ON THE INTERIOR OF “FAT” SIERPINSKI TRIANGLES

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Abstract

For $0 < \lambda < 1$ we consider the compact invariant set ("attractor") Λ_λ of the iterated function system $\mathscr{F}_\lambda$ defined by the three maps $f_i = \lambda I + p_i$ on $\mathbb{R}^2$, where $p_0 = (0,0)$, $p_1 = (1-\lambda, 0)$, $p_2 = (1-\lambda)\cdot(1/2, 1)$. $\Lambda_{1/2}$ is the Sierpinski triangle.

For $\lambda \in [.6439, .6441] \cup [.6458, .6466] \cup [.6470, .6472]$ standard techniques for determining the Hausdorff dimension (or Lebesgue measure) of the "fat" Sierpinski triangle Λ_λ do not apply, and the Hausdorff dimension of Λ_λ has not been known for any specific such λ. For all these λ we show that Λ_λ has nonempty interior and hence positive Lebesgue measure and Hausdorff dimension 2.

The novelty of our approach is that instead of extending techniques developed for small λ we significantly extend geometric methods developed by Broomhead, Montaldi and Sidorov for larger values of λ.

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1 Introduction

The focus of this thesis is the geometry of fractals arising from iterated function systems. Fractals generated by these procedures have become common in artwork that many are familiar with. A good example of this is a fern leaf which upon examination it is seen that each part is roughly the same as the whole. In other words each leaf along the main stem looks identical to that of the entire fern and each leaf is composed of smaller leaves that again have the same shape. In real-life examples this pattern ends at some finite stage. This property of self-similarity consistently shows up in the fractals arising from iterated function systems, however, unlike the fern example, it continues indefinitely. Below is an example of a fern arising from an iterated function system.

Figure 1: Fractal Fern

1.1 Iterated function systems

The present form of iterated function systems was first presented by John E. Hutchinson in his 1981 paper "Fractals and Self Similarity" [5]. The idea was popularized by Michael Barnsley's book *Fractals Everywhere* and Kenneth Falconer's book *Fractal Geometry* [1, 4].

Definition 1.1. *An iterated function system is a finite family* $\{f_0, f_1, .., f_k\}$ *of contractions on* $\mathbb{R}^n$ *with* $k \geq 1$, *i.e., there are* $c_i \in (0,1)$ *such that*

$$|f_i(x) - f_i(y)| \leq c_i |x - y| \text{ for all } x, y$$

Associated to each iterated function system is an invariant set we call the attractor.

Theorem 1.2. *For any iterated function system, there exists a unique nonempty, compact set* Λ*, called the attractor of the iterated function system, such that*

$$\Lambda = \bigcup_{i=0}^{k} f_i(\Lambda)$$

The original research in this area was done on iterated function systems for which there is no significant overlap of the components of the corresponding fractal [5]. For example the standard Sierpinski triangle consists of three copies of itself that do not overlap.

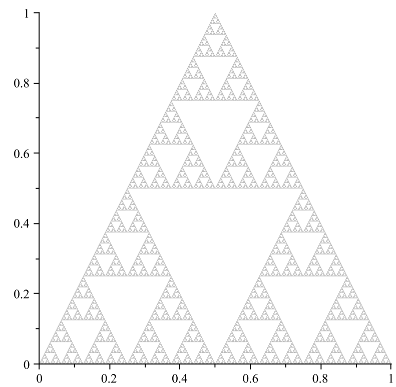


Figure 2: "Standard Sierpinski Triangle"

Broomhead, Montaldi, and Sidorov appear to be the first to have considered a family of iterated function systems for which substantial overlaps occur [2]. For such systems the properties of the resulting fractal are not completely understood. In particular, there are no standard techniques for computing the Hausdorff dimension or Lebesgue measure of such fractals. Therefore our approach to investigate these two properties for attractors with significant overlap focuses on studying the attractors interior. In this way we are able to provide new information about the Hausdorff measure and Lebesgue measure of the attractor we study.

1.2 The "fat" Sierpinski triangle

We consider the family of iterated function systems consisting of the following three linear contractions in the plane:

$$
\begin{aligned}
f_0(x,y) &= (\lambda x, \lambda y) \\
f_1(x,y) &= (\lambda x + (1-\lambda), \lambda y) \\
f_2(x,y) &= (\lambda x + \frac{1}{2}(1-\lambda), \lambda y + (1-\lambda))
\end{aligned}
\tag{1}
$$

with $\lambda \in (0,1)$. Alternatively we can write this as

$$
\begin{aligned}
f_0 &= \lambda I \\
f_1 &= \lambda I + (1-\lambda, 0) \\
f_2 &= \lambda I + (1-\lambda)(1/2, 1)
\end{aligned}
\tag{2}
$$

We denote the attractor of the iterated function systems by Λ_λ. We also denote by Δ the convex hull of the fixed points of this system, i.e., the triangle with vertices at $(0,0)$, $(1,0)$, and $(\frac{1}{2}, 1)$.

1.2.1 Thin attractor: Open-set condition

$\Lambda_{\frac{1}{2}}$ is the standard Sierpinski triangle; it consists of three copies of itself each shrunk by a factor of $\frac{1}{2}$. Here we see that for each pair of distinct indices i, j, the intersection $f_i(\Delta) \cap f_j(\Delta)$ is a single point. For $\lambda < \frac{1}{2}$, the intersections $f_i(\Delta) \cap f_j(\Delta), i \neq j$ are empty. This implies that for $\lambda \in (0, \frac{1}{2})$, the attractor is totally disconnected and has zero Lebesgue measure and thus empty interior, i.e. it is a Cantor set.

The following open-set condition on an iterated function system guarantees that all overlaps between the individual components of the attractor are insignificant.

Definition 1.3. *An iterated function system $\{f_i\}_{i=0}^n$ satisfies the* **open-set condition** *if there exists a nonempty open-set O such that $\bigcup_{i=0}^n f_i(O) \subset O$, the union being disjoint.*

The open-set condition is central to the standard approaches of calculating properties such as Hausdorff dimension. For instance,

Theorem 1.4 (see [4]). *Suppose the open-set condition (1.3) holds for the iterated function system* $\{f_0, f_1, \ldots, f_k\}$ *on* $\mathbb{R}^n$ *with contraction rates* $c_i \in (0,1)$ *such that*

$$|f_i(x) - f_i(y)| = c_i|x - y| \textit{ for all } x, y \in \mathbb{R}^n.$$

Then the Hausdorff dimension of its attractor is equal to the box counting dimension and is given by s where s is the unique solution of

$$\sum_{i=0}^{n} c_i^s = 1.$$

For $\lambda \in [0, \frac{1}{2}]$ the iterated function systems given by (1) satisfy the open-set condition [2], so the Hausdorff dimension of Λ_λ is well known by Theorem 1.4.

