PRETENTIOUSLY DETECTING POWER CANCELLATION

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ABSTRACT. Granville and Soundararajan have recently introduced the notion of *pretentiousness* in the study of multiplicative functions of modulus bounded by 1, essentially the idea that two functions which are similar in a precise sense should exhibit similar behavior. It turns out, somewhat surprisingly, that this does not directly extend to detecting power cancellation - there are multiplicative functions which exhibit as much cancellation as possible in their partial sums that, modified slightly, give rise to functions which exhibit almost as little as possible. We develop two new notions of pretentiousness under which power cancellation can be detected, one of which applies to a much broader class of multiplicative functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

In a series of papers, Granville and Soundararajan ([1], [3], [4], [5], [6] as a few examples) recently introduced the notion of *pretentiousness* in the study of multiplicative functions taking values in the complex unit disc, essentially the idea that if two functions are "close" in some sense, they should exhibit the same behavior. One striking example of this philosophy is a theorem of Halász [7], which can be interpreted as saying that given a multiplicative function $f(n)$ with $|f(n)| \leq 1$ for all n, the partial sums

$$S_f(x) := \sum_{n \leq x} f(n)$$

are large if and only if $f(n)$ "pretends" to be n^{it} for some $t \in \mathbb{R}$ (possibly 0). To make this precise, define the distance between two multiplicative functions $f(n)$ and $g(n)$ taking values in the complex unit disc to be

$$\mathbb{D}(f, g)^2 := \sum_p \frac{1 - \mathrm{Re}(f(p)\bar{g}(p))}{p},$$

where here and throughout, the summation over p is taken to be over primes. This distance is typically infinite, but in the event that it is finite, we follow Granville and Soundararajan and say that $f(n)$ and $g(n)$ are pretentious to each other, or that $f(n)$ is $g(n)$-pretentious. Halász's theorem then says that if $S_f(x) \gg x$, then $f(n)$ must be n^{it}-pretentious for some t. In other words, Halász's theorem classifies those $f(n)$ for which $S_f(x)$ is as large as possible. It is therefore natural to ask for which $f(n)$ we have that $S_f(x)$ is exceptionally small. Since for generic $f(n)$ taking values in the complex unit disc, the best we can typically hope for is $S_f(x) \ll_\epsilon x^{1/2+\epsilon}$, we are interested in when $S_f(x)$ exhibits more than square root cancellation. In particular, we ask the following question.

Question 1. *If $f(n)$ is a completely multiplicative function, bounded by 1 in absolute value, such that both $\sum_{n\leq x} |f(n)|^2 \gg x$ and $S_f(x) \ll x^{\frac{1}{2}-\delta}$ hold for some fixed $\delta > 0$, must $f(n)$ be $\chi(n)n^{it}$-pretentious for some Dirichlet character χ and some $t \in \mathbb{R}$?*

The reason for the condition that

$$\sum_{n\le x} |f(n)|^2 \gg x$$

is twofold. First, we wish to exclude functions like $f(n) = n^{-a}$ for some $a > 0$, and second, this condition is necessary for $\mathbb{D}(f,f)$ to be finite, and therefore for $f(n)$ to be pretentious to any function. In other words, this condition is necessary for $f(n)$ to fit into the context of pretentiousness.

To study Question 1, we first ask that if $f(n)$ is $\chi(n)$-pretentious for some character χ, must $S_f(x)$ be small? This turns out to not be the case - by taking $f(p)$ to be 1 for primes lying in one of a suitably sparse set of dyadic intervals and to be $\chi(p)$ otherwise, one obtains a function which is $\chi(n)$-pretentious, but for which $S_f(x) \gg x/\log x$ for infinitely many x. Thus, we have a function, $f(n)$, which is pretentious to a function, $\chi(n)$, which exhibits as much cancellation as possible in its partial sums, and yet $S_f(x)$ is almost as large as possible. We therefore must ask whether there is a stronger notion of pretentiousness which preserves power savings.

To this end, given any two multiplicative functions $f(n)$ and $g(n)$, not necessarily bounded by 1, define the multiplicative function $h(n)$ by

$$g(n) = (f * h)(n),$$

where $(f * h)(n)$ represents the Dirichlet convolution of $f(n)$ and $h(n)$, and, for any $\beta > 0$, define the (possibly infinite) quantity $H_\beta(f,g)$ by

$$H_\beta(f,g) := \sum_p \sum_{k=1}^{\infty} \frac{|h(p^k)|}{p^{k\beta}}.$$

We caution that the convergence of this quantity is potentially asymmetric in $f(n)$ and $g(n)$. Motivated by the idea that if $f(n)$ and $g(n)$ are close, then each should need to be modified only slightly to obtain the other, we say that $f(n)$ and $g(n)$ are strongly β-pretentious to each other if both $H_\beta(f,g)$ and $H_\beta(g,f)$ are finite. If $f(n)$ and $g(n)$ are strongly β-pretentious for each $\beta > 0$, then we say that they are totally pretentious to each other.

Theorem 1.1. *Suppose that $f(n)$ and $g(n)$ are multiplicative functions, and that $S_f(x) \ll x^\alpha$ for some $\alpha > 0$. If $f(n)$ and $g(n)$ are totally pretentious to each other, then $S_g(x) \ll x^\alpha$. If, however, $f(n)$ and $g(n)$ are only strongly β-pretentious to each other, then we have that $S_g(x) \ll x^{max(\alpha,\beta)}$.*

Two remarks: First, it is apparent that the first statement of Theorem 1.1 regarding total pretentiousness is an immediate corollary to the second statement by taking $\beta < \alpha$. However, we consider its merit to be that it presupposes no knowledge of α to deduce that $S_g(x)$ and $S_f(x)$ exhibit the same level of cancellation.

Second, to obtain the conclusions of Theorem 1.1, it would suffice to suppose only that $H_\beta(f,g)$ is finite, with no hypothesis necessary on $H_\beta(g,f)$. We have chosen this formulation so that strong β-pretentiousness, and hence also total pretentiousness, is an equivalence relation. However, it is only the symmetry requirement that fails if we rely only on the finiteness of $H_\beta(f,g)$, in that if both $H_\beta(f,g)$ and $H_\beta(g,r)$ are finite, then so is $H_\beta(f,r)$.

