Averaged Null Energy Condition

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Abstract

The achronal averaged null energy condition (ANEC) is a sufficient condition to rule out exotic spacetimes such as time machines and wormholes. We review how achronal ANEC is used to prove such restrictions, and the status of the condition. We find counterexamples to achronal ANEC using conformally coupled scalar test fields in a conformally flat background. These examples involve rapid variation in the stress-energy tensor in the vicinity of the geodesic under consideration, suggesting that averaging in additional dimensions would yield a principle universally obeyed by quantum fields. However, we further develop counterexamples to alternative transversely averaged energy conditions. In order to arrive at a valid energy condition, we must then either restrict to minimal coupling or to self consistent rather than arbitrary background metrics. We construct a state as a candidate reference state for quantum difference inequalities, which could then be used to prove ANEC in these conditions. The proposed recently massless vacuum is defined for a region of spacetime which may be embedded in a manifold which for all times prior to some recent Cauchy surface has vanishing potential energy. It is similar to the standard in vacuum, with a modified spacetime taking the place of an asymptotically distant massless region. This allows us to focus on the contribution of local terms to the stress tensor.

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I. INTRODUCTION

Wheeler's succinct summary of general relativity is that "spacetime tells matter how to move, and matter tells spacetime how to curve." This is the direct message of Einstein's equation

$$G_{ab} = 8\pi T_{ab}. \tag{1.1}$$

General relativity places no restrictions on the form of either the curvature or the stress tensor, merely that they be proportional to each other. Nothing in general relativity prevents an arbitrary spacetime geometry or topology, but merely requires a certain stress tensor to support it. However, we consider certain configurations, such as closed timelike curves, wormholes, and superluminal communication, to be unnatural. In order to restrict them we must appeal to an appropriate energy condition derived from quantum field theory, which would limit the behavior of T_{ab}. Theorems are then proven which rely on these conditions and rule out the existence of certain perverse spacetimes.

Early work limiting exotic phenomena utilized classical energy conditions such as the weak energy condition (WEC),

$$T_{ab} l^a l^b \geq 0 \tag{1.2}$$

for all timelike vectors l^a, evaluated at any point. This condition has the interpretation that all observers see a positive energy density. If instead the stress tensor is contracted with a null vector k^a, it is known as the null energy condition (NEC). Roughly, such a condition can prove exotic phenomena impossible because they have as a common feature null geodesics converging as they enter the exotic region and diverging as they exit. This anti-lensing would require negative energy. Classical fluids and fields[1] obey this constraint, though it can be trivially violated in flat space quantum field theory (for example, with the vacuum plus two photon system. The Casimir two plate system is another more complicated but important example.) Graham and Olum [26] present proofs that the theorems which exclude wormholes, time machines and superluminal communication may follow from the achronal averaged null energy condition, which is the focus of this thesis:

[1] Non-minimally coupled scalar fields are an exception [1, 2, 17]. In this case the classical field can easily violate all pointwise energy conditions.

Achronal ANEC *For every null, complete, achronal geodesic γ with tangent l^a and affine parameter λ,*

$$\int_\gamma T_{ab} l^a l^b d\lambda \geq 0 \tag{1.3}$$

Achronal null geodesics are those which do not contain points also joined by any timelike paths. Complete geodesics extend to infinite positive and negative values of λ. Restricting attention to these null geodesics allows the condition to be true in spacetimes which contain incomplete or chronal geodesics that support states which violate the ANEC integral. ANEC does not have violations in flat space quantum field theory.[2]

Without specifying achronal and complete geodesics, ANEC would be false. A simple example of ANEC violation for chronal geodesics occurs in Minkowski space with one dimension compactified. This produces a Casimir effect of negative energy density and pressure in the compactified direction, violating ANEC along those geodesics. The compactification also renders all geodesics chronal, however. Another violation of ANEC along chronal geodesics is in the Boulware vacuum of Schwarzschild spacetime [56]. The geodesics which avoid the singularity are complete, but they are chronal. Onemli and Woodard [42, 43] find an ANEC violation for ϕ^4 theory in de Sitter space, but it occurs only along incomplete geodesics.

We are able to restrict our attention to achronal geodesics because, heuristically, all problematically exotic spacetimes which we wish to exclude involve paths which are in some sense a 'shortcut.' They either reach an otherwise causally disconnected region of spacetime or travel superluminally. Chronal geodesics have at least two points which are also connected by a timelike geodesic, and are thus not involved in creating exotic spacetimes. Achronal geodesics are highly nongeneric, and by restricting the condition to complete achronal geodesics we avoid most counterexamples.

In the first chapter we present the proofs limiting exotic behavior using the achronal averaged null condition. Many of these theorems were originally proven using more stringent restrictions on the stress tensor, and we review the development. The usefulness of ANEC for these theorems motivates its further study. We then present aspects of quantum field theory,

[2] Classical violations of ANEC for non-minimally coupled scalar fields in curved space [1, 2] are possible only if the field takes on Planck-scale values, which lead the effective Newton's constant to first diverge and then assume negative values. This may mean that such states are not physically realizable.

in order that our presentation be self contained. In particular we analyze the role of curvature coupling and conformal transformations, and introduce the algebraic perspective on field theory. This formalism is well suited to investigations in curved space. We then discuss renormalization in detail, using the Hadamard subtraction method. Explicit calculations are presented, to serve as a more general reference in addition to their use in further sections. Proofs of ANEC in limited situations are collected: those of Klinkhammer and Wald and Yurtsever in flat space, Fewster, Olum and Pfenning in flat tubes with arbitrary curvature outside, Wald and Flanagan in perturbatively flat spaces and Kontou and Olum in more general curved spaces.

The remaining sections contain our original work in collaboration with Ken Olum. First we present counterexamples to ANEC in the regime of a conformally coupled test scalar field in a conformally and asymptotically flat background [52]. We then discuss several possible extensions of ANEC to include additional transverse averaging, but find that for all of these proposals there are counterexamples in the same regime [53]. This work demonstrates that in order to make further progress we must either restrict attention to quantum fields and metrics that are consistent with each other according to general relativity, or perhaps rule conformal coupling for scalar fields unphysical. In the last section we introduce a state constructable for a scalar field with arbitrary potential energy, intended to be used for proving quantum energy inequalities. After defining this state we present calculations which bound its two point function. We hope in the future to develop from this to an argument in curved space. If the difference energy inequality between an arbitrary state and the recently flat vacuum we introduce can take the form of the energy inequality postulated by Kontou and Olum [37], ANEC will be proven for minimally coupled scalar fields on a classical background.

We use the $(+++)$ sign convention, in the classification of Misner, Thorne and Wheeler [39]. This means that the metric signature is $(-+++)$, the Riemann tensor is ${R_{abc}}^d\omega_d = (\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c$ and the Ricci tensor is $R_{ab} = {R_{acb}}^c$. This contrasts with the usual convention in field theory, and the massive wave equation is $(\Box - m^2)\phi = 0$.

II. ENERGY CONDITIONS AND EXOTIC PHENOMENA

Careful arguments are necessary to reason from the properties of the stress tensor to the global nature of causality. The assumed properties of the stress tensor are achronal ANEC, defined at eq. (1.3), and the null generic condition. This is defined below, and roughly means that *somewhere* along each null geodesic, normal matter is encountered. These conditions, together with the equations governing the propagation of null geodesics, form the basis for the particular theorems concerning topological censorship, wormholes, time machines, mass positivity, and superluminal communication, which are each treated independently.

The evolution of a family of geodesics is governed by the Raychaudhuri equation for expansion, and the related shear and vorticity equations. The expansion θ is the scalar measuring the increase in cross sectional area of the congruence. The shear tensor σ_{mn} is defined to be symmetric in its indices, and measures the distortion of the cross section, while the vorticity or twist tensor ω_{mn} is defined to be antisymmetric. Both tensors are spacelike. The formal definitions may be found in [57] or other standard references. We establish a pseudo-orthonormal basis (e_1, e_2, e_3, e_4), where e_1 and e_2 are spacelike orthonormal directions, e_4 is in the direction of the null geodesic with tangent vector l^a, and e_3 is a second null direction whose inner product with e_4 is -1. We denote the affine parameter by λ. Indices m, n and p only run over the two spatial directions, 1 and 2. Then we have

$$\frac{d\theta}{d\lambda} = -R_{ab}l^a l^b + 2\omega^2 - 2\sigma^2 - \frac{1}{2}\theta^2 \tag{2.1}$$

$$\frac{d\sigma_{mn}}{d\lambda} = -\theta\sigma_{mn} - C_{m4n4} - \sigma_{mp}\sigma_{pn} - \omega_{mp}\omega_{mn} + \delta_{mn}(\sigma^2 - \omega^2) \tag{2.2}$$

$$\frac{d\omega_{mn}}{d\lambda} = -\theta\omega_{mn} + 2\omega_{p[m}\sigma_{n]p} \tag{2.3}$$

From eq. (2.1) we see that vorticity causes expansion, but if initially $\omega_{mn} = 0$ then according to eq. (2.3) it never develops. Shear causes contraction, as does any nonzero value for θ. Thus only curvature can cause defocusing; this will not occur unless $R_{ab}l^a l^b$ is negative. By the Einstein equation,

$$R_{ab}l^a l^b = 8\pi T_{ab}l^a l^b, \tag{2.4}$$

and the null energy condition would guarantee that all points experience a focusing of geodesics. Violations of the pointlike NEC require us to consider whether sufficient defocus-

ing can accumulate over the course of a geodesic. If ANEC is obeyed but NEC is violated, along the ANEC integral there must be positive contributions which equal or exceed the negative ones possible.

We next introduce the null generic condition, which describes the typical influence of matter on curvature. If the spacetime obeys both ANEC and the generic condition, there are strong limits on exotic phenomena. The definition is

Null Generic Condition *A spacetime satisfies the condition if every null geodesic possesses at least one point where*

$$l_{[e}R_{a]bc[d}l_{f]}l^b l^c \neq 0. \tag{2.5}$$

Unlike a condition required to hold at every point or in aggregate over many points, this inequality would only need to hold at a *single* point along every geodesic for the spacetime to be considered generic. Because normal matter would generally introduce a nonzero tidal effect, passing through any sort of dust somewhere in the trajectory would yield the inequality at some points. It is thus aptly called generic. A lemma due to Borde [5] proves that in a spacetime which obeys the null generic condition and the null energy condition averaged over some interval, that interval must contain conjugate points.[3] Containing conjugate points implies that a geodesic is chronal. Therefore, when the generic condition holds, if ANEC is obeyed over a complete geodesic it cannot also be achronal. It strengthens the classical result on the necessary existence of conjugate points using the null energy condition found in Hawking and Ellis [29] or Wald [57].

We have highlighted the special status of complete achronal geodesics for energy conditions. In the following sections we cite the modifications of classical theorems that allow achronal ANEC to suffice. The restriction to complete achronal geodesics is particularly relevant for motivating study in conformally flat spacetimes, which do contain many such

[3] Essentially, conjugate points are such that a null geodesic infinitesimally deviated away from γ at $\gamma(\lambda_1)$ returns to $\gamma(\lambda_2)$. Borde defines conjugate points as a pair of points $\gamma(\lambda_1)$ and $\gamma(\lambda_2)$ on a geodesic where $\theta(\lambda) \to +\infty$ as $\lambda \to \lambda_1$ from above and $\theta(\lambda) \to -\infty$ as $\lambda \to \lambda_2$ from below. Wald [57] defines them equivalently but with more general language as points where some Jacobi field that is not identically zero vanishes at $\gamma(\lambda_1)$ and $\gamma(\lambda_2)$.

paths, and demands care in doing any perturbative analysis, as generic perturbations typically introduce chronality.

A. Topological Censorship

Topological censorship states that a causal path cannot traverse non-trivial topology. Any exotic topological structures introduced will be censored by some horizon, and fail to evolve to infinity. As described by Graham and Olum [26], we must assume that our manifold is simply connected, which differs from the formulation of topological censorship given by Friedman, Schleich and Witt [19]. Our version is

Topological Censorship *Let $\mathcal{M}$,g be a simply-connected, asymptotically flat, globally hyperbolic spacetime. If it satisfies achronal ANEC and the generic condition, then every causal curve from past null infinity ($\mathcal{I}^-$) to future null infinity ($\mathcal{I}^+$) can be deformed to a curve in the infinite, asymptotically flat region.*

The proof is by contradiction. Suppose there is a causal curve γ from $\mathcal{I}^-$ to $\mathcal{I}^+$ that could not be deformed to a curve at infinity, because it threaded a wormhole or other nontrivial topology. There then exists some curve γ' which arrives at $\mathcal{I}^+$ later and departs from $\mathcal{I}^-$ earlier than any other in the homotopy class of γ. This curve γ' must be a complete null geodesic because it is extremal. If it were chronal, it could be deformed to a timelike curve, and then the process repeats until a fastest achronal null geodesic is found. But a complete, achronal, null geodesic is forbidden by Borde's lemma, and we have the contradiction.

This modifies the proof of Friedman and Higuchi [20]. Their result applied to manifolds which are not necessarily simply connected, and required ANEC to hold on both chronal and achronal geodesics. Because ANEC does not universally hold on chronal geodesics, we must limit the theorem. However, this version, as given in [26], only fails to censor wormholes which do not threaten causality. If the manifold is not required to be simply connected, one could allow a wormhole that joins a single asymptotically flat region to itself with a throat that is longer than distance between the mouths measured outside of the wormhole. There

would be a fastest curve γ' among the homotopy class that traverses the wormhole, but it would be a chronal complete null geodesic, and is permitted to exist. A wormhole such as this is not censored, but it does not carry the exotic implications we normally associate with wormholes.

B. Chronology Protection

There is a rich literature attempting to rule out closed timelike curves or other exotic chronologies based on difficulties with formulating physics consistently at all. Friedman and Higuchi [20] summarize and give references to research along these lines. Classical dynamical evolution from a Cauchy initial data surface already only appears possible for carefully contrived initial conditions, and maintaining unitary quantum mechanics introduces further difficulty. Hawking conjectured that the presence of closed timelike curve would introduce divergence into the quantum mechanical vacuum [30], though interpretation of his result varies. We adopt the perspective, however, that objections to exotic chronology based on inconsistency or paradox may not rule it out by themselves, but instead motivate us to search for a proof that such a thing cannot be introduced.

Energy conditions may be used to limit the development of closed timelike curves within a conventional universe. ANEC was first introduced by Tipler [51] for this purpose.

Time Machine Restriction *Let $\mathcal{M}$,g be an asymptotically flat spacetime which obeys achronal ANEC and the generic condition. If it is partially asymptotically predictable from a partial Cauchy surface S and the chronology condition is violated in $J^+(S) \cap J^-(\mathcal{I}^+)$ (the causal future of S which is in the past of null infinity), then it cannot be null geodesically complete.*

Violating the chronology conditions means that there exists a Cauchy horizon $H^+(S)$ which is the boundary of the region $D^+(S)$ which is the region predictable from initial data on the Cauchy surface S. The Cauchy horizon consists of a set of null generators, and Tipler proved that at least one such generator η is contained entirely within $H^+(S)$. If the

spacetime is null geodesically complete, η must be complete. No point in the horizon can be in the chronological future of any other point in the horizon, and because η is contained entirely within $H^+(S)$, it is achronal. But, if our spacetime satisfies achronal ANEC and the generic condition, geodesics such as η cannot exist. Although the proof presented in [51] utilizes a formulation of ANEC which must hold for chronal geodesics, no modification of the result is necessary. Hawking proved a further result showing that a compactly generated Cauchy horizon $H^+(S)$ cannot exist in an asymptotically flat spacetime obeying the generic condition and achronal ANEC [26, 30].[4]

C. Positive Mass

A positive mass theorem proves that the Arnowitt-Deser-Misner (ADM) mass is positive for all observers. This mass is well defined for foliated, asymptotically flat spaces, where an observer at infinity measures the strength of the gravitational field. For the ADM mass to be positive means that negative contributions from classical gravitational binding energy and pressures and quantum mechanical negative energies are compensated or exceeded by positive energy sources. Penrose, Sorkin and Woolgar prove a version of the positive mass theorem which relies on achronal ANEC and the generic condition [45]. This differs in character distinctly from the earlier proofs by Schoen and Yau and by Witten, which are purely results of classical general relativity that could not extend to semiclassical gravity. The theorem of [45] specified using achronal ANEC, and so we outline their proof without modification.