Corollary 1.5. *For* $\lambda \in [0, \frac{1}{2}]$ *the Hausdorff dimension of* Λ_λ *is given by*

$$\dim_H(\Lambda_\lambda) = -\frac{\log 3}{\log \lambda}$$

For iterated function systems that do not satisfy the open-set condition it takes a considerable amount of work to calculate quantities like the Hausdorff dimension or the Lebesgue measure of the corresponding attractor. In fact for many iterated function systems that do not satisfy the open-set condition it is not possible to calculate these properties with current techniques. We make some progress in this problem by presenting a new way to determine whether the attractor for an iterated function system has nonempty interior.

1.2.2 When the open-set condition does not hold

For values of $\lambda > \frac{1}{2}$, the intersection $f_i(\Delta) \cap f_j(\Delta)$ is always a triangle. Thus for these values of λ the components of the attractor overlap and compared to the standard Sierpinski trianglethe attractor appears "fat". This changes the attractor significantly. In fact, for $\lambda \in (\frac{1}{2}, 1)$ it was shown that this family of iterated function systems does not satisfy the open-set condition [2]. Therefore the standard techniques used to investigate the attractor do not apply, and we restrict attention to $\lambda > \frac{1}{2}$.

Although the open-set condition does not hold for $\lambda > \frac{1}{2}$, a standard calculation shows that the Lebesgue measure is 0 for a substantially larger set of parameters than $(0, \frac{1}{2}]$. This calculation may be applied to any iterated function system and thus is included below.

Proposition 1.6 (see [4]). *Let $\{f_0, f_1, \ldots, f_k\}$ be an iterated function system on $\mathbb{R}^n$, and $\bar{\lambda} := \max\{c_0, c_1, \ldots, c_k\} < k^{-n}$. Then its attractor Λ has zero Lebesgue measure.*

Proof. $\Lambda = \bigcup_{i=0}^{k} f_i(\Lambda)$ implies

$$m(\Lambda) \leq (\sum_{i=0}^{k} m(f_i(\Lambda)) \leq k\bar{\lambda}^n m(\Lambda)$$

Since Λ is bounded we know that $m(\Lambda) < \infty$. Therefore if $m(\Lambda) \neq 0$ then $k\bar{\lambda}^n \geq 1$. □

Corollary 1.7. *By Proposition 1.6 the attractor Λ_λ has zero Lebesgue measure and thus empty interior for $\lambda < \frac{1}{\sqrt{3}}$.*

By Corollary 1.7 we see that the range of parameters where Λ_λ may have nonempty interior is for $\lambda \in [\frac{1}{\sqrt{3}}, 1]$. We define a hole in the attractor to be a bounded connected component of its complement. Upon inspection it is easy to see that for $\lambda \geq \frac{2}{3}$ there are no holes in the attractor, that is $\Lambda_\lambda = \Delta$. For these values of λ the union of the three images of Δ under the iterated function system is itself. We therefore further restrict attention to the parameters $\lambda \in (\frac{1}{\sqrt{3}}, \frac{2}{3})$ for which the attractor is most interesting.

Broomhead, Montaldi, and Sidorov [2] were able to show that the attractor has nonempty interior for $\lambda \geq \lambda^*$ where $\lambda^* \approx 0.647798$ is the real root of $x^3 - x^2 + x = \frac{1}{2}$. They showed that for $\lambda \in [\lambda^*, \frac{2}{3})$ the holes in the attractor are in a radial position: they are all centered on three radial lines originating from the center of the attractor and extending to the three corners. It follows that the attractor has nonempty interior.

Below we define formally what it means for the attractor Λ_λ to be radial if $\lambda < \frac{2}{3}$. First we define three cones contained in Δ.

1. Let the *central hole* in Λ_λ be defined by $H_0 := \Delta \setminus \bigcup_{i=0}^{2} f_i(\Delta)$.

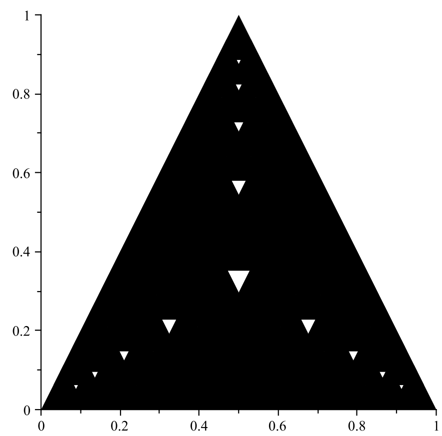


Figure 3: Radial Case

2. For $i \in \{0,1,2\}$ let C_i be the cone whose vertex is at the fixed point of the map f_i, which contains the central hole H_0, and whose edges contain two of the vertices of H_0.

Definition 1.8. *The attractor Λ_λ is said to be radial if*

$$\Lambda_\lambda \cup \bigcup_{i=0}^{2} C_i = \Delta,$$

that is, if all holes in Λ_λ are contained in one of the three cones C_0, C_1, or C_2.

Proposition 1.9 ([2])**.** *Λ_λ is radial for $\lambda \in [\lambda^*, \frac{2}{3})$.*

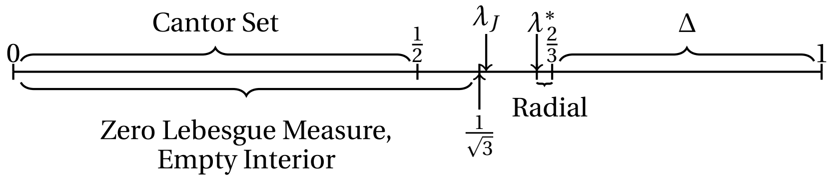
1.2.3 Remaining parameters

Including the interval in Proposition 1.9 it is known that the attractor has nonempty interior for $\lambda \in [\lambda^*, 1)$. We aim to extend this interval for which the attractor is known to have nonempty interior. For a countable set of parameter values called "multinacci" numbers $\lambda = \omega_m$, Broomhead, Montaldi, and Sidorov have shown that Λ_λ has zero Lebesgue measure and thus empty interior [2]. The interval we are interested in, $[\frac{1}{\sqrt{3}}, \lambda^*)$, contains only one multinacci number, given by the reciprocal of the golden ratio $\lambda = \omega_2 \approx 0.618$. Sidorov conjectures that the largest value of λ such that the attractor has empty interior is ω_2 [8].

Proposition 1.10 ([2]). *For $m \geq 2$ the attractor Λ_{ω_m} has zero Lebesgue measure, where ω_m is the unique positive root of $\lambda^m + \lambda^{m-1} + \cdots + \lambda = 1$.*

In this thesis we find nontrivial intervals below λ^* contained in the set of parameters for which the attractor has nonempty interior. We do this in a way that may be applied to the attractor associated to any iterated function system.

We can represent prior knowledge pertinent to the interior of fat Sierpinski triangles by the diagram below.



Jordan has given a result which is encouraging if we hope to show nonempty interior on $[\frac{1}{\sqrt{3}}, \lambda^*)$.

