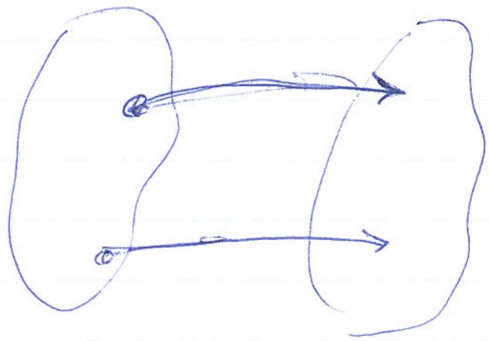
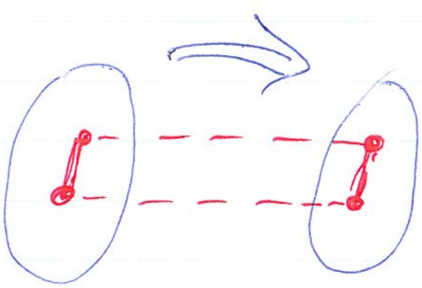
Strains - I

Strains

In *any* motion : each point of body gets displacement

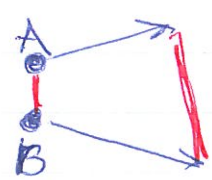


Rigid body motion : material lines do not elongate (or shorten)



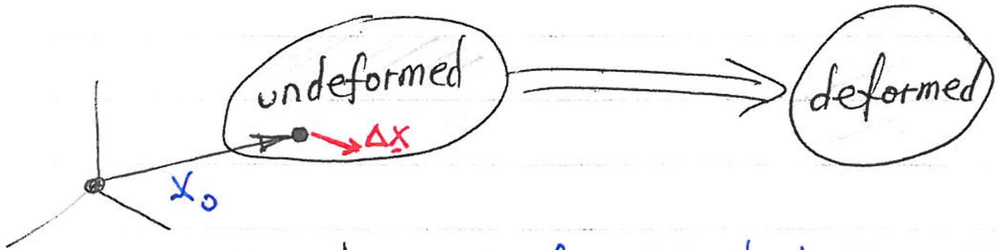
Deformation (strain) : material lines change length

because :



different displacements of points A, B

Non-uniformity of displacement field



<u>Displacement</u> of material pt $\underset{\sim}{x}_0$: $u\,(x_0)$

at <u>neighbouring</u> pt: $\underset{\sim}{u}\,(\underset{\sim}{x}_0 + \Delta \underset{\sim}{x})$

If $\Delta \underset{\sim}{X}$ is <u>small</u>

$$u_i\,(\underset{\sim}{x}_0 + \Delta \underset{\sim}{x}) \doteq u_i\,(\underset{\sim}{x}_0) + \left.\frac{\partial u_i}{\partial x_j}\right|_{\underset{\sim}{x}_0} \Delta x_j \qquad (i = 1, 2, 3)$$

<u>or</u>

$$\boxed{\Delta u_i = \left.\frac{\partial u_i}{\partial x_j}\right|_{x_0} \Delta x_j}$$

describes non-uniformity of displacements, <u>near</u> x_0

components of:

$\Rightarrow$ <u>Displacement gradient tensor</u> $\underset{\sim}{D}$

displacement gradient tensor

$$\underset{\sim}{D} = \frac{\partial u_i}{\partial x_j} \underset{-}{e}_i \underset{-}{e}_j$$

so that

$$\Delta \underset{\sim}{u} = \underset{\sim}{D} \cdot \Delta \underset{\sim}{x}$$

difference in displac. of two points

Decompose into sym & antisym.

$$\underset{\sim}{D} = \underset{\sim}{\epsilon} + \underset{\sim}{\omega} = \left[\frac{1}{2}(D_{ij} + D_{ji}) + \frac{1}{2}(D_{ij} - D_{ji})\right] e_i e_j$$

$$= \Bigg[\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\epsilon_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\omega_{ij}}\Bigg] \underset{-}{e}_i \underset{-}{e}_j$$

$\underset{\sim}{\epsilon}$ - strain tensor $= \epsilon_{ij} \underset{\sim}{e}_i \underset{\sim}{e}_j$