Heuristically, the proof proceeds by contradiction. Assume that a space did have negative net energy. Then, if a family of geodesics left a spacelike foliation at time $t = 0$ together, those that were in the infinitely distant (asymptotically flat) region would have an infinite

[4] Hawking [30] also contained a result suggesting that vacuum divergences along "fountains," closed null geodesic generators of the Cauchy horizon, would render the theory unphysical. Although Chrusciel and Isenberg [9] have proven that such fountains are not generic features of spacetimes with a compactly generated Cauchy horizon, Kay, Radzikowski and Wald [33] have shown that any spacetime with a compactly generated $H^+(S)$ does not admit Hadamard states at all. These results and many others argue that, even if they could be constructed, spacetimes containing time machines are inherently flawed. All results which follow from ANEC, our topic, address the question of construction.

lapse or phase delay relative to those that pass through the negative region. Thus, there are some regions at future times which are not causally related to every point at a sufficiently distant past. There must be null geodesics which thus form a boundary between causally unrelated regions, and these must pass through the matter sources present. One such geodesic must remain in the boundary region for infinite affine length. It would need to be achronal, because if it possessed conjugate points anywhere it would cease to be in the boundary of causally disconnected regions. Under the conditions of ANEC and genericness, however, such a geodesic cannot exist.

D. Incomplete Geodesics

Two classes of theorems require versions of achronal ANEC extended to incomplete or partial geodesics [26]: the singularity theorems of Galloway [23] and Roman [48, 49], and the superluminal communication restriction proven by Olum [41]. The singularity theorems show that if ANEC holds on an achronal geodesic originating on a closed trapped surface and extending to the future, the spacetime must contain a singularity. In [41] it was shown that a null geodesic leaving a flat surfaces could arrive at a future flat surface faster than other any other null geodesic only if the weak energy condition were violated, though it may easily be adapted to requiring non-negativity of achronal ANEC integrated between the two surfaces [26].

On the one hand it is easy to violate ANEC along incomplete geodesics, and counterexamples abound. The earlier mentioned example of Schwarzschild space contains ANEC violation along radial geodesics. A plate in the x-y plane has negative energy density in the $t - z$ null direction, and if the ANEC integral is taken starting a little bit above the plane extending to infinity it can be arbitrarily negative. A path with both start and end in a region containing negative energy due to the Casimir effect of distant plates has abundant negative energy. Thus as a general statement the modified ANEC clearly fails, but on the other hand a limitation to only consider incomplete geodesics with starting and ending points in regions that are Minkowskian seems quite reasonable. All these counterexamples would be eliminated if the integrals were extended into flat regions.

E. Modifying ANEC

ANEC on complete, achronal null geodesics does have known violations in the regime of conformally coupled scalar quantum field theory on a fixed background [52, 55]. Wald and Yurtsever [58] demonstrated a scaling argument which would require ANEC violation for scalar fields regardless of coupling, though not necessarily along achronal geodesics. No specific counterexamples to achronal ANEC in minimal coupling are presently known, though it may be necessary to modify ANEC from the formulation as stated originally. The primary avenue towards proof of ANEC appears to rely on quantum energy inequalities, first introduced by Ford [22] (see [18] for a comprehensive review.)

Fewster and Galloway [12] articulated a version directly motivated by the quantum energy inequalities. They prove versions of the Hawking and Penrose theorems based on the requirement that for a future complete null geodesic γ, the inequality

$$\int_0^\infty e^{-c\lambda} T_{ab} k^a k^b d\lambda - \frac{c}{2}. \tag{2.6}$$

is finite for some $c > 0$. This condition relaxes strict positivity and past completeness. This condition does not immediately follow from the energy inequalities either, but may be closer to them while still sufficient to rule out some exotic phenomena.

In [53], developing an idea introduced by Flanagan and Wald [21], we considered different schemes for transverse averaging. These energy conditions would be sufficient for many theorems, as a modification which only forbids exotic phenomena with some finite spatial extent are adequate. However, we constructed explicit counterexamples to all of them with a conformally coupled scalar field in a fixed background, as will be detailed in Chapter V.

The most reasonable modification of achronal ANEC is to restrict attention to self consistent spacetimes, as discussed in [21, 26, 45]. Here one requires not only that the quantum field present is a solution to the wave equation given a fixed background metric, but that its stress tensor (along with that that of any other matter) produce the curvature present. Simultaneously solving the Einstein and Klein Gordon equations is technically quite difficult. One may also proceed by considering the quantum fields present to be a perturbation on a classical background, as in [37]. The background is considered to be generated by classical matter, obeying the generic condition and achronal ANEC. With certain assumptions on

quantum energy inequalities, one can then prove ANEC holds in the next order of quantum effects on such a spacetime. In Section VII we describe a candidate for a reference state. The restricted consideration to self consistent metrics and fields is difficult to analyze, but it avoids all known counterexamples of other energy conditions while still allowing all proofs ruling out exotic phenomena to proceed.

III. QUANTUM FIELD THEORY

In this chapter we review quantum field theory, focusing on those aspects which are necessary for the further results, or may be unfamiliar. More complete treatments covering quantum field theory in curved space include the books by Birrell and Davies [3] and Wald [59].

A. Basic Constructions

The dynamics of the field operator ϕ are given by the Klein Gordon equation,

$$\left(\Box - m^2 - \xi R\right)\phi^2 = 0. \tag{3.1}$$

The parameter ξ gives the coupling to curvature. The choice $\xi = 0$ is minimal coupling, and $\xi = 1/6$ is conformal coupling. If $R = 0$ the choice of coupling does not effect the form of ϕ, but it does alter the stress tensor. Solutions are specified by weighting the sum over orthonormal modes

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3(2\omega)^{1/2}}\left(a_k e^{-ik_a x^a} + a_k^\dagger e^{ik_a x^a}\right). \tag{3.2}$$

Canonical quantization proceeds via the commutator conditions

$$\left[a_k, a_{k'}\right] = 0 \tag{3.3}$$

$$\left[a_k^\dagger, a_{k'}^\dagger\right] = 0 \tag{3.4}$$

$$\left[a_k, a_{k'}^\dagger\right] = \delta(k - k'). \tag{3.5}$$

The raising and lowering operators act on the vacuum state, and thereby construct a Fock space. The existence of a unique vacuum fails in curved space, and the sophistication of the algebraic construction is often necessary for this reason.

The classical stress tensor[5] is defined by

$$\begin{aligned} T_{ab} &= \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\left(\nabla_c\phi\nabla^c\phi + m^2\phi^2\right) \\ &\quad +\xi\left(g_{ab}\Box - \nabla_a\nabla_b - G_{ab}\right)\phi^2. \end{aligned} \tag{3.6}$$

To calculate the quantum mechanical stress tensor, one first requires the two point function, $\langle\phi^2\rangle$. In canonical flat space quantum theory, this is renormalized through the procedure of normal ordering, which is the dictum to order all creation operators to the left of annihilation operators. The normal ordering of an operator is denoted $\langle : A : \rangle$. Normal ordering of the stress tensor is equivalent to vacuum subtraction:

$$\langle\psi| : T_{ab} : |\psi\rangle = \langle\psi| T_{ab} |\psi\rangle - \langle 0| T_{ab} |0\rangle \tag{3.7}$$

B. Conformal Transformations

A conformal transformation is one which rescales the metric at all points by a function $\Omega^2(x)$. The various quantities in the new spacetime, denoted with a bar, are mapped from their values in the original spacetime. In this section, we work specifically in 4 spacetime dimensions.

$$\bar{g}_{ab} = \Omega^2 g_{ab} \tag{3.8}$$

$$\bar{R}^a_{\ b} = \Omega^{-2}R^a_{\ b} - 2\Omega^{-1}g^{ca}\nabla_c\nabla_b(\Omega^{-1}) + \frac{1}{2}\Omega^{-4}\Box(\Omega^2)\delta^a_b \tag{3.9}$$

$$\bar{R} = \Omega^{-2}R + 6\Omega^{-3}\Box\Omega. \tag{3.10}$$

[5] Some authors reduce this to the form

$$\begin{aligned} T_{ab} &= (1-2\xi)\nabla_a\phi\nabla_b\phi + \left(2\xi - \frac{1}{2}\right)g_{ab}\nabla_c\phi\nabla^c\phi - 2\xi\phi\nabla_a\nabla_b\phi \\ &\quad +\frac{1}{2}\xi g_{ab}\phi\Box\phi - \frac{1}{2}(1-3\xi)m^2 g_{ab}\phi^2 + \xi\left(G_{ab} + \frac{3}{2}\xi R g_{ab}\right)\phi^2 \end{aligned}$$

when ϕ is a solution to the equation of motion (on shell.) This reveals a ξ^2 dependence.

Conformal transformations preserve causal structure; that is, they map null geodesics to null geodesics. The affine parametrization and tangent vector are changed,[6]

$$d\bar{\lambda} = \Omega^2 d\lambda \tag{3.11}$$

$$\bar{l}^a = \Omega^{-2} l^a \tag{3.12}$$

For the particular choice of parameters $m = 0$ and $\xi = 1/6$, the scalar field is conformally invariant if we define $\bar{\phi} = \Omega^{-1}\phi$. This function $\bar{\phi}$ is a solution to the transformed Klein Gordon equation,

$$\left(\bar{\Box} - \frac{1}{6}\bar{R}\right)\bar{\phi} = 0. \tag{3.13}$$

This relationship is valid whether ϕ and $\bar{\phi}$ are thought of as classical solutions or the quantum mechanical field operator. We may also map a quantum state $|\psi\rangle$ to a state in the new spacetime, specifying the state with boundary conditions such that the propagators are conformally related,

$$\bar{G}(x, x') = \Omega(x)^{-1}\Omega(x')^{-1}G(x, x'). \tag{3.14}$$

It is particularly worth noting that if we take as our initial space and state the Minkowski vacuum, the transformation will not in general be to a vacuum of the new space. In the special case where the region affected by the transformation is only compactly different from Minkowski space (the function $\Omega^2 - 1$ has compact support), then the vacuum is transformed to the vacuum. This is because a conformally coupled field does not experience any particle creation moving through a conformally flat region of curvature.

The classical or unrenormalized stress tensor is also transformed with a simple conformal factor. However, renormalization breaks conformal invariance even when the action is

[6] These results are found in Appendix D of [57]. In general for a vector, we have $v^a\bar{\nabla}_a v^b = 2v^b v^c \nabla_c \ln\Omega - (g_{ac}v^a v^c)g^{bd}\nabla_d \ln\Omega$. The second term only vanishes for null geodesics, and the first only for the parametrization in eq. (3.11).

invariant. The anomalous contribution to the stress tensor[7] was found by Page [44]:

$$\begin{aligned}\bar{T}^a{}_b - \Omega^{-4}T^a{}_b &= -\frac{1}{480\pi^2}\Omega^{-4}\left[2\nabla^d\nabla_c(C^{ca}{}_{db}\ln\Omega) + R^d{}_c C^{ca}{}_{db}\ln\Omega\right] \\ &\quad -\frac{1}{5760\pi^2}\left[\left(4\bar{R}^d{}_c\bar{C}^{ca}{}_{db} - 2\bar{H}^a{}_b\right) - \Omega^2\left(4R^d{}_c C^{ca}{}_{db} - 2H^a{}_b\right)\right] \\ &\quad -\frac{1}{17280}\left[\bar{I}^a{}_b - \Omega^{-4}I^a{}_b\right]. \end{aligned} \tag{3.15}$$

The additional curvature tensors are

$$H_{ab} = -R^c{}_a R_{cb} + \frac{2}{3}RR_{ab} + \frac{1}{4}\left(2R^c{}_d R^d{}_c - R^2\right)g_{ab} \tag{3.16}$$

$$I_{ab} = 2R_{;ab} - 2RR_{ab} + \frac{1}{2}\left(R^2 - 4\Box R\right)g_{ab}. \tag{3.17}$$

The coefficient of the I_{ab} term in eq. (3.15) may be modified by adding terms proportional to R^2 to the action, though changes would not effect the character of our results.[8] If we specialize to the case where our initial spacetime is Minkowski space, the Weyl curvature and all unbarred curvatures vanish, and we have

$$\bar{T}_{ab} - \Omega^{-2}T^a_{ab} = \frac{1}{2880\pi^2}\left[\bar{H}_{ab} - \frac{1}{6}\bar{I}_{ab}\right] \tag{3.18}$$

$$\begin{aligned}\bar{T}_{ab} - \Omega^{-2}T^a_{ab} = \frac{1}{2880\pi^2}\Bigg[& -\bar{R}^c{}_a\bar{R}_{cb} + \bar{R}\bar{R}_{ab} - \frac{1}{3}\bar{R}_{;ab} \\ & + \frac{1}{6}\left(3\bar{R}^c{}_d R^d{}_c - 2\bar{R}^2 + 2\bar{\Box}\bar{R}\right)\bar{g}_{ab}\Bigg] \end{aligned} \tag{3.19}$$

Conformally flat spacetimes are non-generic, and all null geodesics are achronal, as these are properties of Minkowski space that are not changeable via local rescalings. Such a spacetime must violate the achronal null convergence condition, which requires that

$$\int_\gamma R_{ab}l^a l^b d\lambda \geq 0. \tag{3.20}$$

[7] This is not the same as what many sources call the conformal anomaly, also the trace anomaly. The conformal anomaly occurs when calculating the renormalized stress tensor in a curved space of any theory with a conformally invariant action, such as the scalar case here, a massless spin 1/2 system or sourceless electromagnetism. Although the classical stress tensor is always tracefree due to conformal symmetry, the trace of the renormalized stress tensor of the quantum theory is nonvanishing and depends on local curvature. These calculations make no reference to a conformal transformation between spaces. Both effects are due to the arbitrary symmetry breaking scale which renormalization procedures introduce. See §6.3 of [3]

[8] Our value corresponds to having no R^2 term in the action, but gives a contribution to the trace $T^a{}_a$. For other purposes, such as the work on the holography and ANEC recently undertaken by Nakayama [40], it is useful to introduce the counterterms necessary for the trace contribution to be set to zero.

C. Algebraic Formulation

The algebraic formulation of quantum field theory provides a powerful insight into the nature of quantum field theories. Haag [27] provides a thorough description. In curved spaces, especially those which lack a timelike Killing symmetry, it is essential. Wald [59] is a standard reference, and Kay [32] provides a recent review. Fewster [18] also introduces the algebraic formulation at some length, aiming towards deriving quantum energy inequalities. The algebraic approach to quantum field theory can provide powerful results in mathematical and axiomatic field theory in curved space, such as generalizations of the Spin-Statistics [54] and PCT [31] theorems, which were originally derived with methods inextricable from Poincare symmetry [50]. In this section we will not at all attempt to cover the full intricacies of the algebraic approach, but will focus on the role of Green's functions and the Hadamard condition for states.