Theorem 1.11 (Jordan [6]). *There is a parameter $\lambda_J \approx 0.585$ such that Λ_λ has positive Lebesgue measure and Hausdorff dimension 2 for a.e. $\lambda \geq \lambda_J$.*

Jordan conjectures that the set of values in this interval for which Λ_λ has zero measure and Hausdorff dimension less than 2 is much more than just the multinacci numbers defined above. One previously open question asks if this set of exceptional values is dense in $[\lambda_J, \lambda^*]$. In this thesis we provide an answer to this question with our Main Theorem. Regardless of whether Jordan's conjecture is true, the existence of the multinacci numbers in the interval he found implies that the Lebesgue measure of the attractor does not increase monotonically as λ increases.

Although Theorem 1.11 may sound promising, there are iterated function systems in the plane for which the attractor has empty interior and positive Lebesgue measure. Two different examples have been given using 10 contractions in one case and 6 in the other [3]. It is interesting to note that no similar results have been shown for iterated function systems in $\mathbb{R}$. To construct an iterated function system in $\mathbb{R}$ having a self-similar attractor with positive measure but empty interior, the system must have components which overlap. Sidorov has made some progress with the understanding of iterated function systems in the one dimensional case, however, the question whether or not there exists one whose attractor has positive measure and empty interior remains open [9]. This illustrates an important fact about the attractors resulting from our family of iterated function systems. Although we know the attractor has positive measure for a.e. $\lambda \geq \lambda_J$ it is not known if it has nonempty interior. The smallest value for which it was previously known Λ_λ had nonempty interior was given by the radial case with $\lambda = \lambda^*$.

We push the idea behind the proof that the attractor is radial for $\lambda \in [\lambda^*, \frac{2}{3})$ to show that the attractor has nonempty interior in a larger range of values. In the proof of Proposition 1.9 Broomhead, Montaldi, and Sidorov show λ must be larger than two distinct parameters in order for the holes in the attractor to be radial. The larger of the two parameters is λ^* and thus Λ_λ is radial for all $\lambda \geq \lambda^*$. The smaller of the two parameters is given by $\lambda_* \approx .6422$ where λ_* is the positive root of $3\lambda^4 - 3\lambda^3 + 2\lambda - 1 = 0$. Our inspiration for looking at the attractor in greater detail for $\lambda \in [\lambda_*, \lambda^*]$ came by observing computer-generated approximations to the attractor for λ in this range. For these values we noticed that the attractor appears to have holes in what we call a generalized radial pattern. However, for $\lambda < \lambda_*$ we observed that this is not true.

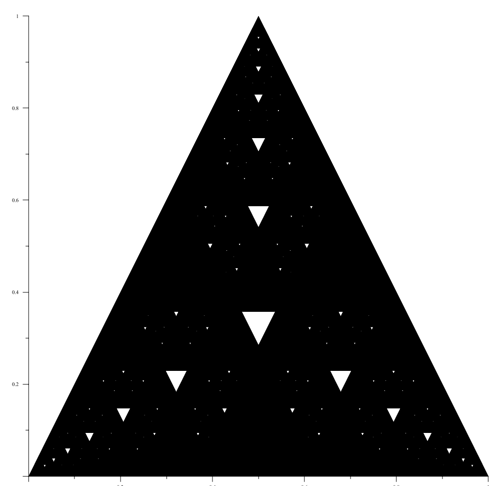


Figure 4: Generalized Radial Case

Definition 1.12. *We say that the attractor Λ_λ is generalized radial if $f_i(\Delta) \cap f_j(\Delta) \subset \Lambda_\lambda$ for $i \neq j$.*

Figure 4 shows no holes in $f_i(\Delta) \cap f_j(\Delta)$ for $i \neq j$. The fact that the attractor has this property implies that it has nonempty interior. It is also known that for $\lambda < \lambda_*$ there is a hole contained in $f_0(\Delta) \cap f_1(\Delta)$.

Proposition 1.13. *For $\lambda < \lambda_*$ the attractor Λ_λ is not generalized radial.*

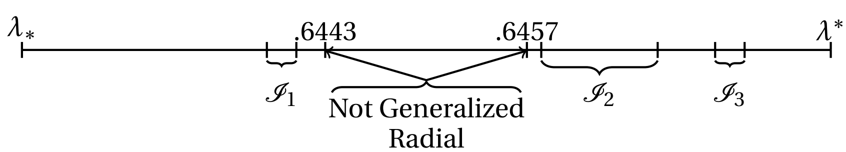
We now give our main result which shows the attractor of this family of iterated function systems has nonempty interior on a substantial subset of $[\lambda_*, \lambda^*)$.

Main Theorem. *Consider the iterated function system defined by the three maps in equation* (1)*. Let $\mathscr{I}_1 := [.6439, .6441]$, $\mathscr{I}_2 := [.6458, .6466]$, and $\mathscr{I}_3 := [.6470, .6472]$. Then for $\lambda \in \mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3 \subset [\lambda_*, \lambda^*)$, Λ_λ is generalized radial and hence has nonempty interior. For $\lambda = .6443$ and $\lambda = .6457$, Λ_λ is not generalized radial.*

Corollary 1.14. *For $\lambda \in \mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3$ the attractor Λ_λ has positive Lebesgue measure and*

$$\dim_H(\Lambda_\lambda) = 2.$$

This gives a set of intervals contained in $[\lambda_J, \lambda^*]$ for which $\dim_H(\Lambda_\lambda) = 2$. This answers the previous question that asked if the set of values for which the Hausdorff dimension is less than 2 is dense in $[\lambda_J, \lambda^*]$. The range of values for λ given in the Main Theorem and Corollary 1.14 is pictured below.



We notice that $\lambda = .6443$ lies close to the right endpoint of the interval $\mathscr{I}_1$ and that $\lambda = .6457$ lies close to the left endpoint of the interval $\mathscr{I}_2$. These two values of λ seem to give rise to the same hole that is not part of the generalized radial pattern. Upon inspection this hole appears to be in the attractor for all values of $\lambda \in (.64415, .64578)$.

Conjecture 1.15. *For $\lambda \in (.64415, .64578)$ the holes in the attractor Λ_λ of the iterated function system given by equation* (1) *are not in a generalized radial pattern.*

However, proving this conjecture requires showing that the hole is persistent throughout the conjectured values for λ. If this conjecture is true it does not imply that the attractor has empty interior. In fact, even though the attractor may not be generalized radial it still may have nonempty interior. Thus the attractor Λ_λ changes from generalized radial to not generalized radial and back as we change the parameter λ. It is an open question whether the interior of an attractor can vary between empty and nonempty in ways as intermittent as we saw previously the Lebesgue measure does.

To prove our main result we use the following construction of the attractor.

