Brill-Noether Theory of Graphs

Ben Hansel
Tufts University

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1 Introduction

The universe is extraordinarily diverse. Even the matter on the surface of our planet takes a seemingly infinite number of forms, and can thus be measured in several different ways. Since matter follows set laws, physical quantities have mathematical relationships between them, which themselves are mysterious and often surprising. The simplest and most ubiquitous of these relationships can be calculated only with addition and multiplication. A body in free fall travels a distance proportional to the square of the time elapsed since the moment it started falling. Light, sound, and the force of gravity all diminish proportional to the square of the distance from their source. We measure space with area and volume, which are respectively the square and cube of length. In fact, addition and multiplication are the only kind of functions that computers can actually calculate, and everything else is simply estimates. These functions are called polynomials, and their general properties have been studied for thousands of years.

A **polynomial** in a single variable is a function of the form:

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n \tag{1}$$

where $a_0, a_1, \ldots, a_n$ and the variable x are all real numbers, which means they lie on the number line and may be positive or negative. We may also assume that $a_n \neq 0$, because if $a_n = 0$, you could then define n to be one less. When given an input x, the output $P(x)$ is calculated using the right-hand side of the equation. The relevant question is: for which inputs x does $P(x) = 0$? If we could identify those x for any polynomial, then by varying the value of a_0 we can determine exactly when $P(x)$ takes any given value. The values of x for which $P(x) = 0$ are called the *roots* of $P(x)$.

When $n = 2$ in Equation 1, the polynomial $P(x)$ is called a **quadratic**. Methods for finding the roots of quadratics can be found in Babylonian and Egyptian tablets dating back to 2000 BC, in Greek literature from 300 BC, in Chinese mathematical treatises circa 200 BC, and in the revolutionary work of the Indian mathematician Brahmagupta circa 600 AD. The quadratic formula, given to us in its present day form by Descartes, but is arguably the most famous formula in all of mathematics:

Theorem 1.1. *(Quadratic formula) If a, b, c, and x are real numbers such that $ax^2 + bx + c = 0$ and $a \neq 0$, then*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The ancient mathematicians relied heavily on geometry to solve, and even to state, these algebraic equations. As the relationship between the visual and the symbolic deepened together with the knowledge of polynomials, their study came to be known as algebraic geometry.

Though developments in mathematics made algebraic manipulations easier to perform, polynomials remained elusive and mysterious. In the 16th century, explicit (albeit cumbersome) formulas for the roots of $P(x)$ when $n = 3$ or 4 were discovered by Italian mathematicians Tartaglia and Cardano, but it wasn't until the early 18th century that Évariste Galois proved that there was no such formula in the case that $n \geq 5$.

In Equation 1, we assumed that the coefficients $a_0, a_1, \ldots, a_n$ and the variable x were all **real** numbers–points on the number line. This sometimes leads to situations where $P(x)$ returns a nonzero value for any real number input. For example, the function $P(x) = x^2 + 1$ never equals 0 for any real number x. This is because such an x would have to satisfy $x^2 = -1$, but both positive and negative real numbers become positive when squared. In order to make sense of this, a new number system was necessary to understand the behavior of polynomials. In the 17th century, René Descartes was the first to call these new numbers **imaginary**, which led to the introduction of the symbol i to denote $\sqrt{-1}$. An imaginary number is any real number times i, and the sum of a real number and an imaginary number is called a **complex** number.

Although it seems as if introducing a new number to an impossible problem is a futile and meaningless action, the complex numbers have found a wide variety of useful applications throughout science and engineering. In fact, the complex numbers can be thought of as a completely geometric object–if the real numbers lie on a number line, then the complex numbers lie on the

complex plane, where the point (x, y) on the plane represents the complex number $x + yi$. So impossible quantities can in fact be useful without really "existing," just like with negative numbers. The introduction of complex numbers was a huge leap in the field of algebraic geometry (and mathematics in general), and paved the way for subsequent generations of mathematicians to develop more and more sophisticated and abstract tools to understand how polynomials behaved.

If we take Equation 1, but instead assume that the coefficients are all complex numbers (often denoted by z), we arrive at a surprising and useful property:

Theorem 1.2. *If $P(z)$ is a polynomial of degree n with complex coefficients $a_0, a_1, \dots, a_n$, then there exists a list of n complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = a_n(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)$$

This is the first of the many great miracles that make algebraic geometry such an alluring and mysterious subject.

We only have to add and subtract using one extra symbol i, where $a + bi = 0$ only when both a and b are 0. This is often referred to as the **fundamental theorem of algebra**. Since these early developments, the tools used to understand polynomials have been continuously refined and the field of algebraic geometry remains perhaps the most developed field of mathematics.

The polynomials we have considered so far only have one input variable and only return 0 for a finite number of inputs. When we allow for more input variables, the solution sets to polynomials become higher-dimensional objects, such as surfaces and solids. There is a special class of surfaces called **Riemann surfaces** with the property that each Riemann surface is the solution set to a complex polynomial, which makes their study quite fruitful in gaining insight to the inner workings of polynomial equations. One effective method of better understanding the geometry of these surfaces is to immerse them in an abstract space called **projective space**, the geometry of which is already well understood.

We must then ask, for a given Riemann surface, how many ways does it fit into projective space? It turns out that most curves are fairly predictable and follow general rules about which ones can be placed in projective space. However, certain Riemann surfaces have more symmetries than others and can therefore be placed in projective space in unexpected ways. The field of study concerned with identifying these surfaces and studying their properties is called **Brill-Noether theory**.

2 Discrete Graphs

For a Riemann surface X, elements of the free abelian group on the points of X are called divisors, and are used to encode a specific kind of data about which meromorphic functions exist on X. This data is encoded in an equivalence relation on the group of divisors. For each meromorphic function $f : X \to \mathbb{C}$, the divisor $\mathrm{div}(f)$ associated to f keeps track of the locations and orders of the zeros and poles of f. Two divisors are equivalent if X admits a meromorphic function whose associated divisor is equal to their difference. Tropical geometry gives us the tools to make powerful connections between the properties of Riemann surfaces, discrete graphs, and metric graphs. It turns out that these three wildly different classes of objects are highly analogous, and several concepts that guide our study of Riemann surfaces can be extended to the theory of graphs, including the concept of a divisor.

Divisors on discrete graphs and metric graphs encode information in the same way–through an equivalence relation. The stark simplicity of this construction makes it useful to consider the idea of a divisor on an abstract set. This treatment not only avoids needless repetition of definitions, but also highlights the minimal amount of information (only one choice, in fact) needed to totally determine the structure of the divisors on any object. The approach we take here is loosely based on the progression in the Yale course [JP17].

2.1 Divisors on sets

Let S be a set, assumed to be nonempty unless otherwise specified.

Definition 2.1. A **divisor** D on S is a function $D : S \to \mathbb{Z}$ that returns 0 for all but a finite number of elements of S. For an element $s \in S$, the value $D(s)$ is called the **degree of** D **at** s.

Divisors on S may be conveniently expressed as a formal sum of elements of S, so that:

$$D = \sum_{s \in S} D(s)s$$

For example, for $s, t \in S$, the divisor $D = 2s - 4t$ has degree 2 at s, degree -4 at t, and degree 0 everywhere else. In fact, under pointwise addition the divisors form the **free abelian group** on S, denoted $\text{Div}(S)$. When dealing with an arbitrary choice of underlying set, we will omit the argument of Div and other associated objects.

Definition 2.2. The **degree** $\deg(D)$ of a divisor D on a set is the sum of the degrees of D over all elements, and it is clearly seen that $\deg : \text{Div} \to (\mathbb{Z}, +)$ is a group homomorphism. For an integer k, let Div_k denote the set of divisors of degree k. Note that Div_0 is a subgroup of Div.

Definition 2.3. For a set S, choosing a subgroup $\text{Prin} \leq \text{Div}_0(S)$ gives S a **divisor structure**. Elements of Prin are called **principal divisors**.

It is interesting that a nuanced study of divisors arises from such a simple choice to be made. We now examine the way that Prin is defined on Riemann surfaces, the example that first motivated the study of divisors.

Example 2.4. In the theory of compact Riemann surfaces, divisors are used to encode information about which meromorphic functions exist on a given surface. Recall that for a compact Riemann surface X, a meromorphic function $f : X \to \mathbb{C}$ is holomorphic everywhere on X but a finite set of points, but is not a "function" in the strictest sense. Not only does it diverge to infinity at certain points called the **poles** of f. However, every such point is **isolated**, which means that each point of X has a punctured neighborhood on which f is defined and holomorphic. For any point p, this allows us to express f via a **Laurent series about** p–an infinite sum that can express any function that is holomorphic on an annulus (or punctured circle, or punctured plane) encircling p.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (p - z)^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)dz}{(z - p)^{n+1}}.$$

Here, γ is a rectifiable loop going counterclockwise around p exactly once that is contained in an annulus on which f is holomorphic. This allows us to define the **order of f at** p, the smallest exponent with a nonzero coefficient in the Laurent expansion of f, denoted $\operatorname{ord}_f(p) = \min\{n : a_n \neq 0\}$. The Laurent series of f contains no negative terms (and is thus a Taylor series) exactly when f is holomorphic at the point p. Furthermore, the order of f at such p is negative when p is a pole, positive when p is a zero, and 0 otherwise [Mar73]. It is possible that a function is holomorphic around an isolated point, but the order does not exist because an infinite number of $a_n \neq 0$ for $n < 0$. These points are called **essential singularities**, and occur when $|f(z)|$ has no limit as z approaches p. We exclude these cases from the definition of meromorphicity [Mir95, pp. 24].

