

SOCIAL NETWORKING AND INDIVIDUAL OUTCOMES BEYOND THE MEAN FIELD CASE

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Abstract. This paper studies individual outcomes in a dynamic environment in the presence of social interactions when outcomes are individually optimized and vary continuously, and the social interactions topology may be either exogenous and time varying, or endogenous. The model accommodates contextual effects and endogenous local and global social effects, both with a lag, that are more general than of the mean-field type. Individual outcomes obey a structural VARX(1,0) model with time-varying coefficients, which also involves expectations of future outcomes only when global interactions are included. A planner's problem demonstrates the suboptimality of individual outcomes. The resulting linear stochastic system with expectations, in effect a second-order system, is amenable to solution by means of standard methods. The system highlights an asset-like property of socially optimal outcomes in every period. This helps characterize the shadow values of connections among agents. The paper addresses endogenous networking by assuming that each individual chooses weights that she attaches to characteristics and to decisions of other agents, which amounts to choice of the elements of a row of a weighted adjacency matrix. Social networking thus involves simultaneity between decisions and patterns of directed social interactions. The paper poses an inverse social interactions problem, that is whether it is possible to design a social network that is comprised of agents whose decisions obey an arbitrarily specified mean and variance covariance matrix. An attractive interpretation of the model links social interactions with the cross-sectional income distribution.

Keywords: Social Interactions, Social Networks, Neighborhood Effects, Endogenous Networking, Social Intermediation, Econometric Identification, Strong versus Weak Ties, Value of Social Connections.

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1 Introduction

Most of the empirical work on social interactions analyzes and seeks to identify econometrically interactions of a mean-field type. This literature typically assumes that individual agents are influenced by the average behavior of the social group to which they belong. Such an emphasis on a broad class of *mean field* models has shifted attention away from other features of social interactions, such as individuals' deliberate efforts to acquire desirable, or avoid undesirable, social connections.

The present paper differs from other approaches to endogenous group formation by its emphasis on macroeconomic aspects of network structure associated with the social groups to which individuals belong. The paper also allows individuals to choose their social networks. The paper investigates the role of social interactions when patterns of social interactions may be more complex than in the mean field case or when they are subject to choice. If individuals in choosing their own actions, which we interpret as incomes, are influenced by the incomes of their social contacts but do not react strategically to it, individual income outcomes in the presence of social interdependence may be restricted with respect to the universe of social outcomes that may be reached. Patterns of interdependence may also reflect self-selection associated with individuals' choice of groups. The particular approach adopted by this paper is in part motivated by an interest to explore in greater depth different varieties of social settings. E.g., in some settings some individuals may be more "influential" than others and in ways that vary systematically across groups or cliques.

The behavioral model of this paper represents situations where individuals' decisions are subject to externalities. Several models of interdependent preferences are compatible with this notion. For example, in a model like Pollak (1976), the parameters of an individual's utility function are functions of the consumptions of other individuals. In such a setting, individuals' choice of their own consumption bundles, taking the decisions of others as given, gives rise to interdependence of incomes. We model exactly such a situation. In addition, we introduce costs which may be interpreted as costs of acquiring income. Deliberately "acquiring" externalities give rise to benefits but also costs. Both benefits and costs are expressed in terms of utility.

A key consequence of social interactions is that even if individuals are subject to random shocks that are independent, their individual actions are simultaneously determined and may end up being stochastically dependent. Precisely because individual reactions reflect influences from others, directly and indirectly, and individual actions reflect characteristics

of others that influence behavior. Interdependence may also follow from the fact individuals take deliberate efforts to form social groups [Moffitt (2001)]. When association with others is valuable, it is also appropriate to wonder about existence of markets through which such “social networking” may be mediated. In fact, networking markets do exist [Rivlin (2005a; 2005b)].

The remainder of the paper is organized as follows. The paper first introduces a model of individual decisions that allows for local and global interactions. The model assumes social interactions which are continuous outcomes. A number of researchers adopt similar approaches, including notably Bisin *et al.* (2004), Brock and Durlauf (2001), Glaeser, Sacerdote and Scheinkman (2003), Glaeser and Scheinkman (2001), Manski (1993), Moffitt (2001) and Weinberg (2004) in the economics literature. Our paper is most notably related to Weinberg (2004) who allows for utility to depend on social interactions with others and emphasizes the role of the interaction structure and of inputs, like own time, that an individual may associate with a given set of neighbors. Weinberg shows that endogeneity of association introduces additional structure that aids identification and thus relaxes the Manski “social reflection” problem.¹ Weinberg also provides empirical results with data from *Add Health* from the National Longitudinal Study of Adolescent Health. Suitable specification of such interactions makes it possible to express a rich variety of patterns of social interdependence that may have important consequences in the context of the life cycle model as well. Earlier research by Binder and Pesaran (1998; 2001) is quite important for the approach taken in our paper.

Unlike Weinberg, but similarly to Binder and Pesaran, we explore dynamic aspects of social interactions and social networking. We go beyond those authors in obtaining precise analytical results also when networking is endogenous. The paper is conceptually related to the problems addressed by Bala and Goyal (2000) and Goyal and Vega-Redondo (2004). Like Binder and Pesaran (1998; 2001) and Weinberg (2004), the paper uses a regression-like framework in order to obtain results that are more amenable to empirical analysis. This, in particular, would allow one to test, in the context of the data analyzed by Weinberg, whether individuals engage in deliberate networking with others.

We explore the properties of the model first in the steady state for individually optimal outcomes. This allows us to examine its implications for interdependence of individual outcomes in the long run. It also allows us to examine the “inverse social interactions” problem,

¹This is, in principle, the same effect as the one that endogeneity of neighborhood choice introduces into even a linear model [Brock and Durlauf (2001; Durlauf (2004)].

that is whether or not an arbitrary specified social outcome, that is one that has been defined as a random vector with a given probability distribution, may be reached as a limit state of social interactions in the long run. Next we formulate the planner’s problem and demonstrate the suboptimality of individually optimal outcomes. Finally, the paper turns to investigation of the range of social outcomes that may be reached when individuals choose deliberately how much they wish to be influenced by other individuals. Conceptually, this is related to individuals’ choice of social groups or neighborhoods, but we do not explore this link further in the present paper.

2 Social Structure and Preferences

We employ the following concept of social networking. Individuals acquire access to external benefits that emanate from activities of other individuals within a particular social setting. Put differently, individuals seek access to desirable while avoiding undesirable social interactions. We specify social settings in terms of social interactions topologies. A social interactions structure, or topology, is defined in terms of the adjacency matrix $\mathbf{\Gamma}$ [Wasserman and Faust (1994)] of the graph \mathcal{G} of connections among individuals indexed by $i = 1, \dots, N$ as follows:

$$\gamma_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are neighbors in } \mathcal{G}; \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The adjacency matrix will be time-subscripted when it is time-varying. If social interactions are assumed to be undirected, the adjacency matrix (or sociomatrix) $\mathbf{\Gamma}$ is symmetric. We define an individual i ’s neighborhood as the set of other agents she is connected with in the sense of the adjacency matrix, $j \in \nu(i), \forall j, \Gamma_{ij} = 1$. $\mathbf{\Gamma}_i$ denotes row i of the adjacency matrix, whose nonzero elements correspond to individual i ’s neighbors. Further below, it will be important to allow for a weighted adjacency matrix.

As we discussed earlier in the introduction, social interactions generate externalities in a variety of settings, such as consumption activities, human capital accumulation, or more generally income-generating activities. We find this last setting particularly apt and assume that individual i derives utility U_{it} from an income generating activity, income² for short, y_{it} in period t and incurs the (utility) costs for maintaining her connections. Utility U_{it} is assumed to be increasing concave in y_{it} . It also depends on the incomes and the characteristics

²Having established a motivation in terms of incomes, later in the paper we refer to the y_{it} s more generally as outcomes.

of i 's neighbors in the sense of the social structure and possibly on the average income and characteristics in the entire economy. Specification of the dependence on income and characteristics of others allows us to express a variety of social interactions, neighborhood and peer effects. We discuss these further after we introduce additional notation. Our specification of preferences owes heavily to Weinberg (2004).

Let

$\mathbf{Y}_t := (y_{1t}, y_{2t}, \dots, y_{Nt})'$ denote the (column) vector with elements all individuals' incomes at time t ;

\mathbf{x}_i : a row K -vector with i 's own characteristics;

\mathbf{X} : a $N \times K$ matrix of characteristics for all individuals;

$\mathbf{x}_{\nu(i)}$: a row K -vector with the mean characteristics of i 's neighbors, $j \in \nu(i)$, $\mathbf{x}_{\nu(i)} = \frac{1}{|\nu(i)|} \mathbf{\Gamma}_i \mathbf{X}$;

ι : a column N - vector of ones.

ϕ, θ, α and ω are column K -vectors of parameters.

We assume individual i at time t enjoys socializing with individuals with specific characteristics. This is expressed through taste over the mean *characteristics* of i 's neighbors $j \in \nu(i)$, $\frac{1}{|\nu_t(i)|} \mathbf{\Gamma}_{i,t} \mathbf{X} \phi$, a local *level contextual* effect. We allow for the marginal utility of own income to depend on neighbors' mean characteristics, which is expressed through the term $\frac{1}{|\nu_t(i)|} \mathbf{\Gamma}_{i,t} \mathbf{X} \theta y_{it}$, which may be referred to as a local *marginal contextual* effect, and separately, on her own characteristics, through a term $\mathbf{x}_i \alpha y_{it}$.