Proposition 1.16 (See [4])**.** *Let Δ be the triangle with vertices at the fixed points of the iterated function system. For $0 \leq n < \infty$ define the n-th level set of the attractor by*

$$\Delta_n := \bigcup_{i=0}^{2} f_i(\Delta_{n-1})$$

where $\Delta_0 = \Delta$. Then,

$$\bigcap_{n=0}^{\infty} (\Delta_n) = \Lambda_\lambda.$$

The criterion we use to prove our main result involves the construction of a subset of the attractor that maps onto itself by the iterated function system. Although we used this result to show nonempty interior for the fat Sierpinski triangle this idea can be applied to the attractor of any iterated function system.

Inclusion Theorem. *If $N \subseteq \Delta$ and $N \subseteq f_0(N) \cup f_1(N) \cup f_2(N)$, then*

$$N \subseteq \Lambda_\lambda.$$

Proof. By Induction:

We show $N \subseteq \Delta_n$ for all $n \geq 0$ because this implies $N \subseteq \bigcap_{n=0}^{\infty} \Delta_n = \Lambda_\lambda$.

Base Case: Let $n = 0$. Then $N \subseteq \Delta_0 = \Delta$.

Induction step: If k is such that $N \subseteq \Delta_k$, then

$$N \subseteq f_0(N) \cup f_1(N) \cup f_2(N) \subseteq f_0(\Delta_k) \cup f_1(\Delta_k) \cup f_2(\Delta_k) = \Delta_{k+1} \qquad \square$$

We apply the Inclusion Theorem in the proof of the Main Theorem by checking the following:

Criterion 1.17. *There exists $N \subset \Delta$ with nonempty interior such that $N \subseteq f_0(N) \cup f_1(N) \cup f_2(N)$.*

2 Proof of the Main Theorem

2.1 The attractor is not generalized radial for $\lambda \in \{.6443, .6457\}$

We first prove the second part of the Main Theorem which states that for $\lambda = .6443$ and $\lambda = .6457$ the attractor Λ_λ is not generalized radial. A calculation using the mathematics program Maple 13 shows

- If $\lambda = .6443$ then $(\frac{1}{2}, .1345) \in I_0$ is not in $\Delta_8 \supset \bigcap_{n=0}^{\infty} \Delta_n = \Lambda_\lambda$.
- If $\lambda = .6457$ then $(\frac{1}{2}, .1351) \in I_0$ is not in $\Delta_8 \supset \bigcap_{n=0}^{\infty} \Delta_n = \Lambda_\lambda$.

We discuss in greater detail the accuracy of these calculations in section 2.4.

2.2 Choice of the included set

The majority of the work involved to prove the Main Theorem goes into proving that a subset N satisfies Criterion 1.17 even as the contraction rate is varied. Figure 5(a) shows an approximation to the set N that we found.

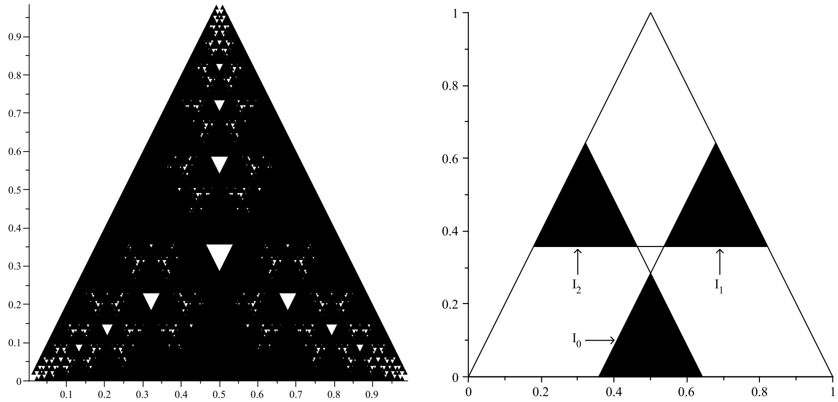
We now begin the construction of the set N which satisfies Criterion 1.17. For $\lambda \in (\frac{1}{2}, \frac{2}{3}]$ let I_0, I_1, I_2 be the Primary Overlaps of the iterated function system as in Figure 5(b), that is,

$$I_0 = f_0(\Delta) \cap f_1(\Delta),$$

$$I_1 = f_1(\Delta) \cap f_2(\Delta),$$

$$I_2 = f_2(\Delta) \cap f_0(\Delta).$$

Now we define N to be the union of all forward images by our iterated function system of the primary overlaps $I_0, I_1,$ and I_2.



(a) Subset of N to 8 levels (shown in black) (b) Primary Overlaps I_0, I_1, I_2 (shown in black)

Figure 5: $N \subset \Lambda$

Definition 2.1. *We call each forward image of I_0, I_1, and I_2 by our iterated function system a tile.*

Upon completion of showing this union of tiles satisfies Criterion 1.17 we will have shown N is contained in the attractor Λ_λ. In particular, the primary overlaps I_0, I_1, and I_2 are contained in Λ_λ and therefore by Definition 1.12 the attractor is generalized radial.

In order to apply Criterion 1.17 we must show that $N \subseteq f_0(N) \cup f_1(N) \cup f_2(N)$. We continue with the proof of the Main Theorem by showing that N has this property.

We examine the set N by breaking it down into tiles of different rank. We say I_0, I_1, and I_2 are tiles or rank 0, tiles of the form $f_j(I_i)$ are tiles of rank 1, tiles of the form $f_k(f_j(I_i))$ are tiles of rank 2, and so on where $i, j, k \in \{0, 1, 2\}$. It is clear that under the iterated function system the set of all rank n tiles maps onto the set of all rank $n+1$ tiles. To show N fully maps onto itself it remains to show that there exists tiles that map onto the rank 0 tiles I_0, I_1, and I_2. Thus we need to find a covering of I_0, I_1, I_2 by tiles, that is by images of I_0, I_1, I_2 under the iterated function system. Since the iterated function system is symmetric about 3 axes we need only look for a cover of any one of the three Primary Overlaps. In fact we only need to find a cover for the left half of I_0.

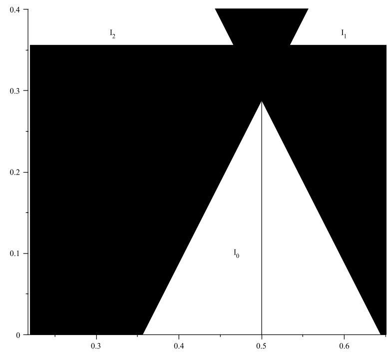


Figure 6: It remains to cover the left half of I_0

2.3 Notations and parametrization

2.3.1 Addresses

In order to clarify the notion of tiles we introduce the following notation.

Definition 2.2. *For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1,2\}^n$ we define*

$$f_\epsilon = f_{\epsilon_1} \circ \cdots \circ f_{\epsilon_n}$$

We say ϵ is the address for the map f_ϵ.

Thus each tile in N is of the form $f_\epsilon(I_l)$ where $l \in \{0,1,2\}$. Many of the tiles in N are associated to maps with very long addresses.