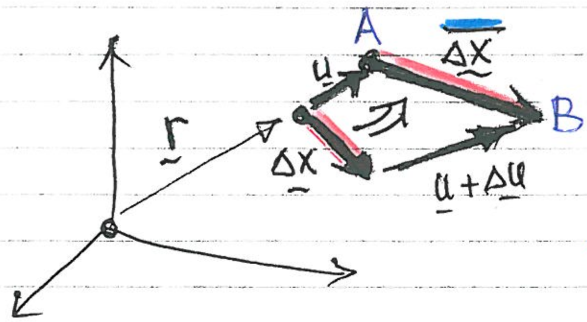
$\underset{\sim}{\omega}$ - rotation tensor $= \omega_{ij} \underset{-}{e}_i \underset{-}{e}_j$

(Have to justify these names)

Mechanical Meaning of strain components ϵ_{ij}

Consider material line $\Delta \underset{\sim}{X}$ that transforms into $\overline{\Delta \underset{\sim}{X}} = \underset{\sim}{AB}$

$\Delta \underset{\sim}{u}$: Displac. vector change (along $\Delta \underset{\sim}{X}$)



position vector of point B

position vector of point A

$$\overline{\Delta \underset{\sim}{X}} = \left(\underset{\sim}{r} + \Delta \underset{\sim}{X} + \underline{u} + \Delta \underline{u}\right) - \left(\underset{\sim}{r} + \underline{u}\right) = \Delta \underline{X} + \Delta \underline{u}$$

$$\Delta S^2 = (\text{Length before})^2 = \Delta \underset{\sim}{X} \cdot \Delta \underset{\sim}{X}$$

$$\overline{\Delta S}^2 = (\text{length after})^2 : \overline{\Delta \underset{\sim}{X}} \cdot \overline{\Delta \underset{\sim}{X}} = \left(\Delta \underline{X} + \Delta \underline{u}\right) \cdot \left(\Delta \underline{X} + \Delta \underline{u}\right) =$$

$$= \underbrace{\Delta \underline{X} \cdot \Delta \underline{X}}_{\Delta S^2} + \underbrace{2\, \Delta \underline{u} \cdot \Delta \underline{X}}_{2\, \Delta u_i \Delta X_i} + \underbrace{\Delta \underline{u} \cdot \Delta \underline{u}}_{\Delta u_i \Delta u_i}$$

Use:
$$\Delta u_i = \frac{\partial u_i}{\partial x_j} \Delta x_j$$

$$\Rightarrow \overline{\Delta S}^2 - \Delta S^2 = 2 \underbrace{\frac{\partial u_i}{\partial x_j}}_{D_{ij}} \Delta x_i \Delta x_j + \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} \Delta x_k \Delta x_j$$

Assumption (throughout the course) : small $\partial u_i / \partial x_j$ — Small geometry changes

$\Rightarrow$ second term neglected

Use $D = \epsilon + \omega$

$$= 2\, \epsilon_{ij} \Delta x_i \Delta x_j + \underbrace{2 \omega_{ij} \Delta x_i \Delta x_j}_{= 0} \quad \text{since } \omega_{ij} = -\omega_{ji}$$

$\Rightarrow \omega_{ij}$ does not affect length changes

Comment on ω - term:

$$\omega_{11} = \omega_{22} = \omega_{33} = 0$$

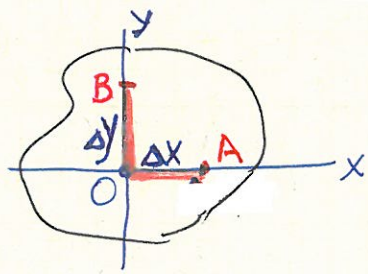
$$\omega_{ij} \Delta X_i \Delta X_j = \underbrace{\omega_{12} \Delta X_1 \Delta X_2 + \omega_{21} \Delta X_2 \Delta X_1}_{=0} + \dots$$

$\Rightarrow$ ω_{ij} term does not affect length changes

ω_{ij} describe <u>rotations</u>

closer look at $\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$

Consider a deformable particle



Planar motion of XY cross-section

How do we define angle ω of rotation of the particle?