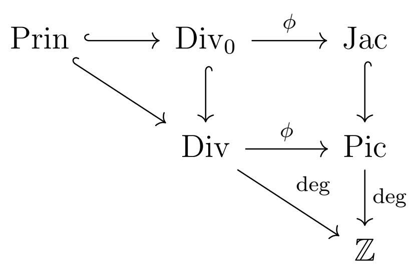
It is a convenient fact that for a meromorphic function on a compact Riemann surface, the orders of all zeros and poles sum to zero [Mir95]. Since the set of zeros and poles is finite, we may thus imbue X with a divisor structure by setting the subgroup $\operatorname{Prin} \leq \operatorname{Div}_0(X)$ to the image of the map $\operatorname{div} : \mathscr{M}(X) \to \operatorname{Div}_0(X)$ from the group $\mathscr{M}(X)$ of meromorphic functions on X to the group of divisors given by

$$\operatorname{div}(f) = \sum_{p \in X} \operatorname{ord}_f(p) p \qquad \Leftrightarrow \qquad \operatorname{div}(f)(p) = \operatorname{ord}_f(p).$$

The set of zeros and poles is finite. So a divisor $D \in \operatorname{Div}(X)$ is principal if and only if X admits a meromorphic function f such that $D = \operatorname{div}(f)$.

Definition 2.5. For a set with divisor structure Prin, the **Picard group** Pic is the quotient $\operatorname{Div} / \operatorname{Prin}$ and the **Jacobian group** Jac is the quotient $\operatorname{Div}_0 / \operatorname{Prin}$. The equivalence class of a divisor D is denoted $[D]$. If $[D] = [D']$ for some $D, D' \in \operatorname{Div}$, then we write $D \sim D'$ and say D and D' are **equivalent**. That is, $D \sim D'$ if and only if $D - D' \in \operatorname{Prin}$.

We have described several groups at this point, but they are all very closely related. The following commutative diagram illustrates their relationship, with $\hookrightarrow$ denoting inclusion, and ϕ denoting the map that takes a divisor D to its equivalence class $[D]$.



Definition 2.6. A divisor is **effective** if it has nonnegative degree at all points, and the set of effective divisors is denoted Div^+. A divisor class $[D] \in \operatorname{Pic}$ is **effective** if it contains an effective divisor, and the set of effective divisor classes is denoted $\operatorname{Pic}^+ \subseteq \operatorname{Pic}$.

Remark 2.7. The symmetric groups $\operatorname{Sym}(k)$ of permutations of k objects under composition acts on S^k by permuting the coordinates of each element, so that

$$\sigma \cdot (p_1, \ldots, p_k) = (p_{\sigma(1)}, \ldots, p_{\sigma(k)})$$

for $\sigma \in S_k$ and $(p_1, \ldots, p_k) \in S^k$. If we take the quotient of S^k by this action, we obtain $\mathrm{Sym}_k(S)$. Since the elements of $\mathrm{Sym}_k(S)$ are uniquely expressed as unordered k-tuples of elements $p_1, \ldots, p_k \in S$, and $p_i = p_j$ is possible when $i \neq j$, that means that the elements of $\mathrm{Sym}_k(S)$ are **multisets** of k elements of S. A multiset is a mathematical object with the same properties as a set, except that it may contain multiple copies of the same element. Effective divisors are the same, since they are also unordered sums of points of S.

Definition 2.8. The **rank** of a divisor D is defined as:

$$\mathrm{rk}(D) = \max\{k \in \mathbb{Z} : [D - E] \in \mathrm{Pic}^+ \quad \forall E \in \mathrm{Div}_k^+\}$$

and if no such maximum exists, we set $\mathrm{rk}(D) = -1$.

Although the definition of rank is rather difficult to parse, it can be thought of as encoding the effectiveness of a divisor (actually divisor class, as we will see). A divisor is of rank at least k if it remains equivalent to an effective divisor no matter what combination of k points (effective divisor of degree k) is subtracted from it. In other words, for any multiset $\wr p_1, \ldots, p_k \wr$ of points in S, all divisors D of rank at least k have $D \sim \sum_i p_i + E$ for some effective E.

Notice that the rank increases with the size of Prin; that is, for two divisor structures Prin and Prin' with respective rank functions $\mathrm{rk}, \mathrm{rk}'$, if $\mathrm{Prin} \subseteq \mathrm{Prin}'$, then $\mathrm{rk}(D) \leq \mathrm{rk}'(D)$ for all $D \in \mathrm{Div}(S)$. This is simply because the larger Prin is, the fewer divisor classes there are in $\mathrm{Pic}(S)$, so the more divisors are equivalent to effective divisors.

Proposition 2.9. Equivalent divisors have equal rank and degree.

Proof. Let $D_1 \sim D_2$. Since $\mathrm{Prin} \subseteq \mathrm{Div}_0$, $\deg(D_1) = \deg(D_2)$. Let $\mathrm{rk}(D_1) = k$, and let $E \in \mathrm{Div}_k^+$ be an effective divisor of degree k, so that there exists $E' \in \mathrm{Div}^+$ with $D_1 - E - E' \in \mathrm{Prin}$. Since $D_1 - D_2 \in \mathrm{Prin}$ and Prin is a subgroup, $D_2 - D_1 \in \mathrm{Prin}$ and thus $(D_2 - D_1) + (D_1 - E - E') = D_2 - E - E' \in \mathrm{Prin}$. Therefore, $\mathrm{rk}(D_2) \geq \mathrm{rk}(D_1)$. If we simply switch D_1 and D_2 in the above argument, we then obtain $\mathrm{rk}(D_1) \geq \mathrm{rk}(D_2)$, so we must have $\mathrm{rk}(D_1) = \mathrm{rk}(D_2)$. □

Since equivalent divisors have equal rank and degree, that allows us to adopt a useful notation for all of the sets and groups we have dealt with so far.

Definition 2.10. For $r, d \in \mathbb{Z}$ and $A = \mathrm{Div}, \mathrm{Div}^+, \mathrm{Pic}, \mathrm{Pic}^+$

1. A_d is the set of degree d divisors (or divisor classes).

2. A_d^r is the set of degree d and rank r divisors (or divisor classes).

2.2 Preliminaries of Graphs

A **graph** G is an ordered pair $(V(G), E(G))$ of sets of vertices and edges respectively. We often simply write V for $V(G)$. The graphs considered in this paper are **loopless** (each edge connects two distinct vertices); however, there may still be multiple edges between two vertices. The number of edges that meet at a vertex v is called the **valency** of v and is denoted $\mathrm{val}(v)$. For two vertices

$v, w \in V(G)$, let $vw \subseteq E(G)$ denote the set of edges with endpoints v and w. A **path** is a finite sequence of edges $e_1, \ldots, e_n$ such that there exist distinct vertices $w_0, w_1, \ldots, w_{n-1}, w_n$ satisfying $e_i \in w_{i-1}w_i$ for each $1 \leq i \leq n$. Such a path is said to **connect** the vertices w_0 and w_n. A path where $w_0 = w_n$ (but all the other vertices remain distinct) is called a **cycle**. All graphs considered in this paper are also **connected** by assumption, which means that any two vertices are connected by at least one path.

Definition 2.11. A **divisor** on a graph G is a function $D : V(G) \to \mathbb{Z}$.

There are several equivalent ways to conceptualize the equivalence relation of divisors on discrete graphs. In order to gain a thorough understanding of this elementary concept, we will first consider the point of view optimized for developing our intuition. This will give us easier access to the other two points of view, one optimized for performing explicit calculations, and one optimized for generalizing to metric graphs and beyond.

As is the tradition in the purely combinatorial setting in which this theory was developed, we begin by thinking of divisors as expressing the number of "chips" on each vertex of the graph. We then define the way that chips may move around the graph that preserve the equivalence of divisors. Vertices with a negative number of chips may be thought of as being "in debt," or having "antichips." Under this interpretation, chips and antichips can annihilate with each other, and also be generated in chip-antichip pairs.

Definition 2.12. A **chip-firing move** is made by selecting any vertex v and "firing" one of the chips on v down each edge incident to v, preserving the total number of chips. A divisor is **principal** if it is chip-firing equivalent to 0.

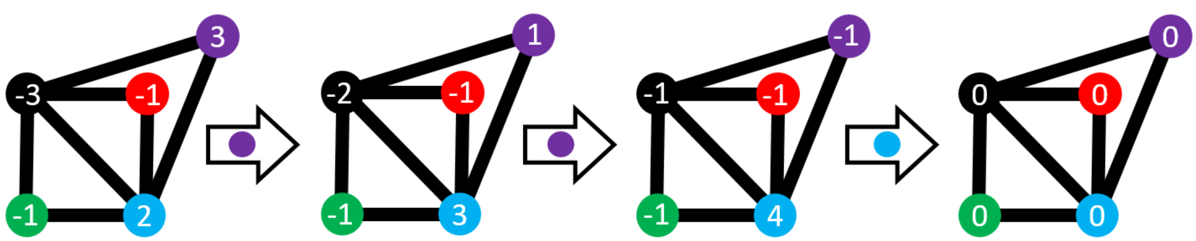


Figure 1: Three chip-firing moves

We now wish to verify that the principal divisors form a subgroup of $\mathrm{Div}_0(G)$. Since chip-firing moves preserve the total number of chips, principal divisors always have degree 0, so $\mathrm{Prin} \subseteq \mathrm{Div}_0(G)$. Furthermore, since the order in which the chips are fired doesn't matter, for two principal divisors P and P' obtained by to two sequences of chip firing moves, $P + P'$ corresponds to the concatenation of these sequences. This property highlights how the free abelian group $\mathrm{Div}(G)$ is straightforward to work with, since it is a direct sum of $|V|$ copies of $\mathbb{Z}$. Finally, what is the inverse of a chip-firing move? In fact, firing every vertex besides some vertex v has the exact opposite effect as firing just v. Each pair of vertices not containing v exchanges a chip for each edge connecting them–with no net loss or gain–and each vertex adjacent to v gives one of its chips to v.

While this definition provides invaluable intuition, we must use more precise and powerful notation to learn about the algebraic properties of divisors. Recall that divisors may be expressed as formal sums of points with integer coefficients.