We allow, symmetrically, individual i at time t to be affected by her neighbors' income in two ways. One way may be thought of as conformism: individual i suffers disutility from a gap between her own income and the mean income of her neighbors in the previous period, $y_{it} - \frac{1}{|\nu_{t-1}(i)|} \mathbf{\Gamma}_{i,t} \mathbf{Y}_{t-1}$. A second way allows for the marginal effect of neighbor's income to depend on i 's own characteristics, $\frac{1}{|\nu_t(i)|} \mathbf{\Gamma}_{i,t} \mathbf{Y}_{t-1} \mathbf{x}_i \omega$. Modifying the adjacency matrix to allow for $\gamma_{ii} \neq 0$ would introduce an "own conformity" effect, which may be thought of as addiction. This allows the effect of neighbors' income to depend on individual background characteristics, e.g. younger people may experience a greater disutility from living in a high-income neighborhood. We also allow for a global conformist effect, through the divergence between an individual's current income and the mean income among all individuals in the

previous period, $y_{it} - \bar{y}_{t-1}$. This implies *endogenous global interactions* albeit with a lag.³ Lagging the effect of neighbors' decisions, while keeping the contemporaneous neighborhood structure, is analytically convenient but not critical.

We assume that each individual i incurs a (utility) cost for maintaining a connection with individual j at time t is a quadratic function of the “intensity” of the social attachment as measured by γ_{ij} , that is, $-c_1\mathbf{\Gamma}_{i,t}\boldsymbol{\iota} - c_2\mathbf{\Gamma}_{i,t}\mathbf{\Gamma}'_{i,t}$, with $c_1, c_2 > 0$, where $\boldsymbol{\iota}$ denotes an N -column vector of 1s.⁴ This component of the utility function is critical when we come to consider weighted adjacency matrices in section 6. Finally, we allow for an individual's own marginal utility to include an additive stochastic shock, ε_{it} , which will be discussed in further detail below. All these effects are represented in a quadratic utility function, because it is the simplest possible functional form that yields linear first order conditions:⁵

$$\begin{aligned}
U_{it} = & \mathbf{\Gamma}_{i,t}\mathbf{X}\phi + (\alpha_0 + \mathbf{x}_i\alpha + \mathbf{\Gamma}_{i,t}\mathbf{X}\theta + \varepsilon_{i,t})y_{it} - \frac{1}{2}(1 - \beta_g - \beta_\ell)y_{it}^2 \\
& - \frac{\beta_g}{2}(y_{it} - \bar{y}_{t-1})^2 - \frac{\beta_\ell}{2}\left(y_{it} - \frac{1}{|\nu_t(i)|}\mathbf{\Gamma}_{i,t}\mathbf{Y}_{t-1}\right)^2 \\
& + \frac{1}{|\nu_t(i)|}\mathbf{\Gamma}_{i,t}\mathbf{Y}_{t-1}\mathbf{x}_i\omega - c_1\mathbf{\Gamma}_{i,t}\boldsymbol{\iota} - c_2\mathbf{\Gamma}_{i,t}\mathbf{\Gamma}'_{i,t}. \tag{2}
\end{aligned}$$

As we see below, some of these terms do not contribute to individuals' reaction functions, but do affect the level of utility an individual derives from networking with particular other individuals. This allows for social effects in the endogenous social networking process.

Our specification of the utility function is closely related to a number of existing approaches in the literature, including Durlauf (2004), Glaeser, Sacerdote and Scheinkman (2003), Horst and Scheinkman (2004), and Weinberg (2004). It is also related to the formulation of the social interactions problem by Binder and Pesaran (1998; 2001). Binder and Pesaran (2001) assume that the social weights are constant across individuals, whereas we posit an arbitrary adjacency matrix which may accommodate the full range between local and global interactions. Second, those authors do not examine deliberate social networking. Yet,

³The concept of global interactions is not very rigorous with a finite number of individuals, because any pattern of interactions may be expressed by a suitably defined interactions matrix.

⁴When entries of the adjacency matrix are restricted to zero-one values, the total cost to individual i can be written simply as $-(c_1 + c_2)\mathbf{\Gamma}_{i,t}\boldsymbol{\iota}$.

⁵An additional advantage of quadratic utility functions is that they lend themselves quite easily to formulation of decision problems as *robust control* problems, that is when agents make decisions without knowledge of the probability model generating the data. See Backus *et al.* (2004) for a simple presentation and Hansen and Sargent (2005) for a thorough development of robust control as a misspecification problem.

they do provide an exhaustive analysis of solutions of rational expectations models for certain classes of social interactions. They also address the infinite regress problem (of forecasting what the average opinion expects the average opinion to be, etc.) by conditioning on public information only. By appropriate redefinition of the adjacency matrix it may accommodate habit persistence. It is possible that individuals may differ with respect to attitudes towards conformism or altruism, which would require a more general specification of preferences than (2).⁶

3 Individual Decision Making

At the beginning of period t , individual i augments her information set Ψ_{it-1} with individual-specific information about the own preference shock, ϵ_{it} , and public information that has become available during period $t-1$, like her neighbors' actual incomes, $\mathbf{\Gamma}_{i,t} \mathbf{Y}_{t-1}$, and the mean lagged income for the entire economy, \bar{y}_{t-1} . She then chooses an income plan $\{y_{it}, y_{it+1}, \dots | \Psi_{it}\}$, so as to maximize expected lifetime utility conditional on Ψ_{it} ,

$$\mathcal{E} \left\{ \sum_{s=t} \delta^{s-t} U_{i,s} | \Psi_{it} \right\}, \quad (3)$$

where δ , $0 < \delta < 1$, is the rate of time preference and individual utility is given by (2). We note that individual i in choosing her plan recognizes that y_{it} enters $U_{i,t+1}$ only if $\beta_g \neq 0$, that is only if endogenous global interactions are assumed. That is due to the fact that in our definition of the adjacency matrix, $\gamma_{ii,t} = 0, \forall t$, and the term $\mathbf{\Gamma}_{i,t+1} \mathbf{Y}_t$ does not contain y_{it} .⁷

⁶That is, following Binder and Pesaran (2001), Equ. (14), would produce quadratic components of the form

$$\frac{1}{2} (y_{it} - \eta y_{i,t-1})^2 - \frac{\beta}{2} \left(y_{it} - \eta y_{i,t-1} - \left(\frac{1}{|\nu(i)|} \mathbf{\Gamma}_i \mathbf{Y}_t - \eta \frac{1}{|\nu(i)|} \mathbf{\Gamma}_i \mathbf{Y}_{t-1} \right) \right)^2,$$

with parameter $\beta > 0$ measuring conformism. They model conformism by expressing a tradeoff between individual-specific growth in consumption, on one hand, and the gap between that growth and its counterpart among an individual's neighbors, on the other. They model altruism by expressing a tradeoff between the excess of one's own current income over target lagged income and its counterpart among one's neighbors.

Following Equ. (15) in Binder and Pesaran (2001) would yield the quadratic component

$$-\frac{1}{2} \left(y_{it} - \eta y_{i,t-1} + \tau \left(\frac{1}{|\nu(i)|} \mathbf{\Gamma}_i \mathbf{Y}_t - \eta \frac{1}{|\nu(i)|} \mathbf{\Gamma}_i \mathbf{Y}_{t-1} \right) \right)^2,$$

with parameter $\tau \in (0, 1)$ reflecting altruism and $\tau \in (-1, 0)$ reflecting jealousy.

⁷See Bisin, Horst and Özgür (2004), who assume only one-sided interactions but allow for a global effect.

The first order condition with respect to y_{it} , given the social structure, is:

$$y_{it} = \frac{\beta_\ell}{|\nu_t(i)|} \mathbf{\Gamma}_{i,t} \mathbf{Y}_{t-1} + \beta_g \bar{y}_{t-1} + \delta \frac{\beta_g}{N} \mathcal{E} \{ (y_{i,t+1} - \bar{y}_t) | \Psi_{it} \} + \alpha_0 + \mathbf{x}_i \alpha + \mathbf{\Gamma}_{i,t} \mathbf{X} \theta + \varepsilon_{it}. \quad (4)$$

The interdependencies between individuals' decisions are clearer when the first order conditions for all individuals are put concisely in matrix form. First, we simplify by setting $\beta_g = 0$, which excludes endogenous global interactions, and get:

$$\mathbf{Y}_t = \mathcal{A}_t + \beta_\ell \mathbf{D}_t \mathbf{\Gamma}_t \mathbf{Y}_{t-1} + \varepsilon_t, \quad (5)$$

where $\varepsilon_t \equiv (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{Nt})'$, is the N -vector of shocks, \mathbf{D}_t is an $N \times N$ diagonal matrix with elements the inverses of the size of each individual's neighborhood,

$$\mathbf{D}_{ii,t} = 1/|\nu_t(i)| = 1/(\mathbf{\Gamma}_t \mathbf{\Gamma}_t)_{ii},$$

and

$$\mathcal{A}_t \equiv (\dots, \alpha_0 + \mathbf{x}_i \alpha + \mathbf{\Gamma}_{i,t} \mathbf{X} \theta, \dots)'$$

is a column N -vector of individual and contextual effects.

The evolution of the state of the economy, defined in terms of \mathbf{Y}_t , the vector of individuals' incomes, is fully described by (5), a VARX(1, 0) model, given the information set $\Psi_t = \bigcup_i \Psi_{i,t}$, and provided that the sequence of adjacency matrices $\mathbf{\Gamma}_t, t = 0, 1, \dots$, is specified. Intuitively, the economy evolves as a Nash system of social interactions that adapts to external shocks of two types, deterministic ones, as denoted by the evolution of the social structure $\mathbf{\Gamma}_t$, and stochastic ones, as denoted by the vectors of shocks ε_t , as $t = 1, \dots, \infty$.