Not obvious: it's different for different lines

Reasonable definition: take ω for two perpendicular lines (say, X, Y) and average ω's for them

$$u_y(0) + \omega_{0A} \cdot \Delta x = u_y(A) \Rightarrow \omega_{0A} = \frac{\partial u_y}{\partial x}$$

$$u_x(0) - \omega_{0B} \cdot \Delta y = u_x(B) \Rightarrow \omega_{0B} = -\frac{\partial u_x}{\partial y}$$

clockwise

Average : $\frac{1}{2}(\omega_{0A} + \omega_{0B}) = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \omega_{xy}$

$\therefore$ ω_{ij} gives a physically reasonable definition of angle of rotation of a deformable particle

<u>Comment</u> : the key assumption was :

$$\frac{\partial u_i}{\partial x_j} \left(= \epsilon_{ij} + \omega_{ij}\right) \text{ are } \underline{\text{all}} \text{ small}$$

(small geometry changes)

This means : <u>both</u> - strains ϵ_{ij} <u>are small</u>
- rotations ω_{ij}

excludes : <u>flexible</u> members (membranes, etc)

(strains may be small, but rotations are not)

Such as thin <u>steel</u> sheets

Thus

$$\overline{\Delta s}^2 - \Delta s^2 = 2\,\epsilon_{ij}\, \underbrace{\Delta X_i\, \Delta X_j}$$

identifies the element $\Delta \underset{\sim}{X}$.

Transform this formula, to get relative elongation $\dfrac{\overline{\Delta s} - \Delta s}{\Delta s}$

$$\overline{\Delta s}^2 - \Delta s^2 = \left(\overline{\Delta s} - \Delta s\right)\underbrace{\left(\overline{\Delta s} + \Delta s\right)}_{2\Delta s + (\overline{\Delta s} - \Delta s)}$$

Neglecting the term of second of smallness $(\overline{\Delta s} - \Delta s)^2$:

$$\simeq 2\Delta s\left(\overline{\Delta s} - \Delta s\right)$$

$$\Rightarrow \quad \frac{\overline{\Delta s} - \Delta s}{\Delta s} = \frac{\epsilon_{ij}\, \Delta X_i\, \Delta X_j}{\Delta s^2}$$

represent

$$\Delta \underset{\sim}{X} = \Delta s\, \underset{\sim}{n}$$

orientation info.

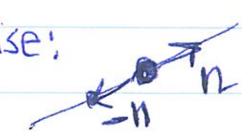
directional info

$$\Rightarrow \quad \boxed{\frac{\overline{\Delta s} - \Delta s}{\Delta s} = \epsilon_{ij}\, n_i\, n_j}$$

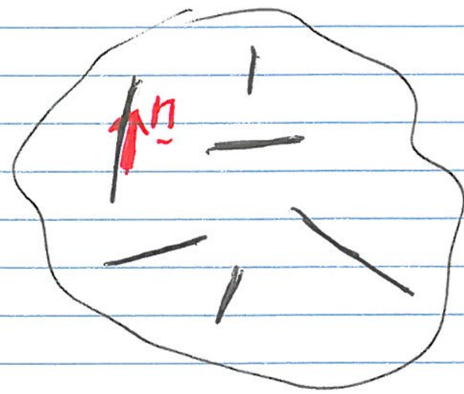
- basic formula of strain analysis

Note : independent of choice of sense:

$$n \to -n$$



Application : Composite material, with fibers



deformed,
strain tensor ϵ_{ij}

Find : relative elongations (strains)
in fibers of different orientations

— use basic formula —

Diagonal components of strain : for example ϵ_{11}

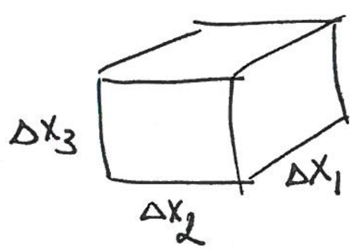
Choose mat'l element along X_1 axis : $n_1 = 1$, $n_2 = n_3 = 0$

Its relative elongation :

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_{ij} n_i n_j = \epsilon_{11}$$

$\Rightarrow$ its new length $\overline{\Delta S} = (1 + \epsilon_{11}) \Delta S$