Remark 2.3. *Equation* (2) *shows that $f_\epsilon = \lambda^n I + p(\lambda, \epsilon)$, where $p(\lambda, \epsilon) \in \Delta$ is given by an $\mathbb{R}^2$-valued polynomial in λ of degree at most n. Therefore any two tiles of the same rank are translates of each other.*

Because the rank of some tiles is very large and we use few of them, we avoid referring to the tiles by their address and instead adopt a two-letter label for each tile. The two letters have a different meaning and to distinguish this we capitalize the first letter in the label. This capital letter refers to the rank of each tile. Thus any two tiles whose labels start with the same capital letter are of the same rank and therefore the same size. As we advance through the alphabet by one letter the rank of the corresponding tiles increases by at least one. For example a tile whose label starts with B has a greater rank than one whose label starts with A. This implies that as the first capital letter in the label advances through the alphabet the corresponding tiles decrease in size.

We use the second (and lower case) letter in the label to distinguish which tile of a given rank we are referring to. For example Aa is the label for a tile of rank 1 in our cover N. In particular Aa is the label for the tile given by $f_0(I_1)$. The labels for each tile and their associated address are listed together in a table at the end of this thesis.

2.3.2 Vertices

We will also adopt the following notation for each vertex of a tile. Given a tile we will denote using subscripts the bottom left vertex by l, the bottom right vertex by r, and the top vertex by t. For example Ba_l is the bottom left vertex of the tile Ba.

We also use the projections $\pi_x : (x, y) \mapsto x$ and $\pi_y : (x, y) \mapsto y$. For example $\pi_x Ba_l$ is the x-coordinate of the bottom left vertex of the tile Ba.

2.3.3 Sides

To designate each side of a given tile we will use the subscripts ls, rs, and bs to stand for the left side, right side, and bottom side respectively. For example, $Ba_{rs} \cap Aa_{bs}$ represents the point of intersection between the right side of tile Ba and the bottom side of tile Aa, and $\pi_y(Ba_{rs} \cap Aa_{bs})$ is the y-coordinate of the point given by this intersection of lines.

2.3.4 Overlaps

We clarify what is meant by an overlap of two tiles in N. First note that any tile in N is an image of one of the primary overlaps by f_ϵ for some $\epsilon \in \{0,1,2\}^n$. Since each map in the iterated function system is a contraction, this implies that f_ϵ is also a contraction as it is a composition of those maps. Therefore any tile in N has sides with slope -2, 2, and 0. This implies that there are only three possibilities for the intersection of two tiles in N. Either the area of intersection is empty, a singleton, or has interior. We say that two tiles in N overlap when their intersection has interior.

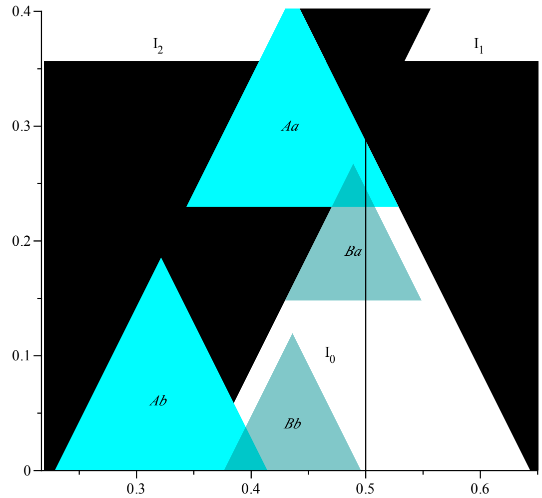


Figure 7: First 4 tiles: $Aa = f_0(I_1)$, $Ab = f_0(I_0)$, $Ba = f_{10}(I_2)$, $Bb = f_{01}(I_0)$

2.4 Initial covering

We can see from Figure 7 that a large portion of the left half of I_0 is covered by four tiles Aa, Ab, Ba, and Bb. However, a single picture only represents Λ_λ for one specific value of λ. For consistency, all pictures of the covering N were computed for $\lambda = 0.6439$ except where otherwise noted. We will check that that the position of the tiles with respect to each other is as pictured in Figure 7 and remains the same for all values of $\lambda \in \mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3$. The relative position of many of the tiles with respect to each other remains the same over the entire interval $\mathscr{I} := [.6439, .6472] \supset \mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3$.

In order to compare the relative position of any two tiles in N we compare their vertices to each other. Since all tiles are iterates by the iterated function system of the primary overlaps I_0, I_1, and I_2 we first investigate the vertices of those three triangles. Since the primary overlaps are defined by the intersection of any two images of Δ by maps from our iterated function system, we observe that their vertices are all given by linear expressions in λ. For example the three vertices of I_0 are given by

$$I_{0_l} = (1-\lambda, 0)$$
$$I_{0_r} = (\lambda, 0)$$
$$I_{0_t} = (\frac{1}{2}, 2\lambda - 1)$$

Since each tile in N is of the form $f_\epsilon(I_l)$, where $l \in \{0,1,2\}$, this implies that the x and y coordinates of its vertices are given by polynomials in λ. Since the vertices of rank 0 tiles are given by linear expressions in λ it follows that the vertices of rank n tiles are given by polynomials of degree at most $n+1$.

Example 1. *$\pi_x Ab_r = \lambda^2$, and $\pi_y Ab_r = 0$ for all $\lambda \in (\frac{1}{2}, \frac{2}{3}]$.*

Example 2. *$\pi_x Bb_l = -\lambda^3 + \lambda$, and $\pi_y Bb_l = 0$ for all $\lambda \in (\frac{1}{2}, \frac{2}{3}]$.*

Thus, we can check the relative position of any two tiles simply by comparing polynomials in λ and we can recast this as verifying the positivity of polynomials. For example, we verify in Claim 2.5 that the bottom right vertex of Ab is to the right of the bottom left vertex of Bb and that they have the same y-coordinate by verifying the following:

$$\pi_x Ab_r > \pi_x Bb_l \text{ and } \pi_y Ab_r = \pi_y Bb_l.$$

We do so using that

$$\lambda^3 + \lambda^2 - \lambda > 0 \text{ and } \pi_y Ab_r = 0 = \pi_y Bb_l \tag{3}$$

for $\lambda \in \mathscr{I}$.

For each of the claims that follow in this thesis we give the inequality that we check and the corresponding polynomial that we show to be positive across a given interval. Each polynomial may be verified by hand by calculating the vertices of the corresponding tiles.

2.4.1 Polynomial inequalities

Checking for positivity of a degree-3 polynomial across an interval is straightforward. However, to completely cover I_0 we will need to check for positivity of polynomials of much higher degree. Each polynomial encountered in this thesis has coefficients at most 2. This bound on the coefficients gives us the following upper bound on the absolute value of the derivative of any polynomial we encounter.

$$\frac{d}{d\lambda}\left(\sum_{n=0}^{\infty} 2\lambda^n\right)\Big|_{\lambda=\frac{2}{3}} = \frac{2}{(1-\lambda)^2}\Big|_{\lambda=\frac{2}{3}} = 18 < 20 \tag{4}$$

We use this bound on the derivative to confirm the minimum value of a polynomial $p(\lambda)$ is positive across any interval say $[L1, L2]$. This is done using the following algorithm.

