An Investigation of Negative Energy Densities in Quantum Field Theory: Diagonalization Method and Worked Examples

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1 Introduction

While most energy quantities are positive (like kinetic, or rest), certain types of energy are considered negative. Potential energy, for example, is often written as negative energy.

This is not what we are talking about here when we talk about negative energy. Here, we are generally referring to a negative energy density – energy per unit volume. The energy density is the (0,0) component of the stress-energy tensor, and is classically found to be positive. For example, the stress-energy tensor of the electromagnetic field alone depends on the electric field squared and the magnetic field squared, multiplied by positive constants.

In quantum field theory, however, it is possible to construct a state that exhibits a negative energy density at certain points in space. In this thesis, we will examine different states that exhibit a negative energy density, and discuss the requirements for a maximally negative energy density. We will also discuss limits on negative energy, more commonly known as "quantum inequalities". Finally, we will examine physical systems corresponding to negative energy densities.

There are many reasons to study negative energy. A current popular topic of discussion is superluminal, or "faster than light" travel. It has been shown that superluminal travel would require a negative energy density along the path traveled by the particle [1]. The way this works is similar to the Shapiro time delay effect. In the Shapiro effect, as a particle travels past a body of some positive energy (such as a planet or a sun), its travel is slowed [2]. However, if a particle were instead traveling past a body of negative energy, its travel would actually be sped up – and if this particle were already traveling close to or at the speed of light (if it were a photon, for example), then an outside observer would actually see the particle travel faster than the speed of light. Superluminal travel is of pop culture interest currently for its potential applications in space exploration. However, superluminal travel can also lead to backwards time travel [3].

Negative energy also offers the curious possibility of repulsive gravity. Just as the regular attractive gravitational force is caused by positive energy, negative energy would

imply a repulsive force instead. So rather than a body falling into orbit around a planet, for example, a particle instead would be pushed away from an accumulation of negative energy.

There is also the possibility of negative energy allowing for traversable wormholes [4]. However, certain limitations on the amount of negative energy it is possible to have (which will be discussed in more detail in later sections) limit the possible size of the wormholes. In general, the possible quantities of negative energy limit the size of the theoretical wormholes to size scales around the Planck length [4]. Macroscopic wormholes are extremely unlikely based on the allowed energy densities, which limits the utility of traversable wormholes.

One common example of negative energy that is often discussed is the Casimir effect, which occurs when you have a pair of uncharged conducting parallel plates separated by a small distance [5]. Since the plates are neutral, classically you would expect no force between the plates. However, in practice there is an attractive force between the plates. This attractive force can only be explained quantum mechanically [5].

In the simplest model, this force can be explained entirely by looking at the zero-point energy fluctuations of the vacuum between and outside the plates. The plates providing conducting boundary conditions for the electric field, which in turn polarizes the vacuum and causes the energy density between the plates to be smaller than the energy density of free space – which causes the attractive force between the plates [6].

In general, however, the negative energy density described by the Casimir effect is different than the types of negative energy densities we will be examining in this thesis. The negative energy density described by the Casimir effect only occurs due to the boundary conditions of the space constrained by the plates. The negative energy densities we will be looking at have no such restriction, although other limitations will apply.

In this thesis, we will not be discussing negative energy from the Casimir effect, but negative energy densities that arise independently of any boundary conditions in the system. In Section 2, we will discuss what negative energy is, and show an example of a negative energy density in a system. In Section 3, we will present a diagonalization method for finding maximally negative energy states. By diagonalizing a general form of an energy density operator, we can construct a method for finding a maximally negative energy state which we can apply to a variety of systems. In Section 4 and 5 we will present some examples where we apply our diagonalization method. Then in Section 6 we will discuss quantum inequalities, and in Section 7 we will discuss the physical relevance of our results.

2 What is Negative Energy?

First, it is important to clarify what we mean when we talk about "states". In quantum field theory, states are defined by the number of particles in them. So the vacuum state is the state with 0 particles, $|0\rangle$. In quantum field theory, states can have an infinite number of modes, where each mode has some number of particles. In a sense, each mode is its own harmonic oscillator, where the number of particles in the mode are analogous to the

energy levels in the regular quantum harmonic oscillator.

In this thesis, we will look at examples of states that only have one mode excited, where that one mode has some number of particles. This simplifies the mathematics, while also still allowing for an examination of negative energy phenomena.

However, in order to find a negative energy density, we need to understand what it even means for a state to have negative energy. Negative energy arises when the fluctuations of the vacuum are suppressed, such as in a "squeezed vacuum state", which we will talk about later in this thesis [7]. In quantum field theory, the energy of the vacuum is in a sense infinite. This is undesirable for many reasons, one of which being that current gravitational theory requires the energy density of flat space to be 0. Setting the vacuum energy equal to zero is the process of normal ordering, which essentially subtracts off that infinite energy. So when we talk about "negative" energy, what we really mean is an energy density that is lower than the usual vacuum energy density.

In a sense, then, negative energy can be thought of as an energy debt – energy owed by the state to the vacuum. Given that we have just said that the vacuum energy is infinite, this might lead one to think that we can have arbitrarily large amounts of negative energy. However, this is not the case. Local averages of energy density in fact do have a lower bound – these are called quantum inequalities [8, 15]. So while you might have a large spike in negative energy, it must be for a correspondingly short time, and large time scales require the magnitude of the negative energy to be small, going to zero as the time goes to infinity.

In a quantum harmonic oscillator, there is no upper limit on energy. However, there is of course a lower limit (the ground state). This lower limit represents the lowest amount of energy a system can have. However, there are restrictions on how low this ground state energy can be. In classical mechanics, the lowest amount of energy a system can have is 0. In quantum mechanics, we see that the harmonic oscillator actually has a ground state energy of $\frac{\hbar\omega}{2}$. In quantum field theory, however, this lowest energy can occasionally be negative (for the energy densities of certain states at certain points – not for a state universally). Later in this thesis we will discuss in more detail how negative this energy can be.

The first example we will be looking at is a superposition of the vacuum state and the two particle state [7]. By including an extra parameter we can vary, we can then choose the parameter such that the state gives a negative energy density. So we want a state of the form

$$|0\rangle + \epsilon\,|2\rangle \tag{1}$$

but of course, this needs to be normalized. So our state ends up looking like

$$|\psi\rangle = \frac{1}{\sqrt{1+\epsilon^2}}(|0\rangle + \epsilon\,|2\rangle). \tag{2}$$

Now let our energy density T be normally ordered such that

$$\langle 0|T|0\rangle = 0. \tag{3}$$

Thus we can see that the expectation value of T with respect to $|\psi\rangle$ is

$$\langle\psi|T|\psi\rangle = \frac{1}{1+\epsilon^2}(\langle 0| + \epsilon\langle 2|)\ T\ (|0\rangle + \epsilon|2\rangle). \tag{4}$$

Here the $\langle 0|T|0\rangle$ term will give us 0, while the other three terms remaining give us

$$\langle\psi|T|\psi\rangle = \frac{1}{1+\epsilon^2}[\epsilon(\langle 0|T|2\rangle + \langle 2|T|0\rangle) + \epsilon^2\langle 2|T|2\rangle]. \tag{5}$$

Now we want to write our energy density operator, T, in a general form. Writing

$$T = Aaa^\dagger + Ba^2 + Ba^{\dagger 2} \tag{6}$$

we can see that both the ϵ and the ϵ^2 terms will contribute, specifically because we are looking at the vacuum and two particle state. We can then choose a negative ϵ such that the first term in Eq. 5 dominates over the second, leaving $\langle T\rangle_\psi < 0$ at certain points. Here, A and B are some constant, real-valued functions of x and t. Since A and B are space and time dependent, the points at which a negative energy density is exhibited will depend on A and B.