To fix ideas, let us assume that the random vectors $\varepsilon_t, t = 1, \dots$, are i.i.d. over time and drawn from a normal distribution with mean $\mathbf{0}$ and variance covariance matrix \mathbf{Q} , $\varepsilon_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$. Then, by a standard derivation from stochastic systems theory [Kumar and Varaiya (1986), p. 27] of the t -step transition probability, the distribution for \mathbf{Y}_t , at time t , is characterized by the following proposition, whose proof is elementary that thus not repeated here.

Proposition 1.

Given the initial state of the economy, \mathbf{Y}_0 , and under the assumption that the shocks ε_t are independent and identically distributed with a multivariate normal distribution $\mathcal{N}(\mathbf{0}, \mathbf{Q})$, then the distribution of the state of the system at time t , \mathbf{Y}_t , is normal,

$$\mathbf{Y}_t \sim \mathcal{N} \left(\beta_\ell^t \prod_{s=1}^t \mathbf{A}_s \mathbf{Y}_0 + \sum_{j=0}^{t-1} \left(\beta_\ell^{t-j-1} \prod_{s=j+1}^t \mathbf{A}_s \right) \mathcal{A}_j, \mathbf{\Sigma}_{t|0} \right), \quad (6)$$

where \mathbf{A}_t is the $N \times N$ positive symmetric matrix defined as

$$\mathbf{A}_t \equiv \mathbf{D}_t \mathbf{\Gamma}_t, \quad (7)$$

which is in effect the adjacency matrix normalized by the size of each individual's neighborhood, and the variance covariance matrix $\mathbf{\Sigma}_{t|0}$ above is given from the time-varying linear difference equation:

$$\mathbf{\Sigma}_{k+m|k} = \beta_\ell^2 \mathbf{A}_{k+m-1} \mathbf{\Sigma}_{k+m-1|k} \mathbf{A}'_{k+m-1} + \mathbf{Q}, \quad m > 1; \mathbf{\Sigma}_{k|k} = [\mathbf{0}]. \quad (8)$$

$\mathbf{\Sigma}_{k+m|k}$ is the matrix of mean-squared errors of the m -step predictor for \mathbf{Y}_t .

Given a starting point \mathbf{Y}_0 , the mean vector of individuals' incomes after t periods reflect the full sequence of contextual effects, $\{\mathcal{A}_0, \dots, \mathcal{A}_{t-1}\}$, weighted by the respective adjacency matrices. The dispersion of individual incomes, on the other hand, reflects the compound effect of weighted interactions as they modify the dispersion of the underlying shocks.

4 Properties of Individually Optimal Outcomes at the Steady State

It is easy to study the properties of the state of the economy at the limit when $t \rightarrow \infty$. It is important to stress right at the outset that this case involves no intertemporal optimization on behalf of individuals. The dynamics come entirely from the lag properties of the preference structure. Then under the assumption of a time-invariant adjacency matrix, $\mathbf{A}_t = \mathbf{A}$, and vector of contextual effects, $\mathcal{A}_t = \mathcal{A}$ and the limit of the mean of \mathbf{Y}_t , \mathbf{Y}^* , is well defined and given by the unique solution to

$$\mathbf{Y}^* = \mathcal{A} + \beta_\ell \mathbf{A} \mathbf{Y}^*. \quad (9)$$

A solution for the vector of mean individual incomes exists, provided that matrix $\mathbf{I} - \mathbf{A}$ is invertible. For this, it suffices that \mathbf{A} be stable, namely that all its eigenvalues have magnitudes that are strictly smaller than one. Since \mathbf{A} comes from an adjacency matrix, "normalized" by each row sum and scaled by β_ℓ , its stability is ensured by assuming $0 < \beta_\ell < 1$. So we have:

$$\mathbf{Y}^* = \left[\mathbf{I} + \beta_\ell \mathbf{A} + \beta_\ell^2 \mathbf{A}^2 + \dots \right] \mathcal{A}. \quad (10)$$

Since matrix \mathbf{A} is symmetric, positive and each of its rows sum up to 1, all of its eigenvalues are inside the unit circle. Therefore, the power series in the right hand side of (10) converges.

The intuition of the solution is straightforward. The vector of individual outcomes reflects the infinite sequence of feedbacks from the vector of contextual effects \mathcal{A} .

The asymptotic behavior of the variance covariance matrix of the vector of individuals' incomes, according to Equ. (5), is also easy to study provided that the adjacency matrix $\Gamma_{i,t}$ is time-invariant. The limit Σ^* from Equ. (8) of the variance covariance matrix Σ_t , as $t \rightarrow \infty$, exists. Theorem (3.4), in Kumar and Varaiya, *op. cit.*, or Proposition 4.1 of Bertsekas (1995), as a special case, ensure that if matrix $\beta_\ell^t \mathbf{A}$ is stable then the limit solution of (8) is a unique positive semidefinite matrix. We may state this in a slightly more general case by relaxing the assumption that the random vectors ε_t are independent draws from a given multivariate distribution and instead assume that they are defined in terms of factor loadings,

$$\varepsilon_t = \mathbf{G}\mathbf{w}_t, \quad (11)$$

where \mathbf{G} is an $N \times M$ mixing matrix and \mathbf{w}_t a random column M -vector, that is independently and identically distributed over time and obeys a normal distribution with mean $\mathbf{0}$ and variance covariance matrix \mathbf{R} , $\mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$. This case is of particular interest, because it allows us to express the contemporaneous stochastic shock in a factor-analytic form of contemporaneously interdependent components and therefore shocks to different individuals may be correlated. This approach also allows for a general form of serial correlation in the \mathbf{w}_t s [*ibid.*, p. 27].⁸

Put succinctly in a proposition, whose proof readily follows from (8) and (11), we have:
Proposition 2.

The variance covariance matrix of vector of individuals' incomes when $t \rightarrow \infty$, that is the limit of (8) when ε_t assumes the factor-analytic form according to (11), satisfies

$$\Sigma_\infty = \beta_\ell^2 \mathbf{A} \Sigma_\infty \mathbf{A}' + \mathbf{G} \mathbf{R} \mathbf{G}'. \quad (12)$$

The matrix Σ_∞ in the equation above would not be a proper variance covariance matrix unless it is positive definite. This is ensured, provided the random error \mathbf{w}_t along with the matrix $\beta_\ell \mathbf{A}$ “allow” the economy to reach any arbitrarily specified social outcomes in a well-defined sense. Clarifying this requirement involves the concept of *controllability*, according to

⁸Such a factor-analytic approach links social interaction models with factor models in large cross-sections of time series that have been employed in generalizing the dynamic index models of business cycle research. See Reichlin (2002) for an excellent review of this recent literature.

which a pair of matrices $\beta_\ell \mathbf{A}, \mathbf{G}$ are *controllable*, if the matrix $[\mathbf{G}, \beta_\ell \mathbf{A}\mathbf{G}, \dots, \beta_\ell^{N-1} \mathbf{A}^{N-1} \mathbf{G}]$ with N rows and NM columns has rank N , that is all its rows are fully independent.

Proposition 3.

If the pair of matrices $\beta_\ell \mathbf{A}, \mathbf{G}$ is controllable, then stability of matrix $\beta_\ell \mathbf{A}$ is equivalent to existence of a unique solution to Equ. (12) that is positive definite and vice versa.

Proof. For stable matrices $\beta_\ell \mathbf{A}$ Theorem (3.9) in Kumar and Varaiya (1986) applies. Alternatively, by applying Equ. (12) recursively we have that if $\lim_{t \rightarrow \infty} \beta_\ell^{2t} \mathbf{A}^t \boldsymbol{\Sigma}_\infty' (\mathbf{A}')^t = [\mathbf{0}]$, then

$$\boldsymbol{\Sigma}_\infty = \sum_{j=0}^{\infty} \beta_\ell^{2j} \mathbf{A}^j \mathbf{G} \mathbf{R} \mathbf{G}' (\mathbf{A}')^j.$$

It then follows that stability of matrix $\beta_\ell \mathbf{A}$ is necessary and sufficient for the existence of $\boldsymbol{\Sigma}_\infty$. Q.E.D.

The interpretation of the condition for controllability is quite revealing in our context. If the $N \times NM$ matrix $[\mathbf{G} | \beta_\ell \mathbf{A}\mathbf{G} | \dots | \beta_\ell^{N-1} \mathbf{A}^{N-1} \mathbf{G}]$ has rank N , then there exists a sequence of vectors $\tilde{\mathbf{w}}_0, \dots, \tilde{\mathbf{w}}_{N-1}$, that take the economy from any arbitrary initial point to a designated final point in N steps. Let $\tilde{\mathbf{Y}}_0$ and $\tilde{\mathbf{Y}}_N$ be these points, respectively, defined as deviations from \mathbf{Y}^* , the unique solution of (9).

By applying Equ. (5), transformed in terms of deviations from \mathbf{Y}^* , $\tilde{\mathbf{Y}}_t \equiv \mathbf{Y}_t - \mathbf{Y}^*$,

$$\tilde{\mathbf{Y}}_t = \beta_\ell \mathbf{A} \tilde{\mathbf{Y}}_{t-1} + \mathbf{G} \mathbf{w}_t, \tag{13}$$

successively N times we stack the N equations so as to have $\tilde{\mathbf{Y}}_N - \beta_\ell^N \mathbf{A}^N \tilde{\mathbf{Y}}_0$ on the LHS and the column NM -vector obtained by stacking up the input vectors as the unknowns, $\tilde{\mathbf{w}}_0, \dots, \tilde{\mathbf{w}}_N$. Controllability ensures that the solution for the input vectors, the $\tilde{\mathbf{w}}_t$'s, is unique. Put differently, for the economy to move from anywhere to anywhere in N steps, as many input vectors are needed as there are members of the economy.