Smeared local observable operators are taken as the fundamental objects, and they form an algebra, $\mathfrak{A}$. Quantization is achieved not through the canonical commutation relationships for raising and lowering operators, eq. (3.5), but via the field operator

$$[\phi(x), \phi(x')] = -iE(x, x') \tag{3.21}$$

or, more properly, its smeared variant, where $f(x)$ and $g(x)$ are functions of spacetime

$$[\phi(f), \phi(g)] = -iE(f, g). \tag{3.22}$$

E is known as the Pauli-Jordan or Lichnerowicz commutator, and is the advanced minus retarded Green's functions: $E = G_A - G_R$. This is equivalent to the Peierls bracket.[9] The familiar advanced (retarded) Green's functions satisfy the equations

$$(\Box - m^2 - \xi R)u(x) = f(x) \tag{3.23}$$

$$u(x) = \int G_{\mathrm{A,R}}(x, x') f(x') d^4(x') \tag{3.24}$$

subject to the boundary condition that $\psi(x) = 0$ for all points outside the causal past (future) of $f(x)$. Every classical solution with bounded support u may be written as $u = Ef$ for

[9] The Peierls bracket can be defined more generally, and take the role of the Poisson bracket in classical mechanics as well. See Haag [27] for the relationship to Poisson, and Marolf [38] for generalizations.

some initial data. To interpret a Green's function as a bidistribution, there is the additional smearing

$$E(f,g) = \int d^4x' E(x,x') f(x') g(x'). \tag{3.25}$$

The anticommutator distribution is defined in canonical quantum field theory for a state ψ

$$\mu(f,g) = \frac{1}{2} \langle \phi(f)\phi(g) \rangle_\psi + \langle \phi(g)\phi(f) \rangle_\psi \tag{3.26}$$

and has the following properties:

(a) Symmetry: $\mu(f,g) = \mu(g,f)$

(b) Weak Bisolution: $\mu(Pf,g) = \mu(f,Pg) = 0$, where $P = \Box - m^2 - \xi R$

(c) Positivity: $\mu(f,f) \geq 0$ and $\mu(f,f)\mu(g,g) \geq \frac{1}{4}(E(f,g))^2$

The algebraic approach utilizes a powerful existence result: if a function G satisfies the above criteria, there exists a quasifree[10] state ψ such that its two point function is given by

$$\langle \phi(f)\phi(g) \rangle_\psi = \mu(f,g) + \frac{i}{2} E(f,g). \tag{3.27}$$

In this way, states are defined as maps from operators to complex numbers:

$$\psi : \mathfrak{A} \rightarrow \mathbb{C} \tag{3.28}$$

$$A \rightarrow \langle A \rangle_\psi \tag{3.29}$$

The positivity condition above can be written in terms of the two point function rather than μ

$$\langle \phi(f)\phi(f) \rangle_\psi \langle \phi(g)\phi(g) \rangle_\psi \geq \frac{1}{4} |E(f,g)|^2 . \tag{3.30}$$

Written in this way, we note that the left side of the equation is state dependent, whereas the Green's functions are determined only by the form of the wave equation. But as E is the imaginary part of the two point function, this becomes

$$\langle \phi(f)\phi(f) \rangle_\psi \langle \phi(g)\phi(g) \rangle_\psi \geq \left(\mathrm{Im} \langle \phi(f)\phi(g) \rangle \right)^2 . \tag{3.31}$$

[10] A quasifree state is one which contains no new information in its n point functions for $n > 2$; all even higher degree point functions can be defined as products of the 2 point function, and all odd point functions vanish. In curved space field theories, the term "vacuum" is generally used to refer to quasifree states that are also pure (as in the preface of Wald [59])

If we knew that $\langle \bullet, \bullet \rangle$ was an inner product, this would be the Cauchy-Schwarz inequality. However, in the algebraic formulation, positivity of G must be taken as a *defining* property, and as a *result* we show that familiar states can be constructed.

We will also denote by μ the function defined on classical solutions, which are be written $u = Ef$ for some distribution f, via the obvious identification

$$\mu(u_1, u_2) = \mu(f_1, f_2). \tag{3.32}$$

IV. HADAMARD RENORMALIZATION

There are a variety of techniques used to renormalize quantum fields. In flat space, the standard technique is vacuum subtraction, which is equivalent to normal ordering. All states are represented as excitations of the Poincare invariant vacuum. This state is defined to have zero energy, with all other states measured with respect to. In a general curved space there is not a unique vacuum, even in spaces which do admit pure quasifree states that share many properties of the vacuum. Even in spacetimes that admit a natural sense of a vacuum (such as stationary spacetimes), this vacuum may be polarized and we would not wish to implement a renormalization procedure that, by fiat, sets $\langle T_{ab} \rangle = 0$. A variety of techniques, including counterterms, Pauli-Villars, zeta function and dimensional regularization, can render a smooth, finite stress tensor. Birrell and Davies [3] describe these, and provide references to the considerable scholarship in the seventies which proved that these procedures coincide, up to state independent, local, conserved curvature terms. The technique which is most suited to an algebraic perspective and to fully general spacetimes and which we use is Hadamard subtraction, and is detailed by Wald in [59].

In this procedure, the two point function of the state, $G(x, x')$, is renormalized by subtracting the Hadamard distribution $H(x, x')$. The Hadamard distribution is also known as the fundamental solution, and was introduced by Hadamard [28] in his study of the Cauchy problem. We denote the renormalized two point function by

$$F(x, x') = \langle \phi(x)\phi(x') \rangle_\psi - H(x, x'). \tag{4.1}$$

F is now smooth in the limit as $x \to x'$. The two point function is a bisolution to the

equation of motion for ϕ, and by following a particular method for constructing $H(x,x')$we can ensure that $F(x,x')$ is also. We obtain the renormalized stress tensor by performing the appropriate operations, eq. (3.6), to this renormalized two point function.

$$T_{ab}^{\mathrm{RN}} = \lim_{x\to x'} \Bigg[\nabla_a \nabla_{b'} - \frac{1}{2} g_{ab} \left(\nabla_c \nabla^{c'} + m^2 \right) + \xi \left(g_{ab'} \nabla_c \nabla^{c'} - \nabla_a \nabla_{b'} - G_{ab} \right) \Bigg] F(x,x'). \tag{4.2}$$

Hadamard's fundamental solution captures all of the possible short range singular behavior for the two point function in solutions to the wave equation. In a broad class of states, known as Hadamard states, there are no further singularities. This was conjectured by Kay and proven by Radzikowski [47]. All familiar states from a Fock space construction are Hadamard states, and in more general contexts it is reasonable to define the set of "physically reasonable" states as those which are Hadamard [32].[11]

Constructing and calculating explicit Hadamard functions can be more difficult than the existence of a formal procedure might suggest. Details are best provided by Garabedian [24] and the original by Hadamard [28]. We begin with an equation of motion for ϕ in its most general form

$$L[\phi] \equiv a_{ij} \frac{\partial^2 \phi}{\partial x^i \partial x^j} + b_k \frac{\partial \phi}{\partial x^k} + c\phi = 0. \tag{4.3}$$

The method for constructing the Hadamard solution outlined here may fail to be a solution in x', but this can be corrected afterward by adding the local curvature term necessary for $T_{\mu\nu}$ to be conserved. This is the same term which causes the conformal anomaly. For the massive Klein Gordon equation we have the special case of $a_{ij} = g_{ij}$, $b_k = 0$, $c = m^2$. We define σ as the square[12] of the geodesic distance between x and x', and let s be a parameter measuring length between them, running from 0 to $\sqrt{|\sigma|}$ Then

$$a_{ij} \frac{\partial \sigma}{\partial x^i} \frac{\partial \sigma}{\partial x^j} = 4\sigma \tag{4.4}$$

We also define $q = (n-2)/2$, where n is the number of spacetime dimensions involved.[13]

[11] A class of examples of non-Hadamard "states" are the α vacua in de Sitter space.
[12] Many references define $\sigma = \frac{1}{2} x^a x_a$.
[13] Mathematical references use m in place of q, which have reserved to denote mass.

Then, we can write

$$H = \frac{U}{A\sigma^q} + V \log \sigma + W \tag{4.5}$$

where U, V, W are all regular functions, and may be expressed as series in σ. For our purposes the series do not need to converge, because only two derivatives will be taken before the coincidence limit. The square of U is known as the van Vleck-Morette determinant, and $U_0 = 1$. When n is even, U may be truncated, and terms may be grouped into W. A is a constant that depends on the dimension, necessary to match the field normalization.[14] In 2+1 dimensions $A = 4\pi$ and in 3+1 $A = 4\pi^2$. Writing U, V, and W as series allows us to acquire recursive solutions. Using

$$C = \frac{1}{4} a_{ij} \frac{\partial^2 \sigma}{\partial x^i \partial x^j} + \frac{1}{4} b_k \frac{\partial \sigma}{\partial x^k} \tag{4.6}$$

We have

$$s\frac{dU_0}{ds} + (C - q - 1)U_0 = 0 \tag{4.7}$$

$$s\frac{dU_l}{ds} + (C - q - 1 + l)U_l = \frac{-1}{4(l-q)} L\left[U_{l-1}\right] \tag{4.8}$$

When n and hence q is odd, $(l - q)$ is always nonzero. When q is even, the series for U terminates at $l = q - 1$ and then V is given by

$$s\frac{dV}{ds} + (C - 1)V = -\frac{1}{4} L\left[U_{q-1}\right] \tag{4.9}$$

Some specific examples of $H(x, x')$ are calculated in Hadamard's lecture notes [28], including the telegraphist, cylindrical wave, and spherical wave equations, all in flat space. We will derive the massless Klein Gordon field in arbitrary dimensions, the constant mass and the transversely flat potential in 2+1 dimensions, which was first renormalized in [25]. For the latter, we will also calculate the renormalized energy density of the vacuum state, to compare with the result in the paper and show agreement. Although these are known to coincide with the standard vacuum two point function obtained by mode sums, we demonstrate their derivation with the above algorithm. We then give the general schematic for a spatially varying potential in 3+1 dimensions.

[14] Garabedian [24] simply has $A = 1$ as he investigates the singularity structure of $H(x, x')$ without any attention to quantum field theory. The overall constant is set for all spaces of the same dimension by comparison to the Minkowski vacuum.

A. Massless field in n-dimensional Minkowski space

For the massless Klein Gordon field

$$C = \frac{n}{2} \tag{4.10}$$

Thus, eq. (4.7) becomes

$$s\frac{dU_0}{ds} = 0. \tag{4.11}$$

This yields a constant value for U_0, which we take to be 1. Subsequently, all U_l must, for $l > 0$, satisfy

$$s\frac{dU_l}{ds} + lU_l = L\left[U_{l-1}\right]. \tag{4.12}$$

The first one is given by $U_1 = cs^{-1}$, and so the constant must be set to zero in order to have U_1 nonsingular. Likewise, each subsequent U_l must also be zero.

If n is odd we have a V term, but it is given by

$$s\frac{dV}{ds} + (C-1)V = 0 \tag{4.13}$$

which has solution

$$V = cs^{1-C}. \tag{4.14}$$

Because V must be nonsingular, the only acceptable value for the integration constant is $c = 0$. Thus we have $H \sim \sigma^{-q}$, with the proportionality constant undetermined by this method.

B. Massive field in (2+1) dimensional Minkowski space

For a massive field, we have the same C, and thus the same constant term for U_0. We can set it to unity. But now, $L\left[U_0\right] = -m^2$. Then the next term in the series is given by

$$s\frac{dU_1}{ds} + U_1 = \frac{1}{2}m^2 \tag{4.15}$$

which has solution

$$U_1 = \frac{1}{2}m^2 + \frac{c}{s}. \tag{4.16}$$

Again the free constant c must be set to zero to maintain regularity. Next, we must solve

$$s\frac{dU_2}{ds} + 2U_2 = \frac{m^4}{12} \tag{4.17}$$

which has solution

$$U_2 = \frac{m^4}{24} + \frac{c}{s}. \tag{4.18}$$

Once more we set the constant is set to zero, and the pattern continues, with

$$U_l = \frac{(m^2/2)^l}{l!(2l-1)!!} \tag{4.19}$$

where !! denotes the double factorial $(2l-1)(2l-3)(2l-5)...$ and not the iterated factorial. Then, taking the sum $U = U_l s^l = \cosh(ms)$. This gives for $H(x, x')$

$$H(x, x') = \frac{\cosh(ms)}{4\pi s} \tag{4.20}$$

C. Stress tensor for a transversely varying potential in 2+1 dimensional Minkowski space

Here we treat the case with a potential $Q(x)$ that does not depend on y. This aims to recapture the result of Graham and Olum [25], who used dimensional regularization rather than Hadamard subtraction. The momentum component in the y direction is denoted p, and in the x direction k. We start by calculating F. Then the mode sum will be

$$\begin{aligned} \langle \phi^2 \rangle &= \left\langle \phi_+^\dagger(x)\phi_+(x') \right\rangle + \left\langle \phi_-^\dagger(x)\phi_-(x') \right\rangle && (4.21) \\ &= \int \frac{dpdk}{(2\pi)^2\omega} e^{ip\Delta y} \left[\psi_-^\dagger(k,x)\psi_-(k,x') + \psi_+^\dagger(k,x)\psi_+(k,x') \right] && (4.22) \end{aligned}$$

We do the integral in p first.

$$\int_{-\infty}^{+\infty} \frac{dp}{\sqrt{p^2+k^2}} e^{ip(y-y')} = 2K_0(\sqrt{k^2(y-y')^2}) \tag{4.23}$$

We note that $2K_0(k\Delta y)$ in the limit of small Δy is $-\ln(k\lambda)^2$, plus terms which do not depend on k and vanish due to the properties of orthogonal functions. Here λ is an arbitrary length. Now we express the mode sum in terms of the Green's function

$$\psi_-^\dagger(k,x)\psi_-(k,x') + \psi_+^\dagger(k,x)\psi_+(k,x') = 2k \operatorname{Im} G(x,x',k) \tag{4.24}$$

which satisfies

$$-G''(x,x',k) + \left(Q(x) - k^2\right) G(x,x',k) = \delta(x-x') \tag{4.25}$$

with only outgoing waves at infinity. This gives

$$\left\langle \phi^2 \right\rangle = -\int_0^\infty \frac{dk}{2\pi^2} k \operatorname{Im} G(x,x',k) \ln k^2\lambda^2 \tag{4.26}$$

Because $G(x,x',-k) = G(x,x',k)^\star$ we can extend the integral to $-k$ as well. Then

$$\left\langle \phi^2 \right\rangle = \int_{-\infty}^\infty \frac{dk}{4\pi^2} ikG(x,x',k) \ln k^2\lambda^2 \tag{4.27}$$

This can be extended to a contour at infinity in the upper half plane which only has contributions along a branch cut on the positive imaginary axis. Off this axis the integrand goes to zero for large magnitudes of complex momentum. In the case of either a massless field or a repulsive $Q(x)$ there will not be any poles. The logarithms cancel leaving a constant term of $2i\pi$. There are four factors of i: that already present, one from each $k = i\kappa$ and one from the $2i\pi$ so there is no sign change.

$$\left\langle \phi^2 \right\rangle = \int_0^\infty \frac{d\kappa\kappa}{2\pi} G(x,x',k) \tag{4.28}$$

We want to renormalize using the Hadamard subtraction. For a massive field with no potential, the Hadamard distribution was found to be $(4\pi s)^{-1}\cosh(ms)$, eq. (4.20). For the massless field with square barrier it is the same, but with $\sqrt{Q}$ in the role of m. Since the Hadamard function only depends on the local conditions at (x,y), this should be the case for any suitably flat potential. We use an integral identity to split

$$H(x,x') = \frac{\cosh(\sqrt{Q}s)}{4\pi s} = \int_0^\infty \frac{d\kappa}{4\pi}\frac{\kappa}{\kappa'} e^{-\kappa' s} + \frac{\sinh(\sqrt{Q}s)}{4\pi s} \tag{4.29}$$

which isolates the divergent portion as an integral. For concise notation we introduced $\kappa' = \sqrt{\kappa^2 + Q}$. The integrand is however not divergent as $s \to 0$, and neither is the function $\sinh(s)/s$.