This of course suggests that states of this type, where ϵ is a free parameter, will allow for negative energy for certain values of ϵ. However, while this does give us a range of states, it is very limited. While we have maximized the magnitude of the negative energy for the vacuum and two particle state, we have no way of knowing if this is even optimal of the states that are superpositions of two states. Therefore, some other method for finding negative energy states is required, beyond guess-and-check.

In the next section, we will construct a diagonalization method such that the lowest eigenvalue of the energy density for some generic state will give us the value of the energy density in the ground state. This means that we can construct a system specifically so that it exhibits a negative energy density at certain points.

3 Constructing Maximally Negative Energy States

While there are many states that will give negative energy, we are most interested in states that give a maximally negative energy density. While negative energy states in general are all interesting, since negative energy is not allowed classically, the maximally negative energy states will give us a better idea of how much negative energy we can expect to find in a given system.

We will consider systems where the energy density can be described as in Eq. 6, in terms of A and B. By leaving A and B general for now, we can develop a method that works for a variety of systems, subject to some constraints, so that we can apply our method to multiple problems. Later in this thesis we will do some examples applying our method to different systems, at which point we will define A and B explicitly.

To construct a general procedure, instead of starting with an explicit state, we consider a generic state in the $a, a^\dagger$ basis, where a and $a^\dagger$ are the usual annihilation and creation operators. This means that the state $a^\dagger |0\rangle_a$ represents a single particle in a given mode in the a basis. Working in the a basis, finding the parameters that describe a negative energy state requires trial and error. Here we will find a basis in which T is diagonal (in that it depends only on $b^\dagger b$), which will allow us to find the lowest energy eigenstate of a system. If this state is negative, then we know it is the maximally negative energy from a system because it is the lowest energy eigenstate for that system. Then we can transform this state back into the a basis, which corresponds to the physical basis.

To do this, we will transform the a basis using a generic Bogulobov transformation

$$a = \alpha b + \beta b^\dagger \tag{7}$$

thus writing a into the b basis. For simplicity, we will take α and β to be real.

Now we can look at the energy density in the a basis, Eq. 6, and transform it into the b basis using our Bogulobov transformation. Here we are taking A and B to be real numbers, and we are taking T to be normally ordered. We know that b and $b^\dagger$ must satisfy the same commutation relations as a and $a^\dagger$, so we can see that

$$[a, a^\dagger] = [\alpha b + \beta b^\dagger, \beta b + \alpha b^\dagger] = \alpha^2 - \beta^2 = 1. \tag{8}$$

This means that when we write T in the b basis instead of the a basis, we can use this relation later to express T back in the a basis and solve for what α and β would have to be in order to make the expectation value of T in the ground state negative. So, substituting and simplifying we get

$$T = (A\alpha\beta + B\alpha^2 + B\beta^2)(b^2 + b^{\dagger 2}) + (A\alpha^2 + 2B\alpha\beta)b^\dagger b + (A\beta^2 + 2B\alpha\beta)bb^\dagger. \tag{9}$$

In order to diagonalize T, we want both the b^2 and the $b^{\dagger 2}$ terms to be zero. So, we can set another restriction on α and β such that

$$(A\alpha\beta + B\alpha^2 + B\beta^2) = 0. \tag{10}$$

For this to be true, given that α and β are real, $A\alpha\beta$ and B must be of opposite sign (since α^2 and β^2 will always be positive). So taking A to be positive, then α and β must be of opposite sign, or B must be negative.

Starting from our commutation relations, which give us $\alpha = \sqrt{1 + \beta^2}$, where we are taking α to be positive, we get

$$\begin{aligned} \alpha &= \sqrt{\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{A^2}{A^2 - 4B^2}}} \\ \beta &= \pm\sqrt{-\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{A^2}{A^2 - 4B^2}}} \end{aligned} \tag{11}$$

where the sign of β will be plus if B is negative, and minus if B is positive.

We can see immediately that in order for this transformation to be possible, $A^2 > 4B^2$, and since we are taking A and B to be real, this means that we need $A > 2|B|$. Otherwise, we will get an infinite or imaginary value for α and β, which is not allowed given our requirement that α and β be real.

However, we also need either α or β to go to 0 when B is 0, so that the $A\alpha\beta$ term also goes to zero. Without loss of generality, we can choose the sign inside the square root such that $\beta \to 0$ as $B \to 0$. This gives us:

$$\begin{aligned} \alpha &= \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{A^2}{A^2 - 4B^2}}} \\ \beta &= \pm\sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{A^2}{A^2 - 4B^2}}} \end{aligned} \tag{12}$$

where again, the sign of β will be determined by the sign of B.

Looking at T again, we can use the commutation relation for b and $b^\dagger$ to rewrite the $bb^\dagger$ term in terms of $b^\dagger b$ and get

$$T = (A\alpha^2 + A\beta^2 + 4B\alpha\beta)b^\dagger b + A\beta^2 + 2B\alpha\beta. \tag{13}$$

Next, we look at the constant term of the expectation value of the energy density, T, in the ground state, $\langle 0|T|0\rangle_b$. This will be the lowest eigenvalue for the state, or λ_T. We can see that

$$\lambda_T = A\beta^2 + 2B\alpha\beta$$

since the operator b will annihilate the vacuum. So plugging in our values for α and β and using that $\alpha\beta = -\frac{B}{A}(\alpha^2 + \beta^2)$, we get

$$\lambda_T = \frac{1}{A}\left[-\frac{A^2}{2} + \frac{A^2}{2}\sqrt{\frac{A^2}{A^2 - 4B^2}} - 2B^2\sqrt{\frac{A^2}{A^2 - 4B^2}}\right] \tag{14}$$

which simplifies to

$$\lambda_T = \frac{1}{A}\left[-\frac{1}{2}A^2 + \frac{A}{2}\sqrt{A^2 - 4B^2}\right]. \tag{15}$$

Now we want to test to see if it's possible for the energy density to ever be negative. If the energy density can be negative, then since λ_T is the lowest eigenvalue for the state, λ_T will be negative. If λ_T is positive, then the energy density will always be positive. To check this, we can assume the condition $\lambda_T < 0$ and see if the system is solvable. So, assuming $\lambda_T < 0$, we see that

$$\sqrt{A^2 - 4B^2} < A \tag{16}$$

and given that $A > 0$, we see that this condition requires $|B| \leq \frac{A}{2}$, which is possible given our initial assumptions about A and B. However, from our conditions for α and β, we know that $2|B| \neq A$. This means that we need $2|B| < A$ in order to find the maximally negative energy density for this system.

However, this just tells us that we have a transformation such that the energy density in the b basis can exhibit a negative energy density. Since the a basis is the basis in which the creation and annihilation operators correspond to the creation and destruction of physical particles, we need to transform this back into the a basis to understand what physical states correspond to our derived negative energy. In order to do this, we can start by writing b in terms of a, using the relation $\alpha^2 - \beta^2 = 1$, which gives us

$$b = \alpha a - \beta a^\dagger. \tag{17}$$

In the b basis we were looking at the ground state, $|0\rangle_b$, which is the state with the lowest eigenvalue, λ_T. When acted upon by b, this of course must give 0. So, we can rewrite this in terms of a as

$$b\,|0\rangle_b = 0 = (\alpha a - \beta a^\dagger) \sum_n c_n\,|n\rangle_a \tag{18}$$

then we can rewrite the right-hand side as

$$\sum_n \left[\alpha c_n \sqrt{n}\,|n-1\rangle_a - \beta c_n \sqrt{n+1}\,|n+1\rangle_a\right]. \tag{19}$$

Now for the first term we can let $n \to n+1$ and for the second term we can let $n \to n-1$ so that we get

$$\sum_n \left[\alpha c_{n+1} \sqrt{n+1}\,|n\rangle_a - \beta c_{n-1} \sqrt{n}\,|n\rangle_a\right]. \tag{20}$$

This gives us the recurrence relation

$$c_{n+1} = \frac{\beta}{\alpha}\sqrt{\frac{n}{n+1}}c_{n-1} \tag{21}$$

which means that each c_n depends only on c_{n-2}, not c_{n-1}.