Having established the mathematical existence and uniqueness of the mean and the variance covariance matrix of the set of individually optimal outcomes, we turn next to their economic properties. Suppose first that the social structure is disconnected, in which case the adjacency matrix is a matrix of zeroes. In that case, the mean outcome is the vector of contextual effects, $\mathbf{Y}^* = \mathcal{A}$, and its variance covariance matrix is simply the variance covariance matrix of individual shocks, $\mathbf{G} \mathbf{R} \mathbf{G}'$.

Let us consider next that the social structure is arbitrary but connected. In that case, the normalized adjacency matrix \mathbf{A} is non-singular. This allows us to solve (12) for $\boldsymbol{\Sigma}_\infty$:

$[\mathbf{I} - \beta_\ell^2 \mathbf{A}^2] \boldsymbol{\Sigma}_\infty = \mathbf{G} \mathbf{R} \mathbf{G}'$. Or:

$$\boldsymbol{\Sigma}_\infty = \left[\mathbf{I} + \beta_\ell^2 \mathbf{A}^2 + \beta_\ell^4 \mathbf{A}^4 + \dots \right] \mathbf{G} \mathbf{R} \mathbf{G}'. \quad (14)$$

The properties of the power series on the right hand sides of (10) and (14) depend on the spectral properties of the respective normalized adjacency matrix for the social structure. These are known for a great variety of social topologies of interest [Cvetković et al. (1995); Ioannides (2003)]. For example, these are known for the case of the complete graph, where everyone is connected with everyone else. It may be considered as the maximal social structure and therefore appropriate to think of it as a benchmark case. Enriching the social structure affects both the mean and the variance covariance matrix. When the social interactions topology is not connected and the adjacency matrix is singular, (12) still applies but (14) does not.

4.1 Cross-sectional Income Distribution

Returning to the interpretation of individual outcomes as individual incomes we wish to analyze the implications of the model as a model of the income distribution. Individual incomes at the steady state obey a multivariate normal $\mathcal{N}(\mathbf{Y}^*, \boldsymbol{\Sigma}_\infty)$. Consider the distribution that describes Y_{it} , $i = 1, \dots, N$, when we do not distinguish to whom it accrues and instead look at the distribution of the Y 's as t tends to ∞ . The resulting distribution is obtained by recognizing that in general the probability that the value of Y_i falls in an interval (z, dz) may be conditioned on the values of all other incomes in the previous period. So, let \mathbf{Y}_{-i} denote the subvector of \mathbf{Y} resulting from partitioning out Y_i . Then the distribution of income is given by:

$$g(z) = \frac{1}{N} \sum_{i=1}^N \int f_{i|-i}(z|\mathbf{Y}_{-i}) f(\mathbf{Y}_{-i}) d\mathbf{Y}_{-i}. \quad (15)$$

The densities in the above integral may be written in terms of the mean and the variance covariance matrix in the standard fashion [Anderson (1958), p. 27–30]. Since the mean of the conditional distribution $f_{i|-i}$ of the i th component conditional on \mathbf{Y}_{-i} may be written as a linear function of the individual components of \mathbf{Y}_{-i} , $\mathcal{E}(Y_i|\mathbf{Y}_{-i}) = Y_i^* + \boldsymbol{\Sigma}_{i,-i} \boldsymbol{\Sigma}_{-i,-i}^{-1} (\mathbf{Y}_{-i} - \mathbf{Y}_{-i}^*)$. Its variance is given by: $\sigma_i^2 - \boldsymbol{\Sigma}_{i,-i} \boldsymbol{\Sigma}_{-i,-i}^{-1} \boldsymbol{\Sigma}_{-i,i}$. It is thus possible to express each of the terms in the sum in the RHS of (15) in terms of $(\mathbf{Y}^*, \boldsymbol{\Sigma}_\infty)$.

Therefore, in this sense, it is appropriate to interpret the solution to the entire system as a model of the income distribution in the presence of social interactions. So, by varying

the social interactions topology we may study its consequences for the income distribution.⁹ Unfortunately, this is too complicated to pursue further analytically. Since any changes in the interaction topology affects the distribution of the entire vector of social outcomes, it is perhaps more appropriate to evaluate it in terms of a social welfare function. We return to this matter further below after we formulate the problem of endogenous social structure.

4.2 Extensions

We address next a number of possible extensions and generalizations. *First*, the stochastic shocks do not need to be Gaussian. As long as their second moments exist, the counterpart of (8) may be derived.

Second, we may generalize the model and interpret \mathcal{A}_t as a vector of decision variables that contributes to the evolution of individual outcomes in the manner indicated by Equ. (5). That is, \mathcal{A}_t can assume a form of closed-loop control and reflect the state of the economy as of time $t-1$, $\mathbf{Y}_t = \mathbf{B}_t \mathcal{A}_t + \beta_\ell \mathbf{A}_t \mathbf{Y}_{t-1} + \varepsilon_t$, where $\mathbf{A}_t, \mathbf{B}_t$ are given matrices and the vector \mathcal{A}_t , the *control*, is optimally chosen so as to optimize an objective, which also includes a quadratic term in \mathcal{A}_t . Naturally, this would make the model very similar to a life cycle optimization of consumption decisions in the presence of social interactions, as in the work by Binder and Pesaran (1998; 2001), discussed earlier. The optimal control, in this case, is defined through a matrix that satisfies an *algebraic Riccati equation* [Bertsekas (1995), 132–133].

Third, the properties that are summarized by Proposition 1 really pertain to a *sequence* of static problems that evolve over time. This is also evidenced by the absence of the time preference parameter δ in (6) – (8). Straightforward generalizations that imply intertemporal tradeoffs render the model genuinely dynamic. The simplest possible one, as we discussed above, would be to allow for a global effect by letting $\beta_g \neq 0$.

Fourth, an interesting extension would be to allow for the stochastic shocks too to be decision variables. The simplest way to do so would be via endogenizing the factor loadings, that is by letting individuals set the elements of the mixing matrix in Equ. (11), \mathbf{G} , which could also be time-varying. This would resemble a dynamic portfolio analysis problem, which we know to be quite tractable when individuals' utility functions are quadratic. We note the special interpretation that specification of a portfolio of factors would confer: individuals choose factor loadings in order to mitigate shocks associated with decisions of others that enter directly their own utility functions and therefore influence their own decisions as well.

⁹We thank a referee for urging us to pursue this point further.

The extent to which these generalizations can be implemented in empirical work is dependent on the quality of the available data. Compared to the time series used in optimal control and financial markets analysis, the time-dimension of most panel data sets currently used to study social networking seems to be insufficient to implement these extensions. Weinberg (2004) for example disregards the panel data aspect of the Add Health data set he uses.

Fifth, suppose that an arbitrarily chosen state of the economy may be specified via a random N -vector \mathcal{Y} , with a given distribution function, $\mathcal{Y} \sim \mathcal{N}(\mathbf{b}, \varsigma)$, where \mathbf{b} denotes an N -column vector and ς an $N \times N$ positive definite matrix. That is, the incomes of the individual members of the economy are assumed to be stochastically interdependent and distributed in the above fashion. In view of the discussion of the model as a model of income determination in the presence of social interactions at the end of the previous section, we define the *inverse social interactions problem* as follows: Is it possible for any arbitrary state \mathcal{Y} be reached as an outcome of individuals' choices in the presence of social interactions, that is as the limit vector of individuals' incomes, satisfying (5), when $t \rightarrow \infty$? That is, given (\mathbf{b}, ς) and the distributional characteristics of the random vector of factors \mathbf{w}_t , $\mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$, can we find a social interactions structure, defined by a vector of contextual effects \mathcal{A} , an adjacency matrix $\mathbf{\Gamma}$, and a mixing matrix \mathbf{G} of dimension $N \times M$, such that the random vector \mathcal{Y} is the limit, when $t \rightarrow \infty$, of the vector of individual outcomes, \mathbf{Y}_t ,

$$\mathbf{Y}_t = \mathcal{A} + \beta_\ell \mathbf{A} \mathbf{Y}_{t-1} + \mathbf{G} \mathbf{w}_t. \quad (16)$$

Intuitively, the inverse social interactions problem is the counterpart here of problems posed by the business cycle literature, that is whether it is possible to define vector autoregressive systems that yields any desirable pattern of correlations among the shocks [Jovanovic (1987); Sargent (1987)]. We note that this problem is more general than examining the limit of the distribution of income in this economy.

First we note that given \mathbf{b} , the pre-specified vector of mean outcomes, (16) implies that $\mathbf{b} = [\mathbf{I} - \beta_\ell \mathbf{A}]^{-1} \mathcal{A}$. If a mixing matrix \mathbf{G} is given, then we may define the square root \mathbf{S} of $\mathbf{G} \mathbf{R} \mathbf{G}'$, $\mathbf{S} \mathbf{S}' = \mathbf{G} \mathbf{R} \mathbf{G}'$. The problem of finding a positive symmetric matrix of full rank N , $\beta_\ell \mathbf{A}$, is overdetermined: there exist $\frac{1}{2}N(N+1) + N$ equations and $\frac{1}{2}N(N-1)$ unknowns. Therefore, there exist up to $2N$ degrees of freedom which may be used for the determination of the mixing matrix.

This result is readily interpretable in terms of weighted adjacency matrices. Imposing the restriction that the social interaction matrix takes the form of a matrix of zeros and ones, with $\mathbf{A} = \mathbf{D} \mathbf{\Gamma}$, translates to the requirement that given ς , a matrix $\mathbf{\Gamma}$ and a social

interactions coefficient β_ℓ may be found that provide sufficient spanning for ς to be the limit distribution satisfying Equ. (12). Relaxing the requirement would be to allow for individuals' interaction coefficients to differ, $(\beta_1, \dots, \beta_N)$, or alternatively, for the adjacency matrix to assume a weighted form with arbitrary positive entries, instead of just 0s and 1s. The case of a weighted adjacency matrix is taken up further below in section 6.