Now, we wish to consider the extent to which strong and total pretentiousness relate to the traditional notion defined by $\mathbb{D}(f,g)$. We begin with the observation that, if $f(n)$ and

$g(n)$ are bounded by 1 in absolute value, then we have that

$$\left(\sum_{p,k} \frac{\left|g(p^k) - f(p^k)\right|}{p^{k\gamma}}\right)^2 \leq \left(\sum_{p,k} \frac{1}{p^{k(2\gamma-\beta)}}\right)\left(\sum_{p,k} \frac{\left|g(p^k) - f(p^k)\right|^2}{p^{k\beta}}\right)$$
$$\ll \sum_{p,k} \frac{1 - \mathrm{Re}\left(f(p^k)\overline{g(p^k)}\right)}{p^{k\beta}},$$

assuming that $\gamma > (1+\beta)/2$. This last quantity is a kind of generalized distance considered by Granville and Soundararajan in their book [2], and so the convergence of the initial series is, in this way, dictated by whether $f(n)$ and $g(n)$ are pretentious in a more traditional sense (although this observation is valid only if $\gamma > 1/2$). Moreover, since we have that $h(p) = g(p) - f(p)$, it is perhaps not unreasonable to hope that the convergence of this series is also related to the convergence of $H_\beta(f,g)$. Thus, define a distance $\widehat{\mathbb{D}}_{\beta,k}(f,g)$ by

$$\widehat{\mathbb{D}}_{\beta,k}(f,g) := \sum_p \sum_{j \leq k} \frac{|g(p^j) - f(p^j)|}{p^{j\beta}},$$

and additionally define $\widehat{\mathbb{D}}_\beta := \widehat{\mathbb{D}}_{\beta,\infty}$. Our next theorem shows that, while $\widehat{\mathbb{D}}_\beta(f,g) < \infty$ does not imply that $H_\beta(f,g) < \infty$, it does imply the convergence for sufficiently large primes. We also consider what power cancellation can be deduced directly from assuming that $\widehat{\mathbb{D}}_{\beta,k}(f,g) < \infty$.

Theorem 1.2. *Let $f(n)$ and $g(n)$ be multiplicative functions satisfying $f(n), g(n) = o(n^\delta)$ for some $\delta > 0$.*

1. If $\widehat{\mathbb{D}}_\beta(f,g) < \infty$, there is a $Y > 0$ such that if

$$H_\sigma(f,g;Y) := \sum_{p<Y} \sum_k \frac{\left|h(p^k)\right|}{p^{k\sigma}}$$

converges for some $\sigma \geq \beta$ and $\sigma > \delta$, then $H_\sigma(f,g) < \infty$.

2. Suppose that $S_f(x) \ll x^\alpha$ and that $\widehat{\mathbb{D}}_{\beta,k}(f,g) < \infty$. There is a $Y > 0$ such that if $H_\sigma(f,g;Y) < \infty$ for some $\sigma > 1/(k+1) + \delta$ also satisfying $\sigma \geq \max(\alpha,\beta)$, then $S_g(x) \ll x^\sigma$.

While it is unfortunate that we are unable to go from $\widehat{\mathbb{D}}_\beta(f,g) < \infty$ to $H_\beta(f,g) < \infty$ without checking the convergence of $H_\beta(f,g;Y)$, it is, in fact, generically necessary. If we let $f(n) = (-1)^{n+1}$, so that $f(2^k) = -1$ and $f(p^k) = 1$ for all $p \neq 2$ and all $k \geq 1$, and we let $g(n) = 1$, then we of course have that $S_f(x) \ll 1$ and that $S_g(x) \gg x$. However, since neither function is large and they differ only at the prime 2, we also have that $\widehat{\mathbb{D}}_\beta(f,g) < \infty$ for every $\beta > 0$, and we do not want to deduce any cancellation in $S_g(x)$. The reason that the theorem does not apply is that $|h(2^k)| = 2^k$, so that $H_\sigma(f,g;Y)$ diverges for every $\sigma \leq 1$.

Despite the above discussion, for certain classes of functions, we do not have to check the convergence of $H_\sigma(f,g;Y)$. We now present two such classes. The first class is motivated by the properties of the normalized coefficients of automorphic forms.

Definition 1. *Given a positive integer d, let $\mathcal{S}_d$ denote the set of "degree d" multiplicative functions, those functions $f(n)$ such that $f(n) = (f_1 * f_2 * \cdots * f_d)(n)$, where each $f_i(n)$ is a completely multiplicative function of modulus bounded by 1.*

As mentioned above, we are able to deduce a nice statement about pretentiousness in the context of degree d functions. Moreover, since the values at prime powers of a degree d function are determined by its values on the first d, it should stand to reason that the convergence of $\widehat{\mathbb{D}}_\beta(f,g)$ should be dictated by the convergence of $\widehat{\mathbb{D}}_{\beta,d}(f,g)$. We are able to show this as well. Thus, we have the following.

Theorem 1.3. *Let $f(n)$ and $g(n)$ be two degree d multiplicative functions such that $\widehat{\mathbb{D}}_{\beta,d}(f,g) < \infty$. We then have that both $\widehat{\mathbb{D}}_\beta(f,g)$ and $H_\beta(f,g)$ are finite. In particular, if we also know that $S_f(x) \ll x^\alpha$, then $S_g(x) \ll x^{\max(\alpha,\beta)}$.*

In the next class of functions, we return to the original setting of pretentiousness, functions of modulus bounded by 1.

Definition 2. *Let $f(n)$ be a mutiplicative function of modulus bounded by 1. We say that $f(n)$ is* good *at a prime p if there is no choice of $g(n)$ with modulus bounded by 1 for which the series*

$$\sum_k \frac{|h(p^k)|}{p^{k\sigma}}$$

fails to converge for some $\sigma > 0$. We say that $f(n)$ is good if it is good at every prime p.

This definition is, of course, exactly what we need to remove the condition on $H_\sigma(f,g;Y)$. However, we note two things: first, it is easy to give examples of good functions – any completely multiplicative function, say, since we have that $|h(p^k)| \leq 2$ – and, second, that it is possible to classify those functions which are good, which we do in Theorem 1.4. Also, we note that for any $f(n)$ and $g(n)$ bounded by 1, to check the convergence of $\widehat{\mathbb{D}}_\beta(f,g)$, it suffices to check the convergence of $\widehat{\mathbb{D}}_{\beta,\beta^{-1}}(f,g)$.

Theorem 1.4. *Let $f(n)$ and $g(n)$ be multiplicative functions of modulus bounded by 1.*

1. If $f(n)$ is good and $\widehat{\mathbb{D}}_\beta(f,g) < \infty$, then $H_\beta(f,g) < \infty$. Thus, if $S_f(x) \ll x^\alpha$, we have that $S_g(x) \ll x^{\max(\alpha,\beta)}$.

2. $f(n)$ is good at p if and only if the function

$$F_p(z) := \sum_{k=0}^{\infty} f\left(p^k\right) z^k$$

has no zeros in the open unit disc.