However, there are additional conditions we must consider. We know that when $\beta \to 0$, then $b \to \alpha a$, and $|0\rangle_b \to |0\rangle_a$. When we look at even c_n we can see that

$$c_2 = \frac{\beta}{\alpha}\sqrt{\frac{1}{2}}c_0 \tag{22}$$

and

$$c_4 = \frac{\beta^2}{\alpha^2}\sqrt{\frac{3}{8}}c_0 \tag{23}$$

and so on for higher c_n.

However, for odd c_n, we see that

$$c_3 = \frac{\beta}{\alpha}\sqrt{\frac{2}{3}}c_1 \tag{24}$$

and

$$c_5 = \frac{\beta^2}{\alpha^2}\sqrt{\frac{8}{15}}c_1. \tag{25}$$

Which depends on c_1. For simplicity, we can choose $c_1 = 0$, which means that all the odd $c_n = 0$, so we have a superposition of even occupation numbers. This will give us the following, which we will see later is another form of what is known as a "squeezed vacuum state":

$$|\phi\rangle = c_0\big(\,|0\rangle + \frac{\beta}{\alpha}\sqrt{\frac{1}{2}}\,|2\rangle + \frac{\beta^2}{\alpha^2}\sqrt{\frac{3}{8}}\,|4\rangle + ... + c_{2n}\,|2n\rangle + ...\big) \tag{26}$$

where we are using the c_{2n} as described above, and c_0 can be found by normalizing the state.

When $|\frac{\beta}{\alpha}| \ll 1$, then we can see that this approximates the vacuum and two particle state. This suggests that for energy densities where the vacuum and two particle state produces a negative energy, in fact the squeezed vacuum state would also work, and possibly give a larger magnitude of negative energy.

We will discuss the squeezed vacuum state in more detail in a later section.

4 Applying the Diagonalization Method

In this section, we will re-examine our vacuum and two particle state example, and then see how it matches to our result using the diagonalization method.

First, consider a massless scalar field. The scalar field will have normally ordered energy density

$$: T_{00} := \frac{1}{2}\left[: \dot{\phi}^2 : + : (\nabla\phi)^2 :\right] \tag{27}$$

where $\phi = \sum(a_k f_k + a_k^\dagger f_k^*)$, and $f_k(\vec{x}, t)$ are the usual mode functions normalized in a box of volume V (and so using periodic boundary conditions) such that

$$f_k(\vec{x}, t) = \frac{1}{\sqrt{2\omega V}}e^{i(kx-\omega t)}. \tag{28}$$

Substituting and simplifying, we see that

$$: T_{00} := \frac{\omega^2 + k^2}{4\omega V} \left[-a^2 e^{2i(kx-\omega t)} - a^{\dagger 2} e^{-2i(kx-\omega t)} + 2a^\dagger a \right]. \tag{29}$$

Now we can take $x = t = 0$, and take the expectation value of T_{00}. Once we simplify, our general form looks like

$$\langle T_{00} \rangle = \frac{\omega^2 + k^2}{4\omega V} \left[-\left\langle a^2 \right\rangle - \left\langle a^{\dagger 2} \right\rangle + 2 \left\langle a^\dagger a \right\rangle \right] \tag{30}$$

where we have not yet put in what state we are looking at.

We will use the vacuum and two particle state from Eq. 2. When we put in this state, we know that some terms will be 0, namely, any terms that have $a\,|0\rangle$ or $\langle m|n\rangle$ when $m \neq n$. So simplifying everything and dropping the 0 terms, we get

$$\langle \psi | T_{00} | \psi \rangle = \frac{\omega^2 + k^2}{4\omega V(1+\epsilon^2)} \left[-\epsilon \left(\langle 2|a^{\dagger 2}|0\rangle + \langle 0|a^2|2\rangle \right) + 2\epsilon^2 \langle 2|a^\dagger a|2\rangle \right] \tag{31}$$

Doing this out, we can simplify this to

$$\langle \psi | T_{00} | \psi \rangle = \frac{\omega^2 + k^2}{4\omega V(1+\epsilon^2)} \left[-2\sqrt{2}\epsilon + 4\epsilon^2 \right]. \tag{32}$$

Graphing $\langle \psi | T_{00} | \psi \rangle = -2\sqrt{2}\epsilon + 4\epsilon^2$ in units of $\frac{\omega^2+k^2}{4\omega V(1+\epsilon^2)}$ where ϵ is the independent variable (see Fig. 1) will show us the possible values for the energy density. Clearly, there is a value of ϵ for which $\langle \psi | T_{00} | \psi \rangle < 0$. The minimum value occurs at $\epsilon = \sqrt{3} - \sqrt{2}$. In these units, we get a maximally negative energy of $2 - \sqrt{6}$. The exact value will depend on ω, k, and V, but since they are constants, we can express the energy density as some multiple of them.

Next, it is important to see how this example fits in with our original diagonalization method for finding negative energy states. We want to apply our diagonalization method to this system and see what it gives us for the energy density. However, we can immediately see a problem.

We know the minimum eigenvalue for the energy density operator looks like

$$\lambda_T = \left[-\frac{1}{2}A + \frac{1}{2}\sqrt{A^2 - 4B^2} \right] \tag{33}$$

where A and B are given by the form of $T = Ba^2 + Ba^{\dagger 2} + Aa^\dagger a$.

So for our example above,

$$A = \frac{\omega^2 + k^2}{2\omega V}. \tag{34}$$

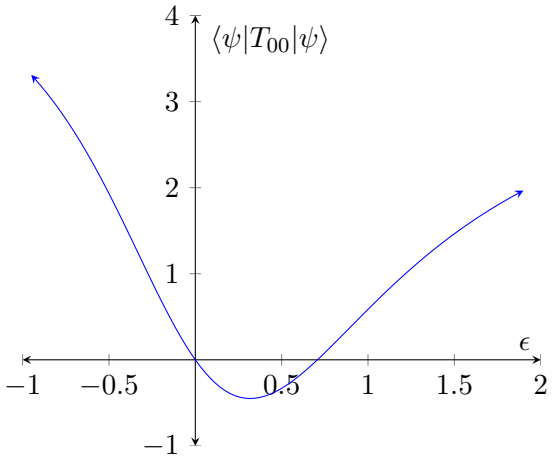

Figure 1: A graph of $\langle\psi|T_{00}|\psi\rangle = -2\sqrt{2}\epsilon + 4\epsilon^2$ in units of $\frac{\omega^2+k^2}{4\omega V}$, where we can clearly see that $\langle\psi|T_{00}|\psi\rangle$ will be negative for some values of ϵ.

$$B = -\frac{\omega^2 + k^2}{4\omega V}. \tag{35}$$

So we can see that $2|B| = |A|$! This means our diagonalization method does not work here, since β and α will blow up.