Proposition 2.13. The subgroup $\mathrm{Prin}(G) \leq \mathrm{Div}(G)$ of principal divisors on a graph G is the image of the map $\mathrm{div} : \mathrm{Div}(G) \to \mathrm{Div}_0(G)$ defined by

$$\mathrm{div}(D) = \sum_{v,w \in V} |vw|(D(w) - D(v))v \qquad \Leftrightarrow \qquad \mathrm{div}(D)(v) = \sum_{w \in V} |vw|(D(w) - D(v)).$$

Here, $|vw|$ denotes the number of edges connecting vertices v and w.

Proof. For a divisor consisting of a single vertex v,

$$\mathrm{div}(v) = \sum_{w,w' \in V} |vw|(v(w) - v(w'))w = \sum_{w,w' \in V} \begin{cases} 0 & \text{for } w, w' = v \\ -e_{w,w'}v & \text{for } w = v, w' \neq v \\ e_{w,w'}v & \text{for } w \neq v, w' = v \\ 0 & \text{for } w, w' \neq v \end{cases}$$

$$= \left(\sum_{w \in V} |vw|w \right) - \left(\sum_{w \in V} |vw|v \right) = \left(\sum_{w \in V} |vw|w \right) - \mathrm{val}(v)v.$$

This is consistent with the definition of a chip-firing move–the term $\left(\sum_{w \in V} |vw|w\right)$ means that for each vertex w adjacent to v, degree (chips) equal to $|vw|$ is moved to w, and consequently $\mathrm{val}(v)$ chips are subtracted from v. Furthermore, for any two divisors $D, D' \in \mathrm{Div}(G)$,

$$\mathrm{div}(D + D') = \sum_{v,w \in V(G)} |vw|((D + D')(w) - (D + D')(v))v$$

$$= \left(\sum_{v,w \in V(G)} |vw|(D(w) - D(v))v \right) + \left(\sum_{v,w \in V(G)} |vw|(D'(w) - D'(v))v \right) = \mathrm{div}(D) + \mathrm{div}(D').$$

This amounts to the property of $\mathrm{div} : \mathrm{Div}(G) \to \mathrm{Div}_0(G)$ being a group homomorphism. So if a divisor can be transformed into another by a sequence $v_1, v_2, \ldots, v_n$ of chip-firing moves, then $\mathrm{div}\left(\sum_i v_i\right)$ gives their difference. □

While this definition is rather difficult to parse, writing divisors using formal sums is unambiguous and easy to manipulate algebraically. This exactness makes this method of characterizing principal divisors especially helpful for encoding their algebraic properties. Since divisors are expressed as formal sums of points, manipulating divisors algebraically using chip-firing often involves switching around the domains over which sums are calculated. As such, we will introduce some notation to facilitate these calculations.

Definition 2.14. For a subset A of a set S with a divisor structure, denote by $\overline{A}$ the **complement** of A in S.

Remark 2.15. For a subset $A \subseteq S$, any divisor that can be expressed as the sum of the values of a function $f : S \times S \to \mathrm{Div}(A)$ over the domain $S \times S$ may be decomposed via a simple rule:

$$(2) \qquad \sum_{v,w \in S} f(v,w) = \left(\sum_{v,w \in A} f(v,w) \right) + \left(\sum_{\substack{v \in A \\ w \in \overline{A}}} f(v,w) + f(w,v) \right) + \left(\sum_{v,w \in \overline{A}} f(v,w) \right).$$

It is often the case that we wish to fire each vertex in a given subset $A \subseteq V$ exactly once. We thus introduce notation for this maneuver:

Definition 2.16. The **firing divisor** div_A of A is the divisor obtained as a result of firing each vertex in A exactly once. We obtain the explicit formula via Equation 2:

$$(3) \quad \mathrm{div}_A = \mathrm{div}\left(\sum_{v \in A} v \right) = \sum_{\substack{v \in A \\ w \in \overline{A}}} |vw|(w - v) \qquad \Leftrightarrow \qquad \mathrm{div}_A(v) = \begin{cases} -\sum_{w \in \overline{A}} |vw| & v \in A \\ \sum_{w \in A} |vw| & v \in \overline{A} \end{cases}.$$

Definition 2.17. The **outdegree divisor** outdeg_A of A is the divisor with degree 0 outside of A, and for $v \in A$ has degree equal to the number of edges between v and points in $\overline{A}$. That is,

$$\mathrm{outdeg}_A = \sum_{\substack{v \in A \\ w \in \overline{A}}} |vw| v \qquad \Leftrightarrow \qquad \mathrm{outdeg}_A(v) = \begin{cases} \sum_{w \in \overline{A}} |vw| & v \in A \\ 0 & v \in \overline{A} \end{cases}.$$

Remark 2.18. It is a useful to note that for any subset $A \subseteq V$, the outdegree divisor outdeg_A is effective and $\mathrm{div}_A = \mathrm{outdeg}_{\overline{A}} - \mathrm{outdeg}_A$. Furthermore, if $v \in A \subseteq B$, then $\mathrm{outdeg}_A(v) \geq \mathrm{outdeg}_B(v)$.

Example 2.19. A graph G is a **tree** if for each pair of vertices $v, w \in V(G)$, there is exactly one path connecting v and w. For all trees T, the Jacobian group Jac(T) is trivial.

Let T be a tree with vertices $x, y \in V(T)$, and let $e \in xy$ be an edge. Let $X \subseteq V(T)$ (resp. Y) be connected component of x (resp. y) in the subgraph of T obtained by removing e. Since T is a tree, e is the unique edge connecting X and Y. The divisor div_X is principal by definition, and $\mathrm{div}_X = \mathrm{outdeg}_{\overline{X}} - \mathrm{outdeg}_X = y - x$. Therefore, any two adjacent points x, y have $x \sim y$. Since T is connected, for any two points $v, w \in V(T)$ there is a path from v to w passing through points $p_1, \ldots, p_n$ for some n. We then have $v \sim p_1 \sim \ldots \sim p_n \sim w$, which means that every pair of vertices is equivalent as divisors. This is the same as saying that the set of divisors of the form $D = v - w$ is contained in $\mathrm{Prin}(T)$. Since this set is a generating set for the group $\mathrm{Div}_0(T)$, that means $\mathrm{Div}_0(T) = \mathrm{Prin}(T)$ and $\mathrm{Jac}(T) = \mathrm{Div}_0(T)/\mathrm{Prin}(T) = 1$.

2.3 Dhar's Algorithm and Reduced Divisors

To determine the rank of a given divisor, we must now develop a method to test whether or not a divisor remains effective when you subtract a given point. In fact, there is a general method of moving as much degree as possible to a given vertex while keeping the divisor effective. The resulting divisor is called **reduced**. In this section, we give two conditions for a divisor to be reduced, prove their existence and uniqueness, and finally use reduced divisors to prove some general results about graphs.

Definition 2.20. For a graph G and a vertex $q \in V(G)$, a divisor D is q**-reduced** iff:

1. $D(v) \geq 0$ for all vertices $v \neq q$, and

2. each nonempty subset $A \subseteq V(G)\backslash\{q\}$ contains a point v such that $\text{outdeg}_A(v) > D(v)$.

In the language of chip-firing, the second condition is simply stating that firing all vertices in A will make some vertex other than q go into debt. Notice that the property of being q-reduced does not depend on the degree of the divisor at q. That is, if a divisor D is q-reduced, then so is $D + nq$ for all $n \in \mathbb{Z}$.

In fact, there is an algorithm for determining whether a divisor D is q-reduced. It is called **Dhar's burning algorithm**, introduced first by Deepak Dhar in [Dha90], and first used in the setting of divisor theory on graphs by Ye Luo in [Luo11]. The effectiveness of this algorithm is truly remarkable–not only does it determine if a divisor is q-reduced, it constructively locates the "loose chips" that can be pulled closer to q. It even generalizes naturally to the case of metric graphs. Dhar's algorithm is defined recursively. We think of each chip on $V(G)\backslash\{q\}$ as a firefighter that is able to stave off the flames of exactly one adjacent edge. We then set a fire at the vertex q. The fire then spreads until the whole graph burns, or a team of firefighters halts its advance. That is:

1. The vertex q burns.

2. If a vertex burns, all adjacent edges burn.

3. A vertex v burns if $D(v)$ is less than the number of adjacent burning edges.

Definition 2.21. The set of burned vertices after n steps of Dhar's algorithm is $\Delta_n^q(D) \subseteq V(G)$ (or simply Δ_n) given by

$$\Delta_0 = \{q\} \qquad \text{and} \qquad \Delta_{n+1} = \{v \in V(G) : \text{outdeg}_{\overline{\Delta_n}}(v) > D(v)\} \cup \Delta_n.$$

Furthermore, we denote $N = \min\{n : \Delta_n = \Delta_{n+1}\}$ and $\delta(v) = \min\{n : v \in \Delta_n\}$ for vertices $v \in \Delta_N$.

Proposition 2.22. The whole graph burns if and only if D is q-reduced.

Proof. Suppose the whole graph burns, and let $A \subseteq V\backslash\{q\}$ be nonempty, and let $\Delta_n = \Delta_n^q(D)$ as written above. We may define a function

$$\delta(v) = \max\{n : v \notin \Delta_n\},$$

and let $\{v_1, \ldots, v_n\}$ be an enumeration of $V(G)$ such that $v_1 = q$ and $\delta(v_i) \leq \delta(v_j)$ for any indices $i < j$. Let $v \in A$ be the vertex with lowest index in A and let $n = \delta(v)$, so that $A \subseteq \overline{\Delta_n}$ and thus

$$D(v) < \text{outdeg}_{\overline{\Delta_n}}(v) \leq \text{outdeg}_A(v),$$

guaranteeing that D is q-reduced.

Otherwise, let $A \subset V\backslash\{q\}$ be the set of unburnt vertices. By virtue of being unburnt, each vertex $v \in A$ has $D(v) \geq \text{outdeg}_A(v)$, which means that

$$D(v) + \text{div}_A(v) = D(v) - \text{outdeg}_A(v) \geq 0.$$

Furthermore, each vertex $w \in \overline{A}\backslash\{q\}$ has $D(w)+\mathrm{div}_A(w) = D(w)+\mathrm{outdeg}_{\overline{A}}(w) \geq 0$. Therefore, $D + \mathrm{div}_A$ is effective outside q, and D is not q-reduced.