Together with the mode sum, eq. (4.28), we have the renormalized two point function

$$F = \left\langle \phi^2 \right\rangle - H = \frac{1}{4\pi}\int_0^\infty d\kappa \left[2\kappa G(x,x',k) - \frac{\kappa}{\kappa'} e^{-\kappa' s}\right] - \frac{\sinh(\sqrt{Q}s)}{4\pi s}. \tag{4.30}$$

Using this in the definition

$$T_{\mu\nu} = \lim_{x\to x'} \left[\nabla_\mu \nabla_{\nu'} - \frac{1}{2} g_{\mu\nu'} \left(\nabla_\alpha \nabla^{\alpha'} + Q \right) \right] F(x, x') \tag{4.31}$$

gives

$$\langle \mathcal{H} \rangle = T_{tt'} = \lim_{x\to x'} \frac{1}{2} \left[F_{,tt'} + F_{,xx'} + F_{,yy'} + QF \right]. \tag{4.32}$$

We first show that the $\partial_{yy'} \langle \phi^2 \rangle + \partial_{tt'} = 0$. We consider the boost of $K_0(k\Delta y)$ to $K_0(k\sigma)$, with $\sigma = \sqrt{\Delta y^2 - \Delta t^2}$. We consider, for some *general* coordinates x, x' and z a function $z(s)$

$$\begin{aligned}
\frac{\partial^2 f(z)}{\partial x \partial x'} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x'} \partial_z f(z) \right) && (4.33)\\
&= \frac{\partial^2 z}{\partial x \partial x'} \partial_z f(z) + \frac{\partial z}{\partial x} \frac{\partial z}{\partial x'} \partial_z^2 f(z) && (4.34)\\
&= \frac{\partial s}{\partial x \partial x'} \frac{\partial z}{\partial s} \partial_z f(z) + \frac{\partial s}{\partial x} \frac{\partial s}{\partial x'} \left[\frac{\partial^2 z}{\partial s^2} \partial_z f(z) + \left(\frac{\partial z}{\partial s} \right)^2 \partial_z^2 f(z) \right]. && (4.35)
\end{aligned}$$

This is simply the chain rule. In the first of these, we note that $\partial_{xx'} s = -2 g_{xx'}$. Since we are considering $\partial_{yy'} + \partial_{tt'}$ we will have these cancel exactly.[15] The other term is direction dependent, with $\partial_x s = \hat{x}/\sqrt{s}$, but we will see it does not contribute either. Our particular functions are $f = K_0(z)$ and $z = k\sqrt{s}$, and thus the direction dependent part becomes

$$\hat{x}\hat{x}' \sigma \left[\frac{-k}{4\sigma^{3/2}} K_0'(z) + \frac{k}{4\sigma} K_0''(z) \right]. \tag{4.36}$$

The derivatives of the Bessel function

$$\begin{aligned}
\frac{\partial}{\partial z} K_0(z) &= -K_1(z) && (4.37)\\
\frac{\partial}{\partial z} K_1(z) &= -K_0(z) - z^{-1} K_1(z) && (4.38)
\end{aligned}$$

combine to give

$$\hat{x}\hat{x}' \frac{k}{4\sqrt{\sigma}} \left[2K_1(z) + zK_0(z) \right]. \tag{4.39}$$

Now we take the small σ and hence small z expansions

$$\begin{aligned}
2K_1 &\approx \frac{2}{z} + z \left(\ln(z) - \ln(2) + \gamma - \frac{1}{2} \right) && (4.40)\\
K_0 &\approx -z \left(\ln(z) - \ln(2) + \gamma \right). && (4.41)
\end{aligned}$$

[15] In four dimensions such a transverse component would survive.

This gives for the directional dependent part

$$4\hat{x}\hat{x}'\left[\frac{1}{2s}-\frac{k^2}{8}+\mathcal{O}(s)\right] \tag{4.42}$$

The first term diverges as s^{-1} but does not depend on k so it vanishes on integration, by completeness. The second term also vanishes upon the contour integration in k, and all other terms drop as $s \to 0$. Thus, the transverse and time derivative parts of the unrenormalized energy density vanish entirely, and we are left with

$$\langle \mathcal{H}_0 \rangle = \frac{1}{2}\left[Q + \partial_{xx'}\right]\left\langle \phi^2 \right\rangle \tag{4.43}$$

This is, in terms of the Green's function

$$\langle \mathcal{H}_0 \rangle = \frac{1}{4\pi}\int d\kappa\kappa\left(QG(x,x',k)+\frac{\partial^2}{\partial x\partial x'}G(x,x',k)\right) \tag{4.44}$$

To take the $x \to x'$ limit, we write $\kappa G(x,x')$ as $\psi(x)\psi(x')$ and suppress writing the sum over modes. Then we have

$$\lim_{x\to x'}\frac{\partial^2}{\partial x\partial x'}\psi(x)\psi(x') = \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x)^2 - \psi''(x)\psi(x). \tag{4.45}$$

Because ψ is a solution to the equations of motion, this becomes

$$\lim_{x\to x'}\frac{\partial^2}{\partial x\partial x'}\psi(x)\psi(x') + Q\psi(x)^2 = \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x)^2 - \kappa^2\psi(x)^2. \tag{4.46}$$

Altogether this gives

$$\langle \mathcal{H}_0 \rangle = -\frac{1}{8\pi}\int d\kappa\kappa\left((2\kappa^2)G(x,x',k) - \frac{\partial^2}{\partial x\partial x}G(x,x',k)\right). \tag{4.47}$$

Now we calculate the derivatives of $H(x,x')$. The derivatives of the exponential give $-\kappa'^2$, and $-\kappa'^2 + V = -\kappa^2$, so this term enters neatly. For the sinh part, we have

$$\lim_{s\to 0}\frac{1}{2}\frac{\partial^2}{\partial x\partial x'}\frac{\sinh(\sqrt{Q}s)}{4\pi s} = \frac{-1}{8\pi}\frac{Q^{3/2}}{3}. \tag{4.48}$$

The y, y' contribution is the same, and t, t' differs by a sign. Thus

$$\lim_{s\to 0}\frac{Q\sinh(\sqrt{Q}s)}{9\pi s} = \frac{1}{8\pi}Q^{3/2}. \tag{4.49}$$

This means the total contribution of the sinh term is $\frac{-Q^{3/2}}{12\pi}$ and we thus have

$$\langle \mathcal{H}_0 \rangle = -\frac{1}{8\pi} \int d\kappa \left((2\kappa^3) G(x, x', k) - \frac{\kappa^3}{\kappa'} - \kappa \frac{\partial^2}{\partial x \partial x} G(x, x, k) \right) - \frac{2Q^{3/2}}{12\pi}. \tag{4.50}$$

Note that for large κ,

$$\frac{\kappa}{\kappa'} \approx 1 - \frac{Q}{2\kappa^2} + \frac{3Q^2}{8\kappa^4} \tag{4.51}$$

In eq. 15 of [25], only the first two terms in this expansion are present. The difference is upon integration precisely the constant that resulted from the sinh,

$$\int_0^\infty d\kappa \kappa^2 \left[\frac{\kappa}{\kappa'} - 1 + \frac{Q}{2\kappa^2} \right] = \frac{2Q^{3/2}}{3}, \tag{4.52}$$

and thus our Hadamard renormalization manifestly agrees with the dimensional regularization.

D. Schematic for a general Q in 3+1 dimensions

The Hadamard form will be

$$H = \frac{U}{\sigma} + V \log \sigma + W. \tag{4.53}$$

As before, the quantity $C - q - 1 = 0$ depends only dimension, and $U_0 = 1$. Because $l - q = 0$ for $l = 1$, U_0 is the only term in the series for U that exists.

Then, we go on to calculate V:

$$V_0 = -\frac{U_0}{4s} \int_0^s \frac{L[U_0]}{U_0} ds \tag{4.54}$$

In our case this is simply

$$V_0 = -\frac{1}{4s} \int_x^y Q ds \tag{4.55}$$

$$= -\frac{1}{4s} \int_0^s Q ds \tag{4.56}$$

This is symmetric, written as an integral. To first order, this is $-\bar{Q}/4$, where $\bar{Q} = (Q(x) + Q(y))/2$.

The next term in V is

$$V_1 = -\frac{1}{4s^2} \int_0^s s L[V_0] ds \tag{4.57}$$

This is

$$V_1 = -\frac{1}{16s^2}\int_0^s s\left(\Box Q + Q^2\right) ds \tag{4.58}$$

$$= \frac{1}{8s^2}\left(\Box Q + \bar{Q}^2\right) \tag{4.59}$$

So, we have:

$$H(x,x') = \frac{1}{4\pi^2}\left(\frac{1}{\sigma} + \left(\frac{\bar{Q}}{4} + \frac{\Box Q + \bar{Q}^2}{8}\sigma\right)\ln(\bar{Q}\sigma)\right). \tag{4.60}$$

We will use this to renormalize the recently massless vacuum in the last section. The $\sigma \ln \sigma$ terms have a vanishing contribution to the two point function, but they are singular and need to be explicitly subtracted when calculating the stress tensor.

E. Curved Space

The Hadamard form in curved space is given in series, as a reference. DeWitt and Brehme [11] established this approach to renormalization, and found for a massless field with minimal coupling

$$4\pi^2 H(x,x') = \frac{\left(1 - \frac{1}{6}R^{ab}\sigma_{,a}\sigma_{,b}\right)^{1/2}}{\sigma^2} + \frac{1}{12}R\ln|\sigma| + \ldots \tag{4.61}$$

Christensen[16] [7] finds, for a massive field with arbitrary coupling

$$\begin{aligned} 4\pi^2 H(x,x') &= \frac{1}{\sigma^2} + \left[m^2 - \left(\frac{1}{6} - \xi\right)R\right]\left[\gamma + \frac{1}{2}\ln\left|\frac{1}{2}m^2\sigma^2\right|\right] - \frac{1}{2}m^2 + \frac{1}{12}R_{ab}\frac{x^a x^b}{\sigma^2} \\ &+ \frac{1}{4m^2}\left[\left(\frac{1}{6} - \xi\right)^2 R^2 + \frac{1}{90}\left(R^{abcd}R_{abcd} - R^{ab}R_{ab}\right) + \frac{1}{3}\left(\frac{1}{5} - \xi\right)\Box R\right] \\ &+ \mathcal{O}(m^{-4}). \end{aligned} \tag{4.62}$$

Note that this contains a term which depends on the direction of the vector x^a which connects x to x'. Christensen's result is a series in inverse powers of the mass, and so the massless case must be treated separately and not as a limit. We will not invoke these curved space forms further, but include them for completeness.

[16] Christensen further considers fields of higher spin, giving a unified treatment [8].

V. PROOFS OF ANEC

We here summarize the current status of proofs of the average null energy condition, based on the properties of quantum field theory.[17] Klinkhammer [35] proved ANEC in flat space using the plane wave decomposition. This requires the full spacetime to be Minkowski. Wald and Yurtsever [58] present their proof in full Minkowski as well, but their techniques only require Fourier transforms along the null geodesic over which the ANEC integral is taken. This philosophy is adopted in the proof by Fewster, Olum and Pfenning [13] for geodesics in a tubular region of flat space, which permits curvature some finite distance away. As we have found counterexamples to ANEC in the regime of test fields on a fixed background for the conformally coupled scalar field, it is likely that the much anticipated proof for curved spaces may require limiting to minimal coupling, or restriction to self consistent spacetimes and fields. Flanagan and Wald [21] considered perturbations of flat space, finding no violations of achronal ANEC. Kontou and Olum [37] have presented a proof for achronal ANEC for fields which exist on more general classical backgrounds, which relies on a conjectured quantum energy inequality.

A. Flat Space

For reference, the stress tensor, given earlier at eq. (3.6), is

$$\begin{aligned} T_{ab} &= \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\left(\nabla_c\phi\nabla^c\phi + m^2\phi^2\right) \\ &\quad +\xi\left(g_{ab}\Box - \nabla_a\nabla_b - G_{ab}\right)\phi^2. \end{aligned} \tag{5.1}$$

It takes a much simpler form as a contribution to the flat space ANEC integral. Curvature dependent terms all vanish in flat space. Upon null projection, all terms whose tensor structure is given by the metric vanish. Thus we have

$$T_{ab}l^al^b = \left[\nabla_a\phi\nabla_b\phi - \xi\nabla_a\nabla_b\left(\phi^2\right)\right]l^al^b. \tag{5.2}$$

[17] Wall has presented a proof of ANEC as a corollary to a certain generalized second law of thermodynamics. It would then remain to establish the range of conditions under which that generalized second law holds [60].

When we consider ANEC, we will integrate over a complete geodesic. The ξ dependent term here is a total derivative, and so will not contribute at all. Thus, the flat space ANEC integral simply becomes

$$\int d\lambda T_{ab} l^a l^b = \int d\lambda \nabla_a \phi \nabla_b \phi l^a l^b. \tag{5.3}$$

Note in particular that all explicit dependence on ξ has dropped. Furthermore the equation of motion, which determines the normal modes and thus the construction of states, does not depend on ξ, and therefore all flat space results are independent of coupling. If this were a classical theory with ϕ treated as a simple wave function solution, the above would be $\int (\phi')^2$, a manifestly positive quantity. The case in quantum field theory requires more finesse.

Klinkhammer [35] proceeds with the decomposition into normal modes using the definition of the field operator in terms of raising and lowering operators, eq. (3.2):

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 (2\omega)^{1/2}} \left(a_k e^{-ik_a x^a} + a_k^\dagger e^{ik_a x^a} \right). \tag{5.4}$$

With this, we have

$$\begin{aligned} \int d\lambda T_{ab} l^a l^b = \int d\lambda \int \frac{d^3k d^3k'}{2(2\pi)^6 \sqrt{\omega_k \omega_{k'}}} \left(-i a_k k_a e^{-ik_a x^a} + i a_k^\dagger k_a e^{ik_a x^a} \right) \\ \times \left(-i a_{k'} k'_a e^{-ik'_a x'^a} + i a_{k'}^\dagger k'_b e^{-ik'_b x'^b} \right) l^a l^b. \end{aligned} \tag{5.5}$$

We must renormalize, which we achieve via normal ordering of the operators a and $a^\dagger$

$$\begin{aligned} \int d\lambda : T_{ab} : l^a l^b = \int d\lambda \int \frac{d^3k d^3k'}{2(2\pi)^6 \sqrt{\omega_k \omega_{k'}}} \Big(2 a_k^\dagger a_{k'} e^{i(k_a x^a - k'_b x^b)} \\ - a_k a_{k'} e^{-i(k_a x^a + k_b x^x)} - a_k^\dagger a_{k'}^\dagger e^{+i(k_a x^a + k_b x^x)} \Big) k_a l^a k'_b l^b. \end{aligned} \tag{5.6}$$

The first term is a product of $\int d^k k_a l^a a^\dagger e^{-ik_a x^a}$ and its complex conjugate, and thus it is manifestly positive.

To analyze the other two terms, we write $l^a = \lambda(-e^0 + e^1)$, for timelike and spacelike basis elements 0 and 1. We then perform the integral in λ first

$$\int d\lambda e^{i\lambda(k_a + k_b)(-e^0 + e^1)} = \delta(-\omega_k - \omega_{k'} + k_1 + k'_1). \tag{5.7}$$

For a massive field, ω_k is strictly greater than the first component of momentum, and so this delta function never has support. For a massless field, it is possible that k could lie

completely in the $(0,1)$ direction and thus satisfy the delta function, but then $k_a l^a = 0$ and the entire ANEC integral is equal to zero. Thus, the only possible contribution is the positive one.