In order to apply our diagonalization method, we can instead look at the averaged energy density sampled by some function. For our sampling function, we can use the Lorentzian which has been normalized so that its integral is 1. Then, once we integrate over time, we will get a new function for B, Fourier transformed.

$$f(t) = \frac{\tau}{\pi}\frac{1}{t^2 + \tau^2} \tag{36}$$

Then again taking $x = 0$, we can write the averaged energy density as

$$\overline{T_{00}} = \frac{\omega^2 + k^2}{4\omega V}\int_{-\infty}^{\infty}\left[-a^2 e^{-2i\omega t} - a^{\dagger 2}e^{2i\omega t} + 2a^\dagger a\right]\frac{\tau}{\pi}\frac{1}{t^2+\tau^2}dt \tag{37}$$

We know that $\tau > 0$, since τ represents a time constant. We can separate out this integral and take $t = -t$ in the second term, since the sampling function is even. Then we can simplify this to:

$$\overline{T_{00}} = \frac{\omega^2 + k^2}{4\omega V}\int_{-\infty}^{\infty}\left[-e^{-2i\omega t}(a^{\dagger 2} + a^2) + 2a^\dagger a\right]\frac{\tau}{\pi}\frac{1}{t^2+\tau^2}dt \tag{38}$$

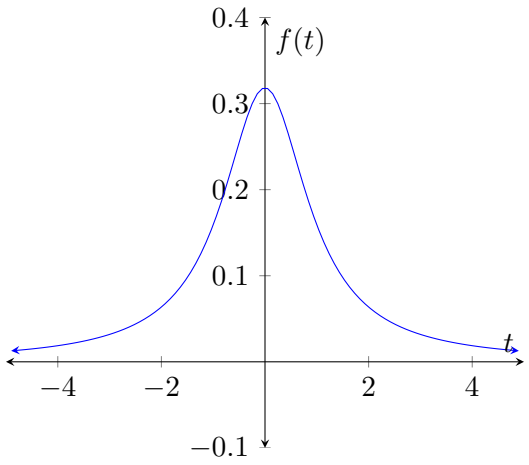

Figure 2: Our sample function using the Lorentzian, taking $\tau = 1$.

Doing this out, we get

$$\overline{T_{00}} = \frac{\omega^2 + k^2}{4\omega V}\left(2a^\dagger a - (a^2 + a^{\dagger 2})e^{-2\tau\omega}\right) \tag{39}$$

So now we have A the same as before (Eq. 34), but

$$B = -\frac{\omega^2 + k^2}{4\omega V}e^{-2\tau\omega}. \tag{40}$$

This means that $2|B| < A$ as long as $\tau\omega > 0$, which is trivially true. So now we can apply our diagonalization method.

Plugging A and B into Eq. 33, our formula for λ_T, we get

$$\lambda_T = \frac{\omega^2 + k^2}{4\omega V}(\sqrt{1 - e^{-4\tau\omega}} - 1) \tag{41}$$

So we can see that although $\tau\omega = 0$ is not allowed by our diagonalization conditions, λ_T will approach a minimum value as $\tau\omega \to 0$. Additionally, we can see that in units of $\frac{\omega^2+k^2}{4\omega V}$, as for the vacuum and two particle state example, the value for the negative energy density will approach -1, instead of $2 - \sqrt{6}$ as it did earlier. So clearly, we have found a state with a greater negative energy density.

Now suppose we want to find explicitly what this state is, rather than just the value of the negative energy density. Using our formulas for α and β in terms of A and B (Eq. 12), and looking at units of $\frac{\omega^2+k^2}{4\omega V}$, we get:

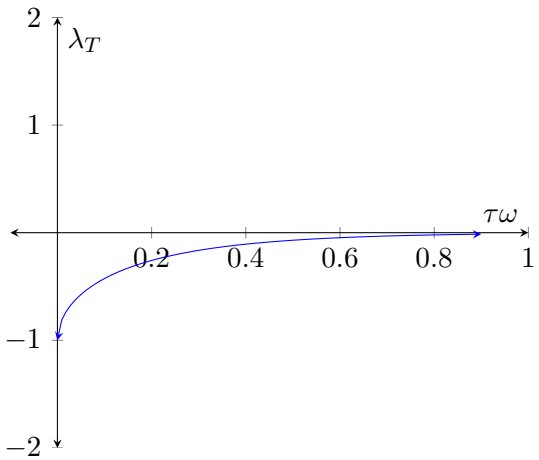

Figure 3: The energy density as a function of $\tau\omega$. As $\tau\omega \to 0$, $\lambda_T \to -1$.

$$\alpha = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{1 - e^{-4\tau\omega}}}}$$
$$\beta = \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{1 - e^{-4\tau\omega}}}} \tag{42}$$

where we take β to be positive, since our B is negative.

Then we can plug these values into our formula for the squeezed state, Eq. 26, and find specifically the state that satisfies this. However, we can see that since α and β are dependent on $\tau\omega$ as $\tau\omega \to 0$, we really get a range of states that approaches our maximal negative energy condition, rather than a single state.

5 A Closer Look at the Squeezed Vacuum State

First, however, we must understand what is meant by a squeezed vacuum state.

A squeezed state is essentially a minimum uncertainty state, where the uncertainty in one variable is "squeezed" so that the uncertainty in the other variable may be expanded [9]. An illustration of the idea of a squeezed state can be found by looking at the quantum harmonic oscillator. Consider the wave function of the QHO in the ground state, in the dimensionless position basis:

$$\psi_0(\mathbf{x}) = \frac{1}{\pi^{1/4}} e^{-\mathbf{x}^2/2} \tag{43}$$

Then after transforming this to the momentum basis we get:

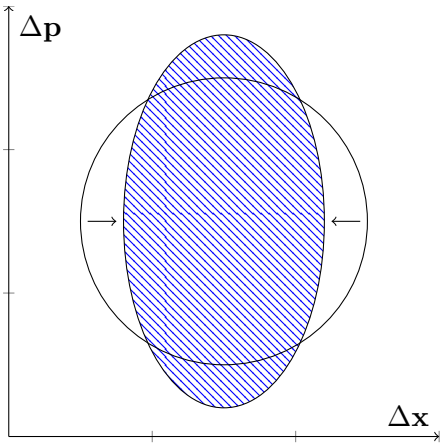

Figure 4: A visual representation of squeezing the uncertainties. As the uncertainty in $\mathbf{x}$ shrinks, the uncertainty in $\mathbf{p}$ grows.

$$\tilde{\psi}_0(\mathbf{p}) = \frac{1}{\pi^{1/4}} e^{-\mathbf{p}^2/2} \tag{44}$$

where this is the same form of the wave function as in the position basis [9]. So the expectation values of $\Delta\mathbf{x}^2$ and $\Delta\mathbf{p}^2$ in the ground state are both $1/2$.

So now suppose we want to introduce squeezing into this. Then let $R > 0$ be some squeezing factor. The squeezed ground state will look like:

$$\psi_R(\mathbf{x}) = \frac{\sqrt{R}}{\pi^{1/4}} e^{-(R\mathbf{x})^2/2} \tag{45}$$

and

$$\psi_R(\mathbf{p}) = \frac{1}{\pi^{1/4}\sqrt{R}} e^{-(\mathbf{p}/R)^2/2} \tag{46}$$

for the position and momentum bases [9].