Sixth, were we not to exclude a global effect and allow $\beta_g \neq 0$, then instead of (5) we have:

$$\left[\mathbf{I} + \delta \frac{\beta_g}{N} \bar{\mathbf{I}} \right] \mathbf{Y}_t = \mathcal{A}_t + \left[\beta_\ell \mathbf{A}_t + \frac{\beta_g}{N} \bar{\mathbf{I}} \right] \mathbf{Y}_{t-1} + \delta \frac{\beta_g}{N} \mathcal{E} [\mathbf{Y}_{t+1} | \Psi_t] + \varepsilon_t, \quad (17)$$

where $\bar{\mathbf{I}}$ is an $N \times N$ matrix of ones. This system is more complicated to deal with because of the simultaneous presence of the vector of current and of lagged individual decisions, $\mathbf{Y}_{t-1}, \mathbf{Y}_t$ and of the expectation of the future ones, $\mathcal{E} [\mathbf{Y}_{t+1} | \Psi_t]$. Mathematical techniques that allow us to handle this system are introduced below immediately following introduction of the planner's problem (although we retain the assumption of $\beta_g = 0$ in our exposition of the planner's problem.) The consequences of allowing for a global effect are quite evident by comparing (5) with (17). If β_ℓ, β_g both were zero, that is in the absence of all social effects, then the vector of individual outcomes simply equals the contextual effects plus the contemporaneous random shock. Otherwise, in the presence of social effects, the left and right hand sides of (17) denote marginal lifetime utility costs and benefits for each individual from an additional unit of the respective outcome. An additional unit of \mathbf{Y}_t increases marginal costs now by \mathbf{Y}_t and marginal costs next period via the conformism effect, which when discounted to the present is equal to $\delta \frac{\beta_g}{N} \bar{\mathbf{I}} \mathbf{Y}_t$. Similarly, the terms on the r.h.s admit interpretations as marginal benefits.

5 A Social Planner's Problem

Next we introduce a social planner who recognizes the interdependence of agents and takes it into consideration in setting individuals' outcomes, while respecting an existing topology of social interactions. This analysis helps assess the scope for social intermediation, as we shall see shortly below.

The planner seeks to choose outcomes for individuals so as to maximize the sum of individuals' expected lifetimes utilities.¹⁰ The planner is forward-looking and in recognizing

¹⁰We eschew introduction of arbitrary positive weights for individuals' utilities, $\lambda_1, \dots, \lambda_N$, because they are not really necessary for expressing the planner's problem. There are sufficient degrees of freedom elsewhere in the individuals' utility functions.

individuals' interdependence she conditions her decisions on knowledge of all outcomes in the previous periods, $\{\mathbf{Y}_0, \mathbf{Y}_1, \dots, \mathbf{Y}_{t-1}\}$, and of all individual preference information available as of the beginning of every time period, Ψ_t . At the beginning of each period t , the planner chooses a plan $\{\mathbf{Y}_t, \mathbf{Y}_{t+1}, \dots\}$ so as to maximize the expectation of individuals' lifetime utility, according to

$$\sum_{i=1}^N \left[U_{it} + \mathcal{E} \left\{ \sum_{s=t+1} \delta^{s-t} U_{i,s} | \Psi_t \right\} \right]. \quad (18)$$

The planner's maximization is conditional on information available as of time period t and on exogenous evolution of the adjacency matrix Γ_t . We have again simplified utility function (2) by assuming no global effect, $\beta_g = 0$ and no effect from neighbors' outcomes interacted with one's characteristics, $\omega = \mathbf{0}$.

Differentiating objective function (18) with respect to $y_{it}, i = 1, \dots, N$, yields the first order conditions:

$$\begin{aligned} -\delta\beta_\ell \sum_{j \neq i} \left(\frac{\gamma_{ji,t+1}}{|\nu_{t+1}(j)|} \mathcal{E} \{y_{j,t+1}|t\} - \frac{\gamma_{ji,t+1}}{|\nu_{t+1}(j)|} \frac{1}{|\nu_{t+1}(j)|} \Gamma_{j,t+1} \mathbf{Y}_t \right) \\ = \mathcal{A}_{it} + \mathbf{G}_i \mathbf{w}_t - \left(y_{it} - \frac{\beta_\ell}{|\nu_t(i)|} \Gamma_{it} \mathbf{Y}_{t-1} \right). \end{aligned} \quad (19)$$

The left hand side is the social cost of an additional unit of y_{it} through the utility of all of i 's neighbors next period. The right hand side is the net benefit to individual i herself which is comprised of the contextual effect minus the marginal disutility of effort.

Rewriting conditions (19) in matrix form,

$$\begin{aligned} [\mathbf{I} + \delta\beta_\ell \Gamma'_{t+1} \mathbf{D}_{t+1} \mathbf{D}_{t+1} \Gamma_{t+1}] \mathbf{Y}_t \\ = \mathcal{A}_t + \beta_\ell \mathbf{D}_t \Gamma_t \mathbf{Y}_{t-1} + \delta\beta_\ell \Gamma'_{t+1} \mathbf{D}_{t+1} \mathcal{E} \{ \mathbf{Y}_{t+1} | \Psi_t \} + \mathbf{G} \mathbf{w}_t, \end{aligned} \quad (20)$$

also helps clarify the social interactions effects. A change in \mathbf{Y}_t affects marginal utility costs of all individuals now, the first term in the LHS of (20), and via the conformism effect on all of her neighbors in the following period, discounted to the present, the second term in the LHS of (20). Since conformity enters via a quadratic function, this effect operates essentially through walks of length 2, normalized by the size of each individual's neighborhood.

To solve for \mathbf{Y}_t we recognize that in contrast to (5) but like (17) the model is no longer recursive; \mathbf{Y}_{t-1} , \mathbf{Y}_t , and $\mathcal{E} \{ \mathbf{Y}_{t+1} | \Psi_t \}$ all appear in (20), rendering it in effect a second-order linear stochastic system because of backward and forward dependence. There is a standard

approach for such problems that goes as follows. We simplify (20) by assuming that the adjacency matrix and the vector of contextual effects are time invariant, use (7) and define:

$$\mathbf{H} = \mathbf{I} + \delta\beta_\ell \mathbf{A}'\mathbf{A}, \quad \mathbf{W}_t = \mathbf{H}^{-1}[\mathcal{A} + \mathbf{G}\mathbf{w}_t].$$

Under the assumption that \mathbf{H} is invertible we solve Equ. (20) for \mathbf{Y}_t and have:

$$\mathbf{Y}_t = \mathbf{H}^{-1}\beta_\ell \mathbf{A}\mathbf{Y}_{t-1} + \delta\beta_\ell \mathbf{H}^{-1}\mathbf{A}' \mathcal{E}\{\mathbf{Y}_{t+1}|\Psi_t\} + \mathbf{W}_t. \quad (21)$$

Solutions for this class of models have been studied most recently by Binder and Pesaran; see in particular, Binder and Pesaran (1997). The properties of the solutions depend critically on whether the matrix $\mathbf{H}^{-1}\mathbf{A}'$ is nonsingular. Their solution method involves the introduction of an auxiliary matrix \mathbf{C} as the solution to the quadratic matrix equation

$$\delta\beta_\ell \mathbf{H}^{-1}\mathbf{A}'\mathbf{C}^2 - \mathbf{C} + \mathbf{H}^{-1}\beta_\ell \mathbf{A} = \mathbf{0}. \quad (22)$$

Under special assumptions about the properties of matrices $\delta\beta_\ell \mathbf{H}^{-1}\mathbf{A}'$, \mathbf{C} , like those discussed in Proposition 2.2 in Binder and Pesaran (1997), the solution to the above system may be written as the sum of a backward component and a forward component. Otherwise, we need to consider the general set of possibilities discussed by Binder and Pesaran (1997). Drawing from *ibid.*, we can put the results concisely in the form of a proposition as follows:

Proposition 4.

Part A. *Maximization of expected social welfare, defined as the sum of expected life time utilities of all agents, requires that the vector of outcomes \mathbf{Y}_t satisfies the system of linear difference equations (20) with expectations.*

Part B. *The solution for the socially optimal outcomes depends upon whether or not matrix $\mathbf{H}^{-1}\mathbf{A}'$ is nonsingular.*

Define

$$\mathbf{F} = [\mathbf{I} - \delta\mathbf{H}^{-1}\beta_\ell \mathbf{A}'\mathbf{C}]^{-1} \delta\beta_\ell \mathbf{H}^{-1}\mathbf{A}', \quad \widehat{\mathbf{W}}_t = [\mathbf{I} - \delta\mathbf{H}^{-1}\beta_\ell \mathbf{A}'\mathbf{C}]^{-1} \mathbf{W}_t,$$

and assume that $\widehat{\mathbf{W}}_t' \widehat{\mathbf{W}}_t$ is bounded upwards by a finite constant. We distinguish two cases.
(i). *If matrix $\mathbf{H}^{-1}\mathbf{A}'$ is nonsingular so will \mathbf{F} , defined above. If all the eigenvalues of \mathbf{F} fall inside the unit circle then the solution of (20) may be written as*

$$\mathbf{Y}_t = \mathbf{C}\mathbf{Y}_{t-1} + \sum_{k=0}^{\infty} \mathbf{F}^k \mathcal{E}[\widehat{\mathbf{W}}_{t+k}|\Psi_t] + \mathbf{F}^{-t}\mathcal{M}_t. \quad (23)$$

where and \mathcal{M}_t is a vector of N distinct martingale processes with respect to the information set Ψ_t . If all the eigenvalues of \mathbf{F} fall outside the unit circle then solution of (20) may be written as

$$\mathbf{Y}_t = \mathbf{C}\mathbf{Y}_{t-1} - \sum_{k=0}^{\infty} \mathbf{F}^{-k} \mathcal{E} \left[\widehat{\mathbf{W}}_{t-k} | \Psi_t \right] + \mathbf{F}^{-t} \mathcal{M}_t^b. \quad (24)$$

where and \mathcal{M}_t^b is a vector of N distinct martingale processes with respect to the information set Ψ_t . If some of the eigenvalues fall within and some outside the unit circle, the solution may be adapted accordingly.