□

In fact, a strong result that we can prove about q-reduced divisors which will guide our study from this point onward. The following three proofs are based on [BN06, Proposition 3.1].

Lemma 2.23. Each divisor D is equivalent to a divisor effective outside q.

Proof. For $n \geq 0$, let $Q_n = \{w \in V | d(q, w) \leq n\}$ be the set of vertices whose distance from q is at most n. Since G is finite and connected, we may let N be the smallest integer with $Q_N = V(G)$. For $1 \leq n \leq N$, each vertex in $Q_n\backslash Q_{n-1}$ is adjacent to at least one vertex in Q_{n-1} and 0 vertices in $\overline{Q_n}$. By Proposition 13, $\mathrm{div}(Q_{n-1}) = \mathrm{outdeg}(\overline{Q_{n-1}}) - \mathrm{outdeg}(Q_{n-1})$, so $\mathrm{div}(Q_{n-1})$ has positive nonzero degree on $Q_n\backslash Q_{n-1}$ and degree 0 on $\overline{Q_n}$. Therefore, given a divisor D effective on $\overline{Q_n}$, there is some positive integer m such that $D' = D + \mathrm{div}\,(m(Q_{n-1}))$ is effective on $(Q_n\backslash Q_{n-1}) \cup \overline{Q_n} = \overline{Q_{n-1}}$. Since each divisor D is effective on $\overline{Q_n} = \emptyset$, we may repeat this process N times to obtain a divisor effective on $\overline{Q_0} = V(G)\backslash\{q\}$. □

Lemma 2.24. Each divisor D is equivalent to a q-reduced divisor.

Proof. By the preceding lemma, we may select a divisor D effective outside of q. Let $A_1, \ldots, A_n$ be an enumeration of the nonempty subsets of $V\backslash\{q\}$. If for each index i there is a vertex $v \in A_i$ with $\mathrm{outdeg}_A(v) > D(v)$, then D is q-reduced. Otherwise we cycle through the indices and for each index i, fire all points in a set A_i if the resulting divisor remains effective outside of q. Each iteration of this process sends degree to q (since G is connected), but since q itself is never chip-fired, the sum $\left(\sum_{v\in V\backslash\{q\}} D(v)\right)$ decreases with each iteration, and since it must remain nonnegative, this procedure terminates with a q-reduced divisor. □

Proposition 2.25. There is exactly one q-reduced divisor in each divisor class.

Proof. Let $D \sim D'$ be two q-reduced divisors with $D \neq D'$, so that $D' = D + \mathrm{div}(C)$ for some non-constant divisor C, chosen so that $C(q) > C(v)$ for some v. Let m be the minimum of C, and let $A = \{v \in V : C(v) = m\} \subset V\backslash\{q\}$ be the set on which C achieves its minimum. For each $v \in A$,

$$\mathrm{outdeg}_A(v) = \sum_{w\in\overline{A}} |vw| \leq \sum_{w\in\overline{A}} |vw|(C(w) - C(v)) = \sum_{w\in V} |vw|(C(w) - C(v)) = \mathrm{div}(C)(v).$$

This means that

$$0 \leq D(v) = D'(v) - \mathrm{div}(C)(v) \leq D'(v) - \mathrm{outdeg}_A(v).$$

Since $D'(v) \geq \mathrm{outdeg}_A(v)$ is true for all $v \in A$, then D' can not be q-reduced. Therefore there can be at most one q-reduced divisor in each divisor class, and by the preceding lemma, there is exactly one. □

This is an essential result, and an invaluable way to characterize the Picard group $\mathrm{Pic}(G)$ of divisor classes.

Proposition 2.26. A divisor class is effective if and only if its q-reduced representative is effective.

Let D be q-reduced. If D is effective, then $[D]$ is by definition effective. Now assume that D is not effective, let $C \in \mathrm{Div}(G)$ be any divisor, and let $A \subseteq V$ be the set at which C achieves its maximum, so that $\mathrm{div}(C)(v) \leq -\mathrm{outdeg}_A(v)$ for all $v \in A$. If $q \in A$, then $(D + \mathrm{div}(C))(q) < 0$. If $q \notin A$, since D is q-reduced, there is some $v \in A$ with $\mathrm{outdeg}_A(v) > D(v)$, so $(D + \mathrm{div}(C))(v) < 0$, so $D + \mathrm{div}(C)$ is not effective. □

2.4 Riemann-Roch for Sets

This section uses results and proofs from [BN07, Section 2], in which Matthew Baker and Serguei Norine use the theory of divisors on abstract sets to formulate a condition equivalent to the Riemann-Roch formula. They then show that graphs satisfy this condition using the results concerning reduced divisors.

Definition 2.27. For a divisor D, define the **plus-degree** (resp. **minus-degree**) as the sum of the degrees of D at all points where it is positive (resp. negative). That is:

$$\deg^+(D) = \sum_{\substack{v \in V \\ D(v) > 0}} D(v) \qquad \text{and} \qquad \deg^-(D) = \sum_{\substack{v \in V \\ D(v) < 0}} D(v).$$

Definition 2.28. In this section, let $\mathcal{N}$ denote $\mathrm{Div}_{g-1}^{-1}(S)$, as in Definition 2.10. We say a divisor structure Prin on a set S is **special** if for all $D \in \mathrm{Div}(S)$, there exists a divisor $\nu \in \mathcal{N}$ such that exactly one of the two divisor classes $[D]$ and $[\nu - D]$ is effective. z

Lemma 2.29. For divisor $D \in \mathrm{Div}(S)$, if S has a special divisor structure then $\mathrm{rk}(D) = r$, where

$$r = \min_{\substack{D' \in [D] \\ \nu \in \mathcal{N}}} \{\deg^+(D' - \nu)\} - 1.$$

Proof. If $\mathrm{rk}(D) < r$, then there exists an effective divisor $E \in \mathrm{Div}_r^+$ such that $[D - E]$ is not effective. By assumption, there is then a divisor $\nu \in \mathcal{N}$ such that $\nu - D + E$ is equivalent to some effective $E' \in \mathrm{Div}^+$. However, this means that $E - E' + \nu = D' \sim D$, and

$$\deg^+(D' - \nu) = \deg^+(E - E') < \deg(E) + 1 = \min_{\substack{D' \in [D] \\ \nu \in \mathcal{N}}} \{\deg^+(D' - \nu)\},$$

which is a contradiction. Therefore, $\mathrm{rk}(D) \geq r$.

Now, let $D' \in [D]$ and $\nu \in \mathcal{N}$ be divisors such that $\deg^+(D' - \nu) = r + 1$. Therefore, there exists an effective divisor $E \in \mathrm{Div}_{r+1}^+$ such that $D' - \nu = E - E'$ for another effective $E \in \mathrm{Div}^+$. However, this means that $D - E \sim \nu - E'$, and since $[\nu]$ and thus $[\nu - E']$ are ineffective, so is $[D - E]$. Therefore, $\mathrm{rk}(D) \leq r$. □

Theorem 2.30. *For a set S with special divisor structure and a divisor $K \in \mathrm{Div}_{2g-2}^{g-1}(S)$,*

$$\mathrm{rk}(D) - \mathrm{rk}(K - D) = \deg(D) - g + 1 \tag{4}$$

for all $D \in \mathrm{Div}(S)$.

Proof. For $\nu \in \mathcal{N}$ and $D' \in [D]$, write $\overline{\nu} = K - \nu$ and $\overline{D'} = K - D'$. Since $\nu - D' = \overline{D'} - \overline{\nu}$, we then have:

$$\deg^+(D'-\nu) - \deg^+(\overline{D'}-\overline{\nu}) = \deg^+(D'-\nu) - \deg^+(\nu - D') = \deg(D'-\nu) = \deg(D) - g + 1.$$

Since $\deg(\overline{\nu}) = g-1$ and $\mathrm{rk}(K) = g-1$, if $\overline{\nu} \sim E$ for some $E \in \mathrm{Div}^+_{g-1}(S)$, then $K - \overline{\nu} = \nu \sim E'$ for some $E' \in \mathrm{Div}^+$, which is a contradiction. Therefore, $\mathrm{rk}(\overline{\nu}) = -1$, which means that $\overline{\nu} \in \mathcal{N}$. By Lemma 2.25, we may choose $D_1', D_2' \in [D]$ and $\nu_1, \nu_2 \in \mathcal{N}$ such that

$$\deg^+(D'-\nu) \geq \deg^+(D_1'-\nu_1) = \mathrm{rk}(D) \qquad \text{and} \qquad \deg^+(\overline{D'}-\overline{\nu}) \geq \deg^+(\overline{D_2'}-\overline{\nu_2}) = \mathrm{rk}(K-D)$$

for all $\nu \in \mathcal{N}$ and $D' \in [D]$. We may then combine the inequalities above to obtain

$$\deg^+(D_1' - \nu_1) - \deg^+(\overline{D'} - \overline{\nu}) \leq \mathrm{rk}(D) - \mathrm{rk}(K-D) \leq \deg^+(D'-\nu) - \deg^+(\overline{D_2'} - \overline{\nu_2}),$$

Finally, taking $D = D_1$ and $\nu = \nu_1$ on left hand side and next taking $D = D_2$ and $\nu = \nu_2$ on the right hand side gives us the necessary bounds to complete the proof. □

2.5 Riemann-Roch for Discrete Graphs

An **orientation** on a graph G is a function $\mathcal{O} : E(G) \to V(G)$ that "orients" each edge $e \in E(G)$ to one of its endpoints $\mathcal{O}(e)$. A graph together with an orientation is called a **directed graph**. A path $e_1, \ldots, e_n$ connecting vertices w_0 and w_n with $e_i \cap e_{i+1} = \{w_i\}$ is a **directed path** if it satisfies $\mathcal{O}(e_i) = w_i$ for each $1 \leq i \leq n$. A cycle satisfying the same condition is called a **directed cycle**. An orientation is **acyclic** if it contains no directed cycles. For an orientation $\mathcal{O}$, the **dual orientation** $\overline{\mathcal{O}}$ is the orientation obtained by reversing the direction of each edge. A **source** (resp. **sink**) is a vertex with all adjacent edges oriented outward (resp. inward). Given an orientation $\mathcal{O}$ with no sinks, the finiteness of $V(G)$ allows us to extend each directed path until it becomes a directed cycle. Therefore each acyclic orientation $\mathcal{O}$ has at least one sink, and if we reverse the orientation, the same argument also guarantees at least one source.