Sum of the diagonal elements ϵ_{ii} $(= \epsilon_{11} + \epsilon_{22} + \epsilon_{33})$



$$\Delta V = \Delta X_1 \, \Delta X_2 \, \Delta X_3$$

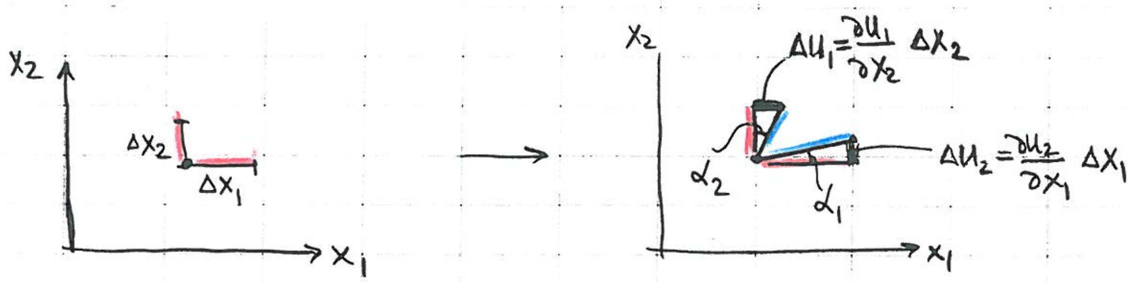
$$\overline{\Delta V} = \Delta X_1 (1 + \epsilon_{11}) \Delta X_2 (1 + \epsilon_{22}) \Delta X_3 (1 + \epsilon_{33})$$

$$= \Delta V (1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \text{higher order terms})$$

$\Rightarrow$ $\epsilon_{ii} = \dfrac{\overline{\Delta V} - \Delta V}{\Delta V}$ – relative volume change

(dilatation)

Off-Diagonal components of ϵ_{ij} . For example ϵ_{12}



$$\tan \alpha_1 = \frac{\partial u_2/\partial x_1 \, \Delta x_1}{\Delta x_1} = \frac{\partial u_2}{\partial x_1} \quad ; \quad \tan \alpha_2 = \frac{\partial u_1}{\partial x_2}$$

$$\therefore \quad 2\epsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \tan \alpha_1 + \tan \alpha_2 \simeq \alpha_1 + \alpha_2$$ for small geom. changes

$\therefore \quad 2\epsilon_{12}$ = distortion of the (originally $90°$) angle between material lines along x_1 & x_2 directions

Analogously, ϵ_{23}, ϵ_{31}.

Principal Form of Strain

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ - symmetric tensor $\Rightarrow$ 3 real roots in eigenv. problem

$$\Rightarrow \underset{\sim}{\epsilon} = \epsilon_{I}\, \underset{\sim}{e}_{I}\, \underset{\sim}{e}_{I} + \epsilon_{II}\, \underset{\sim}{e}_{II}\, \underset{\sim}{e}_{II} + \epsilon_{III}\, \underset{\sim}{e}_{III}\, \underset{\sim}{e}_{III}$$

Any deformation: equivalent to elongations & contractions in 3 principal directions; angles do not change between them

Summary on strains

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \epsilon_{ji}$$

symmetric tensor

$\epsilon_{12} = \epsilon_{21}$, etc

Diagonals:

$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$ – (relative) elongation in x_1-directio

off-diagonals:

$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}$ distortion of (originally 90°) angle $x_1 \hat{\ } x_2$

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}$$

do not have to distinguish between ϵ_{12}, ϵ_{21}

Symmetric tensor $\Rightarrow$ 3 real eigenvalues & 3 eigenvectors (princ. axes of $\underset{\sim}{\epsilon}$)

$\Rightarrow$ has principal representation:

$$\underset{\sim}{\epsilon} = \epsilon_{I}\, \underset{\sim}{e}_{I} \underset{\sim}{e}_{I} + \epsilon_{II}\, \underset{\sim}{e}_{II} \underset{\sim}{e}_{II} + \epsilon_{III}\, \underset{\sim}{e}_{III} \underset{\sim}{e}_{III}$$

$\Downarrow$

Any deformation is equivalent to elongations or contractions in 3 orthogonal princ. directions