(ii). If matrix $\mathbf{H}^{-1}\mathbf{A}'$ is singular so will \mathbf{F} . Expressions for the solution of (20) may still be written, involve the Jordan normal form of \mathbf{F} , and distinguish between blocks corresponding to the zero and non-zero eigenvalues of \mathbf{F} . Let \mathbf{F} have N_1 nonzero eigenvalues inside the unit circle, N_2 falling outside the unit circle and N_3 being zero, $N_1 + N_2 + N_3 = N$, and let the Jordan normal form be $\mathbf{F} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}$, where the block-diagonal matrix $\mathbf{\Lambda}$ is defined as:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{0}_{m_1 \times m_2} & \mathbf{0}_{m_1 \times m_3} \\ \mathbf{0}_{m_2 \times m_1} & \mathbf{\Lambda}_2 & \mathbf{0}_{m_2 \times m_3} \\ \mathbf{0}_{m_3 \times m_1} & \mathbf{0}_{m_3 \times m_2} & \mathbf{\Lambda}_3 \end{bmatrix}$$

The general solution of (20) is:

$$\mathbf{Y}_t = [\mathbf{C} + \mathbf{Z}] \mathbf{Y}_{t-1} - \mathbf{Z}\mathbf{C}\mathbf{Y}_{t-2} - \mathbf{Z}\widehat{\mathbf{W}}_{t-1} + \sum_{k=0}^{\infty} \mathbf{F}_{1,k} \mathcal{E} \left[\widehat{\mathbf{W}}_{t+k} | \Psi_t \right] + \sum_{k=0}^{N_3-1} \mathbf{F}_{3,k} \mathcal{E} \left[\widehat{\mathbf{W}}_{t+k} | \Psi_t \right] + \mathcal{D}_t, \quad (25)$$

where

$$\begin{aligned} \mathbf{Z} &= \mathbf{T} \begin{bmatrix} \mathbf{0}_{m_1 \times m_1} & \mathbf{0}_{m_1 \times m_2} & \mathbf{0}_{m_1 \times m_3} \\ \mathbf{0}_{m_2 \times m_1} & \mathbf{\Lambda}_2^{-1} & \mathbf{0}_{m_2 \times m_3} \\ \mathbf{0}_{m_3 \times m_1} & \mathbf{0}_{m_3 \times m_2} & \mathbf{0}_{m_3 \times m_3} \end{bmatrix} \mathbf{T}^{-1}, \\ \mathbf{F}_{1,k} &= \mathbf{T} \begin{bmatrix} \mathbf{\Lambda}_1^k & \mathbf{0}_{m_1 \times m_2} & \mathbf{0}_{m_1 \times m_3} \\ \mathbf{0}_{m_2 \times m_1} & \mathbf{0}_{m_2 \times m_2} & \mathbf{0}_{m_2 \times m_3} \\ \mathbf{0}_{m_3 \times m_1} & \mathbf{0}_{m_3 \times m_2} & \mathbf{0}_{m_3 \times m_3} \end{bmatrix} \mathbf{T}^{-1}, \\ \mathbf{F}_{3,k} &= \mathbf{T} \begin{bmatrix} \mathbf{0}_{m_1 \times m_1} & \mathbf{0}_{m_1 \times m_2} & \mathbf{0}_{m_1 \times m_3} \\ \mathbf{0}_{m_2 \times m_1} & \mathbf{0}_{m_2 \times m_2} & \mathbf{0}_{m_2 \times m_3} \\ \mathbf{0}_{m_3 \times m_1} & \mathbf{0}_{m_3 \times m_2} & \mathbf{\Lambda}_1^k \end{bmatrix} \mathbf{T}^k, \end{aligned}$$

and

$$\mathcal{D}_t = \mathbf{T} \begin{bmatrix} \mathbf{\Lambda}_1^{-t} \mathcal{M}_{t,1} \\ \xi_t \\ \mathbf{0}_{m_3 \times 1} \end{bmatrix},$$

where $\mathcal{M}_{t,1}$ is a vector martingale process of dimension N_1 and ξ_t is a martingale-difference process of order N_2 . Stability requires that the subvector of \mathcal{D}_t associated with $\mathcal{M}_{t,1}$ has to be equal to zero almost surely.

In view of our definition of the social structure, it is easy to imagine instances where the matrix $\mathbf{H}^{-1} \mathbf{A}'$ is singular and therefore accounting for all possible forms of the solution is necessary for completeness.

It is straightforward to apply (21) by taking expectations of both sides. The economic interpretation of the resulting system is as follows. At the steady state, the vector of expected outcomes is equal to the vector of contextual effects, of the own lagged feedbacks, and of the net conformity effect on one's neighbors in the following period.

A number of remarks are in order. *First*, expectations enter the planner's problem because the planner recognizes that setting individual i 's period t outcome affects the expectation of the utilities of her neighbors in the successive period. In contrast, individually optimal decisions are backward looking only because we have assumed away an own conformity effect (via the definition of the adjacency matrix). Therefore the first-order conditions yield a simple vector autoregression of order 1.

Second, by employing the methods used earlier and summarized in Proposition 1 above, it is possible, in principle, to extend Proposition 4 so as to clarify the evolution of the variance-covariance matrix for the vector of socially optimal outcomes. However elementary this looks, it appears not to have been done before. If such expressions were easily obtainable, one could express the expectation of the optimal value of planner's objective at the steady state and compare it with the respective one for the individually optimal case. That is, since the planner's objective is a quadratic function, its expected value at the steady state of the optimal solution admits an expression in terms of means and variances and covariances.

Third, when the solution to Equ. (21) involves iterating forwards, we obtain an *asset-like property* of socially optimal period t outcomes, \mathbf{Y}_t . That is, the value from setting an individual's outcome in this period depends upon others' outcomes next period, and so on. This confers an asset element to agents' outcomes and may be interpreted as *an asset theory* of social interactions. At the social optimum, individual outcomes at time t reflect the present value of the future stream of contextual effects of all current neighbors, adjusted by

discount factors that are functions of the rate of time preference and multiplied by the social interactions coefficient and weighted by the directness of the connections between agents. It is not surprising that the planner is forward looking in assessing the effects that different individuals' actions have on those individuals' own outcomes and those of their neighbors. The one period forward dependence may, of course, be translated into the standard infinitely forward dependence familiar from rational expectations models, depending upon parameter values.

Fourth, we note that there is another aspect of the social value of social interactions which the effect discussed above does not capture. That is, the value for an agent of having versus not having a particular connection with another agent. Associated with every solution path is an expected value of the social welfare function, the planner's objective function. Therefore, in principle, we may assess the impact of a change in the adjacency matrix on the socially optimal solution and thus on the expected value of the planner's objective function. That is, adding a new link changes the solution path in a particular way and produces an increment in the expected value of the planner's objective function. In general, this yields a shadow value for the corresponding link which is a benchmark by means of which we may measure the social value of intermediation. This calculation is straightforward conceptually. In practice, it is quite tedious to compute the social value of an additional connection.

Since the individually optimal outcomes are inefficient, the social optimum may be implemented, at least conceptually, through an appropriately chosen set of prices, $\pi_{it}, i = 1, \dots, N$. By adapting individual i 's utility so as to include a term $-\pi_{it}y_{it}$, maximization of $U_{it} - \pi_{it}y_{it}$ brings about the social optimum, provided that the prices reflect the respective marginal social effects, which in view of quadratic utility functions are equal to the difference between the socially optimal and individually optimal outcomes. For completeness, we also need to impose conditions of incentive compatibility and allow for transfers that leave individuals at least indifferent while budget balance is satisfied.

There may be circumstances where the socially optimal outcomes can be implemented via regulation. For example, methods of non-random assignment of students to school classes impose interactions among students with specific characteristics. This is the case of "tracking", in the U.S. context and usage of the term. Another example is the experiments known as Moving to Opportunity (MTO), which have administered by the U.S. Department of Housing and Urban Development. These are large scale social experiments that encourage individuals to move by providing benefits only when individuals relocate to

communities that are “better” from those where they currently reside in.¹¹

So far we have assumed that the network of social connections is given. We next turn to the situation where this network is endogenous. In that case, implementing individual prices π_{it} in general is insufficient as a tool for the social planner to implement the socially optimal outcomes and network connections.

6 Endogenous Networking

How could social network connections come about? This section explores the notion that network connections are initiated by means of individuals’ initiatives, when individuals stand to benefit by doing so. We examine endogenous networking by assuming that individuals choose weights which they associate with their connections with other individuals. This approach assuages the inherent difficulty of dealing with discrete endogenous variables while taking advantage of a natural interpretation of weighted social connections. Individuals’ social attachments do differ: they vary from close friendships to mere acquaintances. And, naturally, weighted adjacency matrices are used in the social science literature. A formulation of endogenous networking in a discrete setting of the model of social connections we have been working with so far is available from the authors upon request. It demonstrates that characterization of individual networking initiatives is quite awkward, and so is the associated welfare analysis.