Wald and Yurtsever's proof of ANEC [58] hinges on the positivity condition, eq. (3.30), obeyed by every state in the algebraic construction of a field theory. In particular we will show to extract ANEC, which appears to be highly nontrivial, from the basic assumption of positivity,

$$\langle \phi(f)\phi(f)\rangle_\psi \langle \phi(g)\phi(g)\rangle_\psi \geq \frac{1}{4}\left|E(f,g)\right|^2. \tag{5.8}$$

We express the positivity condition using classical solutions $u = Ef$ rather than test distributions, giving the form

$$\mu(u_1,u_1)\mu(u_2,u_2) \geq \frac{1}{4}\left|\sigma(u_1,u_2)\right|^2 \tag{5.9}$$

where

$$\sigma(u_1,u_2) = \int \left(u_1 \nabla^a u_2 - u_2 \nabla^a u_1\right) d\Sigma. \tag{5.10}$$

This symplectic inner product is independent of the choice of Cauchy surface Σ. We restrict attention to those which coincide with the null geodesic γ for some compact range, while spacelike apart from this. The class of solutions considered are those u which have compact support, and whose restrictions u_0 to the surface Σ are C^∞. It is proven in [34] that the set of functions μ under consideration does not fail to include the two point function of any Hadamard states that can be defined on the spacetime, and thus may be used for ease of proof.

We divide the real part of the two point function, μ, into the vacuum component μ_0 and a remaining portion, w. We establish coordinates, with v in the direction tangent to γ and (s_1, s_2) spacelike orthogonal to γ. Note that the null derivatives of w give the desired term in the ANEC integral,

$$\frac{\partial^2}{\partial v \partial v'} w(v,s,v',s') = T^{RN}_{ab} l^a l^b \tag{5.11}$$

Appendix B of [34] shows that the positivity condition can be recast as

$$\left[2\mu_0(u_{01},u_{01}) + 8\int_{\Sigma\times\Sigma} \partial_v \partial_{v'} w(v,s,v',s') u_{01}(v,s) u_{01}(v',s') dv d^2s dv' d^2s'\right] \times \left[\text{with } \ u_{02}\right] \geq \left|\sigma(u_1,u_2)\right|^2. \tag{5.12}$$

We use as our convention for the Fourier transform

$$\hat{u}(k) = \int e^{-ikv} u(v) dv. \tag{5.13}$$

Transforming in the v direction, it can be shown that

$$\mu_0(u_{01}, u_{01}) = 4\pi \int \int_0^\infty k \left|\hat{u}_{01}\right|^2 dk d^2 s \tag{5.14}$$

$$\sigma(u_1, u_2) = -8\pi \operatorname{Im} \int \int_0^\infty k \hat{u}_{01}^*(k,s) \hat{u}_{02}(k,s) dk d^2 s \tag{5.15}$$

In Klinkhammer's proof, the mode decomposition effectively requires a Fourier transform over the entire spacetime. The method of Wald and Yurtsever only transforms along the geodesic γ, and thus can more easily be generalized. With these identities, we now have

$$\left[8\pi \int \int_0^\infty k \left|\hat{u}_{01}\right|^2 dk d^2 s + 8 \int_{\Sigma\times\Sigma} \partial_v \partial_{v'} w(v,s,v',s') u_{01}(v,s) u_{01}(v',s') dv d^2 s dv' d^2 s\right] \times \left[\text{with } \ u_{02}\right] \geq 64\pi^2 \left|\operatorname{Im} \int \int_0^\infty k \hat{u}_{01}^*(k,s) \hat{u}_{02}(k,s) dk d^2 s\right|^2 . \tag{5.16}$$

Thus we have an inequality involving the ANEC integrand, $\partial_v \partial_{v'} w$, and Fourier transforms of arbitrary functions. Wald and Yurtsever [58] proceed to isolate the ANEC integral, and use properties of Fourier analysis to show that it must be positive.

B. Flat Tube, Boundary Conditions

Fewster, Olum and Pfenning [13] present a proof of ANEC for any geodesic γ in a spacetime $\mathcal{N}$ which contains an open neighborhood around it that is flat (locally isometric to Minkowski space) which has some minimum finite radius. We denote the region $\mathcal{N}'$, and this radius of the minimal tube r. Outside of the tube there may be any sort of curvature so long as it does not alter the causal structure of γ, which must be achronal, and must be isometric to some subregion of Minkowski space (that is, we can isometrically map $\mathcal{N}' \to \mathcal{M}$).

The result used to limit the quantum field theory is a quantum energy inequality, or QEI, for timelike geodesics proved as Theorem III.1 of Fewster and Roman [14], which was based on the general worldline quantum inequality of Fewster [16]. Fewster presents a general overview of QEIs is in [18]. Fewster, Olum and Pfenning [13] consider a timelike geodesic

η characterized by proper time τ. The vector field l^a is null, *not* a timelike tangent to η, which distinguishes it from both ANEC and AWEC. They prove

$$\int_\eta d\tau \left(\left\langle T^{\mathrm{RN}}_{ab} \right\rangle_\psi - \left\langle T^{\mathrm{RN}}_{ab} \right\rangle_{\psi_0} \right) l^a l^b g(\tau)^2 \geq -2 \int_0^\infty \hat{F}(\alpha, -\alpha) \tag{5.17}$$

where

$$F(\tau, \tau') = g(\tau) g(\tau') \left\langle l^a \nabla_a \phi(\tau) l^b \nabla_b \phi(\tau') \right\rangle_{\psi_0}, \tag{5.18}$$

the state ψ_0 is some reference state, and g is some sampling function with normalization $\int g^2(\tau/\tau_0) d\tau = \tau_0$. We choose a family of timelike geodesics η_v, with each one characterized by a boost velocity v. Comparing to the Minkowski vacuum state gives

$$\int_{\eta_v} d\tau \left\langle T^{\mathrm{RN}}_{ab} \right\rangle_\psi l^a l^b g(\tau/\tau_0)^2 \geq -\frac{(l_a k^a)^2}{12\pi^2 \tau_0^4} \int_{\eta_v} d\tau g''(\tau/\tau_0)^2, \tag{5.19}$$

where Parseval's theorem converted the Fourier transforms to second derivatives of the sampling function. Given certain geometrical caveats [13], the limit to a null geodesic will converge, and $(l_a k^1)^2/\tau_0^4 \to 0$ while $\int g''$ remains finite. Thus, the quantity on the left, which converges to the ANEC integral, must be greater than some quantity arbitrarily close to zero.

C. Self Consistency

There are known violations of ANEC within the regime of conformally coupled quantum field theory on an arbitrary fixed background [52, 53]. It has been proposed [26, 45] to limit consideration to self consistent pairs of spacetimes and stress tensors. An exact semiclassical solution would require one to couple the Einstein and Klein Gordon equations:

$$G_{ab} = 8\pi G \langle T_{ab} \rangle \tag{5.20}$$

$$\left(\Box - m^2 - \xi R \right) \phi = 0. \tag{5.21}$$

The field ϕ must solve a wave equation set by the metric g_{ab}, and the stress tensor of the particular state $|\psi\rangle$ it is in must generate the curvature of the metric. Simultaneously solving the Einstein equation and classical dynamical equations for even simple fluids is a difficult task, and the quantum regime is worse still.

Flanagan and Wald undertook a perturbative analysis of quantum backreaction in flat space [21] of a free, massless scalar field with arbitrary coupling. They found no violations of achronal ANEC at first order. At second order the perturbations studied in general would violate the generic condition, eq. (2.5), and therefore induce chronality. They did find first order violations of ANEC in the case of incoming mixed states, though the chronality of these geodesics was not addressed. Thus, there is no explicit counterexample to achronal, self consistent ANEC.

Kontou and Olum [37] begin with an asymptotically flat background metric that is considered classical; i.e., it obeys the generic condition and is sourced by matter that (near γ, an achronal geodesic) obeys the null energy condition (NEC). This latter statement is equivalent to saying the background metric obeys the null convergence condition

$$R_{ab}k^a k^b \geq 0. \tag{5.22}$$

It must also be required that curvature components be bounded, or else we would be in the regime of quantum gravity. These bounds only need to be kept below the Planck regime. This cannot be stated purely in terms of invariant measures like the Ricci scalar; instead, we require that each component $|R_{abcd}| < R_{\max}$, for some maximum curvature value. These components are calculated with respect to a generalization of Fermi normal coordinates, adapted for the region surrounding the geodesic [36]. This makes the observer who travels along the null geodesic calculating the ANEC integral a privileged observer, but it appears quite natural. Likewise the first and second derivatives of the Riemann tensor with respect to every coordinate must be less than some values $R'_{\max}$ and $R''_{\max}$.

We then consider the expectation value of the stress tensor any quantum state which may then live on this background, $\langle T_{ab}\rangle_\psi$. If such a stress tensor obeys a modified form of eq. (5.19)

$$\int_{-\tau_0}^{+\tau_0} d\tau \left\langle T^{\mathrm{RN}}_{ab}\right\rangle_\psi l^a l^b g(\tau/\tau_0)^2 \geq -\frac{(l_a k^a)^2}{12\pi^2\tau_0^4}\int_{-\tau_0}^{+\tau_0} d\tau g''(\tau/\tau_0)^2\left[1 + c(R_{\max}\tau_0^2)\right], \tag{5.23}$$

where $c(R_{\max}\tau_0^2) \to 0$ as $\tau_0 \to 0$, then they prove ANEC holds in such a spacetime, using a parallelogram construction in the tubular neighborhood of the geodesic similar to the approach in [13].

VI. COUNTEREXAMPLES

We presented two classes of counterexamples to ANEC in [52], within the regime of conformally coupled scalar fields in a fixed conformally and asymptotically flat background. The first example chooses a particular fixed metric and shows that there then exist states with arbitrarily negative ANEC values. The second example exhibits conformal factors such that the transformation of the Minkowski vacuum state will be to one with a negative ANEC integral, due to the anomalous transformation of the stress tensor.

A. State Dependent Violation

We construct our specific violation of ANEC as a conformal transformation of Minkowski space. In flat space, we choose a state which obeys ANEC, violating NEC in some places but compensating with a positive contribution in others. The conformal transformation enhances the contribution to the integral in those places where NEC is violated, so that the overall integral is negative in the transformed spacetime.

We let our transformed metric be $\bar{g}_{ab} = \Omega^2(x)\eta_{ab}$. The stress-energy tensor then transforms as eq. (3.18)

$$\bar{T}_{ab} = \Omega^{-2}T_{ab} + \text{anomaly}. \tag{6.1}$$

The anomalous contribution depends only on local curvature terms and is finite. A null geodesic remains a null geodesic under a conformal transformation, but the parametrization is no longer affine. The new affine parametrization is given by $d\bar{\lambda} = \Omega^2 d\lambda$, eq. (3.11), and $\bar{l}^a = (dx^a/d\bar{\lambda}) = \Omega^{-2}l^a$ The ANEC integral then becomes

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^b d\bar{\lambda} = \int \Omega^{-4}T_{ab}l^a l^b d\lambda + \text{anomaly}\,. \tag{6.2}$$

For a given conformal transformation, we will exhibit a sequence of states in which the nonanomalous term becomes *arbitrarily* negative. Thus, even if the anomalous term is positive, there are states which overcome it and make the ANEC integral negative.

Consider a geodesic γ as above and a smooth conformal transformation $\Omega(x)$, with the

properties that

$$\Omega(\gamma(\lambda)) \leq 1 \,\text{everywhere on } \gamma \tag{6.3}$$

$$\Omega(\gamma(\lambda)) \text{ is bounded from below by some } \epsilon > 0 \tag{6.4}$$

$$\Omega(\gamma(\lambda)) \text{ differs from 1 on a non-empty compact set of } \lambda \tag{6.5}$$

The conformal transformation shrinks the spacetime by some bounded amount over some limited range of the geodesic. For parsimonious notation we define $g(\lambda) = \Omega(\gamma(\lambda))^{-4}$ and $f(\lambda) = g(\lambda) - 1$, and f will then be smooth, bounded, and of compact support.

The ANEC integral in the conformally flat spacetime is

$$\int \bar{T}_{ab}\bar{l}^a\bar{l}^b d\bar{\lambda} = \mathcal{E}[g] + \text{anomaly}\,. \tag{6.6}$$

where $\mathcal{E}[g]$ is defined as the flat-spacetime integral with sampling function g,

$$\mathcal{E}[g] = \int_\gamma g(\lambda) T_{ab} l^a l^b d\lambda \tag{6.7}$$

Following [14], we will now exhibit a sequence of states ψ_α that will make the ANEC integral arbitrarily negative. Since we are concerned only with a counterexample to ANEC, will not attempt to be general but opt instead for simplicity. Our procedure differs from that of [14] in that our field is conformally rather than minimally coupled, and our sampling function g is not compactly supported but rather goes to 1 at large distances.

A massless field ϕ is defined by eq. (3.2)

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3(2\omega)^{1/2}} \left(a(k)e^{-ik_a x^a} + a^\dagger(k)e^{ik_a x^a}\right)\,. \tag{6.8}$$

We define a class of vacuum plus two particle state vectors, which depend on a parameter $\alpha \in (0,1)$. First, given the function f, we will define a momentum parameter Λ_0 by a procedure to be described later. Then we define our states

$$\psi_\alpha = N_\alpha \left[|0\rangle + \frac{\alpha^{-1/4}}{\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^3(2\pi)^3}\sqrt{kk'}\,|k,k'\rangle\right] \tag{6.9}$$

where $\Lambda = \Lambda_0/\alpha$ is a momentum cutoff, N_α is a normalization constant,

$$N_\alpha = \left(1 + \frac{\alpha^{3/2}}{128\pi^4}\right)^{-1/2} \tag{6.10}$$

and

$$\int_\Sigma d^3k \quad \text{denotes} \quad \int_0^\Lambda k^2 dk \int_{1-\alpha}^1 d\cos\theta \int_0^{2\pi} d\phi \tag{6.11}$$

where k is the magnitude of the 3-vector $\mathbf{k}$, θ is the angle between $\mathbf{k}$ and the tangent vector $\mathbf{l}$, and ϕ is the azimuthal angle. These states excite only particles with momentum less than Λ, and directed inside an angle $\cos^{-1}(1-\alpha)$ from the null ray, which puts the four-momentum inside a tightening and lengthening cone as $\alpha \to 0$. Note that as α falls to zero, $N_\alpha \to 1$ and the excitation term in eq. (6.9) goes to zero. Thus the state approaches the vacuum, but we shall see that its stress-energy tensor does not.

In order to find the stress tensor, we need the normal ordered two point function

$$\langle\psi_\alpha| : \phi(x)\phi(x') : |\psi_\alpha\rangle = \frac{2N_\alpha^2}{\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6}\left[\alpha^{-1/4}e^{-i(k\cdot x+k'\cdot x')} + \frac{\alpha^{1/2}}{8\pi^2}e^{i(-k\cdot x+k'\cdot x')}\right]. \tag{6.12}$$

The first term arises from the coupling of the two-particle states to the vacuum. The second arises from the coupling between the two-particle states. In the limit $\alpha \to 0$, the first term is dominant because the admixture of two-particle states becomes very small.