Now when we find the expectation values of $\Delta\mathbf{x}^2$ and $\Delta\mathbf{p}^2$ in the ground state, we get

$$\left\langle \Delta\mathbf{x}^2 \right\rangle = \frac{1}{2R^2} \text{ and } \left\langle \Delta\mathbf{p}^2 \right\rangle = \frac{R^2}{2}. \tag{47}$$

When $R > 1$, then $\Delta\mathbf{x}^2$ will be less than $1/2$, and $\Delta\mathbf{p}^2$ will be greater than $1/2$. This is then called "position-squeezed", because the uncertainty in the position is constricted, while the uncertainty in the momentum is appropriately expanded so as not to violate the uncertainty principle. If $R < 1$, then we would see squeezing in the momentum instead [9].

In quantum field theory, the squeezed vacuum state looks like

$$|\zeta\rangle = S(\zeta)\,|0\rangle$$

where $S(\zeta)$ is the squeeze operator:

$$S(\zeta) = \exp\left[\frac{1}{2}(\zeta^* a^2 - \zeta a^{\dagger 2})\right] \tag{48}$$

and has the property

$$S^\dagger a S = a\cosh r - a^\dagger e^{i\delta}\sinh r. \tag{49}$$

This gives us the following relations for the squeezed vacuum state:

$$\left\langle a^2\right\rangle = \left\langle a^{\dagger 2}\right\rangle^* = -e^{i\delta}\sinh r\cosh r \tag{50}$$

$$\left\langle a^\dagger a\right\rangle = \sinh^2 r$$

The parameter δ is a phase parameter, and r represents the mean number of particles in a state.

We can also write the squeeze operator as:

$$S(r,\delta)\,|0\rangle = (\mathrm{sech}(r))^{1/2}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{n!}\left[-\frac{1}{2}e^{i\delta}\tanh(r)\right]^n |2n\rangle \tag{51}$$

This is the same form as we saw earlier, although the coefficients are written in a different form.

It is important to consider how the squeezed state works in the context of our diagonalization method. We found earlier that using the diagonalization method on a general state to find a negative energy density gave us the squeezed vacuum state. However, what if we want to find negative energy in a system that does not satisfy our diagonalization conditions?

Let's look again at the scalar field scenario we considered in the previous section. Before, we had to average over t. So now, let's consider the same system, without averaging over t. Then we have from Eq. 34:

$$A = \frac{\omega^2 + k^2}{2\omega V} = -2B. \tag{52}$$

In this situation, our β that we were using for the Bogulobov transformation, which has a $\frac{1}{A^2-4B^2}$ dependence, now blows up. So, we can see that our diagonalization method definitely does not apply here.

Now we have normalized energy density

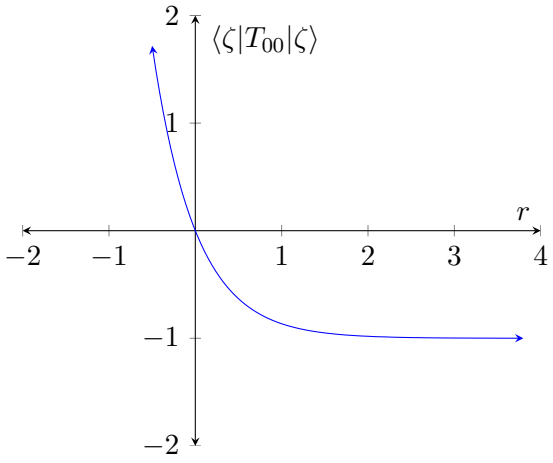

Figure 5: From Eq. 54. Here we can see that as $r \to \infty$, $\langle\zeta|T_{00}|\zeta\rangle$ approaches some maximally negative value, where $\langle\zeta|T_{00}|\zeta\rangle$ is in units of $\frac{\omega^2+k^2}{4\omega V}$.

$$: T_{00} := \frac{\omega^2 + k^2}{4\omega V}\left[-a^2 - a^{\dagger 2} + 2a^\dagger a\right]. \tag{53}$$

Taking the expectation value of this in our squeezed state, using Eq. 50, we get

$$\langle T_{00}\rangle = \frac{\omega^2 + k^2}{4\omega V}\left[e^{i\delta}\sinh r\cosh r + e^{-i\delta}\sinh r\cosh r + 2\sinh^2 r\right].$$

In order to see the most negative energy density, we want to take $\delta = \pi$. Now, taking our energy in units of $\frac{\omega^2+k^2}{4\omega V}$ as we did in the previous section, we can look at the behavior of r and see where $\langle T_{00}\rangle$ is negative.

$$\langle T_{00}\rangle = \frac{\omega^2 + k^2}{4\omega V}\left[-2\sinh r\cosh r + 2\sinh^2 r\right] \tag{54}$$

We can see here that there will be no single state that has the most negative energy. This explains why the diagonalization method doesn't work for this situation, because the diagonalization method is designed to find the single state with the maximally negative energy.

Instead, as $r \to \infty$, $\langle T_{00}\rangle$ approaches a maximally negative value of -1 asymptotically, as shown in Fig. 5.

This is exactly the result we found with our diagonalization method, except here we are looking at a system which does not meet the conditions for diagonalization. That these results match is promising, because it means our diagonalization method was able to account for the fact that there was not a single state that had the maximally negative energy, but instead a series of states.

The fact that the magnitude of the negative energy is proportional to $\omega^2 + k^2$ is enticing in the fact that is suggests we could construct states with arbitrarily large negative energy densities at a specific point. Since ω is the frequency of our state, it is natural to think that the negative energy could be increased by raising the frequency to some arbitrarily large amount.

However, while this may be true for a single point, it is not true for the average energy density of a system. Indeed, quantum inequalities show us that there is a limit on the length of time negative energy can last. So while you can construct a system which has an arbitrarily large amount of negative energy at a certain point, this negative energy will last for a correspondingly short amount of time.

6 Limits on Negative Energy

The problem with negative energy is that it seems to violate what we know of the laws of physics. For instance, the existence of negative energy seems to violate the second law of thermodynamics, which says that entropy can never decrease. It seems on first glance that this negative energy could potentially cool a body and thus decrease its entropy. However, in practice there are certain natural limits on how much negative energy you can have, which prevents any such violation. These natural limits are called quantum inequalities.

How much negative energy you can have over some amount of time is dictated by quantum inequalities. This limit is small enough that it's impossible to use a beam of negative energy (for example) to lower the entropy of a system macroscopically for long enough to measure it before the corresponding positive energy arrives. This neatly prevents any measurable violation of the second law.

For a clearer example of this, consider a beam of negative energy shooting at a black hole. This beam of negative energy would cause the overall mass of the black hole to shrink, which shrinks the horizon area, which then correspondingly lowers the entropy of the black hole. However, the incoming beam of negative energy has no entropy – it is a pure state. This suggests that this beam of negative energy could potentially lower the total entropy of the universe, thus violating the second law of thermodynamics. How is this possible?

As we have discussed, this situation is avoided because of the limits on negative energy described by quantum inequalities. This beam of negative energy is small enough in magnitude that the horizon of the black hole shrinks by such a small amount that it is impossible to measure (because light can not travel across the horizon in the required time) before the fluctuations cause the energy to become positive again and restores the original mass of the black hole. Thus, it is impossible to see any sort of second law violation [14].

There are, however, some limits on the utility of quantum inequalities. Quantum inequalities dictate a theoretical limit for how much negative energy you can have. While it is relatively simple to place some limit on negative energy, it is much more difficult to find an optimal limit. In fact, it is yet unknown if the current inequalities are optimal in four dimensions. This means that while we know you can not have negative energy below

a certain amount, we do not know if you can actually have as much negative energy as the quantum inequalities would imply. Or in other words, it is unknown whether or not it is possible to find a state that saturates the 4D inequality.