Definition 2.31. For an orientation $\mathcal{O}$, the **indegree** divisor $\mathrm{indeg}_{\mathcal{O}}$ counts the number of edges oriented towards each vertex. That is:

$$\mathrm{indeg}_{\mathcal{O}} = \sum_{e \in E} \mathcal{O}(e) \qquad \Leftrightarrow \qquad \mathrm{indeg}_{\mathcal{O}}(v) = |\mathcal{O}^{-1}(v)|$$

Definition 2.32. For an orientation $\mathcal{O}$ on a graph G, the **moderator** $\nu_{\mathcal{O}} \in \mathrm{Div}(G)$ of $\mathcal{O}$ is a divisor defined as:

$$\nu_{\mathcal{O}} = \mathrm{indeg}_{\mathcal{O}} - \sum_{v \in V} v \qquad \Leftrightarrow \qquad \nu_{\mathcal{O}}(v) = \mathrm{indeg}_{\mathcal{O}}(v) - 1$$

Proposition 2.33. For each acyclic orientation $\mathcal{O}$, the moderator $\nu_{\mathcal{O}}$ has degree $g-1$ and rank -1.

Proof. Since each edge is oriented towards precisely one vertex,

$$\deg(v_{\mathcal{O}}) = \deg(\mathrm{indeg}_{\mathcal{O}}) + \deg\left(\sum_{v \in V} v\right) = |E| - |V| = g - 1.$$

We will now show that $\nu_{\mathcal{O}} + \mathrm{div}(D)$ is not equivalent to an effective divisor for all $D \in \mathrm{Div}(G)$. Let $A \subseteq V$ be the set at which D achieves its maximum, so that $\mathrm{div}(D)(v) \leq -\mathrm{outdeg}_A(v)$ for all $v \in A$. The orientation $\mathcal{O}$ is still acyclic when restricted to A, and must therefore have a source vertex $v \in A$, so that $\mathrm{indeg}_{\mathcal{O}}(v) \leq \mathrm{outdeg}_A(v)$. Finally, $(\nu_{\mathcal{O}} + \mathrm{div}(D))(v) \leq \mathrm{indeg}_{\mathcal{O}}(v) - \mathrm{outdeg}_A(v) - 1 < 0$, so $[v_{\mathcal{O}}]$ is not effective. □

Lemma 2.34. If D_1, D_2 are two divisors of nonnegative rank, $\mathrm{rk}(D_1 + D_2) \geq \mathrm{rk}(D_1) + \mathrm{rk}(D_2)$.

Proof. Let divisor D_1 and D_2 have nonnegative ranks k_1 and k_2 respectively. For each effective divisor E of degree $k_1 + k_2$, decompose E into two effective divisors E_1 and E_2 of degrees k_1 and k_2 respectively, so that $E = E_1 + E_2$. By assumption, there exist effective E_1', E_2' such that $D_1 \sim E_1 + E_1'$ and $D_2 \sim E_2 + E_2'$. Let $E' = E_1' + E_2'$, so that $D_1 + D_2 \sim E_1 + E_2 + E_1' + E_2' = E + E'$. □

Proposition 2.35. For each divisor D, there is an orientation $\mathcal{O}$ such that $[\nu_{\mathcal{O}} - D]$ is effective if and only if $[D]$ is not effective.

Proof. By Proposition 2.22, we may choose D to be the unique q-reduced representative of its divisor class. If $[D]$ is effective, then $\mathrm{rk}(D) \geq 0$. If $[\nu_{\mathcal{O}} - D]$ were effective, then $\mathrm{rk}(D - \nu_{\mathcal{O}}) \geq 0$, which by the previous lemma implies $0 \leq \mathrm{rk}(D) + \mathrm{rk}(\nu_{\mathcal{O}} - D) \leq \mathrm{rk}(\nu_{\mathcal{O}}) = -1$, a contradiction.

Conversely, in the case that $[D]$ is not effective, $D(q) \leq -1$ by Proposition 2.23. Since D is q-reduced, if we run Dhar's algorithm, the entire graph will burn. We will now construct an acyclic orientation $\mathcal{O}$ by orienting edges in the direction that the fire spreads. To make this precise, let $\Delta_n = \Delta_n^q(D)$ as in Definition 2.21 and Proposition 2.22, again letting $\{v_1, \ldots, v_n\}$ be an enumeration of $V(G)$ such that $v_1 = q$ and $\delta(v_i) \leq \delta(v_j)$ for any indices $i < j$. Define an orientation $\mathcal{O}$ by $\mathcal{O}(e) = v_{\max\{i,j\}}$ for each edge $e \in v_i v_j$, an orientation clearly seen to be acyclic. For each $v_i \neq q$, since $\{v_j : j \geq i\} \subseteq \overline{\Delta_{\delta(v_i)-1}}$, we have

$$\mathrm{indeg}_{\mathcal{O}}(v_i) = \sum_{j<i} |v_i v_j| = \mathrm{outdeg}_{\{v_j : j \geq i\}}(v_i) \geq \mathrm{outdeg}_{\overline{\Delta_{\delta(v_i)-1}}}(v_i) > D(v_i),$$

which means $\mathrm{indeg}_{\mathcal{O}}(v) - 1 = \nu_{\mathcal{O}} \geq D(v_i)$. Since $D(q) \leq -1 = \nu_{\mathcal{O}}(q)$, we know that $\nu_{\mathcal{O}} - D$ and thus $[\nu_{\mathcal{O}} - D]$ is effective. □

Definition 2.36. For a graph G, the **canonical divisor** K_G is given by:

$$K_G = \sum_{v \in V} (\mathrm{val}(v) - 2) v \qquad \Leftrightarrow \qquad K_G(v) = \mathrm{val}(v) - 2.$$

Proposition 2.37. The canonical divisor K_G has degree $2g - 2$ and rank $g - 1$.

Proof. Let $E \in \mathrm{Div}_{g-1}^{+}(G)$ be an effective divisor of degree $g - 1$, and assume for purposes of contradiction that $[K_G - E]$ is not effective. Then by Proposition 2.32, there exists an orientation $\mathcal{O}$ such that $[v_{\mathcal{O}} - K_G + E]$ is effective. Since $\mathrm{indeg}_{\mathcal{O}}(v) + \mathrm{indeg}_{\overline{\mathcal{O}}}(v) = \mathrm{val}(v)$, we have $[v_{\mathcal{O}} - K_G + E] = [E - \nu_{\overline{\mathcal{O}}}]$, and since $\deg(E - \nu_{\overline{\mathcal{O}}}) = 0$, that means that $v_{\mathcal{O}} \sim E$, which is a contradiction since $\mathrm{rk}(\nu_{\overline{\mathcal{O}}}) = -1$. □

Theorem 2.38. *(Riemann-Roch) For all divisors D on a graph G:*

$$\mathrm{rk}(D) - \mathrm{rk}(K_G - D) = \deg(D) + 1 - g.$$

Proof. By Propositions 2.31 and 2.32, G has a special divisor structure. The theorem then follows from Propositions 2.34 and 2.27. □

3 Metric Graphs

In the previous section, we looked at the equivalence of divisors on discrete graphs from two equivalent points of view: as arrangements of chips that could be transformed into one another via sequences of chip-firing moves, and as the equivalence relation induced by a subgroup of $\mathrm{Div}_0(G)$, which was defined as the image of a map $\mathrm{div} : \mathrm{Div}(G) \to \mathrm{Div}_0(G)$. While these are certainly useful ways to conceptualize the equivalence of divisors, there is another viewpoint that most accurately foreshadows the generalization to metric graphs, and even to Riemann surfaces.

In all three settings, the set of principal divisors is formulated as the image of a map div from the group of rational functions to the group of degree 0 divisors. For discrete graphs, rational functions have the same definition as divisors, so to avoid confusion no distinction between them was made in the previous section. However, they actually encode less information than divisors. We can condense the information by examining the kernel of div, which would be 0 were it an injective map. In the case of discrete graphs, a divisor C is constant if and only if $\mathrm{div}(C) = 0$ if and only if F is a constant divisor; that is, if it has the same degree at each vertex. We can then define the set $\mathrm{Rat}(G)$ of rational functions on G as

$$\mathrm{Rat}(G) = \frac{\mathrm{Div}(G)}{1_G\mathbb{Z}} \qquad \text{where} \qquad 1_G = \sum_{v \in V} v.$$

In fact, a rational function f on a discrete graph is in fact completely determined by the integer slope it takes on each edge, provided that it has the same slope on multiple edges. This information requires an orientation to be expressed.

3.1 Properties of Metric Graphs

Definition 3.1. A **metric graph** is the metric space obtained by gluing together a finite number of closed real intervals at a finite set of points.

While this is arguably the simplest definition of a metric graph, we may think of a metric graph Γ as finite **underlying graph** G together with a function $\ell : E(G) \to \mathbb{R}_{>0}$ assigning to each edge a positive real length, so that Γ is obtained by identifying each edge e of G with an interval of length $\ell(e)$ and gluing the intervals together at the vertices of G. As in the previous section, all metric graphs are assumed to be connected.