The stress tensor for a conformally coupled scalar field is given by eq. (3.18). In Minkowski space, this is just

$$l^a l^b T_{ab} = \frac{2}{3} l^a l^b \phi_{,a}\phi_{,b} - \frac{1}{3} l^a l^b \phi_{,ab}\phi. \tag{6.13}$$

We take the expectation value in the state ψ_α and renormalize by subtracting the vacuum contribution (which is equivalent to normal ordering), then set $x' = x$. The first term becomes

$$\frac{2}{3}\left\langle : \phi_{,a}\phi_{,b} l^a l^b : \right\rangle_\alpha = \frac{4N_\alpha^2}{3\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6} l^a k_a l^b k'_b \left[-\alpha^{-1/4}e^{-ix\cdot(k+k')} + \frac{\alpha^{1/2}}{8\pi^2}e^{ix\cdot(k-k')}\right]. \tag{6.14}$$

The second term is

$$-\frac{1}{3}\left\langle : \phi_{,ab}\phi l^a l^b : \right\rangle_\alpha = \frac{2N_\alpha^2}{3\Lambda^4}\int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6}(l^a k_a)^2 \left[\alpha^{-1/4}e^{-ix\cdot(k+k')} + \frac{\alpha^{1/2}}{8\pi^2}e^{ix\cdot(k-k')}\right]. \tag{6.15}$$

As in eq. (5.13), our convention for the Fourier transform is

$$\hat{f}(u) = \int dt e^{-iut} f(t). \tag{6.16}$$

Since $g(t) = f(t) + 1$, $\hat{g}(u) = \hat{f}(u) + 2\pi\delta(u)$. From the properties of Ω, we see that f is bounded and has a well-defined, positive integral. Thus $\hat{f}$ is continuous and $\hat{f}(0) > 0$.

For any fixed 4-vector K,

$$\int d\lambda g(\lambda) e^{-i\gamma(\lambda)^a K_a} = \hat{g}(l \cdot K) \tag{6.17}$$

so we can write $\mathcal{E}[g] = \mathcal{E}_1[g] + \mathcal{E}_2[g]$, where

$$\mathcal{E}_1[g] = \frac{N_\alpha^2 \alpha^{1/2}}{12\pi^2 \Lambda^4} \int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6} \left[(l \cdot k)^2 + 2(l \cdot k)(l \cdot k')\right] \hat{g}(l \cdot (k - k')) \tag{6.18}$$

$$\mathcal{E}_2[g] = \frac{2N_\alpha^2 \alpha^{-1/4}}{3\Lambda^4} \int_\Sigma \frac{d^3k d^3k'}{(2\pi)^6} \left[(l \cdot k)^2 - 2(l \cdot k)(l \cdot k')\right] \hat{g}(l \cdot (k + k')). \tag{6.19}$$

We will first calculate $\mathcal{E}_2[f]$. Since we are in flat space, the tangent vector l is constant. We can take it to have unit time component, so that $k \cdot l = k(1 - \cos\theta)$. Performing the azimuthal integrations and changing variables to $v = k\alpha$, $u = k \cdot l$, and similarly for v' and u', we find

$$\mathcal{E}_2[f] = \frac{2N_\alpha^2 \alpha^{-1/4}}{3(2\pi)^4 \Lambda_0^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' vv' \int_0^v du \int_0^{v'} du' [u^2 - 2uu'] \hat{f}(u + u'). \tag{6.20}$$

Now $\hat{f} > 0$. Since $\hat{f}$ is continuous, we can choose $\Lambda_0 > 0$ such that $\hat{f}(u)$ is arbitrarily close to $\hat{f}(0)$. Thus we can make the integrals in eq. (6.20) arbitrarily close to

$$\hat{f}(0) \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' vv' \int_0^v du \int_0^{v'} du' [u^2 - 2uu'] = -\frac{13}{1440} \hat{f}(0) < 0. \tag{6.21}$$

As $\alpha \to 0$, the prefactor in eq. (6.20) goes to positive infinity, so we conclude that $\mathcal{E}_2[f] \to -\infty$ in this limit.

The rest of the terms are all finite. Equation (6.18) gives

$$\mathcal{E}_1[f] = \frac{N_\alpha^2 \alpha^{1/2}}{12\pi^2 \Lambda_0^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' vv' \int_0^v du \int_0^{v'} du' [u^2 + 2uu'] \hat{f}(u - u'). \tag{6.22}$$

Since f has compact support, $\hat{f}$ is bounded and the integrals give some finite number independent of α. Since the power of α is positive in this case, we find that $\mathcal{E}_1[f] \to 0$ as $\alpha \to 0$.

In addition we have the delta function in eqs. (6.18),(6.19), which gives the flat-spacetime ANEC integral discussed in Sec. II D of [14]. Since k is restricted to a cone around the direction of l, $l \cdot k \geq 0$, the integrand of $\mathcal{E}_2[\delta]$ has no support except from $k = k' = 0$, in which case the term in brackets vanishes. Thus $\mathcal{E}_2[\delta] = 0$.

Finally we have

$$\mathcal{E}_1[\delta] = \frac{N_\alpha^2 \alpha^{1/2}}{12\pi^2 \Lambda_0^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' v v' \int_0^v du \int_0^{v'} du' [u^2 + 2uu'] \delta(u - u'). \tag{6.23}$$

Again the integrals give a finite number, and the prefactor goes to zero, so $\mathcal{E}_1[f] \to 0$ as $\alpha \to 0$, and finally

$$\lim_{\alpha \to 0} \mathcal{E}[g] \to -\infty. \tag{6.24}$$

Thus we can find a quantum state such that $\mathcal{E}[g]$ is arbitrarily negative by fixing a conformal transformation Ω, adequate to compensate for whatever positive contribution to the ANEC integral the curvature may provide.

B. Curvature Term Violation

The next example of ANEC violation we found comes from the anomalous curvature term in the conformal transformation of T_{ab}. First we will review the curvature anomaly for a general conformally flat space, before specifying a transformation. A conformally coupled field transforms as $\bar{\phi} = \Omega^{-1}\phi$, but the stress tensor has extra terms, given by eq. (3.15). When beginning with Minkowski space, we have eq. (3.18). For the contribution to ANEC, we further eliminate terms proportional to g_{ab} as they vanish upon null projection, and those proportional to $R_{;ab}$ which vanish on integration over a complete geodesic. The remaining terms are

$$\bar{T}_{ab}^{\text{ANEC}} = \Omega^{-2} T_{ab} - \frac{1}{2880\pi^2} \left[-\bar{R}^c_{\ a} \bar{R}_{cb} + \bar{R}\bar{R}_{ab} \right]. \tag{6.25}$$

These curvature quantities can be written in terms of the conformal transformation [57], with $\omega = \ln \Omega$:

$$\bar{R}_{cb} = -2\omega_{,cb} - g_{cb}\Box\omega + 2\omega_{,b}\omega_{,c} - 2g_{cb}\omega_{,\rho}\omega^{,\rho} \tag{6.26}$$

$$\bar{R} = \Omega^{-2} \left[-6\Box\omega - 6\omega_{,\rho}\omega^{,\rho} \right]. \tag{6.27}$$

Thus the relevant contribution to the stress tensor becomes

$$\bar{T}_{ab}^{\text{ANEC}} = \Omega^{-2} T_{ab} - \frac{1}{720\pi^2} \Omega^{-2} \Big[-2 \left(\Box\omega + \omega^{,c}\omega_{,c} \right) \left(\omega_{,ab} - \omega_{,a}\omega_{,b} \right) + \omega^{,c}{}_{a}\omega_{,cb} - \omega^{,c}\omega_{,a}\omega_{,cb} - \omega^{,c}\omega_{,b}\omega_{,ca} \Big]. \tag{6.28}$$

We take as our initial state the Minkowski vacuum, with $T_{ab} = 0$, so the state does not contribute to $\bar{T}_{ab}$. Because we are transforming to an asymptotically flat space, this state is the conformal vacuum. We also have ω much less than one, so we may ignore terms of order ω^3 and take $\Omega \approx 1$. That leaves us with only

$$\bar{T}_{vv}^{\text{ANEC}} = -\frac{1}{720\pi^2}\left[g^{cd}\omega_{,cv}\omega_{,dv} - 2\Box\omega\omega_{,vv}\right]. \tag{6.29}$$

In our coordinates we organize this as

$$\bar{T}_{vv}^{\text{ANEC}} = -\frac{1}{720\pi^2}\left[\omega_{,xv}^2 + \omega_{,yv}^2 - 2\left(\omega_{,uv} + \omega_{,xx} + \omega_{,yy}\right)\omega_{,vv}\right] \tag{6.30}$$

As a counterexample to ANEC, we choose

$$\omega = axr^{-1}e^{-\rho} \tag{6.31}$$

where we define $\rho = (u^2 + v^2 + x^2 + y^2)/r^2$. This gives a localized transformation, so our spacetime is both conformally and asymptotically flat. The stress tensor component at $x = y = 0$ is

$$\bar{T}_{vv}^{\text{ANEC}} = -\frac{2a^2v^2}{45\pi^2r^6}e^{-2\rho}. \tag{6.32}$$

This is always manifestly negative. Thus integrating over γ always yields a negative quantity. This gives a violation of greater magnitude as a grows, but this analysis depends on $\omega \ll 1$ so it is not possible to build an arbitrarily large violation.

VII. TRANSVERSE AVERAGING

Both counterexamples we have constructed in the last chapter rely on effects which are localized near the geodesic in question. Flanagan and Wald [21] found that some transverse averaging eliminated the violations of ANEC they found, and it might be that averaging in additional directions could eliminate all violations and yield a principle that all quantum fields would obey. We explored the possibility of additionally averaged null energy conditions for conformal fields in a fixed background in [53], finding counterexamples to any possible procedure.

First we must specify what exactly is meant by a more general average of NEC. If we establish a null vector field l^a throughout spacetime, we can project the stress-energy tensor on this field and take the average,

$$A_4 = \int \sqrt{-g} d^4x T_{ab} l^a l^b \,, \tag{7.1}$$

but is not clear how we should define l^a.

In the case of the regular ANEC, we can start with a vector l^a tangent to our null geodesic γ at some initial point p. Such a vector is defined only up to rescaling, but such change (equivalent to a change of affine parameter) only affects the magnitude of the ANEC integral, not its sign. We then establish l^a everywhere on the geodesic by parallel transport from p to each destination point x.

We could attempt the same technique for averaging in more dimensions, but now there is more than one choice of path for the parallel transport. In general, when we work in curved space the resulting l^a will depend on the path chosen. Flanagan and Wald [21] make the choice to transport l^a along a geodesic from p to x. This is well defined if one works inside a normal neighborhood. If one considers perturbations of flat space as done in Ref. [21], and as we will do below, one can transport l^a in the unperturbed space-time without ambiguity. But in the general case, there may be no geodesic, or multiple geodesics, connecting p and x, and the procedure does not work.

We can also consider averaging over more than a single geodesic but less than all the dimensions of the manifold. For example, let χ be a timelike line parametrized by proper time τ. Start with a null vector l^a at some point $p \in \chi$, and establish a null vector field l^a on χ by parallel transport. Through each point of χ draw the null geodesic whose tangent vector is l^a. Then we can write

$$A_2 = \int d\tau d\lambda T_{ab} l^a l^b. \tag{7.2}$$

Similarly, we can average over a null surface, but here we will encounter ambiguities. Given a spacelike 2-surface Σ, let us establish a null vector field l^a orthogonal to the surface at each point. These vectors generate a family of geodesics. We can take the integral over each one, to get

$$A_3 = \int_\Sigma \sqrt{g_2}\, d\sigma_1 d\sigma_2 \int T_{ab} l^a l^b d\lambda \,. \tag{7.3}$$

Here σ_1 and σ_2 are the coordinates on the surface and g_2 the induced metric. The inner integral is to be taken over the geodesic generated by l_a at each point.

The direction of l^a is fixed by orthogonality, but we need to fix the magnitude. As before, we could try do to this via parallel transport, but that may depend on the path chosen. This process depends on the choice of the initial surface, even if the resulting null 3-surface is fixed. Suppose we propagate our initial surface an affine distance λ down each geodesic to get a new surface Σ'. The geodesics may spread out or squeeze together between Σ and Σ'. Thus if we started with Σ' instead of Σ, we would have a different weighting of the geodesics.

To avoid this problem, we could integrate over the surface for each λ first and then combine them, giving

$$A_3' = \int d\lambda \int_{\Sigma(\lambda)} \sqrt{g_2}\, d\sigma_1 d\sigma_2 T_{ab} l^a l^b \,. \tag{7.4}$$

However, eq. (7.4), like eq. (7.1), is not in an obvious way an average of ANEC.

We will not attempt to solve these problems, but rather we will exhibit counterexamples that apply to a very wide class of averaging procedures. We are able to do this because we work to first nonvanishing order in a spacetime that is a small perturbation of flat space. As we did in Ref. [52], we work in a conformally flat spacetime with conformal factor $\Omega = e^\omega \sim 1 + \omega$, with $\omega \ll 1$. We define our average by letting l^a be constant in the unperturbed spacetime and find violations of averaged versions of ANEC at order ω^2. Suppose now that we use a different procedure. If we defined l^a by parallel transport along a path which winds many times in the region where ω is largest, we could of course accumulate a large change in l^a. But this procedure is obviously pathological. If we restrict ourselves to a path which is free of such windings, the change in l^a along a path C will be given schematically by

$$\Delta l^a \sim \int_C \Gamma^a_{bc} l^b dC^c. \tag{7.5}$$

If the scale of the curved region is given by r, the magnitude of Γ^a_{bc} is of order ω/r, so $\Delta l^a \sim \omega$. Thus the effect of the choice of path is of higher order in ω than the original effect and can be consistently neglected.

The specific ω of the last section, eq. (6.31)

$$\omega = axr^{-1}e^{-\rho} \tag{7.6}$$

yielded

$$\bar{T}_{vv}^{\text{ANEC}} = -\frac{2a^2v^2\beta}{45\pi^2r^6}e^{-2\rho} \tag{7.7}$$

for $x = y = 0$, with u and v arbitrary. This is negative for all u, and so this example is sufficient to give a negative A_2 as well.

For the transverse averaging prescriptions the above example does not give a negative answer, so instead we use

$$\omega = (bu + cv)\, r^{-1}e^{-\rho}. \tag{7.8}$$

Any term which is odd in x, y, or z will vanish on integration, so we do not write such terms. Including only the even terms, the vv stress tensor component becomes

$$\begin{aligned}\bar{T}_{vv} \;=\; -\frac{2}{45\pi^2r^{10}}\Big\{ & 4b^2u^2\left[(x^2+y^2)(r^2-v^2)+r^2(2v^2-r^2)\right] \\ & -2bc\left[8u^2v^4-2r^2(v^4+5u^2v^2)+r^4(u^2+3v^2)\right] \\ & +c^2\left[(x^2+y^2)(-4v^4+8v^2r^2+r^4)+4r^2(2v^4-3r^2v^2)\right]\Big\}e^{-2\rho}.\end{aligned} \tag{7.9}$$

To calculate A_3 (which coincides with A_3' to first order) we set $u = 0$, and this becomes

$$\begin{aligned}\bar{T}_{vv} = -\frac{2e^{-2\rho}}{45\pi^2r^{10}}\Bigg[\; & c^2\left\{(x^2+y^2)(-4v^4+8v^2r^2+r^4)+4r^2(2v^4-3r^2v^2)\right\} \\ & +2bc(2v^4r^2-3v^2r^4)\Bigg].\end{aligned} \tag{7.10}$$

Note that here, the b^2 term drops completely, but the c^2 term is entirely unchanged. The integral is

$$A_3 = -\frac{\sqrt{2}}{1920\sqrt{\pi}r}c(2b+c). \tag{7.11}$$

So long as $b < -c/2$ this will be a negative quantity.

If instead we average (7.9) over the whole manifold, as in (7.1), we have

$$A_4 = \frac{1}{5760}\left(b^2+6bc+3c^2\right). \tag{7.12}$$

For $-3-\sqrt{6} < b/c < -3+\sqrt{6}$, the average is negative. As discussed earlier, these results still hold even if the integrand differs by any power of Ω. Here all dependence on r has dropped, and thus the sharpness of the curvature does not affect the violation.