Finally, we return to the Granville-Soundararajan distance function, and we ask to what extent the natural modification

$$\mathbb{D}_\beta(f,g)^2 := \sum_p \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p^\beta}$$

allows one to detect power cancellation for multiplicative functions of modulus bounded by 1. For convenience, if $\mathbb{D}_\beta(f,g)$ is finite, we say that $f(n)$ and $g(n)$ are β-pretentious. As in Theorem 1.2, given $f(n)$ and $g(n)$, we will need a consideration of $h(n)$ at small primes, so we define

$$H_\sigma^2(f,g) := \sum_{p \leq 13} \sum_{k=0}^{\infty} \frac{|h(p^k)|^2}{p^{k\sigma}},$$

although, strictly speaking, only the condition $p \leq 4^{1/\sigma}$ is necessary (only $\sigma > 1/2$ will be used, so $p \leq 13$ is, indeed, weaker). We have the following result, establishing both that β-pretentiousness is sufficient to detect some power cancellation, but that it is fundamentally unable to detect to the level we desire.

Theorem 1.5. *Let $f(n)$ and $g(n)$ be multiplicative functions bounded by 1 such that $S_f(x) \ll x^\alpha$ for some $\alpha < 1$, and suppose that $\mathbb{D}_\beta(f,g) < \infty$ for some $\beta \in (0,1]$.*

1. If $\sigma > 3/4$ is such that $\sigma \geq \max(\alpha, (1+\beta)/2)$ and $H^2_{2\sigma-1}(f,g) < \infty$, then $S_g(x) \ll x^\sigma$.

2. If $f(n)$ and $g(n)$ are both completely multiplicative, then $S_g(x) \ll x^{\max(\alpha,(1+\beta)/2)}$.

3. If $f(n)$ is completely multiplicative and $\beta \geq 2\alpha - 1$, there is a completely multiplicative function $f'(n)$ that is β-pretentious to $f(n)$ and is such that $S_{f'}(x)$ is not $O_\epsilon\left(x^{\frac{1+\beta}{2}-\epsilon}\right)$.

Three remarks: While it's perhaps unsatisfying that β-pretentiousness only detects power savings down to $O\left(x^{\frac{1+\beta}{2}}\right)$ even for completely multiplicative functions, the conclusion of the theorem can be strengthened if $f(n)$ and $g(n)$ are assumed to be real-valued. The reason for this is that our proof of optimality relies crucially on the fact that $1 - \mathrm{Re}(f(p)\bar{g}(p))$ can be much smaller than $|f(p) - g(p)|$, which is not the case if $f(n)$ and $g(n)$ take values in $[-1,1]$. Thus, if $f(n)$ and $g(n)$ are β-pretentious and real-valued, then we also have that $\widehat{\mathbb{D}}_{\beta,1}(f,g) < \infty$, and so Theorem 1.2 applies.

Second, as the proof of Theorem 1.5 will show, if we have the stronger condition that $S_f(x) = o(x^\alpha)$, then we may conclude that $S_g(x) = o\left(x^\sigma\right)$.

Third, there are quantitative versions of Halász's theorem, due to Halász [8], Montgomery [10], Tenenbaum [12], and Granville and Soundararajan [2] and [4], but all of these theorems are essentially unable to detect cancellation below $O\left(x\frac{\log\log x}{\log x}\right)$, and so are useless for the question of power cancellation. There is also very recent work of Koukoulopoulos [9], who establishes a variant of Halász's theorem allowing detection of cancellation down to the level of $O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$, but, again, this is insufficient for our purposes.

In view of Theorem 1.5, which implies that β-pretentiousness is enough to detect power savings down to $O(x^{(1+\beta)/2})$, it's natural to ask what happens if $(1+\beta)/2 < \alpha$, so that we can detect below the order of magnitude of $S_f(x)$. That is, supposing we have precise information about $S_f(x)$, can we use β-pretentiousness to deduce precise information about $S_g(x)$? This is the content of our final theorem. For convenience, we state the necessary conditions on $f(n)$ and $g(n)$ here.

First, if $f(n)$ and $g(n)$ are both completely multiplicative, we only require that they are β-pretentious to each other for some $\beta > 0$. If, however, either is not completely multiplicative, we must also have that if $S_f(x) \ll_\epsilon x^{\alpha+\epsilon}$ for all $\epsilon > 0$, then $\alpha > 3/4$, and that both of the series $H^2_{2\sigma-1}(f,g)$ and $H^2_{2\sigma-1}(g,f)$ are convergent for some $\sigma < \alpha$.

Theorem 1.6. *Let $f(n)$ and $g(n)$ be as above.*

1. If $S_f(x) = x^\alpha \xi(x)$ for some function $\xi(x)$ satisfying $\xi(t) \ll_\epsilon t^\epsilon$, then $S_g(x) = x^\alpha \tilde{\xi}(x)$ for an explicitly given function $\tilde{\xi}(x)$ also satisfying $\tilde{\xi}(t) \ll_\epsilon t^\epsilon$.

2. If $\xi(t)$ satisfies the mean-square lower bound

$$\int_1^T |\xi(t)|^2 dt \gg_\epsilon T^{1-\epsilon},$$

then $\tilde{\xi}(t)$ does as well.

We have in mind the following two applications of Theorem 1.6: First, if $S_f(x)$ satisfies an asymptotic formula, then so does $S_g(x)$. For example, if the Dirichlet series associated to f, $L(s,f)$, has a finite number of poles on the line $\mathrm{Re}(s) = \alpha$ and is otherwise analytic on $\mathrm{Re}(s) > \alpha - \delta$ for some δ, then standard Tauberian theorems (for example, see [11]) show that

$$S_f(x) = \sum_{\substack{\rho:\mathrm{Re}(\rho)=\alpha \\ \mathrm{ord}_{s=\rho} L(s,f)<0}} x^{\rho} P_{\rho}(\log x) + O(x^{\alpha-\delta+\epsilon}),$$

where each $P_\rho(\log x)$ is a polynomial in $\log x$. Thus, with the notation of Theorem 1.6, we have that

$$\xi(x) = \sum_{\substack{\rho:\mathrm{Re}(\rho)=\alpha \\ \mathrm{ord}_{s=\rho} L(s,f)<0}} x^{\mathrm{Im}(\rho)} P_{\rho}(\log x) + O(x^{-\delta+\epsilon}),$$

and it is easy to see that $\xi(x)$ satisfies the required upper bound. Thus, we can apply Theorem 1.6, and it turns out that in this application, $\tilde{\xi}(x)$ works out to be

$$\tilde{\xi}(x) = \sum_{\substack{\rho:\mathrm{Re}(\rho)=\alpha \\ \mathrm{ord}_{s=\rho} L(s,f)<0}} x^{\mathrm{Im}(\rho)} Q_{\rho}(\log x) + O(x^{-\delta'})$$

for some suitably small $\delta' > 0$, where $Q_\rho(\log x)$ is a polynomial in $\log x$ of the same degree as $P_\rho(\log x)$. Thus, the explicit nature of $\tilde{\xi}(t)$ is of use.

