -x_1 - \cos\alpha \cdot x_2 - \sin\alpha \cdot x_3 &= 0 \\ x_1 \text{ [redacted] } x_2 &= 0 \\ -\sin\alpha \, x_2 - (1 - \cos\alpha)\, x_3 &= 0 \end{aligned}\right\}$$

Intersection of two planes:

$$\begin{cases} x_1 = x_2 \\ x_1 = -\dfrac{\sin\alpha}{1 + \cos\alpha} x_3 \end{cases}$$

(only two eq-ns are independent, the 3rd is repetitive - check!)

(as expected: det = 0)

Thus: 3 pairs $(\lambda, \underset{\sim}{x})$ — solutions of the eigenvalue problem

$$\left\{ \begin{array}{l} \underset{\sim}{T} \cdot \underset{\sim}{x}^{(1)} = \lambda_1 \underset{\sim}{x}^{(1)} \\ \underset{\sim}{T} \cdot \underset{\sim}{x}^{(2)} = \lambda_2 \underset{\sim}{x}^{(2)} \\ \underset{\sim}{T} \cdot \underset{\sim}{x}^{(3)} = \lambda_3 \underset{\sim}{x}^{(3)} \end{array} \right.$$

Now, consider

$$\underset{\sim}{T} \cdot \underset{\sim}{x}^{(1)} = \lambda_1 \underset{\sim}{x}^{(1)} \quad \text{and} \quad \cdot \underset{\sim}{x}^{(2)}$$

$$T \cdot x^{(2)} = \lambda_2 x^{(2)} \quad \text{and} \quad \cdot \underset{\sim}{x}^{(1)}$$

Subtract:

$$\underbrace{T_{ij} x_i^{(2)} x_j^{(1)} - T_{ij} x_i^{(1)} x_j^{(2)}}_{= 0, \text{ If } \underset{\sim}{T} \text{ is symmetric}} = (\lambda_1 - \lambda_2) \underset{\sim}{x}^{(1)} \cdot \underset{\sim}{x}^{(2)}$$

$(T_{13} x_1^{(2)} x_3^{(1)} - T_{31} x_3^{(1)} x_1^{(2)} = 0)$

$$\Rightarrow (\lambda_1 - \lambda_2) \underset{\sim}{x}^{(1)} \cdot \underset{\sim}{x}^{(2)} = 0$$

If $\lambda_1 \neq \lambda_2$ then $\underset{\sim}{x}^{(1)} \perp \underset{\sim}{x}^{(2)}$

$$\Downarrow$$

Eigenvectors (corresponding to different λ's) are orthogonal

They form principal axes of $\underset{\sim}{T}$

$e_{\mathrm{I}}, e_{\mathrm{II}}, e_{\mathrm{III}}$ - unit vectors

In the principal axes : Matrix of components diagonal

$$\underset{\sim}{T} = \lambda_1 \, \underline{e}_I \underline{e}_I + \lambda_2 \, \underline{e}_{II} \underline{e}_{II} + \lambda_3 \, \underline{e}_{III} \underline{e}_{III}$$

(check : $T \cdot e_I = \lambda_1 e_I$, etc)

- If two eigenvalues coincide (double root of cubic eq.)

$$\lambda_1 = \lambda_2$$

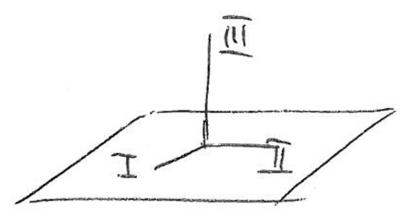
Then

$$\underset{\sim}{T} \cdot \left(a \, \underline{e}_I + b \, \underline{e}_{II} \right) = a \underbrace{T \cdot e_I}_{\lambda e_I} + b \underbrace{T \cdot e_{II}}_{\lambda e_{II}} = \lambda (a e_I + b$$

any vector in plane (e_I, e_{II}) – eigenvector

Example: mirror reflection

any vector in plane
is an eigenvector



- If all three coincide: $\underset{\sim}{T} = \lambda \underset{\sim}{I}$

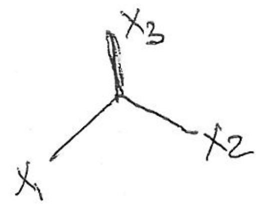
(proportional to the unit tensor)

Any vector is an eigenvector

Example: uniform expansion (heating

Example : using intuition in solving eigenvalue problems

$$T_{ij} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



Eigenvalue λ : root of eq.

$$\det \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\underline{\lambda_1 = 3} \qquad (\lambda_2 = \lambda_3 = 0)$$

Eigenvector ?

in T-matrix : x_1, x_2, x_3 enter the same way

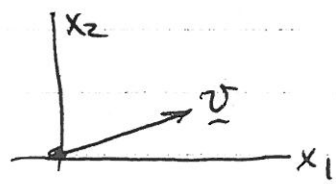
Expect : eigenvector is equally inclined to $x_1\ x_2\ x_3$ axes

Means : $x_1 = x_2 = x_3$

Check:
$$\begin{cases} -2x_1 + x_2 + x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

Two - Dimensional Tensors

2-D vector



$$v = v_1 \underset{\sim}{e}_1 + v_2 \underset{\sim}{e}_2 = v_i \underset{\sim}{e}_i$$

assumes sum from 1 to $\underline{\underline{2}}$!

2-D tensor

$$v_i = T_{ij} u_j$$

$$\begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix}$$

Eigenvalue problem:

$$\det \begin{vmatrix} T_{11}-\lambda & T_{12} \\ T_{21} & T_{22}-\lambda \end{vmatrix} = 0$$

- quadratic eq-n
  <u>Two</u> roots — either <u>both</u> real
  — or, <u>no</u> real roots (← rotation!)

<u>If</u> both roots are real (symm matrix)

$$\underset{\sim}{T} = \lambda_1 \underset{\sim}{e}_I \underset{\sim}{e}_I + \lambda_2 \underset{\sim}{e}_{II} \underset{\sim}{e}_{II}$$ (princ represent)