Thus we see that in order to articulate a viable energy condition, we must either rule out conformally coupled scalar fields in favor of minimal coupling only, or restrict our attention away from the regime of test fields on arbitrary backgrounds.

VIII. BOUNDING THE TWO POINT FUNCTION FOR SELF CONSISTENT STATES

We would like a quantum energy inequality which bounds the stress tensor of a state in a space with small curvature similar, to the one conjectured by Kontou and Olum, here presented as eq. (5.23). We propose to use the difference quantum inequality eq. (5.17) with a specific reference state, which we will call the recently flat vacuum.

This state is defined by its action on the algebra of local operators in a compact, globally hyperbolic region $\mathcal{O}$ (rather than being defined by a mode sum.) We embed this region in some larger spacetime $\mathcal{M}$, which contains a spacelike surface Σ_0 which we identify with coordinate time $t' = 0$, such that there and for all time before it the spacetime is flat. Between $\mathcal{O}$ and Σ_0 there must be an interpolating region. We define the reference state $|\Omega\rangle$ by declaring it to be the state which coincides with the Minkowski vacuum for $t' < 0$. This is known as the in-vacuum of the spacetime $\mathcal{M}$.

This defines a state in the region $\mathcal{O}$, and a state which yields equal expectation values for all local observables over $\mathcal{O}$ exists regardless of what larger spacetime it is embedded in. Thus we may use this recently flat vacuum as a reference state in any spacetime, regardless of whether there is a region of flat space in the distant past, or nowhere at all.

In this chapter we address a recently massless vacuum for a quantum field theory with potential, $Q(x)$. Future work may adapt this approach to curved space quantum field theory, with the recently flat vacuum. We take points (x, y) at time t, and for all times $t' < 0$ we have $Q(x') = 0$. At all points, the two point function $\mu(x, y)$ (defined at eq. (3.26)) is a bisolution. So long as $Qt^2 \ll 1$, we may expand $\mu(x, y)$ as a perturbation of the massless vacuum two point function, $\mu_0 = (4\pi\sigma)^{-1}$. Utilizing $\Box\mu_0 = 0$, we find the next term in the

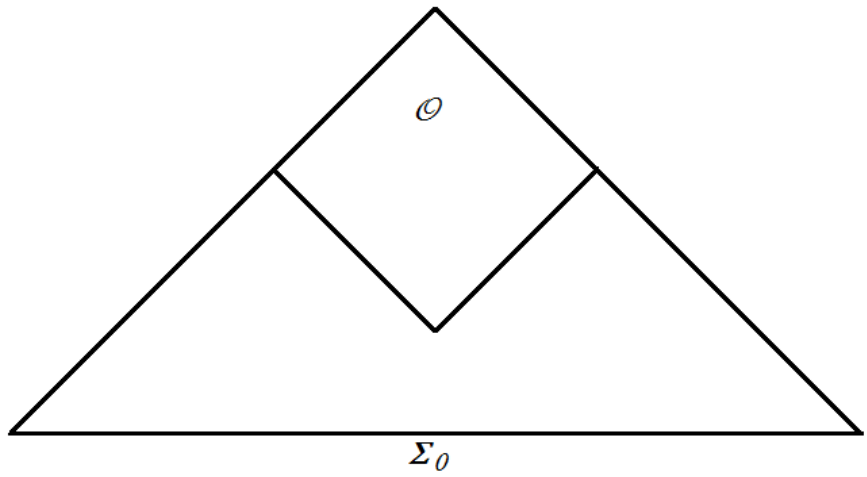

FIG. 1: The region $\mathcal{O}$ is globally hyperbolic, and contains the points x and y. At times prior to Σ_0, where $t' = 0$, the potential $Q(x') = 0$. In between there is a region where $Q(x')$ interpolates.

expansion $\mu = \mu_0 + \mu_1 + \ldots$

$$(\Box_x - Q)\mu = 0 \tag{8.1}$$

$$\Box_x \mu_1 = Q\mu_0. \tag{8.2}$$

This may be solved using the retarded Green's function, showing

$$\mu_1(x,y) = f(x,y) + \int d^4x' G_0(x',x) Q(x') \mu_0(x',y) \tag{8.3}$$

where $f(x,y)$ is some function that obeys $\Box_x f(x,y) = 0$. Likewise we can show a similar result for the integral in y, giving

$$\mu_1(x,y) = f(x,y) + \int d^4x' G_0(x',x) Q(x') \mu_0(x',y) + \int d^4y' G_0(y',y) Q(y') \mu_0(x,y') \tag{8.4}$$

where now $f(x,y)$ must satisfy both $\Box_x f(x,y) = \Box_y f(x,y) = 0$. We specify our state by requiring that in the past, at $t = 0$ where $Q = 0$, $\mu_1 = 0$. This is accomplished only by setting $f(x,y) = 0$. Thus we have

$$\begin{aligned} \mu_1(x,y) = & \int d^4x' G_0(x',x) Q(x') \mu_0(x',y) \\ & + \int d^4y' G_0(y',y) Q(y') \mu_0(x,y') \end{aligned} \tag{8.5}$$

Before evaluating we describe how we first arrived at eq. (8.5). This line of reasoning introduces test functions, which then integrate away. We would like to see how μ is evaluated in the region $\mathcal{O}$. We express $\mu(f,g)$ as the smearing of $\mu(x,y)$ for some test distributions. The conventional choice would be $f(x') = \delta(x'-x)$, and likewise for g. But the test functions are free to be any functions such that they generate the same classical solutions to the wave equation when operated on by E, according to $u = Ef$. Thus we have

$$\mu(x,y) = \int d^4x' d^4y' f(x';x) g(y';y) \mu(x',y') \tag{8.6}$$

We expand both the test distributions and the solutions they generate as series in Qt^2, related such that $u_0 = Ef_0$, $u_1 = Ef_1$, and so on.

We define the first function by

$$(\Box - Q)u = \delta(x) \tag{8.7}$$

$$\Box u_0 = 0 \tag{8.8}$$

$$u_0(x') = G_0(x',x). \tag{8.9}$$

This is the particular solution generated by the test function $\delta(x)$, and f_0 is any test function with support for $t \leq 0$ that also generates it.[18] The next term in ψ is given by

$$\Box u_1 = -Qu_0 \tag{8.10}$$

$$u_1(x') = -\int d^4x'' G(x',x'') Q(x'') u_0(x''). \tag{8.11}$$

The next term in f is given by

$$f_1(x';x) = \int d^4x'' f_0(x';x'') Q(x'') G(x'',x). \tag{8.12}$$

We seek the first correction to the two point function, which is symmetric in f and g

$$\begin{aligned} \mu_1(x,y) = & \int d^4x' d^4y' f_1(x';x) g_0(y';y) \mu_0(x',y') \\ & + \int d^4y' d^4x' f_0(x';x) g_1(y';y) \mu_0(x',y') \end{aligned} \tag{8.13}$$

[18] A concrete example of such a function is given by Wald [59]

$$f_0(x';x) = \psi(x';x)\delta'(t') + 2\partial_{t'}\psi(x';x)\delta(t'),$$

though we do not require a particular functional form of $f(x',t')$ at any point in our calculation.

After substituting eq. (8.12) and the parallel expression for g_1 into eq. (8.13), we then carry out the x' and y' integrals in each. These both act simply, by changing the argument of μ_0:

$$\mu_1(x,y) = -\int d^4x'' Q(x'')G(x'',x)\mu_0(x'',y) - \int d^4y'' Q(y'')G(y'',y)\mu_0(x,y''). \tag{8.14}$$

This recovers eq. (8.5). The primed integrals were an intermediate step. We return to the notation of eq. (8.5), without any double primed variables.

We define the vacuum two point function μ_0 using the prescription of Wald [59]

$$\mu_0(x,y) = \frac{1}{4\pi^2(-(t_x-t_y)^2+|\mathbf{x}-\mathbf{y}|^2}. \tag{8.15}$$

Using the explicit form of the Green's function, we have

$$\begin{aligned}\mu_1(x,y) \;=\; & -\frac{1}{16\pi^3}\int d^4x' \frac{Q(t',\mathbf{x}')\delta(t'-t_x+|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}\frac{1}{|x'-y|^2} \\ & -\frac{1}{16\pi^3}\int d^4y' \frac{Q(t',\mathbf{y}')\delta(t'-t_y+|\mathbf{y}'-\mathbf{y}|)}{|\mathbf{y}'-\mathbf{y}|}\frac{1}{|y'-x|^2}\end{aligned} \tag{8.16}$$

This gives the form which we will integrate. We proceed with a series of calculations. First we find the first order correction to the two point function μ in the case $Q = m^2$, a constant mass potential, and also the first correction to T_{ab} for our state. Then we find bounds for μ in the case of a completely general potential, dependent on the maximum value for for Q'.

A. Two Point Function for $Q = m^2$

With no loss of generality, we choose the two points to lie along the z axis and denote their separation by z. We first perform our calculation with $t_x = t_y$. We will always have the two points spacelike separated. In order to calculate the stress tensor, however, we must be able to take derivatives of the two point function in the time direction. The two point function is rotationally invariant, but it is *not* Lorentz invariant, as the surface where Q steps from 0 to m^2 is a privileged point in time.

$$\begin{aligned}\mu_1(x,y) \;=\; & -\frac{1}{16\pi^3}\int d^4x' \frac{Q(t',\mathbf{x}')\delta(t'-t+|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}\frac{1}{|x'-y|^2} \\ & -\frac{1}{16\pi^3}\int d^4y' \frac{Q(t',\mathbf{y}')\delta(t'-t+|\mathbf{y}'-\mathbf{y}|)}{|\mathbf{y}'-\mathbf{y}|}\frac{1}{|y'-x|^2}\end{aligned} \tag{8.17}$$

Performing the time integrals, we have

$$\begin{aligned}\mu_1(x,y) &= -\frac{1}{16\pi^3}\int d^3x' \frac{Q(t-|\mathbf{x}'-\mathbf{x}|,\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\frac{1}{-|\mathbf{x}'-\mathbf{x}|^2+|\mathbf{x}'-\mathbf{y}|^2} \\ &\quad -\frac{1}{16\pi^3}\int d^3y' \frac{Q(t-|\mathbf{y}'-\mathbf{y}|,\mathbf{y}')}{|\mathbf{y}'-\mathbf{y}|}\frac{1}{-|\mathbf{y}'-\mathbf{y}|^2+|\mathbf{y}'-\mathbf{x}|^2}. \end{aligned} \tag{8.18}$$

For values of $|\mathbf{x}'-\mathbf{x}| < z/2$, poles do not occur and the light cones do not intersect. We choose spherical coordinates centered at the point x for the first integral, and likewise for y which gives an identical contribution. For the range $|\mathbf{x}'-\mathbf{x}| \geq z/2$, there will be poles. We use spherical coordinates centered at the midpoint of x and y, and group the two integrals together in order that points which lie along the same ray may cancel their contributions.

We first address the case where there are no poles. Then, we have

$$-\frac{1}{8\pi^3}\int d^3x' \frac{Q(t-|\mathbf{x}'-\mathbf{x}|,\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\frac{1}{-|\mathbf{x}'-\mathbf{x}|^2+|\mathbf{x}'-\mathbf{y}|^2}. \tag{8.19}$$

With x as our origin, r' the radial coordinate, α' the cosine of the polar angle and ψ' the azimuthal, we have the geometric quantities

$$|\mathbf{x}'-\mathbf{x}| = r' \tag{8.20}$$

$$|\mathbf{x}'-\mathbf{y}| = \sqrt{r'^2+z^2-2r'z\alpha'}. \tag{8.21}$$

This gives

$$-\frac{1}{8\pi^3}\int_0^{2\pi}\int_{-1}^{1}\int_0^{z/2} dr'd\alpha'd\psi' \frac{r'Q(t-r',r',\alpha',\psi')}{z^2-2r'z\alpha}. \tag{8.22}$$

With $Q = m^2$, this simplifies greatly. The azimuthal integration is trivial. Integrating gives

$$-\frac{m^2}{4\pi^2 z}\int_0^{z/2} dr' \int_{-1}^{1} d\alpha' \frac{r'}{z-2r'\alpha'} \tag{8.23}$$

$$-\frac{m^2}{8\pi^2 z}\int_0^{z/2} dr' \left[\ln(z+2r') - \ln(z-2r')\right] \tag{8.24}$$

$$-\frac{m^2}{8\pi^2}\ln(2) \tag{8.25}$$

We return to eq. (8.18) to handle the range with poles. Here, we must treat the two ranges separately

$$\begin{aligned} &-\frac{1}{16\pi^3}\int r_x'^2 dr_x d\alpha_x' d\psi \frac{Q(t-|\mathbf{x}'-\mathbf{x}|,\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\frac{1}{-|\mathbf{x}'-\mathbf{x}|^2+|\mathbf{x}'-\mathbf{y}|^2} \\ &-\frac{1}{16\pi^3}\int r_y'^2 dr_y d\alpha_y' d\psi \frac{Q(t-|\mathbf{y}'-\mathbf{y}|,\mathbf{y}')}{|\mathbf{y}'-\mathbf{y}|}\frac{1}{-|\mathbf{y}'-\mathbf{y}|^2+|\mathbf{y}'-\mathbf{x}|^2}. \end{aligned} \tag{8.26}$$

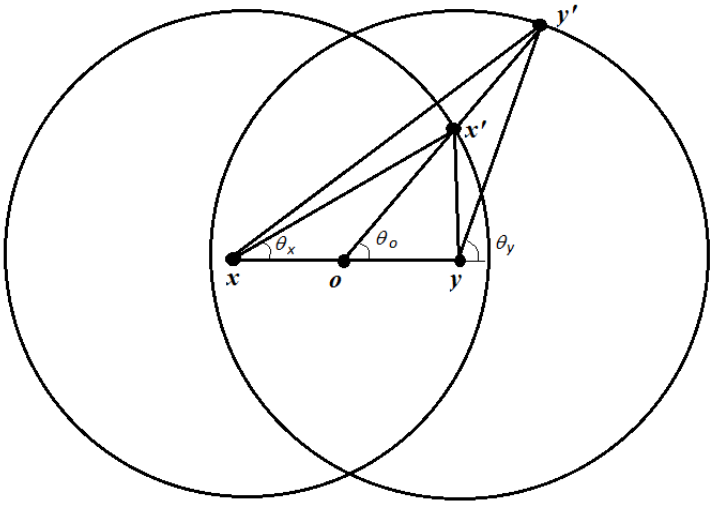

FIG. 2: A slice of constant time, for $r' > z/2$. We define $\alpha'_x = \cos(\theta_x)$, $\alpha'_y = \cos(\theta_y)$, and $\beta' = \cos(\theta_o)$.

For a constant slice in time, shown in fig. 2, $r'_x = r'_y$ and we drop the subscript. We change our polar integration from α_x and α_y, which have as their origin the points x and y respectively, to β, which takes the midpoint between the two as origin $\mathbf{o}$. The points x' and y' now lie on the same ray.