Second, if $S_f(x)$ exhibits a consistent level of cancellation, then so does $S_g(x)$. In the above situation, we made use of the explicit nature of $\tilde{\xi}(x)$ to deduce an asymptotic formula for $S_g(x)$, but in many cases, we would not be lucky enough to have an asymptotic formula for $S_f(x)$ with which to begin. However, it is often possible to deduce the weaker statement that $S_f(x) \not\ll_\epsilon x^{\alpha-\epsilon}$ for any $\epsilon > 0$ – for example, $L(s,f)$ may have infinitely many poles on the line $\mathrm{Re}(s) = \alpha$. In this situation, the use of the mean-square lower bound becomes apparent - because $S_f(x)$ exhibits cancellation without satisfying an asymptotic formula, it is likely that $S_f(x)$ could be exceptionally small, perhaps even 0, for some values of x, but it also seems that this occurrence should be fairly rare. We can therefore deduce from Theorem 1.6 that if x^α is the right order of magnitude of $S_f(x)$ in this sense, then x^α is also the right order of magnitude for $S_g(x)$.

This paper is organized as follows: In Section 2, we consider strong pretentiousness and its relation to the Granville-Soundararajan distances, as discussed in the introduction. Thus, this is where Theorems 1.1-1.4 are proved. In Section 3, we consider the notion of β-pretentiousness, and establish Theorems 1.5 and 1.6

2. Strong pretentiousness

In this section, we consider the distances $H_\beta(f,g)$ and $\widehat{\mathbb{D}}_{\beta,k}(f,g)$ and their relation to each other. Thus, we prove Theorems 1.1-1.4, and we do so, in order, in Sections 2.1-2.4.

2.1. **Detecting power cancellation.**

We now let $f(n)$, $g(n)$, and $h(n)$ be as in the hypotheses of Theorem 1.1. Thus, $f(n)$ and $g(n)$ are multiplicative and $h(n)$ is defined by $g(n) = (f * h)(n)$. We now prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $S_f(x) \ll x^\alpha$ for some $\alpha > 0$. We first claim that the series $\sum_{n=1}^\infty |h(n)|/n^\beta$ is convergent. From this, we conclude that

$$\begin{aligned}\sum_{n\le x} g(n) &= \sum_{m\le x} h(m) \sum_{d\le x/m} f(d) \\ &\ll x^\alpha \sum_{m\le x} \frac{|h(m)|}{m^\alpha} \\ &\ll x^{\max(\alpha,\beta)},\end{aligned}$$

by partial summation. Thus, to establish the theorem, it just remains to show that the series above is convergent. However, this, too, is straightforward, as we have that

$$\begin{aligned}\sum_{n=1}^\infty \frac{|h(n)|}{n^\beta} &= \prod_p \left(1 + \sum_{k=1}^\infty \frac{|h(p^k)|}{p^{k\beta}}\right) \\ &\le \prod_p \exp\left(\sum_{k=1}^\infty \frac{|h(p^k)|}{p^{k\beta}}\right) \\ &= \exp\left(H_\beta(f,g)\right) < \infty.\end{aligned}$$

Thus, we have proved Theorem 1.1. □

2.2. **Relation to Granville-Soundararajan distances: Proof of Theorem 1.2.** We now wish to relate the finiteness of the distance $\widehat{\mathbb{D}}_\beta(f,g)$ to the finiteness of $H_\beta(f,g)$. For convenience, we recall that

$$H_\beta(f,g) := \sum_{p,k} \frac{|h(p^k)|}{p^{k\beta}}, \qquad \widehat{\mathbb{D}}_{\beta,k}(f,g) := \sum_p \sum_{j\le k} \frac{|g(p^j) - f(p^j)|}{p^{j\beta}},$$

and that $\widehat{\mathbb{D}}_\beta(f,g) := \widehat{\mathbb{D}}_{\beta,\infty}(f,g)$.

From the definition of $h(n)$, we have that

$$g(p^k) - f(p^k) = \sum_{j=1}^k f(p^{k-j}) h(p^j),$$

which, by incorporating all the powers up to n, we may express in terms of the $n \times n$ matrix $A := \left(f(p^{i-j})\right)_{i,j\le n}$, as

$$A \cdot \left(h(p), \cdots, h(p^n)\right)^t = \left(g(p) - f(p), \cdots, g(p^n) - f(p^n)\right)^t.$$

where we have set $f(p^j) = 0$ if $j < 0$. For any $k \ge 1$, define $D_f(k,p)$ to be the determinant of the $k \times k$ matrix (a_{ij}) given by

$$a_{ij} = \begin{cases} f(p^{i-j+1}) & \text{if } i-j+1 \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

so that $(-1)^k D_f(k,p)$ is the $(n, n-k)$-th entry of the matrix A^{-1}. We now have that

$$h(p^n) = \sum_{k=0}^{n-1} (-1)^k \left(g(p^{n-k}) - f(p^{n-k})\right) D_f(k,p).$$

Therefore for $\sigma > 0$ sufficiently large, we have that

$$\sum_{n=1}^{\infty} \frac{|h(p^n)|}{p^{n\sigma}} \leq \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} |f(p^k) - g(p^k)| \cdot |D_f(n-k,p)| \right) p^{-n\sigma}$$
$$= \left(\sum_{n=0}^{\infty} \frac{|D_f(n,p)|}{p^{n\sigma}} \right) \left(\sum_{m=1}^{\infty} \frac{|f(p^m) - g(p^m)|}{p^{m\sigma}} \right).$$

Of course, at this stage, we would like to sum over p. Lemma 2.1 below states that the first quantity on the right hand side is uniformly bounded for p sufficiently large, say $p > Y_1$, provided that $f(n)$ is not too big, say $f(n) = o(n^\delta)$, and that $\sigma > \delta$. Thus, if we assume that $H_\sigma(f, g; Y_1)$ is finite, when we sum over p, the second summation on the right hand side will yield $\widehat{\mathbb{D}}_\sigma(f, g)$, and the first part of Theorem 1.2 follows.

Lemma 2.1. *If $f(n) = o(n^\delta)$ and $\sigma > \delta$, then for all but finitely many p, the series*

$$\sum_{n=0}^{\infty} \frac{|D_f(n,p)|}{p^{n\sigma}}$$

is convergent and uniformly bounded.