Analysis of endogenous networking is facilitated by assuming a finite horizon, T , version of the typical individual’s problem and considering the networking decision prior to setting the period T outcome.¹² The impact on utility in period T is easier to obtain when we work with the indirect utility function, given a network topology. We continue excluding a global effect, just as with the planner’s problem, by setting $\beta_g = 0$. Choosing y_{iT} so as to maximize the expectation of period T utility,

$$U_{iT} = \mathbf{\Gamma}_{i,T} \mathbf{X} \phi + (\mathcal{A}_{iT} + \varepsilon_{iT}) y_{iT} - \frac{1}{2} (1 - \beta_\ell) y_{iT}^2$$

¹¹The effect of these programs is that participating individuals change their network of connections. This foreshadows the discussion taken up in section 6.1. Note also that MTO experiments may not only aim at using interaction effects taking place through income, but also at endogenous interactions in education, substance abuse, lifestyle, etc.

¹²The analytics employed and intuition gained by the finite horizon approach may be used to extend the model to an infinite horizon setting. See Bisin, *et al.*, *op. cit.* for an extension with one-sided local but global social interactions, as well.

$$-\frac{\beta_\ell}{2} \left(y_{iT} - \frac{1}{|\nu_T(i)|} \mathbf{\Gamma}_{i,T} \mathbf{Y}_{T-1} \right)^2 + \frac{1}{|\nu_T(i)|} \mathbf{\Gamma}_{i,T} \mathbf{Y}_{T-1} \mathbf{x}_i \omega - c_1 \mathbf{\Gamma}_{i,T} \ell - c_2 \mathbf{\Gamma}_{i,T} \mathbf{\Gamma}'_{i,T}, \quad (26)$$

yields an indirect utility function for agent i :

$$\begin{aligned} \tilde{U}_{iT}(\mathbf{\Gamma}_T; \mathbf{Y}_{T-1}) &\equiv \mathcal{E}_{\varepsilon_t} \left[\max_{y_{iT}} : U_{iT} \mid \Psi_{i,T-1} \right] \\ &= \mathbf{\Gamma}_{i,T} \mathbf{X} \phi + \frac{1}{2} \mathcal{A}_{iT}^2 + (\beta_\ell \mathcal{A}_{iT} + \mathbf{x}_i \omega) \frac{1}{|\nu_T(i)|} \mathbf{\Gamma}_{i,T} \mathbf{Y}_{T-1} \\ &\quad - \frac{1}{2} \beta_\ell (1 - \beta_\ell) \left(\frac{1}{|\nu_T(i)|} \mathbf{\Gamma}_{i,T} \mathbf{Y}_{T-1} \right)^2 - c_1 \mathbf{\Gamma}_{i,T} \ell - c_2 \mathbf{\Gamma}_{i,T} \mathbf{\Gamma}'_{i,T} + \frac{1}{2} \sigma_\varepsilon^2. \end{aligned} \quad (27)$$

A network of connections among N individuals is defined by means of a $N \times N$ *weighted* adjacency matrix $\mathbf{\Gamma}_t$, of intensities of social attachment:

$$\gamma_{ij,t} \begin{cases} \neq 0, & \text{if } i \text{ is influenced by } j \text{ in } \mathcal{G}; \\ = 0, & \text{otherwise.} \end{cases} \quad (28)$$

This formulation combines the notion of an adjacency matrix in graphs with the notion of varying intensities of social contacts¹³ and at the same time allows for the network to be directed. For $c_2 > 0$ the marginal cost of a connection is increasing with intensity. This has an interpretation in terms of opportunity cost: short and superficial encounters, e.g. at parties, bear a lower opportunity cost than spending quality time together. We return below to an interpretation of individuals' choice of intensities of social attachment as neighborhood choice. In the remainder of the paper, it is actually more convenient to revert to the original notation in terms of $\mathbf{\Gamma}_t$.

Individuals are assumed to choose by which other individuals they are influenced under the assumption that they optimize their own outcomes for any given social structure and take all others' decisions as given.¹⁴ All previous derivations in the paper so far readily apply to the case of a weighted adjacency matrix, except that it is no longer necessary to normalize by the size of neighborhood. Formally, individual i seeks to maximize expected period T utility, given by (27), with respect to

$$\mathbf{\Gamma}_{i,T} = (\gamma_{i1,T}, \dots, \gamma_{ij,T}, \dots, \gamma_{iN,T}), \text{ with } \gamma_{ij,T} \geq 0, j = 1, \dots, N,$$

¹³We thank Alan Kirman and Bruce Weinberg who emphasized this interpretation.

¹⁴In the terminology of DeMarzo *et al.* (2003), if $\gamma_{ij} > 0$, then i "listens" to j . A weighted adjacency matrix may come about even when connections are indicated by either 0 or 1. This would be the case when agents update their beliefs using Bayes theorem. See DeMarzo *et al.* (2003). Our approach is conceptually related to Goyal (2005), a paper that we were unaware of in the first version of this paper. However, Goyal assumes a given *balanced* graph and allows agents (firms, in his case) to choose intensity of effort. All agents' efforts affect interagent link quality.

and conditional on her information set Ψ_{it} . The results are summarized in the following proposition.

Proposition 5.

The indirect utility function (27) for individual i is concave with respect to $\Gamma_{i,T}$, individual i 's vector of social weights, provided that $\mathbf{X} = \mathbf{0}$. It admits a unique maximum with respect to $\Gamma_{i,T}$ in that case, given \mathbf{Y}_{T-1} .

Proof. By differentiating (27) twice with respect to $\Gamma_{i,T}$ yields the Hessian,

$$\begin{aligned} \text{Hessian} &= [\mathbf{X}\theta + \beta_\ell \mathbf{Y}_{T-1}] (\mathbf{X}\theta)' + \beta_\ell [\mathbf{X}\theta - (1 - \beta_\ell) \mathbf{Y}_{T-1}] \mathbf{Y}'_{T-1} - 2c_2 \mathbf{I} \\ &= \mathbf{X}\theta(\mathbf{X}\theta)' + \beta_\ell [\mathbf{X}\theta \mathbf{Y}_{T-1} + (\mathbf{X}\theta \mathbf{Y}'_{T-1})'] - \beta_\ell (1 - \beta_\ell) \mathbf{Y}_{T-1} \mathbf{Y}'_{T-1} - 2c_2 \mathbf{I}. \end{aligned}$$

To ensure concavity, the Hessian must be negative semidefinite. In the simple case in which $\mathbf{X} = \mathbf{0}$, Hessian reduces to $\text{Hessian} = -\beta_\ell (1 - \beta_\ell) \mathbf{Y}_{T-1} \mathbf{Y}'_{T-1} - 2c_2 \mathbf{I}$, which is negative semidefinite as long as $\beta_\ell < 1$, which ensures that the only terms in equation (27) that are quadratic in $\Gamma_{i,T}$ receive a negative sign. Of course, this is a very simple case with all individuals ending up with similar outcomes. For $\mathbf{X} \neq \mathbf{0}$, interpretation of the condition is more involved. Since the term $\mathbf{X}\theta(\mathbf{X}\theta)'$ is positive definite and the term $\mathbf{X}\theta \mathbf{Y}_{T-1} + (\mathbf{X}\theta \mathbf{Y}'_{T-1})'$ is ambiguous in sign, a sufficiently large coefficient of the marginal connection cost would be required to make the utility function concave with respect to the weighted adjacency matrix. In case Hessian is negative semidefinite, the first order conditions,

$$\begin{aligned} \Gamma'_{i,T} &= - \{ [\mathbf{X}\theta + \beta_\ell \mathbf{Y}_{T-1}] (\mathbf{X}\theta)' + \beta_\ell [\mathbf{X}\theta - (1 - \beta_\ell) \mathbf{Y}_{T-1}] \mathbf{Y}'_{T-1} - 2c_2 \mathbf{I} \}^{-1} \times \\ &\quad [\mathbf{X}\phi + \mathcal{A}_{iT} \mathbf{X}\theta + (\beta_\ell \mathcal{A}_{iT} + \mathbf{x}_i \omega) \mathbf{Y}_{T-1} - c_1 \iota], \end{aligned} \tag{29}$$

characterize the unique maximum, given \mathbf{Y}_{T-1} .

Q.E.D.

We note that (29) provides an explicit solution for $\Gamma'_{i,T}$ in terms of \mathbf{Y}_{T-1} , of contextual effects and of individual characteristics. This solution does not involve explicit dependence on random shocks, but randomness is present through \mathbf{Y}_{T-1} . A number of additional remarks are in order. *First*, for certain applications, it may be important to require the entries of the adjacency matrix to be positive and to normalize them so as they sum up to one. Restricting $\Gamma_{i,T}$ so that it lies in the positive orthant of R^N , $\gamma_{ij,T} \geq 0$, $j = 1, \dots, N$, a convex set, is straightforward but quite stringent. That is, it would require restrictions on values of the preference parameters. While some of the variables are constant, others depend on the actual

state of the economy as of the preceding period, \mathbf{Y}_{T-1} . These values accumulate past values of stochastic shocks and are therefore random. Consequently, endogenous weighted networking implies that agents in setting their period $T - 1$ decisions must take into consideration that $\mathbf{\Gamma}_T$, the social adjacency matrix in period T , is an outcome of individuals' uncoordinated decisions and stochastic. We note that positivity of the $\gamma_{ij,T}$ s is quite critical for a meaningful interpretation of endogenous networking as being akin to neighborhood choice. If the adjacency matrix were to be restricted to be positive, then particular patterns in $\mathbf{\Gamma}$, like a block-diagonal structure with many small blocks and only a few connections between the blocks would be reminiscent of so-called "small worlds," emerging endogenously.