1. Define $m := \min\{p(L1), p(L2)\}$.
2. Calculate the length of the interval $L := L2 - L1$.
3. Let $n := \lceil \frac{2\times 20L}{|m|} \rceil$ be the number of equal-length steps to evaluate the polynomial across the interval $[L1, L2]$. We call n the mesh size.
4. Define the step size by $\delta := \frac{L}{n}$.
5. Define $m_* = \min_{0\le k\le n} p(L1 + k\delta)$. If $m_* < m$ redefine $m := m_*$ and restart at step 3, otherwise stop and report m.

Theorem 2.4. *If this algorithm terminates and reports $m > 0$ then*

$$\min_{\lambda\in[L1,L2]} p(\lambda) \in [m/2, m].$$

Proof. By assumption

$$m \le p(L1 + k\delta)$$

for $0 \le k \le n$, where $n := \lceil \frac{2\times 20L}{m} \rceil$ and $\delta = \frac{L}{n}$.

Then by equation (4)

$$\begin{aligned} m \geq \min_{\lambda\in[L1,L2]} p(\lambda) \geq \min_k p(L1+k\delta) - 20\delta \geq m - 20\delta \\ = m - 20\frac{L}{n} \\ = m - 20\frac{L}{\lceil \frac{2\times 20L}{m} \rceil} \\ \geq m - \frac{L}{\frac{2\times 20L}{m}} \\ = \frac{m}{2}. \end{aligned}$$

□

This algorithm is used in each of the following claims in this thesis to show each polynomial is positive across a given interval. For each of these polynomials we report the minimum estimate m across that interval as well as the mesh size n. The minimum of each polynomial checked in this thesis was given at one of the two endpoints of each interval. This is a result of the polynomials being essentially monotone on small intervals even though they are not monotone polynomials. Therefore the algorithm we used to determine the minimum estimate always terminated after one run. Thus our minimum estimate for each polynomial on the interval $[L1, L2]$ is given by $\min\{p(L1), p(L2)\}$.

Claim 2.5. *The tiles in the Figure 7 stay in the same relative position as pictured for $\lambda \in \mathscr{I}$.*

Proof.

$$\lambda^3 + \lambda^2 - \lambda > 0$$

(minimum estimate 0.0376727925 with mesh size 4) implies

$$\pi_x Ab_r > \pi_x Bb_l.$$

$$\lambda^2 - 2\lambda + 1 > 0$$

(minimum estimate 0.1244678400 with mesh size 2) implies

$$\pi_x (I_0)_l > \pi_x Ab_l \text{ and } \pi_y Aa_t > \pi_y (I_0)_t.$$

$$-\lambda^3 + \lambda^2 - \lambda + \frac{1}{2} > 0$$

(minimum estimate 0.0005765740 with mesh size 229) implies

$$\frac{1}{2} > \pi_x Bb_r.$$

$$\lambda^3 + \frac{1}{2}\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} > 0$$

(minimum estimate 0.0084191875 with mesh size 16) implies

$$\pi_x(Ba_{rs} \cap Aa_{bs}) > \frac{1}{2}. \qquad \square$$

2.4.2 Recurring ideas

In this subsection we present some recurring ideas that are used to check the relative positions of the tiles in the cover N.

Remark 2.6. *Let $\epsilon, \gamma \in \{0,1,2\}^n$ and write $\epsilon = \epsilon_1\epsilon_2 \dots \epsilon_n$ and $\gamma = \gamma_1\gamma_2 \dots \gamma_n$. If for each $i \in \{1,\dots,n\}$ either $\epsilon_i = \gamma_i = 2$ or $\{\epsilon_i, \gamma_i\} \subset \{0,1\}$ and if ℓ is a horizontal line then*

$$f_\epsilon(\ell) = f_\gamma(\ell).$$

This is so because f_0, f_1 and f_2 map horizontal lines to horizontal lines and $f_0(\ell) = f_1(\ell)$ for any horizontal line ℓ. This remark gives a condition on any two maps which guarantees that the images of a horizontal line by the two maps coincide. This is used throughout this proof to show that the bottom sides of any two tiles in the cover lie on the same horizontal line. We show this fact about pairs of tiles by confirming that the property in Remark 2.6 holds for the addresses associated to the tiles. Another property we consider in more detail involves overlapping tiles in N.

Remark 2.7. *Let $\epsilon, \gamma, \upsilon \in \{0,1,2\}^n$ and $i, j \in \{0,1,2\}$. If $f_\epsilon(I_i)$ and $f_\gamma(I_j)$ overlap then so do $f_\upsilon(f_\epsilon(I_i))$ and $f_\upsilon(f_\gamma(I_j))$.*

This is just the obvious observation that if two tiles overlap then so do their images under the same map. This is used often as many pairs of tiles in the cover are images of the same pair for which we check overlap for $\lambda \geq \frac{1}{2}$. An example of this are the three tiles we call interior primary overlaps below. Many tiles in the cover are images of these three tiles and thus overlap each other.

Remark 2.8. *For $\lambda > \frac{1}{2}$ the three tiles given by $f_0(I_1), f_1(I_2)$, and $f_2(I_0)$ pairwise overlap as in Figure 8. We call these three tiles the interior primary overlaps. Likewise for $\lambda > \frac{1}{2}$ we also know that $f_i(I_i)$ overlaps with I_i and $f_{i+1 \pmod 3}(I_i)$ overlaps with I_i for $i \in \{0,1,2\}$.*

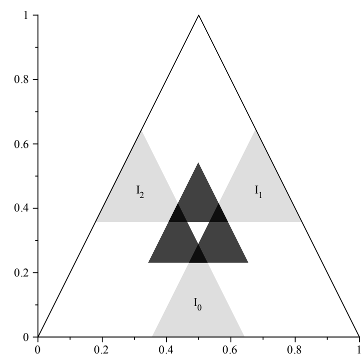


Figure 8: *Interior Primary Overlaps* $f_0(I_1)$, $f_1(I_2)$, and $f_2(I_0)$ shown in dark grey

2.5 Reduction to Areas 1, 2, and 3

We now focus our attention on Figure 9 which includes the tiles Ba, Ca, Da, Db, and Dc. We show that the relative position of these tiles is as pictured in Figure 9 for all $\lambda \in \mathscr{I}$. Since I_2 overlaps $f_0(I_2)$ and $f_0(I_2)$ overlaps $f_{00}(I_2)$ this implies by Remark 2.7 that $Ba = f_{10}(I_2)$ overlaps $Ca = f_{100}(I_2)$ and $Ca = f_{100}(I_2)$ overlaps $Da = f_{1000}(I_2)$. Also, the left sides of Ba, Ca and Da lie along the left side of the primary overlap I_0 since the left sides of $f_0(I_2), f_0(I_2)$ and $f_{00}(I_2)$ lie along the left side of Δ.