Proof. Let $M(k,p)$ be the maximum of the absolute value of the determinants of the $k \times k$ matrices (a_{ij}) which satisfy

$$|a_{ij}| \leq \begin{cases} p^{(i-j+1)\delta} & \text{if } i-j+1 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, we observe that

$$M(k+1,p) \leq 2p^\delta M(k,p)$$

by cofactor expansion, and that $M(1,p) = p^\delta$. It therefore follows that

$$M(k,p) \leq 2^{k-1} p^{k\delta},$$

which implies that the bound

$$|D_f(n,p)| < (2p^\delta)^n$$

holds for all but finitely many p. □

Now, it remains to establish the second part of Theorem 1.2. To do so, we must be able to control the contribution of large prime powers to the sum

$$\sum_{m=1}^{\infty} \frac{|f(p^m) - g(p^m)|}{p^{m\sigma}}.$$

This control is provided by our assumption that $f(n), g(n) = o(n^\delta)$. In particular, it is straightforward to see that

$$\sum_{p > Y_2} \sum_{m=k+1}^{\infty} \frac{|f(p^m) - g(p^m)|}{p^{m\sigma}}$$

must converge for some Y_2, provided that $\sigma > \frac{1}{k+1} + \delta$. Thus, the second part of Theorem 1.2 is obtained with $Y = \max(Y_1, Y_2)$.

2.3. **Degree d functions: Proof of Theorem 1.3.** Suppose that $f(n)$ and $g(n)$ are multiplicative functions of degree d, and that $\widehat{\mathbb{D}}_{\beta,d}(f,g) < \infty$. We first show that $\widehat{\mathbb{D}}_{\beta}(f,g)$ is finite, and then we consider $H_\beta(f,g)$.

Lemma 2.2. *Let $f(n)$ and $g(n)$ be degree d multiplicative functions, and suppose that $\widehat{\mathbb{D}}_{\beta,d}(f,g) < \infty$. Then $\widehat{\mathbb{D}}_{\beta}(f,g) < \infty$.*

Proof. We begin with some general notation. For any given pair of integers $k, d \geq 0$, define the homogeneous symmetric polynomials r_k^d and q_k^d of degree k in d variables by

$$r_k^d(x_1, \cdots, x_d) := \begin{cases} 1, & \text{if } k = 0 \\ \displaystyle\sum_{1 \leq i_1 < \cdots < i_k \leq d} x_{i_1} x_{i_2} \cdots x_{i_k}, & \text{if } 1 \leq k \leq d \\ 0, & \text{if } k > d, \end{cases}$$

and

$$q_k^d(x_1, \cdots, x_d) := \sum_{j_1 + \cdots + j_d = k} x_1^{j_1} x_2^{j_2} \cdots x_d^{j_d}.$$

Then for an auxiliary variable X, we have that

$$\begin{aligned} \sum_{k=0}^{\infty} q_k^d X^k &= \prod_{j=1}^{d} \left(\sum_{k=0}^{\infty} x_j^k X^k \right) \\ &= \prod_{j=1}^{d} (1 - x_j X)^{-1} \\ &= \left(\sum_{k=0}^{d} (-1)^k r_k^d X^k \right)^{-1}, \end{aligned}$$

which implies that the identity

$$\sum_{j=0}^{k} (-1)^j r_{k-j}^d q_j^d = 0$$

holds for all $k \geq 1$.

Now, if $f(n)$ is a multiplicative function of degree d, so that $f = f_1 * \cdots * f_d$ where each f_i is completely multiplicative, we have that $f(p^k) = q_k^d(f_1(p), \ldots, f_d(p))$. Thus, if we set $\alpha_k(f,p) = r_k^d(f_1(p), \ldots, f_d(p))$ for $k = 0, \cdots, d$, we have that

$$\sum_{k=0}^{d} (-1)^k \alpha_k(f,p) f(p^{n-k}) = 0$$

for any $n \geq 0$, where, of course, we have set $f(p^r) = 0$ for $r < 0$. In particular, for any multiplicative functions $f(n)$ and $g(n)$ of degree d, since $\alpha_k(f,p) \ll_d 1$, we have that

$$|\alpha_k(f,p) - \alpha_k(g,p)| \ll_d |f(p) - g(p)| + |f(p^2) - g(p^2)| + \cdots + |f(p^d) - g(p^d)|$$

for any $k = 1, \cdots, d$. We are now ready to prove the lemma. Assume that $n \geq d+1$. Observing that $f(p^n) \ll_d n^{d-1}$ and $\alpha_k(f,p) \ll_d 1$, we have

$$\begin{aligned}
|f(p^n) - g(p^n)| &= \left| \sum_{k=1}^{d} (-1)^k \alpha_k(f,p) f(p^{n-k}) + (-1)^k \alpha_k(g,p) g(p^{n-k}) \right| \\
&\ll_d \sum_{k=1}^{d} \left| \alpha_k(f,p) f(p^{n-k}) + \alpha_k(g,p) g(p^{n-k}) \right| \\
&\ll_d \sum_{k=1}^{d} \left| \alpha_k(f,p) \left(f(p^{n-k}) - g(p^{n-k}) \right) \right| + \left| g(p^{n-k}) \left(\alpha_k(f,p) - \alpha_k(g,p) \right) \right| \\
&\ll_d \sum_{k=1}^{d} \left| f(p^{n-k}) - g(p^{n-k}) \right| \\
&\quad + n^{d-1} \left(|f(p) - g(p)| + \cdots + |f(p^d) - g(p^d)| \right),
\end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \frac{|f(p^n) - g(p^n)|}{p^{n\sigma}} \ll_d \sum_{n=1}^{\infty} \frac{n^{d-1}}{p^{n\sigma}}$$

is convergent, this inequality leads to

$$\begin{aligned}
\sum_{n=d+1}^{\infty} \frac{|f(p^n) - g(p^n)|}{p^{n\sigma}} &\ll_d \sum_{k=1}^{d} \sum_{n=d+1}^{\infty} \frac{|f(p^{n-k}) - g(p^{n-k})|}{p^{n\sigma}} \\
&\quad + \sum_{n=d+1}^{\infty} \frac{n^{d-1}}{p^{n\sigma}} \left(|f(p) - g(p)| + \cdots + |f(p^d) - g(p^d)| \right) \\
&\ll_d \frac{1}{p^{\sigma}} \sum_{n=1}^{\infty} \frac{|f(p^n) - g(p^n)|}{p^{n\sigma}}.
\end{aligned}$$

Therefore for all sufficiently large p, we have

$$\sum_{n=d+1}^{\infty} \frac{|f(p^n) - g(p^n)|}{p^{n\sigma}} \ll_d \sum_{n=1}^{d} \frac{|f(p^n) - g(p^n)|}{p^{n\sigma}}.$$

By summing over p, we get the conclusion. □

It remains to show that $H_\beta(f,g)$ is finite. Recall for each prime p, that

$$\sum_{n=1}^{\infty} \frac{|h(p^n)|}{p^{n\beta}} \leq \left(\sum_{n=0}^{\infty} \frac{|D_f(n,p)|}{p^{n\beta}} \right) \left(\sum_{m=1}^{\infty} \frac{|f(p^m) - g(p^m)|}{p^{m\beta}} \right),$$

where $D_f(n,p)$ is as in Section 2.2. We will show that the first summation on the right hand side is uniformly bounded, so that by summing over p and using the finiteness of $\widehat{\mathbb{D}}_\beta(f,g)$, the result follows.