It is immediately noticeable that there is not a unique underlying discrete graph for a given metric graph. As we move from the world of the discrete to that of the continuous, we treat each point on the interior of an edge in Γ as a vertex of valency 2. For two discrete graphs G, G', we call G' a **subdivision** of G if G' can be obtained by introducing new vertices into edges of G. Two graphs can be the underlying graph for the same metric graph Γ if and only if they have a common subdivision.

As the name would suggest, metric graphs are metric spaces with the distance between two points defined as the length of the shortest path between them. Now that we have defined the set

Γ, we may imbue it with a divisor structure. As we mentioned before, this requires the notion of a rational function on Γ.

Definition 3.2. A **rational function** on a metric graph Γ is a continuous function $f : \Gamma \to \mathbb{R}$ that is linear with integer slope at all but a finite number of points of Γ. Let $\mathrm{Rat}(\Gamma)$ denote the set of rational functions on Γ.

Definition 3.3. For a rational function f on a metric graph Γ, we define the **order** $\mathrm{ord}_x(f)$ of f at a point $x \in \Gamma$ as the sum of the incoming slopes of f at x. The group Prin of **principal divisors** on Γ is the image of the map $\mathrm{div} : \mathrm{Rat}(\Gamma) \to \mathrm{Div}_0(\Gamma)$ given by:

$$\mathrm{div}(f) = \sum_{x\in\Gamma} \mathrm{ord}_x(f)x \qquad \Leftrightarrow \qquad \mathrm{div}(f)(x) = \mathrm{ord}_x(f).$$

Given here are two of the most useful properties of metric graphs, both of which were first proven in [Luo11].

Theorem 3.4. *(Riemann-Roch for metric graphs) For a metric graph Γ, each divisor $D \in \mathrm{Div}(\Gamma)$ satisfies*

$$\mathrm{rk}(D) - \mathrm{rk}(K_\Gamma - D) = \deg(D) - g + 1,$$

*where the **canonical divisor** K_Γ is equal to $\sum_{v\in\Gamma}(\mathrm{val}(v) - 2)v$.*

Definition 3.5. For a divisor $D \in \mathrm{Div}(S)$ and a subset $A \subseteq S$, the A**-rank** of D is defined as:

$$\mathrm{rk}^A(D) = \max\{k \in \mathbb{Z} : [D - E] \in \mathrm{Pic}^+(S) \quad \forall E \in \mathrm{Div}_k^+(A)\}$$

and if no such maximum exists, we set $\mathrm{rk}^A(D) = -1$. The set A is **rank-determining** if $\mathrm{rk}^A(D) = \mathrm{rk}(D)$ for all divisors $D \in \mathrm{Div}(S)$.

Theorem 3.6. *Let Γ be a metric graph with underlying graph G. Then $V(G) \subseteq \Gamma$ is a rank-determining set.* [Luo11, Theorem 1.5]

While we will not examine the proofs of these theorems in this paper, we will rely on the following two results in the final sections.

Definition 3.7. For two distinct points x and y on a set with a divisor structure, the **torsion order** $m(x, y)$ of x and y is defined as

$$m(x, y) = \min\{m \in \mathbb{Z}_{>0} : mx \sim my\},$$

with $m(x, y) = 0$ if no such minimum exists.

Proposition 3.8. On a metric graph Γ, two points $x, y \in \Gamma$ have $x \sim y$ if and only if there is a unique path connecting x and y.

Proof. If there is a unique path γ connecting x and y, the function f with slope 1 on γ, orientation $x \to y$, and slope 0 everywhere else is well defined with $\mathrm{div}(f) = y - x$. Conversely, let γ_1 and γ_2 be two distinct geodesics connecting x and y. Let γ_x, γ_y be the connected components of x and y respectively in $\gamma_1 \cap \gamma_2$, and let x_0 and y_0 be the endpoints of γ_x and γ_y farthest from x and y, so that $x \sim x_0$ and $y \sim y_0$. Let f be a function such that $\mathrm{div}(f) = y_0 - x_0$. There are multiple paths to y_0 emanating from x_0, but $\mathrm{div}(f)(x_0) = -1$, so f must have a slope of 1 on one of the paths but a slope of 0 on the rest. But since $\mathrm{supp}(\mathrm{div}(f)) = \{x_0, y_0\}$, f cannot change slope on these paths and would have to be discontinuous at y_0, so we have arrived at a contradiction. □

Proposition 3.9. Let x and y be two distinct points in the metric graph Γ, let P be the set of all paths from x to y, and let

$$\Gamma' = \bigcup_{p \in P} p \subseteq \Gamma,$$

be the subgraph of Γ made up of all paths in P. Then the torsion order of x and y is the same in Γ and Γ'.

Proof. Let x and y have torsion order m and m' in Γ and Γ' respectively, let $f \in \mathrm{Rat}(\Gamma)$ be a rational function with $\mathrm{div}(f) = m(y - x)$, and let A be the subgraph of Γ obtained by removing each edge of Γ' and vertex $v \in \Gamma'$ with $\mathrm{outdeg}_{\Gamma'}(v) = 0$. Each connected component $U \subseteq A$ is a subgraph of Γ that meets Γ' at a unique point u, since if it met Γ' at two points, it would constitute another path from x to y. Now since $f|_U \in \mathrm{Rat}(U)$, we have $\deg(\mathrm{div}(f|_U)) = 0$, so if f is nonconstant on U, there must be at least two points $v, w \in U$ at which $\mathrm{div}(f|_U)(v), \mathrm{div}(f|_U)(w) \neq 0$. Since $\mathrm{div}(f) = m(y - x)$ however, and we may not have both x, y in any such U, this forces f to have slope zero outside of Γ'. So there is in fact a bijection

$$\{f \in \mathrm{Rat}(\Gamma) : \mathrm{div}(f) = m(y - x)\} \Leftrightarrow \{f' \in \mathrm{Rat}(\Gamma') : \mathrm{div}(f') = m(y - x)\},$$

so that $m = m'$. □

3.2 A Single Loop

In this section, let Γ be the metric graph of a single loop, so that points x, y are connected by clockwise and counterclockwise edges, chosen such that the clockwise edge is longer. Since scaling the loop has no effect on the divisor structure, we set the length of the shorter edge to 1, and denote by $\ell \geq 1$ the length of the longer clockwise edge. Furthermore, for $a \in \mathbb{R}$, we denote by w_a the point a counterclockwise distance

$$d = a - (\ell + 1) \left\lfloor \frac{a}{\ell + 1} \right\rfloor$$

from x so that $x = w_0$, $y = w_1$, and $w_a = w_b$ if $a \equiv b \mod \ell + 1$.

By Dhar's algorithm, a divisor is q-reduced for any point $q \in \Gamma$ if it has at most 1 degree on points other than q.

The following proposition is adapted from [Pfl17].

Proposition 3.10. On the single loop Γ above,

$$m(x, y) = \begin{cases} \alpha + \beta & \text{if } \ell \in \mathbb{Q} \\ 0 & \text{if } \ell \notin \mathbb{Q} \end{cases}.$$

where $\alpha/\beta = \ell$ and $\gcd(\alpha, \beta) = 1$.

Proof. The torsion order $m(x, y)$ is the smallest positive integer m such that there is a rational function $f : \Gamma \to \mathbb{Z}$ such that $\mathrm{div}(f) = m(y - x)$. This function is completely determined by its slopes $\alpha, \beta \in \mathbb{Z}$ on the counterclockwise and counterclockwise edges respectively, oriented

$x \to y$. Since f is continuous, the slopes must satisfy $\alpha = f(y) - f(x) = \beta\ell$. If $\ell \notin \mathbb{Q}$, then no such integers exist and $m(x,y) = 0$. Otherwise, the torsion order is given by

$$m(x,y) = \min\left\{\alpha + \beta : \ell = \frac{\alpha}{\beta}\right\}.$$

This is achieved when α/β is a fraction in lowest terms. □

Remark 3.11. On the single loop Γ above, if ℓ is rational, let α/β express ℓ as a fraction in lowest terms. The expression for the torsion order of x and y is then:

$$m(x,y) = \alpha + \beta = \beta(\ell + 1).$$

Furthermore, if $\ell + 1 \mid k$ for some integer k, we may then express ℓ as a quotient of two integers $(k-n)/n$, where $n = k/(\ell+1) \in \mathbb{Z}$. This means $\beta \mid n$, and thus $\beta(\ell+1) = m \mid k$. So for all integers k, we in fact have $(\ell + 1 \mid k) \Leftrightarrow (m \mid k)$.

The following proof is based on [CDPR10 Example 2.1]. Here we offer explicit constructions of each equivalence, and expand the scope by removing the genericity condition, which ensures $\ell + 1 \nmid \beta k$.

Proposition 3.12. On the single loop Γ above, let $D = kx + u$ be an x-reduced divisor where $k \geq 0$ and u is either the zero divisor or $u = w_a$ for $a \in (0, \ell+1)$, and let $m = m(x,y)$ be the torsion order of x and y. Let D' be the unique y-reduced divisor equivalent to D. We then have

$$D' = \begin{cases} ky & \text{if } u = 0 \text{ and } m \mid k \\ (k-1)y + w_{1-k} & \text{if } u = 0 \text{ and } m \nmid k \\ (k+1)y & \text{if } u = w_{k+1} \\ ky + w_{a-k} & \text{otherwise} \end{cases}.$$

Proof. First, we consider the case that $m \mid k$. By Dhar's algorithm, $ky + u \sim D$ is a y-reduced divisor no matter the value of u. We can simplify this to $(k+1)y$ if and only if $u = w_1 = y$.

In the case that $m \nmid k$ and $u = 0$. The distance going clockwise from x to w_{1-k} is

$$d = 1 - k - (\ell+1)n, \qquad \text{where} \qquad n = \left\lfloor \frac{1-k}{\ell+1} \right\rfloor \in \mathbb{Z}.$$

We obtain $D + \operatorname{div}(f) = (k-1)y + w_{1-k}$, and we know that $w_{1-k} \neq y$, because if $d = 1$ then $\ell + 1 \mid k$, which by the remark above contradicts our assumption that $m \nmid k$.