Similarly by Remark 2.7 $Bb = f_{01}(I_0)$ and $Cb = f_{011}(I_2) = f_{01}(f_1(I_2))$ overlap and their left sides lie on the same line since this is true of $f_1(I_2)$ and I_0.

By Remarks 2.7 and 2.8, since Ca, Db, and Dc are images of the interior primary overlaps, they also pairwise overlap. By Remark 2.6 the bottom vertices of Da, Dc, and Dd all lie on the same horizontal line, and likewise for Ca, Cb, and Db. One last relation that may be seen by the maps is that the right sides of Ca, Dc, and Cc all lie on the same line.

We now check that the relative position of each tile shown in Figure 9 is as pictured.

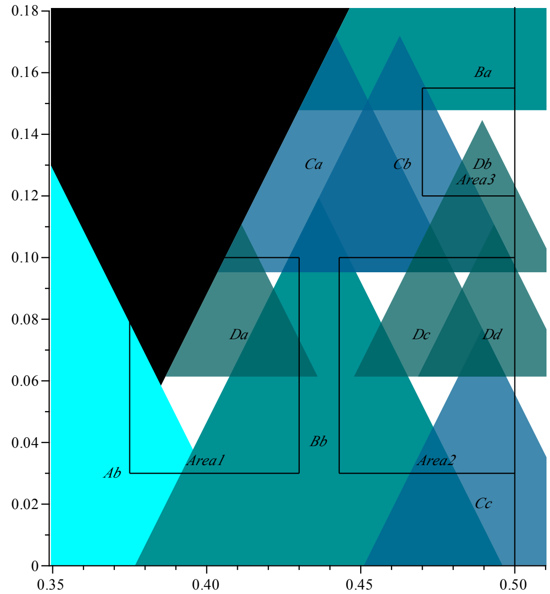


Figure 9: Main Picture

2.5.1 There are no holes outside Areas 1,2, and 3

The addition of rank 3 and 4 tiles in Figure 9 leaves three areas which contain holes, which we cover at a later step. We define

$$\text{Area } 1 := [.375, .43] \times [.03, .1],$$

$$\text{Area } 2 := [.443, .5] \times [.03, .1], \text{and}$$

$$\text{Area } 3 := [.47, .5] \times [.12, .155].$$

We now show the addition of these tiles covers all but Area 1, 2, and 3.

Claim 2.9. *For $\lambda \in \mathscr{I}$ there is no hole between Ca, Cb, and Ba.*

Proof.

$$\lambda^4 + 2\lambda^3 - \lambda^2 - 2\lambda + 1 > 0$$

(minimum estimate 0.0034230936 with mesh size 39) implies

$$\pi_y(Ca_{rs} \cap Cb_{ls}) > \pi_y Ba_l.$$

$$-\lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0208344180 with mesh size 7) implies

$$\pi_x Cb_t > \pi_x Ca_t. \qquad \square$$

Claim 2.10. *For $\lambda \in \mathscr{I}$ the bottom right vertex of Cb lies outside Area 2.*

Proof.

$$\frac{3}{2}\lambda^4 - \frac{3}{2}\lambda^3 + \lambda - \frac{1}{2} > 0$$

(minimum estimate 0.0013003341 with mesh size 102) implies

$$\pi_x Cb_r > \frac{1}{2}. \qquad \square$$

Claim 2.11. *Both Da and Dc overlap Bb.*

Proof.

$$\lambda^5 - \frac{1}{2}\lambda^4 + \frac{1}{2}\lambda^3 - \lambda + \frac{1}{2} > 0$$

(minimum estimate 0.0141719124 with mesh size 10) implies

$$\pi_x Da_r > \pi_x (Da_{rs} \cap Bb_{ls}).$$

$$2\lambda^5 - 2\lambda^4 + \lambda^3 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0177318060 with mesh size 8) implies

$$\pi_x (Dc_{bs} \cap Bb_{rs}) > \pi_x Dc_l. \qquad \square$$

Claim 2.12. *For $\lambda \in \mathscr{I}$ the bottom right vertex of Dc lies within Area 2.*

Proof.

$$-\frac{1}{2}\lambda^5 - \frac{1}{2}\lambda^4 + \lambda - \frac{1}{2} > 0$$

(minimum estimate 0.0026075030 with mesh size 51) implies

$$\frac{1}{2} > \pi_x Dc_r. \qquad \square$$

Claim 2.13. *For $\lambda \in \mathscr{I}$ there is only one hole in Area 3.*

Proof.

$$-2\lambda^5 + 2\lambda^4 - 2\lambda^3 + \lambda^2 > 0$$

(minimum estimate 0.0004830166 with mesh size 274) implies

$$\pi_y Ba_l > \pi_y Db_t. \qquad \square$$

Area 1,2, and 3 are chosen so that we can later focus our attention to covering the remaining holes in each section. The next set of inequalities shows that the holes remaining in the Main Picture stay within their respective area for $\lambda \in \mathscr{I}$.

2.5.2 Holes stay in Areas 1, 2, and 3

Claim 2.14. *The holes shown in Area 1 of Figure 9 are in Area 1 for $\lambda \in \mathscr{I}$.*

Proof.

$$\frac{1}{2}\lambda^4 + \frac{1}{2}\lambda^3 - \lambda + .43 > 0$$

(minimum estimate 0.0055323605 with mesh size 24) implies

$$.43 > \pi_x Cb_l.$$

$$\lambda^4 - \lambda^3 + .1 > 0$$

(minimum estimate 0.0043590014 with mesh size 31) implies

$$.1 > \pi_y Cb_l.$$

$$\frac{1}{2}\lambda^2 - \frac{1}{2}\lambda + .125 > 0$$

(minimum estimate 0.0103536050 with mesh size 13) implies

$$\pi_x(Da_{ls} \cap Ab_{rs}) > 0.375.$$

$$\lambda^3 + \lambda^2 - \lambda - .03 > 0$$

(minimum estimate 0.0076727925 with mesh size 18) implies

$$\pi_y(Ab_{rs} \cap Bb_{ls}) > 0.03. \qquad \square$$

Claim 2.15. *The holes shown in Area 2 of Figure 9 are in Area 2 for $\lambda \in \mathscr{I}$.*

Proof.