Lemma 2.3. *If $f(n)$ is a degree d multiplicative function and $\sigma > 0$, then, for all p, the series*

$$\sum_{n=0}^{\infty} \frac{|D_f(n,p)|}{p^{n\sigma}}$$

converges and is bounded independent of p.

Proof. Recall that we defined $D_f(k,p)$ so that the equation,

$$h(p^n) = \sum_{k=0}^{n-1} (-1)^k (g(p^{n-k}) - f(p^{n-k})) D_f(k,p),$$

holds. We may think of this as a linear polynomial in the variables $g(p^i)$ for $i = 1, \ldots, n$, and we note that the coefficient of $g(p^{n-j})$ is $D_f(j,p)$ for all j. On the other hand, from the definition of $h(n)$, we have the Euler product identity

$$\prod_p \left(\sum_{n=0}^{\infty} h(p^n) p^{-ns} \right) = \prod_p \left(\sum_{n=0}^{\infty} g(p^n) p^{-ns} \right) \left(1 - f_1(p)p^{-s}\right) \ldots \left(1 - f_d(p)p^{-s}\right),$$

where the $f_i(n)$ are the constituent completely multiplicative functions of $f(n)$. Thus, $h(p^n)$ can be expressed as a linear combination of the variables $g(p^i)$ for $i = n-d, \ldots, n$. Combining these two observations, we conclude that $D_f(k,p) = 0$ for $k \geq d+1$. The result follows by noting that each of the $D_f(k,p)$ for $k \leq d$ can be bounded independent of p. $\square$

2.4. **Good functions.** Recall that a multiplicative function $f(n)$ of modulus bounded by 1 is good at p if there are no multiplicative functions $g(n)$, of modulus bounded by 1, such that the series

$$\sum_{k=0}^{\infty} \frac{|h(p^k)|}{p^{k\sigma}}$$

diverges for any $\sigma > 0$, and that $f(n)$ is good if it is good at each prime p. This condition ensures that $H_\sigma(f,g;Y)$ is finite for every $Y > 0$, so the first part of Theorem 1.4 is immediate, that the finiteness of $\widehat{\mathbb{D}}_\beta(f,g)$ implies the finiteness of $H_\beta(f,g)$. The second part, the classification of functions which are good at p, is proved along the following lines.

Recall that we defined

$$F_p(z) := \sum_{k=0}^{\infty} f\left(p^k\right) z^k,$$

and we wish to show that $f(n)$ is good at p if and only if $F_p(z)$ has no zeros in the open unit disc. To do this, we observe that $G_p(z) = F_p(z)H_p(z)$, where $G_p(z)$ and $H_p(z)$ are defined analogously to $F_p(z)$. Since $g(n)$ is bounded by 1, we must have that $G_p(z)$ is holomorphic in the disc. Now, the convergence of

$$\sum_{k=0}^{\infty} |h(p^k)| p^{-k\sigma}$$

is equivalent to the statement that $H_p(z)$ is holomorphic. Thus, the result follows.

3. β-PRETENTIOUSNESS

In this section, we consider the notion of β-pretentiousness in some detail. Recall that two multiplicative functions $f(n)$ and $g(n)$, both of modulus bounded by 1, are such that the series

$$\mathbb{D}_\beta(f,g) = \sum_p \frac{1-\mathrm{Re}(f(p)\bar{g}(p))}{p^\beta}$$

converges, then they are said to be β-pretentious. In Section 3.1, we establish that if $f(n)$ and $g(n)$ are β-pretentious and if $S_f(x) \ll x^\alpha$, then we can detect power cancellation in $S_g(x)$. In Section 3.2, we construct a function $f'(n)$ which is β-pretentious to $f(n)$ and exhibits as little cancellation as possible in view of the estimates established in Section 3.1, thereby establishing their optimality. Thus, these two sections comprise the proof of Theorem 1.5. In Section 3.3, we establish Theorem 1.6 regarding what happens if we are permitted to detect more cancellation than exists.

3.1. Detecting power cancellation.

The key result which we use to exhibit cancellation in Theorem 1.5 is the following proposition, which of course is reminiscent of the proof of Theorem 1.1.

Proposition 3.1. *Let $f(n)$ $g(n)$ be as above, and let $h(n)$ be defined by $g(n) = (f * h)(n)$. If the series*

$$\sum_{n=1}^\infty \frac{|h(n)|^2}{n^\sigma}$$

is convergent for some $\sigma > 0$, then $S_g(x) \ll x^{\max(\alpha,(1+\sigma)/2)}$. Moreover, if $S_f(x) = o(x^\alpha)$, then $S_g(x) = o(x^{\max(\alpha,(1+\sigma)/2)})$.

Proof. From the definition of $h(n)$, we have that

$$\begin{aligned}
\sum_{n\le x} g(n) &= \sum_{m\le x} h(m) \sum_{d\le x/m} f(d) \\
&\ll x^\alpha \sum_{m\le x} \frac{|h(m)|}{m^\alpha} \\
&\le x^\alpha \left(\sum_{m=1}^\infty \frac{|h(m)|^2}{m^\sigma}\right)^{1/2} \left(\sum_{m\le x} \frac{1}{m^{2\alpha-\sigma}}\right)^{1/2} \\
&\ll x^{\max(\alpha,(\sigma+1)/2)}.
\end{aligned}$$

If we have the stronger assumption that $S_f(x) = o(x^\alpha)$, by splitting the sum over m on the first line according to whether m is large and proceeding in the same way, it is easily seen that $S_g(x) = o\left(x^{\max\left(\alpha,\frac{1+\sigma}{2}\right)}\right)$. $\square$

In light of Proposition 3.1, to prove the first part of Theorem 1.5, it suffices to establish the following lemma.

Lemma 3.1. *If $f(n)$, $g(n)$, and $h(n)$ are as above, $|f(n)|, |g(n)| \leq 1$ for all n, $f(n)$ and $g(n)$ are β-pretentious for some $\beta > 0$, and $\sigma > 1/2$ is such that $\sigma \geq \beta$, then the series*

$$\sum_{n=1}^{\infty} \frac{|h(n)|^2}{n^\sigma}$$

converges if the quantity

$$H(\sigma) = \sum_{p \leq 4^{1/\sigma}} \sum_{k=0}^{\infty} \frac{|h(p^k)|^2}{p^{k\sigma}}$$

is finite.