In these coordinates, after applying the law of cosines multiple times, we find that the vector quantities are

$$|\mathbf{x}' - \mathbf{o}| = \frac{1}{2}\left(-z\beta' + \sqrt{4r'^2 - (1-\beta'^2)z^2}\right) \tag{8.27}$$

$$|\mathbf{y}' - \mathbf{o}| = \frac{1}{2}\left(+z\beta' + \sqrt{4r'^2 - (1-\beta'^2)z^2}\right) \tag{8.28}$$

$$|\mathbf{x}' - \mathbf{y}| = \left(r'^2 + z^2\beta'^2 - z\beta\sqrt{4r'^2 - (1-\beta'^2)z^2}\right)^{1/2} \tag{8.29}$$

$$|\mathbf{y}' - \mathbf{x}| = \left(r'^2 + z^2\beta'^2 + z\beta\sqrt{4r'^2 - (1-\beta'^2)z^2}\right)^{1/2} \tag{8.30}$$

With these, we can then determine the Jacobian for the transformation between angles

$$\frac{d\alpha'_x}{d\beta'} = \frac{1}{2r'}\left[-2\beta' z + \frac{(2\beta'^2 - 1)z^2 + 4r'^2}{\sqrt{(\beta'^2 - 1)z^2 + 4r'^2}}\right] \tag{8.31}$$

$$\frac{d\alpha'_y}{d\beta'} = \frac{1}{2r'}\left[+2\beta' z + \frac{(2\beta'^2 - 1)z^2 + 4r'^2}{\sqrt{(\beta'^2 - 1)z^2 + 4r'^2}}\right]. \tag{8.32}$$

Then, the integral becomes

$$-\int \frac{dr'd\beta'd\psi'}{32\pi^3} \left[\left[-2\beta z + \frac{(2\beta^2-1)z^2+4r'^2}{\sqrt{(\beta^2-1)z^2+4r'^2}}\right] \frac{Q(t-r',\mathbf{x}')}{z^2\beta'^2 - z\beta\sqrt{4r'^2-(1-\beta'^2)z^2}} \right.$$
$$\left. + \left[+2\beta z + \frac{(2\beta^2-1)z^2+4r'^2}{\sqrt{(\beta^2-1)z^2+4r'^2}}\right] \frac{Q(t-r',\mathbf{y}')}{z^2\beta'^2 + z\beta\sqrt{4r'^2-(1-\beta'^2)z^2}}\right] \tag{8.33}$$

This can be regrouped to become

$$-\frac{1}{32\pi^3}\int dr'd\beta'd\psi' \left[\frac{Q(t-r',\mathbf{x}')-Q(t-r',\mathbf{y}')}{z\beta'} + \frac{Q(t-r',\mathbf{x}')+Q(t-r',\mathbf{y}')}{\sqrt{(\beta'^2-1)z^2+4r'^2}}\right]. \tag{8.34}$$

With $Q = m^2$, the subtracted terms cancel entirely, and we have

$$-\frac{m^2}{8\pi^2}\int_{z/2}^{t}\int_{-1}^{+1} \frac{dr'd\beta'}{\sqrt{(\beta'^2-1)z^2+4r'^2}} \tag{8.35}$$
$$= -\frac{m^2}{8\pi^2 z}\int_{z/2}^{t} dr' \Big[\ln(2r'+z) - \ln(2r'-z)\Big] \tag{8.36}$$
$$= \frac{m^2}{16\pi^2}\left[\ln\left(\frac{4z^2}{4t^2-z^2}\right) + \frac{2t}{z}\ln\left(\frac{2t-z}{2t+z}\right)\right] \tag{8.37}$$

Combining this with the poleless contribution, eq. (8.25), yields the full unrenormalized two point function

$$\mu_1(z) = \frac{m^2}{16\pi^2}\left[\ln\left(\frac{z^2}{4t^2-z^2}\right) + \frac{2t}{z}\ln\left(\frac{2t-z}{2t+z}\right)\right]. \tag{8.38}$$

From this we subtract the Hadamard function, derived at eq. (4.60),

$$H(x,y) = \frac{1}{4\pi^2}\left(\frac{1}{\sigma} + \frac{\bar{Q}^2}{4}\ln(\bar{Q}\sigma)\right). \tag{8.39}$$

The σ^{-1} term cancels μ_0 and, with $\bar{Q} = m^2$, the $\ln(m^2\sigma)$ removes the logarithmic divergence, yielding the renormalized two point function

$$\mu_{\mathrm{RN}}(z) = \frac{m^2}{16\pi^2}\left[-\ln\big((4t^2-z^2)m^2\big) + \frac{2t}{z}\ln\left(\frac{2t-z}{2t+z}\right)\right] \tag{8.40}$$
$$= -\frac{m^2}{16\pi^2}\left[\ln(4m^2t^2) + 2 - \frac{z^2}{12t^2}\right] + \dots \tag{8.41}$$

B. Bounding μ for Bounded $Q(x)$

We consider a potential $Q(x)$ bounded such that $|Q| < F$ and $|\partial_a Q| < F'$ for all points in the region, and for all directions of the derivative (in unboosted coordinates).

First we address the region without poles, where the contribution is again given by eq. (8.22)

$$-\frac{1}{8\pi^3}\int_0^{2\pi}\int_{-1}^{1}\int_0^{z/2} dr' d\alpha' d\psi' \frac{r' Q(t-r', r', \alpha', \psi')}{z^2 - 2r' z\alpha}. \tag{8.42}$$

We add and subtract from the potential $Q(t, \mathbf{x})$. The added term is taken to give a constant contribution, calculated as eq. (8.25) with $m^2 = Q(t, \mathbf{x})$. If we repeat the procedure at $\mathbf{y}$, we have $\bar{Q}$. The subtraction is arranged as difference terms

$$-\frac{1}{8\pi^3}\int_0^{2\pi}\int_{-1}^{+1}\int_0^{z/2} dr' d\alpha' d\psi' \frac{r' \left(Q(t-r', r', \alpha', \psi') - Q(t, \mathbf{x})\right)}{z^2 - 2r' z\alpha}. \tag{8.43}$$

We bound the absolute of the two point function by bounding the difference in Q by the maximum value the derivative may attain, denoted F', and the Euclidean separation between the points, $\sqrt{2}r'$, finding

$$\frac{\sqrt{2}F'}{8\pi^3}\int_0^{2\pi}\int_{-1}^{1}\int_0^{z/2} dr' d\alpha' d\psi' \frac{r'^2}{z^2 - 2r' z\alpha}. \tag{8.44}$$

which integrates to

$$\frac{\sqrt{2}F' z}{32\pi^2}. \tag{8.45}$$

To calculate the contribution where poles do exist we return to eq. (8.34)

$$-\frac{1}{32\pi^3}\int dr' d\beta' d\psi' \left[\frac{Q(t-r', \mathbf{x}') - Q(t-r', \mathbf{y}')}{z\beta'} + \frac{Q(t', \mathbf{x}') + Q(t', \mathbf{y}')}{\sqrt{(\beta'^2 - 1)z^2 + 4r'^2}}\right]. \tag{8.46}$$

For the first term we bound the difference $Q(\mathbf{x}') - Q(\mathbf{y}')$ by the maximum value of the derivative times the separation, which is $F' z\beta'$. This greatly simplifies things; the $d\beta'$ integral brings a factor 2, $d\psi$ brings 2π, and dz' brings $t - z/2$. Thus its contribution is bounded by

$$\frac{F'}{8\pi^2}(t - z/2) \tag{8.47}$$

To the second term we add and subtract $\bar{Q}/\sqrt{(\beta^2 - 1)z^2 + 4z'^2}$. The added terms again give the constant mass two point function. The subtracted term appears as

$$-\frac{1}{16\pi^3}\int_0^{2\pi}\int_{z/2}^{t}\int_{-1}^{+1} d\beta' dr' d\psi' \left[\frac{Q(t-r', \mathbf{x}') + Q(t-r', \mathbf{y}') - Q(t, \mathbf{x}) - Q(t, \mathbf{y})}{2\sqrt{(\beta'^2 - 1)z^2 + 4r'^2}}\right]. \tag{8.48}$$

We associate the $Q(\mathbf{x})$ subtraction with $Q(\mathbf{x}')$, and $Q(\mathbf{y})$ with $Q(\mathbf{y}')$. We bound each difference with the derivative bound times the Euclidean distance between the points, $\sqrt{2}r'$.

We always use the derivative bound rather than bounding each difference by $2F$ as this gives us only a single parameter to control. The absolute value of this term is thus bounded by

$$\frac{\sqrt{2}F'}{8\pi^2}\int dr' d\beta' \frac{r'}{\sqrt{(\beta^2-1)z^2+4z'^2}} \tag{8.49}$$

$$= \frac{\sqrt{2}F'}{8\pi^2 z}\int dr' \Big[\ln(2r'+z)-\ln(2r'-z)\Big] \tag{8.50}$$

$$= \frac{\sqrt{2}F'}{64\pi^2 z}\Big[4tz-2z^2+(4t^2-z^2)\ln\left(\frac{2t+z}{2t-z}\right)\Big]. \tag{8.51}$$

We renormalize using the Hadamard form defined at the two base points, which is the same as in the constant mass case. Putting everything together, we have at base the renormalized two point function of a constant mass of $\bar{Q}$, found originally at eq. (8.38)

$$\mu_{\text{const}}(z) = \frac{\bar{Q}}{16\pi^2}\Big[-\ln(4\bar{Q}t^2)-2+\frac{z^2}{12t^2}\Big]+\dots \tag{8.52}$$

and a combination of derivative bounds on the absolute value, present in eqs. (8.45), (8.47) and (8.51):

$$\begin{aligned}|\mu_1-\mu_{1\,\text{const}}| < & \frac{\sqrt{2}F'z}{32\pi^2} \\ & +\frac{F'}{8\pi^2}(t-z/2) \\ & +\frac{\sqrt{2}F't}{64\pi^2}\Big[4-\frac{2z}{t}+\frac{4t^2-z^2}{zt}\ln\left(\frac{2t+z}{2t-z}\right)\Big].\end{aligned} \tag{8.53}$$

The distinct contributions from the each range are manifestly positive; $-z$ terms appear only when subtracted from t, as even when don't take the $z\to 0$ limit, we still have $t>z$. Now that this is done, we may associate terms, giving

$$|\mu_1-\mu_{1\,\text{const}}| < \frac{F't}{16\pi^2}\Big[(2+\sqrt{2})-\frac{z}{t}+\frac{4t^2-z^2}{4zt}\ln\left(\frac{2t+z}{2t-z}\right)\Big] \tag{8.54}$$

$$< \frac{F't}{8\pi^2}\Big[\left(1+\sqrt{2}\right)-\frac{z}{2t}-\frac{z^2}{12t^2}-\frac{z^4}{240t^4}\Big]+\dots \tag{8.55}$$

IX. OUTLOOK AND CONCLUSIONS

Exotic spacetime configurations are generally considered to be physically impossible, but this must be proven based on some physical law obeyed by the matter content, such as the achronal averaged null energy condition. We have found violations of achronal ANEC

within the regime of test fields on an arbitrary background for the conformally flat scalar field [52]. Requiring transverse averaging, though reasonable in the sense that only macroscopic exotica need concern us, is not sufficient to give a valid condition [53]. In order to consider achronal ANEC an acceptable basis from which to prove theorems ensuring well behaved causality in the universe, we must either limit attention to systems in which Einstein's equation for general relativity is enforced, or exclude conformally coupled scalars. Requiring self consistency is a significant calculational complication, but is a very natural condition. Kontou and Olum's proof of ANEC [37] for minimally coupled scalars depends on a proposed quantum energy inequality, eq. (5.23). Further progress will be necessary to establish the validity of such a condition.

We have defined a recently massless vacuum state ψ_0, and bounded its two point function μ for a general potential $Q(x)$ in eq. (8.54). We must further bound the derivatives, to be used in the difference quantum energy equality of Fewster and Roman [14], presented here as eq. (5.17):

$$\int_\eta d\tau \left(\left\langle T^{\mathrm{RN}}_{ab} \right\rangle_\psi - \left\langle T^{\mathrm{RN}}_{ab} \right\rangle_{\psi_0} \right) l^a l^b g(\tau)^2 \geq -2 \int_0^\infty \hat{F}(\alpha, -\alpha). \tag{9.1}$$

Thus we must, for our state ψ_0, calculate

$$F(\tau, \tau') = g(\tau) g(\tau') \left\langle l^a \nabla_a l^{b'} \nabla_{b'} \mu(\tau, \tau') \right\rangle_{\psi_0} \tag{9.2}$$

and $\langle T_{ab} \rangle_{\psi_0} l^a l^b$. Only one derivative of each argument is required as can be seen explicitly in F and, for the minimally coupled scalar field, within T_{ab} (eq. (3.6)). This can be found directly by taking derivatives of eq. (8.5)

$$\begin{aligned} \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^b} \mu_1(x, y) = & \int d^4x' \frac{\partial}{\partial x^a} G_0(x', x) Q(x') \frac{\partial}{\partial y^b} \mu_0(x', y) \\ & + \int d^4y' \frac{\partial}{\partial y^b} G_0(y', y) Q(y') \frac{\partial}{\partial x^a} \mu_0(x, y'). \end{aligned} \tag{9.3}$$

One derivative always acts on G_0, and one on μ_0. The δ' portion of G_0' may be integrated by parts to give a Q' term. Then, the entire expression must be bounded.

To modify this procedure for curved space, we may again use a Green's function integrated over a kernel, found perturbatively. For the recently massive potential, we considered the flat space wave equation with an arbitrary potential function

$$\left(\partial_a \partial^a - Q(x) \right) u = 0. \tag{9.4}$$

The curved space wave equation is

$$\left(\partial_a \partial^a - \Gamma^a_{ab} \partial^b\right) u = 0. \tag{9.5}$$

Here we have what is essentially an arbitrary function multiplying ∂u rather than u as before. If we apply this to μ as a bisolution and expand from the flat space μ_0, such that $\partial_a \partial^a \mu_0 = 0$, we have

$$\left(\partial_a \partial^a - \Gamma^a_{ab} \partial^b\right) (\mu_0 + \mu_1 + \ldots) = 0 \tag{9.6}$$

$$\partial_a \partial^a \mu_1 = \Gamma^a_{ab} \partial^b \mu_0. \tag{9.7}$$

This may again be solved using the flat space retarded Green's function, as the differential operators appearing at this order are simple partial derivatives. For the perturbation series to be valid, now we must require that the magnitude of $\Gamma^a_{ab} t$ be small. The requirement that Qt^2 be small has clear physical meaning, but Γ is coordinate dependent. Kontou and Olum's proof of ANEC [37] requires that components of the Riemann tensor in particular coordinates adapted to the null geodesic in question be small, but this may require an additional constraint.

Another approach for curved space utilizes the Kirchhoff representation, detailed in a review by Poisson [46]. For any bisolution, we have

$$\mu(x, y) = \int_\Sigma \Big(G(x, z') \nabla^{a'} \mu(z', y) - \mu(z', y) \nabla^{a'} G(x, z') \Big) \, d\Sigma_{a'} \tag{9.8}$$

where Σ is a spacelike surface in the past of x and y and G is the retarded Green's function. Chu and Starkman [10] obtain a perturbative expression for the retarded Green's function in curved space. For a perturbation around a flat spacetime, with $g_{ab} = \eta_{ab} + h_{ab}$, this simplifies to

$$G(x, x') = G_0(x, x') + \partial_a \partial_{b'} \int d^4 x'' G_0(x, x'') \left(\frac{1}{2} h'' \eta^{ab} - h^{a'' b''} \right) G_0(x'', x') \tag{9.9}$$

with $h'' = h_{ab} \eta^{ab}$. Either calculational method would construct the same state ψ_0.

We have seen that the achronal averaged null energy condition would guarantee that time machines, wormholes, and other strange geometries and topologies could not develop in our universe. By finding explicit counterexamples to achronal ANEC with a conformally coupled scalar field on a conformally and asymptotically flat background, we have demonstrated

that in order to articulate a true energy condition, we must limit our scope: either to minimal coupling, self consistent matter and metric configurations, or both together. Distant curvature and boundary terms cannot by themselves cause an ANEC violation; the geodesic must pass through a region of curvature. By constructing a recently flat reference state ψ_0 we hope to bound the contribution of local curvature terms, proving the intuitive conjecture that for small curvature flat space quantum energy inequalities should hold with corrections whose order is set by the maximum value of the curvature.

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