In this section, we will do some examples using quantum inequalities, looking at a state and then seeing if the inequality is satisfied. However, it is important to note that we will be doing calculations for single mode systems, while the quantum inequalities actually hold for systems with any number of modes excited.

In two dimensions, maximal quantum inequalities have already been found. Consider an example of a 2D inequality, the optimal lower bound on the averaged expected energy density, derived by Flanagan and explained in [10], looks like:

$$\rho_{f,\psi} \geq -\frac{1}{24\pi} \int dt \frac{f'(t)^2}{f(t)} \tag{55}$$

where $\rho_{f,\psi}$ is the sampled expectation value of the energy density sampled by the function f and with respect to the state ψ, and $f(t)$ is any smooth non-negative sampling function.

$$\rho_{f,\psi} = \int f(t) \, \langle\psi| : T_{00}(t) : |\psi\rangle \, dt$$

Essentially, $f(t)$ provides a mathematical description for how the negative energy is being measured. It is impossible to measure the energy instantaneously, so looking at the instantaneous energy density does not have a corresponding physical meaning. Instead, we can look at the average energy density over some time. In order to do this, we must sample the energy density using a sampling function. This sampling function represents our machine that is measuring the energy density. Clearly then, this sampling function must go to 0 as $t \to \pm\infty$, because we can not measure something indefinitely. Additionally, taking $t \to \infty$, we want to see the lower bound on the energy go to 0, since otherwise that would indicate that the total energy of our system is negative.

A sampling function might look something like the Lorentzian, which we used in a previous section:

$$f(t) = \frac{\tau}{\pi} \frac{1}{t^2 + \tau^2} \tag{56}$$

as shown in Fig. 2. We've chosen the constant in front such that $f(t)$ normalizes to 1. The peak here essentially represents when the measurement is being taken.

So plugging this sampling function into (42), we get:

$$\rho_{f,\psi} \geq -\frac{1}{48\pi\tau^2} \tag{57}$$

Note that here, τ represents the width of the peak of $f(t)$, or the time constant. So we can see that as the time gets very small, and the peak gets very high, the minimum

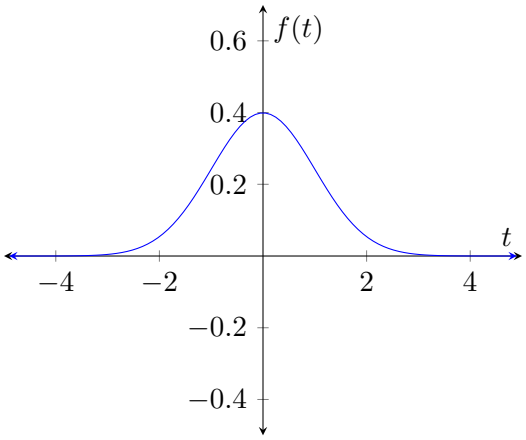

Figure 6: Our Gaussian sampling function.

negative energy can actually get quite large. But if we want a longer time constant, then the amount of negative energy allowed is very small.

Now let's look at another sampling function, this time a simple Gaussian. So

$$f(t) = \frac{1}{\tau\sqrt{\pi}} e^{\frac{-t^2}{\tau^2}} \tag{58}$$

which can be seen in Fig. 6.

Then plugging this sampling function into our formula for $\min \rho_{f,\psi}$, we get:

$$\rho_{f,\psi} \geq -\frac{1}{12\tau^2} \tag{59}$$

which exhibits the same behavior as with the previous sampling function. Namely, as τ gets very small, $\rho_{f,\psi}$ gets large, and as τ goes to infinity, $\rho_{f,\psi}$ goes to 0.

The fact that this is optimal means that this is the smallest lower bound you can have. Namely, that there exists a state that saturates this inequality. However, as discussed, an explicit optimal quantum inequality only exists in two dimensions [10].

Consider now a bound for the massless Klein-Gordon field in four dimensional Minkowski space [8]:

$$\int \langle T_{00} \rangle_\psi (t, \vec{x}) g(t)^2 dt \geq -\frac{1}{16\pi^2} \int_{-\infty}^{\infty} |g''(t)|^2 dt \tag{60}$$

where $g(t)$ is our sampling function and ψ is the quantum state [10]. Note that unlike the left-hand side, the right-hand side is actually state independent, and depends only on the sampling function.

We can see that now in addition to $g(t)$ and $g'(t)$, $g''(t)$ must also exist and be square-integrable, so we have here an additional restriction on the sampling function that we did not have for $f(t)$.

Now we will look at some examples of sampling functions for the 4D case. Consider first the Lorentzian as discussed above (Eq. 56), except now $g(t)$ is the square root of the Lorenztian:

$$g(t) = \sqrt{\frac{\tau}{\pi}\frac{1}{t^2+\tau^2}} \tag{61}$$

So $g''(t)^2$ can be written

$$g''(t)^2 = \frac{\tau(-2t^2+\tau^2)^2}{\pi(t^2+\tau^2)^5} \tag{62}$$

so then our inequality becomes

$$\int_{-\infty}^{\infty} \langle T_{00}\rangle_{\psi}(t,\vec{x}) \left(\frac{\tau}{\pi}\frac{1}{t^2+\tau^2}\right) dt \geq -\frac{1}{16\pi^2}\int_{-\infty}^{\infty} \frac{\tau(-2t^2+\tau^2)^2}{\pi(t^2+\tau^2)^5} dt \tag{63}$$

and the right side simplifies to:

$$\int_{-\infty}^{\infty} \langle T_{00}\rangle_{\psi}(t,\vec{x}) \left(\frac{\tau}{\pi}\frac{1}{t^2+\tau^2}\right) dt \geq -\frac{27}{2048\pi^2\tau^4}. \tag{64}$$

Now we want to know what our $\langle T_{00}\rangle_{\omega}(t,\vec{x})$ looks like. Consider the system we examined before, the massless scalar field. From Eq. 29, we have that

$$\langle\psi|T_{00}|\psi\rangle = \frac{\omega^2+k^2}{4\omega V}\left[-a^2 e^{2i(kx-\omega t)} - a^{\dagger 2} e^{-2i(kx-\omega t)} + 2a^\dagger a\right]. \tag{65}$$

If we assume discrete modes and periodic boundary conditions, like you would see in an E&M cavity, then we can take

$$\omega = k = \frac{n\pi}{l} \text{ and } V = l^3. \tag{66}$$

In the vacuum and two particle state looking at a single mode, this gives us:

$$\int_{-\infty}^{\infty} \frac{n^2\pi^2}{2l^4(1+\epsilon^2)}\left[-2\sqrt{2}\epsilon e^{\frac{2in\pi(x-t)}{l}} + 4\epsilon^2\right]\left(\frac{\tau}{\pi}\frac{1}{t^2+\tau^2}\right) dt \geq -\frac{27}{2048\pi^2\tau^4} \tag{67}$$

which becomes

$$\frac{n^2\pi^2\epsilon\left(2\epsilon - \sqrt{2}e^{-\frac{2n\pi\tau}{l}}e^{\frac{2in\pi x}{l}}\right)}{l^4(1+\epsilon^2)} \geq -\frac{27}{2048\pi^2\tau^4} \tag{68}$$

Since we have the freedom to choose our axes, we can let $x = 0$. Then since we know that the energy density will be at a minimum in the lowest energy level, we can let $n = 1$. This gives us:

$$\frac{\pi^2\epsilon\left(2\epsilon - \sqrt{2}e^{-\frac{2\pi\tau}{l}}\right)}{l^4(1+\epsilon^2)} \geq -\frac{27}{2048\pi^2\tau^4} \tag{69}$$