Proof. Since $|g(n)| \leq 1$ and $|f(n)| \leq 1$, we have that

$$|h(p^k)| \leq 2^k$$

for all p and all k. Therefore for $p > 4^{1/\sigma}$, one has

$$\sum_{k=1}^{\infty} \frac{|h(p^k)|^2}{p^{k\sigma}} \leq \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p^\sigma} + \frac{16}{p^{2\sigma}}(1 - 4/p^\sigma)^{-1}.$$

Thus, our assumption that $\sigma \geq \beta$ and that

$$\mathbb{D}_\beta(f, g) = \sum_p \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p^\beta}$$

is finite, together with the assumptions of the lemma, guarantee that the series

$$\sum_{n=1}^{\infty} \frac{|h(n)|^2}{n^\sigma} = \prod_p \left(\sum_{k=0}^{\infty} \frac{|h(p^k)|^2}{p^{k\sigma}} \right)$$

is absolutely convergent. $\square$

To establish the cancellation for completely multiplicative functions claimed in the second part of Theorem 1.5, we have the following lemma.

Lemma 3.2. *If $f(n)$,$g(n)$, and $h(n)$ are as in Lemma 3.1 and $f(n)$ and $g(n)$ are completely multiplicative, then the series*

$$\sum_{n=1}^{\infty} \frac{|h(n)|^2}{n^\beta}$$

is convergent.

Proof. Since $h(p^k) = g(p^{k-1})(g(p) - f(p))$ for all primes p and all $k \geq 1$, we have that

$$|h(p^k)|^2 \leq |g(p) - f(p)|^2 \leq 2(1 - \mathrm{Re}(f(p)\bar{g}(p))).$$

Therefore, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|h(n)|^2}{n^\beta} &= \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{|h(p^k)|^2}{p^{k\beta}}\right) \\
&\le \prod_p \left(1 + \frac{2(1 - \mathrm{Re}(f(p)\bar{g}(p)))}{p^\beta}(1 - p^{-\beta})^{-1}\right) \\
&\le \exp\left(\sum_p \frac{2(1 - \mathrm{Re}(f(p)\bar{g}(p)))}{p^\beta}(1 - 2^{-\beta})^{-1}\right) \\
&= \exp\left(2(1 - 2^{-\beta})^{-1}\mathbb{D}_\beta(f, g)\right) < \infty,
\end{aligned}$$

exactly as desired. $\square$

To establish Theorem 1.5, it now remains to establish the optimality of the bound for completely multiplicative functions.

3.2. **Optimality.** It is worth noting at this point that there is another natural approach to proving the theorem, albeit one that is not entirely within the bounds of the pretentious philosophy. From the relation $g(n) = (f * h)(n)$, we have the Dirichlet series identity

$$L(s, g) = L(s, f)L(s, h).$$

The assumption that $S_f(x) \ll x^\alpha$ translates to $L(s, f)$ being analytic in the right half-plane $\mathrm{Re}(s) > \alpha$ and the assumption that $g(n)$ is β-pretentious to $f(n)$, in light of Lemma 3.1 and the Cauchy-Schwarz inequality, implies that $L(s, h)$ is analytic in the region $\mathrm{Re}(s) > \max\left(3/4, \frac{1+\beta}{2}\right)$. Standard arguments (e.g. Perron's formula) then imply the desired bound for $S_g(x)$. Our proof of optimality will proceed along similar lines. While it is somewhat unfortunate that we have to use this mildly non-pretentious argument, it is not entirely clear how to avoid its use.

Lemma 3.3. *Given any $\beta > 0$ and a completely multiplicative function $f(n)$ of modulus bounded by 1 such that $f(n)$ is 1-pretentious to itself, there is a completely multiplicative function $g(n)$ that is β-pretentious to $f(n)$, and which does not satisfy $S_g(x) \ll x^{(1+\beta)/2-\epsilon}$ for any $\epsilon > 0$.*

Proof. First, we may assume that $L(s, f)$ is analytic in the region $\mathrm{Re}(s) > (1 + \beta)/2 - \delta$ for some $\delta > 0$, otherwise we could simply take $g(n)$ to be $f(n)$. Let

$$g(p) := e\left(\frac{\omega_p}{p^{\frac{1-\beta}{2}} \log\log p}\right) f(p),$$

where $\omega_p = \pm 1$ is a system of signs to be specified later and, as is standard, $e(x) := e^{2\pi i x}$. It is easy to verify that $g(n)$ is β-pretentious to $f(n)$. Our goal is to force $L(s, h)$ to have a singularity at $s = \frac{1+\beta}{2}$. We compute the Euler product for $L(s, h)$ using the Taylor expansion of $e(x)$, getting that

$$\begin{aligned} L(s,h) &= \prod_p \left(1 + \frac{g(p)-f(p)}{p^s} + O\left(p^{-2s}\right)\right) \\ &= \prod_p \left(1 + \frac{2\pi i \omega_p f(p)}{p^{s+\frac{1-\beta}{2}} \log\log p} + O\left(p^{-2s} + p^{-s-1+\beta}\right)\right). \end{aligned}$$

The convergence of $L(s,h)$ at $s = \frac{1+\beta}{2}$ is thus dictated by the behavior of the series

$$P_f(\tau) := \sum_p \frac{i\omega_p f(p)}{p^\tau \log\log p}$$

as τ tends to 1 from the right. In particular, $L(s,h)$ will have a singularity at $s = \frac{1+\beta}{2}$ if we can force either the real part of $P_f(\tau)$ to tend to infinity, accounting for a (possibly fractional order) pole, or, failing that, to have the real part of $P_f(\tau)$ converge but the imaginary part diverge to infinity, accounting for an essential singularity. Obviously, we now choose ω_p to ensure one of these situations. If the series

$$\sum_p \frac{\mathrm{Im}(f(p))}{p \log\log p}$$

is not absolutely convergent, we choose $\omega_p = -\mathrm{sign}(\mathrm{Im}(f(p)))$, forcing $\mathrm{Re}(P_f(\tau))$ to diverge to infinity. If the series is absolutely convergent, we choose $\omega_p = \mathrm{sign}(\mathrm{Re}(f(p)))$, observing that

$$\begin{aligned} \sum_p \frac{|\mathrm{Re}(f(p))|}{p^\tau \log\log p} + \sum_p \frac{|\mathrm{Im}(f(p))|}{p^\tau \log\log p} &\geq \sum_p \frac{\mathrm{Re}(f(p))^2 + \mathrm{Im}(f(p))^2}{p^\tau \log\log p} \\ &= \sum_p \frac{|f(p)|^2}{p^\tau \log\log p} \\ &\geq \sum_p \frac{1}{p^\tau \log\log p} - \mathbb{D}_1(f,f), \end{aligned}$$

which tends to infinity as $\tau \to 1^+$. We thus have that

$$\mathrm{Im}\left(\sum_p \frac{i\omega_p f(p)}{p \log\log p}\right) = \sum_p \frac{|\mathrm{Re}(f(p))|}{p \log\log p} = \infty,$$

from which we conclude that $\mathrm{Im}(P_f(x))$ tends to infinity. We have thus constructed $g(n)$ so that $L(s,h)$ has a singularity at $s = \frac{1+\beta}{2}$, so provided that $L\left(\frac{1+\beta}{2}, f\right) \neq 0$, we obtain the result. If $L\left(\frac{1+\beta}{2}, f\right) = 0$, there is a $t \in \mathbb{R}$ such that $L\left(\frac{1+\beta}{2} + it, f\right) \neq 0$. We make the obvious modifications to the construction above to force $L(s,h)$ to have a singularity at $s = \frac{1+\beta}{2} + it$. □