In the case that $m \nmid k$ and $u = w_a$ for some $a \in (0, \ell+1)$. The distance going clockwise from x to w_{a-k} is

$$d = a - k - (\ell+1)n \qquad \text{where} \qquad n = \left\lfloor \frac{a-k}{\ell+1} \right\rfloor \in \mathbb{Z}.$$

We obtain $D + \operatorname{div}(f) = ky + w_{a-k}$, where $w_{a-k} = y$ if and only if $d = 1$, which is true exactly when $a \equiv k+1 \mod \ell+1$. Since $a \neq 0$, this is not possible in the case that $k+1 \equiv 0 \mod \ell+1$. If $d = a$ then $k \equiv 0 \mod \ell+1$, again contradicting our assumption that $m \nmid k$.

It can quickly be seen by Dhar's algorithm that the resulting divisors are y-reduced. □

Since this proof is central to the subsequent material, the explicit formula for the rational function f in each separate case is given at the end of the paper. The separate cases depend on which edge of the loops and in which order the points in question fall.

3.3 Chains of Loops

In this section, let the metric graph Γ be a chain of g loops, denoted $\gamma_1, \dots, \gamma_g$. The loops are joined at vertices $v_1, \dots, v_{g-1}$. By [Luo11, Theorem 1.5], for any choice of $v_0 \in \gamma_1 \backslash \{v_1\}$ and $v_g \in \gamma_g \backslash \{v_{g-1}\}$, the set $\{v_0, \dots, v_g\}$ will be a rank-determining set. We therefore choose v_0 and v_g to lie in the midpoints of γ_0 and γ_g respectively. Since the divisor structure of a single loop remains invariant under scaling, by Proposition 3.9, we may set the length of the shorter edge in γ_i to 1, and denote the length of the longer edge by $\ell_i > 1$ for each $i \in \{1, \dots, g\}$, so that $\ell_1 = \ell_g = 1$. Furthermore, for $a \in \mathbb{R}$ and $i \in \{1, \dots, g\}$, we denote by $w_a^i \in \gamma_i$ the point a counterclockwise distance

$$d = a - (\ell_i + 1) \left\lfloor \frac{a}{\ell_i + 1} \right\rfloor$$

from v_{i-1} so that $v_{i-1} = w_0^i$, $v_i = w_1^i$, and $w_a^i = w_b^i$ if and only if $a \equiv b \mod \ell_i + 1$.

Definition 3.13. The **torsion profile** of a chain of loops g loops Γ is the g-tuple $(m_1, \dots, m_g)$, where m_i is the torsion order $m(v_{i-1}, v_i)$ for each index $i \in \{1, \dots, g\}$.

By Dhar's algorithm, it is clearly seen that a divisor is v_i-reduced if and only if it is effective outside of v_i and there is at most one degree on each $\gamma_j \backslash v_j$ for indices $j \leq i$ and also on each $\gamma_j \backslash v_{j-1}$ for indices $j > i$. This gives us the following characterization of v_0-reduced divisors:

Remark 3.14. We may write each v_0-reduced divisor $D \in \mathrm{Div}(\Gamma)$ as $(d_0; a_1, \dots, a_g)$, with $d_0 \in \mathbb{Z}$ and $a_\iota \in \mathbb{R}/((\ell_i + 1)\mathbb{Z})$ for each index $i \in \{1, \dots, g\}$, where

$$D = d_0 v_0 + \sum_{i=1}^{g} \begin{cases} w_{a_i}^i & \text{if } a_i \neq 0 \\ 0 & \text{if } a_i = 0 \end{cases}.$$

This defines a bijection between the following sets:

$$\mathrm{Pic}(\Gamma) \leftrightarrow \{v_0\text{-reduced divisors on } \Gamma\} \leftrightarrow \mathbb{Z} \times \frac{\mathbb{R}}{(\ell_1 + 1)\mathbb{Z}} \times \cdots \times \frac{\mathbb{R}}{(\ell_g + 1)\mathbb{Z}}.$$

Given a v_i-reduced divisor D, Proposition 4.3 allows us to calculate the unique v_{i+1}-reduced divisor equivalent to D. Note that a divisor moving along the chain of loops leaves at most one chip behind on each loop, which is a necessary condition for this technique to work. In order to keep track of how much degree is left behind as we move the divisor along the chain of loops, we introduce a sequence associated to each v_0-reduced divisor.

Definition 3.15. The **lingering lattice path** $P_0, \dots, P_g \in \mathbb{Z}$ associated to a v_0-reduced divisor $D = (d_0; a_1, \dots, a_g)$ on Γ has $P_0 = d_0$ and

$$P_i - P_{i-1} = \begin{cases} -1 & \text{if } m \nmid P_{i-1} \text{ and } a_i = 0 \\ 1 & \text{if } m \nmid P_{i-1} + 1 \text{ and } a_i \equiv P_{i-1} + 1 \mod \ell_i + 1 \\ 0 & \text{otherwise} \end{cases}$$

for each $i \in \{1, \dots, g\}$.

The ith entry P_i of the lingering lattice path associated to a divisor D is the degree at v_i of the unique v_i-reduced divisor equivalent to D. Therefore, a v_0-reduced divisor with associated lingering lattice path $P_0, \dots, P_g$ is rank at least 1 if and only if $P_i > 0$ for each index $0 \le i \le g$. $\alpha_i = 0$.

Definition 3.16. A set with divisor structure is called k**-gonal** if it admits a divisor of rank 1 and degree k. If 2-gonal set is called **hyperelliptic** and a 3-gonal set is called **trigonal**.

Proposition 3.17. A chain of $g \ge 2$ loops Γ is hyperelliptic if and only if $\ell_i = 1$ for all $2 \le i \le g-1$. There is then exactly one divisor class of degree 2 and rank 1.

Proof. Let $D = (d_0; a_1, \dots, a_g)$ be a v_0-reduced divisor with associated lingering lattice path $P_0, \dots, P_g$. If $P_0 = d_0 = 1$, since $P_0 + 1 = 2 \equiv 0 \mod \ell_1 + 1$, the next entry P_1 cannot exceed P_0. In fact, $P_1 = 1$ if and only if $D = (1; 1, 0, \dots, 0)$, in which case $P_2 = 0$, so that $\mathrm{rk}(D) = 0$. If $P_0 = d_0 = 2$, there are no more chips to spare, so D is rank 1 if and only if $m_i = 2$ for all $2 \le i \le g-1$. □

Let $D = (d_0; a_1, \dots, a_g)$ be a v_0-reduced divisor of degree k and rank at least 1 with associated lingering lattice path $P_0, \dots, P_g$. Let $H, I, J \subseteq \{1, \dots, g\}$ be given by:

$$H = \{\iota : a_\iota \neq 0 \text{ and } P_\iota - P_{\iota-1} = 0\}$$

$$I = \{\iota : P_\iota - P_{\iota-1} = 1\}$$

$$J = \{\iota : P_\iota - P_{\iota-1} = -1\}$$

and write

$$\Pi_\iota = d_0 + |\{i \in I : i \le \iota\}| - |\{i \in J : j \le \iota\}|$$

By expressing the changes in the lingering lattice path as sets of indices, it becomes easier to consolidate the conditions that we impose on a chain of loops by assuming the existence of a divisor of rank at least 1. Defined as above, H, I, J satisfy the following conditions for all indices $\iota \in \{1, \dots, g\}$:

1. $m_\iota \nmid \Pi_{\iota-1} + 1$ for all $\iota \in I$,
2. $m_\iota \nmid \Pi_{\iota-1}$ for all $\iota \in J$,
3. $m_\iota \mid \Pi_{\iota-1}$ for all $\iota \notin H \cup I \cup J$,
4. $\Pi_\iota = P_\iota \ge 1$ for all ι.

Searching for a rank 1 divisor can then begin with searching for a triple of sets (H, I, J) satisfying the above conditions.

Remark 3.18. For a v_0-reduced divisor of rank 1 on a chain of g loops Γ with sets $H, I, J \subseteq \{1, \dots, g\}$ defined as above, since $m_\iota \neq 1$ for each index $\iota \in \{1, \dots, g\}$, we have $\iota \in H \cup I$ whenever $P_{\iota-1} = 1$.

Proposition 3.19. Let Γ be a chain of $g \ge 3$ loops with torsion profile $(m_1, \dots, m_g)$, let $\iota, i, j \in \{1, \dots, g\}$.

1. $\mathrm{rk}(d_0 + w^1_a + w^2_2) = 1$ with $a \in (0, \ell_1 + 1)$ if and only if $m_2 \neq 2$ and $m_\iota = 2$ for $\iota \geq 3$.
2. $\mathrm{rk}(2d_0 + w^i_a) = 1$ with $a \in (0, \ell_i + 1)$ if and only if $m_\iota = 2$ for $\iota \neq i$.
3. $\mathrm{rk}(2d_0 + w^\iota_3) = 1$ if and only if there is an index $j > i$ such that
   (a) $m_\iota = 2$ for each index $\iota < i$ and $\iota > j$,
   (b) $m_\iota = 3$ for each index $i < \iota < j$.
   (c) $m_i \nmid 3$.
4. $\mathrm{rk}(2d_0 + w^i_2) = 1$ if and only if
   (a) $m_\iota = 2$ for each index $\iota \neq i, i-1$,
   (b) $m_i \nmid 2$.
5. $\mathrm{rk}(3d_0) \geq 1$ if and only if $m_\iota = 2$ for each ι.

Furthermore, all divisors of rank 1 and degree 3 on Γ are equivalent to one of these Γ is trigonal if and only if either 2, 3, or 4 is satisfied.

Proof. Let $D = (d_0; a_1, \ldots, a_g)$ be a q-reduced divisor on Γ of degree 3 and rank 1 with associated lingering lattice path $P_0, \ldots, P_g$, and let $H, I, J \subseteq \{1, \ldots, g\}$ be defined as above.