Now we want to see if this inequality is satisfied. To do so, we first need to know what τ is. We know that τ is the longest interval when the energy density is negative. Taking the value for ϵ we found earlier, and looking at the energy density with respect to t (from Eq. 64), we see that $\langle\psi|T_{00}|\psi\rangle$ will be negative for $\frac{\tau}{l} < 0.3$, or when $\tau \leq 0.3l$. Now plugging this into our inequality, and minimizing with respect to ϵ, we get:

$$-0.01 \geq -0.1 \tag{70}$$

which is, of course, true, and we can see that the limit is an order of magnitude larger than what we actually found. Recalling our example applying the diagonalization method, we know that the vacuum and two-particle state actually gave us a negative energy density smaller in magnitude than the squeezed vacuum state. So now we will analyze the quantum inequality using Eq. 54, except we are not taking $x = t = 0$, so we have an extra exponential term:

$$\langle T_{00}\rangle = \frac{\omega^2+k^2}{4\omega V}\left[e^{2i(kx-\omega t)}\sinh r\cosh r + e^{-2i(kx-\omega t)}\sinh r\cosh r + 2\sinh^2 r\right]. \tag{71}$$

We can then take $x = 0$, and use our values for ω, k, and V so that our quantum inequality becomes

$$\int_{-\infty}^{\infty}\frac{\pi}{2l^4}\left[(e^{\frac{-2\pi it}{l}} + e^{\frac{2\pi it}{l}})\sinh r\cosh r + 2\sinh^2 r\right]\left(\frac{\tau}{t^2+\tau^2}\right)dt \geq -\frac{27}{2048\pi^2\tau^4} \tag{72}$$

which then becomes

$$\frac{\pi^2}{l^4}\sinh r\left(e^{\frac{-2\pi\tau}{l}}\cosh r + \sinh r\right) \geq -\frac{27}{2048\pi^2\tau^4} \tag{73}$$

Since our energy density is the same as before, only the state has changed, we can use the same value for $\frac{\tau}{l}$, which is around 0.3. Here, we are in fact finding the lowest energy density eigenstate given by the diagonalization method.

Then this function will be at a minimum at $r = -0.06$, which when plugging everything in gives us

$$-0.04 \geq -0.1. \tag{74}$$

which is clearly true. we can see that while our squeezed vacuum state did bring us closer to the value for the inequality, we are still nowhere near saturating it. Since here we are looking at a system with only one mode excited, it is unsurprising that our calculated negative energy is smaller in magnitude than the lower bound, since we know our state will not be maximal across all systems. It is also important to note that this inequality has not been proven to be optimal, suggesting that it might not be possible to find a state that saturates this inequality.

Now suppose we want to look at this example in free space. Now, we assume that $V \to \infty$, which means that instead of discrete modes, we are looking at a continuous band of modes. So our energy density requires an integral over k, and $\langle\psi|T_{00}|\psi\rangle$ becomes

$$\langle\psi|T_{00}|\psi\rangle = \int \frac{\omega^2 + \vec{k}^2}{32\omega\pi^3(1+\epsilon^2)} \left[-2\sqrt{2}\epsilon + 4\epsilon^2\right] d^3k. \tag{75}$$

Then we could use the same procedure to find the quantum inequality for this state, find τ from the length of time that the energy density is negative, and then check to see if the quantum inequality holds. However, complicated nature of this integral puts this procedure outside the scope of this thesis.

The restrictions on $f(t)$ and $g(t)$ mean that these inequalities will not work for a sampling function with instantaneous switching. However, the uncertainty principle in quantum mechanics shows that any physical device is incapable of physical switching, so this is not a problem [10].

7 Examples of Corresponding Physical Systems

While our state earlier was chosen for simplicity, the single mode state does actually have a physical representation. For example, a single standing E&M wave in a cavity. This is analogous to the classic "particle-in-a-box" problem in quantum mechanics, where the boundary conditions on the wave function require that the wave function is 0 at the boundary. Then, when paired with the continuity requirement of the wave function and its derivative, the boundary conditions restrict the shape of the wave function to sinusoidal with an integer number of anti-nodes. This corresponds to the discrete energy levels of the quantum harmonic oscillator. In this section we will examine the physical validity of the theoretical results we have found.

For our standing wave in a cavity example, we must consider how exactly to determine the average energy density, since with current technology it remains impossible to simply take that measurement directly. One way to do this is to look at $\langle E^2 \rangle$, the expectation value of the electric field squared [11]. In certain cases, $\langle E^2 \rangle$ will actually correspond to negative energy density. In general, periods of $\langle E^2 \rangle < 0$ do not directly correspond with a negative energy density, since the magnetic field will also contribute to the energy. However, it is still valuable to look at $\langle E^2 \rangle$ on its own, since this is another quantity that

is classically always positive. In this case, looking at $\langle E^2 \rangle$ is easier than looking at energy density, because looking at the energy density would also require finding $\langle B^2 \rangle$, which is not straightforward for the system we are describing here. A squeezed state such as the one examined earlier, the vacuum and two particle state, will allow for the possibility of a negative $\langle E^2 \rangle$ value [11].

The next thing to consider is how exactly to probe $\langle E^2 \rangle$. Ford and Roman suggested using a beam of excited atoms, which will then decay while going through the cavity [7]. In theory, being exposed to a negative $\langle E^2 \rangle$ will increase the lifetime of the particles (rather than decreasing the lifetimes, as with a regular electric field). Then one can observe the outflux of particles from the other side of the cavity, and see if more remain than one might expect based on a classical analysis.

It is important to pause here for a moment and examine the nature of how vacuum fluctuations relate to spontaneous emission. This relation can be seen by examining the intensity of the spontaneous emission from a purely quantum mechanical standpoint [12]. In a classical or semi-classical formulation, one can see that a highly excited particle will release radiation. However, in a quantum mechanical formulation, one can see that particles at lower energy levels will also undergo spontaneous emission, although it is impossible for a particle in the ground state to due so [12]. Both of these effects are dependent on the vacuum fluctuations of a particle. For example, the intensity of the spontaneous radiation of a two-level system will look like:

$$I = I_0 \left[\frac{1}{2} + (n_+ - n_-) \frac{\overline{p^2} + \omega^2 \overline{q^2}}{2\hbar\omega} \right] \tag{76}$$

where p, q, and ω are the usual variables from the quantum harmonic oscillator, I_0 is the intensity of the spontaneous emission of the system in the + level, and the $n_\pm$ are the number operators of the + and − levels and can be either 0 or 1 [12]. When $n_+ = 0$ and $n_- = 1$, the system is in its ground state and the right hand side (the vacuum fluctuation term) will cancel exactly with the left hand side, leaving no spontaneous emission. However, when $n_+ = 1$ and $n_- = 0$, the vacuum fluctuation term contributes half of the overall intensity of the radiation.

This means that by suppressing the vacuum fluctuations, such as through a negative $\langle E^2 \rangle$, the intensity of the radiation will decrease, and thus the lifetime of the particles will be longer. So if we saw a higher flux of particles out of the cavity than one would expect without an electric field present, that is an indication that $\langle E^2 \rangle$ is negative.

Looking at the outflux of particles requires an understanding of how the probability of the particle decaying will change. This can be derived from perturbation theory, however the derivation is quite complicated and is beyond the scope of this thesis. The following different formulations for the probability are derived under the assumption that the speed of the atoms is such that we can ignore relativistic effects.