3.3. **Asymptotic formulae.** We now suppose we are in the situation of Theorem 1.6. That is, we assume that $f(n)$ is multiplicative, of modulus bounded by 1, and is such that

$$S_f(x) = x^\alpha \xi(x)$$

for some function $\xi(x)$ satisfying $\xi(t) \ll_\epsilon t^\epsilon$ for all $\epsilon > 0$, and we also assume that $\beta < 2\alpha - 1$. In addition, if $f(n)$ is not completely multiplicative, we assume that $\alpha > 3/4$ and that the series $H^2_{2\sigma-1}(f,g)$ and $H^2_{2\sigma-1}(g,f)$ are convergent. To establish a formula for $S_g(x)$, we note that

$$\begin{aligned}\sum_{n \le x} g(n) &= \sum_{m \le x} h(m) \sum_{d \le x/m} f(d) \\ &= x^\alpha \sum_{m \le x} \frac{h(m)}{m^\alpha} \xi(x/m),\end{aligned}$$

and so we naturally define $\tilde{\xi}(x)$ to be the convolution

$$\tilde{\xi}(x) := \sum_{m \le x} \frac{h(m)}{m^\alpha} \xi(x/m).$$

To see that $\tilde{\xi}(x) \ll x^\epsilon$, we merely note that

$$|\tilde{\xi}(x)| \le \sum_{m \le x} \frac{|h(m)|}{m^\alpha} |\xi(x/m)| \ll_\epsilon x^\epsilon \sum_{m \le x} \frac{|h(m)|}{m^{\alpha+\epsilon}}.$$

Our assumptions guarantee that the series on the right is convergent, whence the claimed bound. Now, suppose that

$$\int_1^T |\xi(t)|^2 dt \gg_\epsilon T^{1-\epsilon}.$$

Möbius inversion gives that

$$\xi(x) = \sum_{m \le x} \frac{\tilde{h}(m)}{m^\alpha} \tilde{\xi}(x/m),$$

where $\tilde{h}(n)$ is the Dirichlet inverse of $h(n)$ (i.e., $(h * \tilde{h})(1) = 1$ and $(h * \tilde{h})(n) = 0$ for $n > 1$). Using this and the Cauchy-Schwarz inequality in the above, we obtain that

$$\begin{aligned}T^{1-\epsilon} &\ll_\epsilon \int_1^T \left(\sum_{m \le t} \frac{|\tilde{h}(m)|^2}{m^\beta} \right) \left(\sum_{m \le t} \frac{|\tilde{\xi}(t/m)|^2}{m^{2\alpha-\beta}} \right) dt \\ &\le \sum_{m=1}^\infty \frac{|\tilde{h}(m)|^2}{m^\beta} \int_1^T \sum_{m \le t} \frac{|\tilde{\xi}(t/m)|^2}{m^{2\alpha-\beta}} dt \\ &= \sum_{m=1}^\infty \frac{|\tilde{h}(m)|^2}{m^\beta} \sum_{m \le T} \frac{1}{m^{2\alpha-\beta-1}} \int_1^{T/m} |\tilde{\xi}(t)|^2 dt \\ &\ll T^{2-2\alpha+\beta} \int_1^T |\tilde{\xi}(t)|^2 \frac{dt}{t^{2-2\alpha+\beta}},\end{aligned}$$

where the infinite series is convergent by assumption, so we have absorbed it into the implied constant. Now, let

$$I := \int_1^T |\tilde{\xi}(t)|^2 dt,$$

and apply Hölder's inequality to get that

$$\begin{aligned}\int_1^T |\tilde{\xi}(t)|^2 \frac{dt}{t^{2-2\alpha+\beta}} &\leq I^{\frac{2\alpha-\beta-1}{2}} \left(\int_1^T \frac{|\tilde{\xi}(t)|^2}{t^{\frac{2(2-2\alpha+\beta)}{3-2\alpha+\beta}}} dt \right)^{\frac{3-2\alpha+\beta}{2}} \\ &\ll_\epsilon I^{\frac{2\alpha-\beta-1}{2}} \left(\int_1^T t^{\frac{-2(2-2\alpha+\beta)}{3-2\alpha+\beta}+\epsilon} dt \right)^{\frac{3-2\alpha+\beta}{2}} \\ &\ll I^{\frac{2\alpha-\beta-1}{2}} T^{\frac{2\alpha-\beta-1}{2}+\epsilon}.\end{aligned}$$

Using this in the above, we obtain that

$$I^{\frac{2\alpha-\beta-1}{2}} T^{\frac{3-2\alpha+\beta}{2}+\epsilon} \gg_\epsilon T^{1-\epsilon},$$

and so we have that

$$I^{\frac{2\alpha-\beta-1}{2}} \gg_\epsilon T^{\frac{2\alpha-\beta-1}{2}-\epsilon},$$

and the result follows, concluding the proof of Theorem 1.6.

Since the Dirichlet series $L(s,h)$ for $\mathrm{Re}(s) \geq \alpha$ plays a critical role in the definition of $\tilde{\xi}(x)$, it is useful to know whether it is 0. In particular, in applying Theorem 1.6 in the case when $S_f(x)$ satisfies an asymptotic formula, we might potentially lose a term in our formula if $L(\rho, h) = 0$ for some pole ρ of $L(s,f)$. However, we have the following simple observation.

Lemma 3.4. *If $f(n)$ and $g(n)$ are completely multiplicative and as above, then the Dirichlet series $L(s,h)$ associated to $h(n)$ is non-zero in the region Re$(s) > (1+\beta)/2$.*

Proof. Since $h(n)$ is defined by the relation $g = f * h$, we have the Dirichlet series formula

$$L(s,h) = \frac{L(s,g)}{L(s,f)}.$$

By Lemma 3.2, this is absolutely convergent in the region $\mathrm{Re}(s) > (1+\beta)/2$. If we define $\tilde{h}(n)$ by $f = g * \tilde{h}$, the same argument applies to $L(s,\tilde{h})$. Since we also have that

$$L(s,\tilde{h}) = \frac{1}{L(s,h)},$$

this immediately yields the result. □

Of course, if $f(n)$ and $g(n)$ are not completely multiplicative, the analog of Lemma 3.4 can still be obtained with Lemma 3.1 replacing Lemma 3.2.

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