In the case that $d_0 = 1$, since $m_1 = 2$, we have $1 \notin I$, which forces $1 \in H$. Since $g \geq 3$, we must have $2 \in I$. Since we have no remaining chips, this forces $m_\iota | 2$ for all $\iota \in \{3, \ldots, g-1\}$.

In the case that $d_0 = 2$, if the one extra chip be placed on loop $h \in H$, then $P_\iota = 2$ and thus $m_\iota \mid 2$ for all $\iota \in \{2, \ldots, g-1\}$ other than h. Otherwise if it is placed on $i \in I$, there are two possibilities. We may have some index $i < j \leq g-1$ with $j \in J$, in which case $m_\iota = 2$ for each index $\iota < i$ and $\iota > j$, and $m_\iota = 3$ for each index $i < \iota < j$. However, if $i > 2$ we could also have $j = i - 1$, in which case $m_\iota = 2$ for all indices besides i, j.

In the case that $d_0 = 3$, we must have $1 \in J$, forcing $m_\iota = 2$ for each $\iota \in \{2, \ldots, g-1\}$.

The condition for trigonality follows since $2 \Rightarrow 1, 5$ and $2, 3, 4$ do not imply each other. □

We have arrived at the necessary and sufficient conditions guaranteeing the trigonality of a chain of loops.

4 Appendix: Proposition 3.12

1. If $m \mid k$, then f takes

slope	$k\ell/(\ell+1)$	on	$x \to y$	which has length	1,
and slope	$k(\ell-1)/(\ell+1)$	on	$y \to x$	which has length	ℓ.

2. If $m \nmid k$ and $u = 0$, let

$$d = 1 - k - (\ell+1)n, \qquad \text{where} \qquad n = \left\lfloor \frac{1-k}{\ell+1} \right\rfloor \in \mathbb{Z}.$$

   Since $m \nmid k$, we cannot have $d = 1$. There are three cases:

(a) If $d = 0$, then f takes

$$\begin{array}{rllll} \text{slope} & k+n-1 & \text{on} & x \to y & \text{which has length} & 1, \\ \text{and slope} & n & \text{on} & y \to x & \text{which has length} & \ell. \end{array}$$

(b) If $0 < d < 1$, then f takes

$$\begin{array}{rllll} \text{slope} & k+n & \text{on} & x \to w_{1-k} & \text{which has length} & d, \\ \text{slope} & k+n-1 & \text{on} & w_{1-k} \to y & \text{which has length} & 1-d, \\ \text{and slope} & n & \text{on} & y \to x & \text{which has length} & \ell. \end{array}$$

(c) If $d > 1$, then f takes

$$\begin{array}{rllll} \text{slope} & k+n & \text{on} & x \to y & \text{which has length} & 1, \\ \text{slope} & n+1 & \text{on} & y \to w_{1-k} & \text{which has length} & d-1, \\ \text{and slope} & n & \text{on} & w_{1-k} \to x & \text{which has length} & \ell+1-d. \end{array}$$

3. If $m \nmid k$ and $u = w_a$ for some $a \in (0, \ell+1)$, let

$$d = a - k - (\ell+1)n \qquad \text{where} \qquad n = \left\lfloor \frac{a-k}{\ell+1} \right\rfloor \in \mathbb{Z}.$$

By assumption $a \neq 0$, and since $m \nmid k$, we cannot have $a = d$.

(a) If $d = 1$, there are two cases:

i. If $a < 1$, then f takes

$$\begin{array}{rllll} \text{slope} & k+n & \text{on} & x \to w_a & \text{which has length} & a, \\ \text{slope} & k+n+1 & \text{on} & w_a \to y & \text{which has length} & 1-a, \\ \text{and slope} & n & \text{on} & y \to x & \text{which has length} & \ell+1-d. \end{array}$$

ii. If $a > 1$, then f takes

$$\begin{array}{rllll} \text{slope} & k+n & \text{on} & x \to y & \text{which has length} & 1, \\ \text{slope} & n-1 & \text{on} & y \to w_a & \text{which has length} & a-1, \\ \text{and slope} & n & \text{on} & w_a \to x & \text{which has length} & \ell+1-a. \end{array}$$

(b) If $d \neq 1$, there are eleven cases:

i. If $a < d < 1$, then f takes

$$\begin{array}{rllll} \text{slope} & k+n & \text{on} & x \to w_a & \text{which has length} & a, \\ \text{slope} & k+n+1 & \text{on} & w_a \to w_{a-k} & \text{which has length} & d-a, \\ \text{slope} & k+n & \text{on} & w_{a-k} \to y & \text{which has length} & 1-d, \\ \text{and slope} & n & \text{on} & y \to x & \text{which has length} & \ell. \end{array}$$

ii. If $a < 1 < d$, then f takes

slope	$k+n$	on	$x \to w_a$	which has length	a,
slope	$k+n+1$	on	$w_a \to y$	which has length	$1-a$,
slope	$n+1$	on	$y \to w_{a-k}$	which has length	$d-1$,
and slope	n	on	$w_{a-k} \to x$	which has length	$\ell+1-d$.

iii. If $a = 1 < d$, then f takes

slope	$k+n$	on	$x \to y$	which has length	1,
slope	$n+1$	on	$y \to w_{a-k}$	which has length	$d-1$,
and slope	n	on	$w_{a-k} \to x$	which has length	$\ell+1-d$.

iv. If $1 < a < d$, then f takes

slope	$k+n$	on	$x \to y$	which has length	1,
slope	n	on	$y \to w_a$	which has length	$a-1$,
slope	$n+1$	on	$w_a \to w_{a-k}$	which has length	$d-a$,
and slope	n	on	$w_{a-k} \to x$	which has length	$\ell+1-d$.

v. If $d = 0$ and $a < 1$, then f takes

slope	$k+n-1$	on	$x \to w_a$	which has length	a,
slope	$k+n$	on	$w_a \to y$	which has length	$1-a$,
and slope	n	on	$y \to x$	which has length	ℓ.

vi. If $d = 0$ and $a = 1$, then f takes

slope	$k+n-1$	on	$x \to y$	which has length	1,
and slope	n	on	$y \to x$	which has length	ℓ.

vii. If $d = 0$ and $a > 1$, then f takes

slope	$k+n-1$	on	$x \to y$	which has length	1,
slope	$n-1$	on	$y \to w_{1-k}$	which has length	$d-1$,
and slope	n	on	$w_{1-k} \to x$	which has length	$\ell+1-d$.

viii. If $0 < d < a < 1$, then f takes

slope	$k+n$	on	$x \to w_{a-k}$	which has length	d,
slope	$k+n-1$	on	$w_{a-k} \to w_a$	which has length	$a-d$,
slope	$k+n$	on	$w_a \to y$	which has length	$1-a$,
and slope	n	on	$y \to x$	which has length	ℓ.

ix. If $0 < d < a = 1$, then f takes

slope	$k+n$	on	$x \to w_{a-k}$	which has length	d,
slope	$k+n-1$	on	$w_{a-k} \to y$	which has length	$a-d$,
and slope	n	on	$y \to x$	which has length	ℓ.

x. If $0 < d < 1 < a$, then f takes

slope	$k+n$	on	$x \to y$	which has length	d,
slope	$k+n-1$	on	$y \to w_{a-k}$	which has length	$1-d$,
slope	$n-1$	on	$w_{a-k} \to w_a$	which has length	$a-1$,
and slope	n	on	$w_a \to x$	which has length	$\ell+1-a$.

xi. If $1 < d < a$, then f takes

slope	$k+n$	on	$x \to y$	which has length	1,
slope	n	on	$y \to w_{a-k}$	which has length	$d-1$,
slope	$n-1$	on	$w_{a-k} \to w_a$	which has length	$a-d$,
and slope	n	on	$w_a \to x$	which has length	$\ell+1-a$.

5 Bibliography

[Bak08] M. Baker, Specialization of linear systems from curves to graphs, Algebra Number Theory 2 (2008), no. 6, 613653.

[BN07] M. Baker, S. Norine, Riemann-Roch and Abel-Jacobi theory on a finite graph. Adv. Math. 215 (2007), no. 2, 766-788.

[BN09] M. Baker, S. Norine, Harmonic morphisms and hyperelliptic graphs. International Math. Research Notices 15 (2009), 29142955.

[CDPR10] F. Cools, J. Draisma, S. Payne, and E. Robeva, A tropical proof of the Brill-Noether Theorem, preprint, arXiv:1001.2774v2, 2010.

[Cha11] Chan, Melody, Tropical hyperelliptic curves, preprint. arXiv:1110.0273v1, 2011.

[Dha90] D. Dhar, Self-organized critical state of sandpile automaton models, Physical Review Letters 64 (1990), no. 14, 16131616.

[GH07] P. Griffiths, J. Harris. Principles of Algebraic Geometry. Beijing: World Corporation, 2007. Print.

[HLM+08] A. Holroyd, L. Levine, K. Meszaros, Y. Peres, J. Propp, and D. Wilson, Chip-ring and rotor-routing on directed graphs, In and out of equilibrium. 2, Progr. Probab., vol. 60, Birkhauser, Basel, 2008, pp. 331364.

[JP17] Jensen, David, Sam Payne, Nikolay Malkin, Nicholas Wawrykow, Daniel Corey, Dhruv Ranganathan, Jeremy Usatine, Oleksii Kislinskyi, Tif Shen, and Asher Auel. "Math 665: Tropical Brill-Noether Theory." Sam Payne. Web. 05 May 2017. http://users.math.yale.edu/ sp547/Math665.html.

[LPP11] C. M. Lim, S. Payne, N. Potashnik, A note on Brill-Noether theory and rank determining sets for metric graphs, preprint, arXiv:1106.5519v1, 2011.

[Luo11] Y. Luo, Rank-determining sets of metric graphs, J. Combin. Theory Ser. A 118 (2011), 17751793.

[Mar73] Marsden, J. Basic Complex Analysis. N.p.: W.H. Freeman, San Francisco, 1973. Print.

[Mir95] Miranda, Rick. Algebraic Curves and Riemann Surfaces. Providence, RI: American Math. Soc., 1997. Print.