If the particle being sent through the cavity has only one mode, and the transit time

is short, then the de-excitation probability will look like [11]:

$$P = P_0 \left[1 + \frac{1}{f^2(\vec{x})} \left\langle E^2(\vec{x}, t) \right\rangle \right] \tag{77}$$

where P_0 is the original decay probability in the absence of the electric field and $f(\vec{x})$ is the spatial part of the mode function, which is taken to be real. So we can see that when $\left\langle E^2 \right\rangle$ is less than 0, the decay probability will be smaller (and thus the average particle lifetime extended).

As it turns out, quantum inequalities prevent $\left\langle E^2(\vec{x}, t) \right\rangle$ from being less than $-f^2(\vec{x})$. This means that we have no risk of seeing a negative probability [11]. While at first glance it seems possible to get a P greater than 1, which would suggest a problem in our formulation, in fact this formula is only valid when P_0 and $\left\langle E^2(\vec{x}, t) \right\rangle / f^2(\vec{x})$ are small enough that this is not an issue.

While this equation is only valid near resonance (because the formulation assumes decay into only one mode), which limits the applicability of the equation, the advantage of using this formulation is that there is no dependence on the form of the mode function.

We can also rewrite $\Delta P = P - P_0$ and reformulate it [11]. An advantage of this formulation is that it is valid even when the system is not near the resonant frequency. This new formulation looks like [11]:

$$\Delta P = |\left\langle \psi_2 | d_y | \psi_1 \right\rangle|^2 f^2 \left[\langle n \rangle \left(|I_1|^2 + |I_2|^2 \right) + 2 \sum_n \sqrt{(n+1)(n+2)} Re(c_n c^*_{n+2} I_1^* I_2) \right] \tag{78}$$

where I_1 and I_2 are functions of the time, and the mode functions f are real. ψ_1 and ψ_2 are the two different states of the atom, and d_y is the distance of the cavity in the y-direction.

This formulation will end up taking different forms depending on whether the atom is far below resonance, or far above resonance. This is because the I_1 and I_2 also depend on the energy difference between the two states, $\Delta\epsilon$. In this case, ΔP becomes [11]:

$$\Delta P \approx 2 |\left\langle \psi_2 | d_y | \psi_1 \right\rangle|^2 \frac{[1 - \cos \Delta\epsilon (t_1 - t_0)]}{\Delta\epsilon^2} \left\langle E^2(\vec{x}_0, t_0) \right\rangle . \tag{79}$$

In this scenario, ΔP is depending on the electric field, as expected. We also see that ΔP will be negative, and thus the particle lifetime longer, at the same time as when $\left\langle E^2(\vec{x}, t) \right\rangle$ is negative. Thus this formulation is handy because it offers a direct correlation between a negative $\left\langle E^2(\vec{x}, t) \right\rangle$ and a longer decay lifetime.

However, this equation is only valid below resonance. Now we will examine when the atoms are far above resonance, and when $\Delta\epsilon(t_1 - t_0) << 1$. In this case, ΔP becomes [11]:

$$\Delta P \approx \frac{2}{\omega^2}|\langle\psi_2|d_y|\psi_1\rangle|^2|f|^2\{2\langle n\rangle(1-\cos\omega(t_1-t_0)) \\ -\sum_n \sqrt{(n+1)(n+2)}Re\left(c_n c_{n+2}^*(e^{i\omega t_1}-e^{i\omega t_0})^2\right)\}} \tag{80}$$

where ω is the frequency, and n is the energy level.

It is important to note that this is not dependent on $\left\langle E^2(\vec{x},t)\right\rangle$ at all. So while ΔP can be negative, it is not clear what that means for $\left\langle E^2(\vec{x},t)\right\rangle$.

This suggests that if this experiment were actually being performed, it would be most useful to examine the situation when the frequency is near resonance, since this allows us to relate the probability to $\left\langle E^2(\vec{x},t)\right\rangle$ directly.

If this experiment were actually to be done, however, what else must be taken into consideration? For starters, the size of the cavity needs to be determined. The frequency of the standing wave in the cavity will be related to the size of the cavity. So if we assume the atoms being sent through the cavity are Rydberg atoms, with a transition between the $n = 51$ and $n = 50$ energy levels, then the length and width of the cavity need to be at least 3 mm each [11]. Then the height of the cavity needs to be big enough to allow the atoms to fit through it, which for a Rydberg atom of $n = 50$, requires a height of at least 100 nm [11].

However, it is uncertain whether or not this experimental set-up is possible with current technology. Common cavity sizes for cavity QED experiments are on the order of length ≈ 150 μm [13]. This is about an order of magnitude smaller than what we would require for our experiment. Cavity QED experiments are an area of active research, due to their applications to quantum computing, so we may soon see advances in technology allowing for this experiment to be done.

Now consider the application of this experiment to the example we looked at earlier, of the vacuum and two-particle state, $|\psi\rangle = \frac{1}{\sqrt{1+\epsilon^2}}(|0\rangle + \epsilon|2\rangle)$. This gives us [11]:

$$\left\langle E^2(\vec{x},t)\right\rangle = \frac{2\epsilon}{1+\epsilon^2}f^2(\vec{x}_0)[2\epsilon + \sqrt{2}\cos(2\omega t)] \tag{81}$$

which, of course, reaches it's lowest value when $\cos(2\omega t) = -1$, as we saw earlier.

If the particles are near resonance, then we see

$$P = P_0\left[1 + \frac{2\epsilon}{1+\epsilon^2}[2\epsilon + \sqrt{2}\cos(2\omega t)]\right]. \tag{82}$$

Setting $\cos(2\omega t) = -1$ and taking ϵ to be $\sqrt{3} - \sqrt{2}$, the value we found earlier that minimizes the negative energy density, we get

$$P = 0.6P_0. \tag{83}$$

This means that for our given state, we see approximately a 40% reduction in the decay probability. This is large enough to be visible in an experiment. However, this is for a state with only one mode excited, containing two photons at different energy levels. While in general squeezed states have been created in the lab, whether or not our specific state could be created requires further experimental analysis beyond the scope of this thesis.

8 Conclusions

While the idea of negative energy is enticing in the possibilities it offers, the reality is rather less exciting. Quantum inequalities place a strict limit on the amount of negative energy possible in a system, and these limits prevent any interesting effects, such as a violation of the second law of thermodynamics, from actually manifesting. However, the fact that negative energy is possible at all is an intriguing phenomenon that arises out of the already peculiar theory of QFT, and is worth further study.

Here, we have examined one method of possibly finding maximally negative energy states, and found that it gave reasonable results for the value of the negative energy density even in situations where diagonalization was not possible. Our method was also able to find series of states that would give close to the negative energy value, even when there was not a single maximal state.

While quantum inequalities will give us a limit on how great the magnitude of the negative energy can be, they do not give us the state that corresponds to this value. Additionally, the quantum inequalities have yet to be proven optimal for the 4D case, which means that any estimate of the negative energy density could very well be an overestimate.

We also examined the nature of the quantum inequalities, looking at examples of how they might be applied to actual states, and found that indeed, the limits given by the inequalities seemed to be significantly lower than the values we actually found for our states. Research on quantum inequalities is currently ongoing, and it is possible in the future that someone will find an optimal 4D inequality, which would allow us to examine the limits on negative energy in actual space-time.

At this time, it is unlikely that conducting an experiment to examine the negative energy densities found here is practical or cost effective. However, the fact that it is possible in theory leaves open the possibility for experimental verification in the future. Since cavity QED experiment is an active area of research, what is actually possible in experiments is advancing at a promising rate.

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