Second, endogenous determination of the social adjacency matrix introduces intertemporal linkages in the individual's decision making for two reasons: first, period $T - 1$ decisions affect the period T adjacency matrix $\mathbf{\Gamma}_T$; and second, unlike when the adjacency matrix was taken as given and the assumption was made that all diagonal terms were assumed to be equal to zero, $\gamma_{ii,t} = 0$, now they may well be nonzero, thereby strengthening habit formation. From a psychological perspective, one can argue whether positive $\gamma_{ii,t}$'s should be interpreted as the result of willful acts or are expressing addiction. This intertemporal linkage in turn leads to a clarification of the asset value of a connection between agents. Because of our functional specification for individual utility according to Equ. (2), additional information, in the form of total and marginal contextual effects, is brought to bear upon the problem of determining the social structure and also enters the asset value of social connections. In particular, additional information is included in the reaction function, such as the term $\mathbf{X}\phi$. *Third*, once the connection weights are endogenized, the resulting weights are not necessarily equal, that is the adjacency matrix is not symmetric, let alone constant over time. *Fourth*, and related to the previous remark, a critical feature of our approach is that an individual's choice to network with others is directed. The people I choose to be influenced by are not affected by my networking initiatives and need not reciprocate. The interactions that are initiated here are not being evaluated from the viewpoint of being mutually advantageous. This is, of course, in contrast to the pairwise stability concept that was pioneered by Jackson and Wolinsky (1996) and Jackson and Watts (2002).

We refrain from pursuing further these issues in the present paper. However, in an important sense, networking, once it has been endogenized no longer implies that social interactions assume the form of the mean field case. Furthermore, variable intensities of attachment constitute a continuous version of strong versus weak ties. However, in contrast

to Granovetter (1994), there is no implication here that those who have stronger ties with one another are more likely to have friends in common than those with weaker ties. Nonetheless it is interesting to know under what conditions that is indeed a property of the endogenous social adjacency matrix.

6.1 Endogenous Networking and Individual Decisions

In obtaining the first-order conditions for individual i 's decision in period $T - 1$ we need to recognize that $\gamma_{ii,T}$, which is optimally chosen according to (29) above, need not be equal to zero. Consequently, $y_{i,T-1}$ enters period T indirect utility \tilde{U}_{iT} via the term $\mathbf{\Gamma}_{i,T}\mathbf{Y}_{T-1}$.¹⁵ For the solution for \mathbf{Y}_{T-1} in matrix form:

$$[\delta\beta_\ell(1 - \beta_\ell)[\text{Diag}\mathbf{\Gamma}_T]\mathbf{\Gamma}_T + \mathbf{I}]\mathbf{Y}_{T-1} = \delta[\text{Diag}\mathbf{\Gamma}_T][\beta_\ell\mathcal{A}_T + \mathbf{X}\omega] + \beta_\ell\mathbf{\Gamma}_{T-1}\mathbf{Y}_{T-2} + \mathcal{A}_{T-1} + \epsilon_{T-1}. \quad (30)$$

where $[\text{Diag}\mathbf{\Gamma}_T]$ denotes the diagonal matrix made up of the diagonal terms of $\mathbf{\Gamma}_T$.

Not surprisingly, the strength of the social interaction effect depends on $\delta\gamma_{ii,T}$. This involves the own period T individual and contextual effects $\mathcal{A}_{i,T}$, and the own endogenous local interaction effect on period T utility, as they are added to the period $T - 1$ individual and contextual effects, $\mathcal{A}_{i,T-1}$. It also involves the own conformist effect of the current decision $y_{i,T-1}$ on itself through its presence in period T utility.

Finally, we turn to the joint evolution of individual decisions and the social network. We note that the setting of \mathbf{Y}_{T-1} and $\mathbf{\Gamma}_T$ are based on the same information. Therefore, (30) and (29) may be treated as a simultaneous system of equations: period $T - 1$ individual decisions and period T weights of social attachment are jointly determined. These equations may be rewritten schematically as:

$$\begin{aligned} \mathbf{\Gamma}_T &= \mathcal{G}(\mathbf{Y}_{T-1}|\mathbf{X}), \\ \mathbf{Y}_{T-1} &= \mathcal{Y}(\mathbf{\Gamma}_T, \mathbf{Y}_{T-1}, \mathbf{Y}_{T-2}, \epsilon_{T-1}|\mathbf{X}). \end{aligned} \quad (31)$$

The above system of equations is autonomous with only a contemporaneous stochastic shock, the vector ϵ_{T-1} . It exhibits complicated dynamic dependence. It is recursive in terms of \mathbf{Y}_{T-1} . That is, by substituting for $\mathbf{\Gamma}_T$ from the first in to the second equation, we are left with a first-order system of difference equations with respect to \mathbf{Y}_{T-1} . It is clear from (29) that the resulting equations are highly nonlinear. In principle, system (31) is amenable to

¹⁵It also enters indirectly through the dependence of the endogenous $\mathbf{\Gamma}_{i,T}$ on \mathbf{Y}_{T-1} , but by the envelope theorem, those terms cancel out.

the usual treatment for dynamical systems, but the fact that $\mathbf{\Gamma}_T$ is a matrix makes matters cumbersome. Multiplicity of equilibria with respect to \mathbf{Y}_{T-1} is transmitted to $\mathbf{\Gamma}_T$ as well. One especially has to examine that the system is dynamically stable, perhaps by imposing a kind of transversality condition that restricts the absolute value of entries of matrix $\mathbf{\Gamma}_T$. We plan to pursue further in future research the properties of co-evolution of individual decisions and social structure, as modelled above by (31).

6.2 Evaluation of Alternative Topologies by a Planner

For completeness, it is important to address the counterpart of the planner's problem when networking is endogenous. In contrast to section 5, where the planner was seen as choosing the vector of individual outcomes so as to maximize a utilitarian social welfare function, here we allow for the planner to set social connections while being cognizant of individuals' preferences. Specifically, and in order to simplify the problem, we define the planner's problem as the choice of a social interactions topology $\mathbf{\Gamma}$ so as to maximize a utilitarian social welfare function, defined as the sum of expected indirect utilities, given from (27) for each agent i , under the assumption that the vector of individual outcomes obeys the steady state conditions (10) and (12). That is:

$$\max_{\mathbf{A}} : \sum_i \mathcal{E} [\tilde{U}_i(\mathbf{A}; \mathbf{Y}^*)], \quad (32)$$

where \mathbf{A} denotes the normalized social adjacency matrix corresponding to $\mathbf{\Gamma}$, subject to the constraint that the adjacency matrix is positive and symmetric. For consistency with our definition of endogenous networking, we no longer require that the entries are either 0 or 1 nor that the diagonal elements are all 0. Consequently, even though the conditions for socially optimal networking are the same as for individually optimal networking, (29) above, the socially optimal outcomes for each individual no longer satisfy (30). The socially optimal value of, say, $y_{i,T-1}$ must reflect the spillovers on the utilities of all other individuals.

7 Conclusions

We conclude by first providing a brief summary. This paper examines social interactions in dynamic settings when the social network may evolve exogenously or determined endogenously. The paper employs a linear-quadratic model that accommodates contextual effects and endogenous interactions, that is local ones where individuals react to the decisions of

their neighbors, and global ones, where individuals react to the mean decision in the economy, both with a lag. Unlike the simple VARX (1,0) form of the structural model, the planner’s problem involves intertemporal optimization and leads to a system of linear difference equations with expectations. It also highlights an asset-like property of socially optimal outcomes in every period which helps characterize the shadow values of connections among agents. Endogenous networking is easiest to characterize when individuals choose weights of social influence from other agents. It is much harder to do so when networking is discrete. The paper also poses the inverse social interactions problem, that is whether it is possible to design a social network whose agents’ decisions will obey an arbitrarily specified variance covariance matrix. Creation of networking markets is conceptually related to completing asset markets in an economy, an intuition which should be explored further in future research.

We think of this paper as addressing problems that pursue further the logic of one of Alan Kirman’s greatest contributions. That is, to further study economies with interacting agents in ways that help bridge an apparent gap with mainstream economics [Kirman (1997; 2003)] and do so in a manner that lends itself to regression-type empirical techniques. Although the paper does not deal with another of Kirman’s key areas of interest, aggregation, we hope it contributes indirectly in the following fashion. First, aggregation that ignores the presence of social interactions is a tricky venture [c.f. Kirman (1992)]. Second, social interactions give rise to a *social multiplier*, construed here in the sense of Glaeser, Sacerdote and Scheinkman (2003). That is, a *social multiplier* is defined as the ratio of the coefficient of a contextual effect obtained from a regression with aggregate data to that obtained from a regression with individual data. As a consequence of aggregation, the social multiplier overstates individual effects, in varying degrees that depend on precise patterns of interdependence and sorting in the data. The tools developed in this paper may allow to explore the following hypothesis in the future. Might the social multiplier be due to the exogeneity of social networks as assumed by earlier papers? We know from *ibid.* that sorting across groups on the basis of observables reduces the social multiplier, but sorting on unobservables has an ambiguous effect. Our model involves both effects, by positing what is essentially an optimal “social” portfolio problem. Endogenous social networking makes the case forcefully that non-market, that is social, interactions need not be of the mean field type.

In a formal sense, dependence on neighbors’ income could be a reduced form for benefits derived from trading with others. In our model, agents are seeking to choose a desirable set of others to be influenced from. However, while they incur costs to do so, they are

not really competing against one another. Since our endogenous interactions are directed, no mutual consent is necessary. Several individuals may seek to be influenced by a single other individual without interfering with each other.¹⁶ We would like to think of this as a precursor to opening up routes for trading. That is a much harder problem, but one that promises attractive payoffs. When markets are incomplete, opening up new markets is a tricky business from the viewpoint of social welfare [Newbery and Stiglitz (1984)].

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¹⁶We thank a perceptive referee for emphasizing this point.

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