$$\lambda^4 - \lambda^2 + 2\lambda - 1.03 > 0$$

(minimum estimate 0.0150919290 with mesh size 9) implies

$$\pi_y(Bb_{rs} \cap Cc_{ls}) > 0.03.$$

$$-\frac{3}{2}\lambda^5 + \frac{3}{2}\lambda^4 - \lambda + .557 > 0$$

(minimum estimate 0.0026482815 with mesh size 50) implies

$$\pi_x Dc_l > 0.443.$$

$$\frac{1}{2}\lambda^4 + \frac{1}{2}\lambda^3 - \lambda^2 + \lambda - .443 > 0$$

(minimum estimate 0.0057251505 with mesh size 24) implies

$$\pi_x(Cb_{bs} \cap Bb_{rs}) > 0.443.$$

$$\lambda^5 - \frac{1}{2}\lambda^4 + \frac{1}{2}\lambda^3 - \lambda + \frac{1}{2} > 0$$

(minimum estimate 0.0141719124 with mesh size 10) implies

$$\frac{1}{2} > \pi_x(Db_{bs} \cap Dd_{ls}). \qquad \square$$

Claim 2.16. *The holes shown in Area 3 of Figure 9 are in Area 3 for $\lambda \in \mathscr{I}$.*

Proof.

$$\lambda^3 - \lambda^2 + .155 > 0$$

(minimum estimate 0.0072234260 with mesh size 19) implies

$$.155 > \pi_y Ba_l.$$

$$\lambda^4 - \frac{1}{2}\lambda^3 - \frac{1}{2}\lambda^2 + \lambda - .47 > 0$$

(minimum estimate 0.0050127424 with mesh size 27) implies

$$\pi_x(Ba_{bs} \cap Cb_{rs}) > 0.47.$$

$$\lambda^5 - \lambda^3 + 2\lambda - 1.12 > 0$$

(minimum estimate 0.0115202730 with mesh size 12) implies

$$\pi_y(Cb_{rs} \cap Db_{ls}) > 0.12.$$

$$-\frac{1}{2}\lambda^5 + \lambda^4 - \frac{1}{2}\lambda^3 + .03 > 0$$

(minimum estimate 0.0130734198 with mesh size 11) implies

$$\pi_x(Cb_{rs} \cap Db_{ls}) > 0.47.$$

$$2\lambda^5 - 2\lambda^4 + 2\lambda^3 - 2\lambda + .88 > 0$$

(minimum estimate 0.0037045984 with mesh size 36) implies

$$\pi_y(Db_{rs} \cap \{x = 0.5\}) > 0.12. \qquad \square$$

We have now shown that for all $\lambda \in \mathscr{I}$ all tiles up to rank 4 in the cover remain in the same relative position with respect to each other as pictured in Figure 9. We have also shown for $\lambda \in \mathscr{I}$ that all holes that remain to be covered are either in Area 1, Area 2, or Area 3.

2.6 Area 1

Our aim is to cover the two holes left uncovered in Area 1. To this end we show how to establish that a tile covers a hole: a set of three inequalities implies that the tile contains the hole.

The most common type of hole encountered in the following claims are those which are bounded by three tiles and are therefore triangular. To clarify the notion of a triangular hole we introduce the following definition.

Definition 2.17. *For tiles x, y, and z we define the triangular hole $H(x, y, z)$ to be the bounded connected component of the complement of $x \cup y \cup z$.*

Thus every triangular hole has sides with slope $0, 2$, and -2. In order to cover a triangular hole with a single tile we first draw three lines through the vertices of the hole as in Figure 10 with the corresponding slopes $0, 2$, and -2. We refer to the line with slope -2 as Line 1, the line with slope 2 as Line 2, and the line with slope 0 as Line 3. A single tile covers this hole if

- Its bottom left vertex lies below Line 3
- Its bottom left vertex lies above the Line 2
- Its bottom right vertex lies above Line 1

Satisfying these three conditions guarantees that the top vertex is above both Line 1 and Line 2 and therefore the hole is covered. In the following claims we always check the

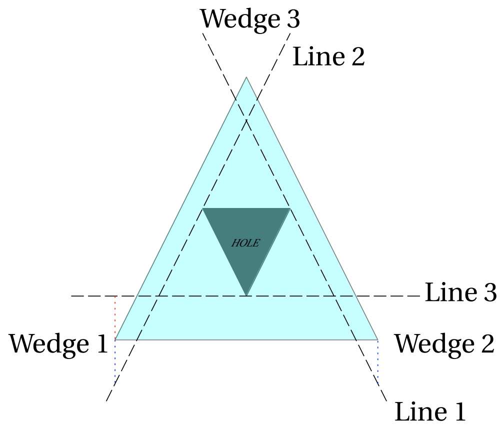


Figure 10: Vertices lie in Wedge 1, 2, and 3 if and only if the tile contains the hole

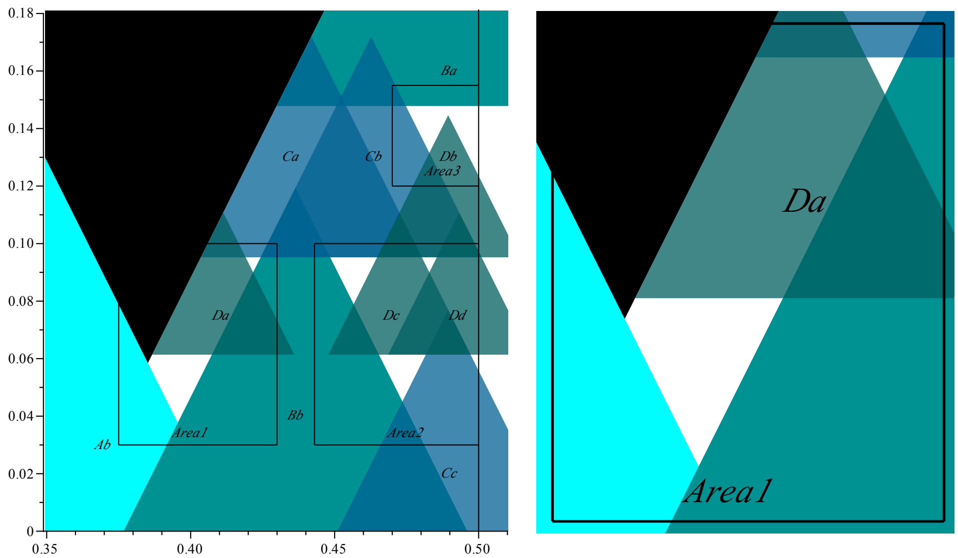
positions of the vertices in the order listed above. That is, we first check the bottom left vertex lies below Line 3, then we check the bottom left vertex lies above Line 2, and lastly we check that the bottom right vertex lies above Line 1. We will call the three areas where the vertices of a covering tile must lie Wedge 1, Wedge 2, and Wedge 3. Therefore the bottom left vertex of a covering tile must lie in Wedge 1, the bottom right vertex in Wedge 2, and the top vertex in Wedge 3. Line 1, Line 2, and Line 3 are defined by vertices which can be written in terms of λ. Thus, in order to check that a tile covers a triangular hole we need to check just three inequalities, and therefore may confirm that a triangular hole is covered as before by checking for positivity of polynomials in λ.

Remark 2.18. *Each hole we aim to cover corresponds to a unique set of wedges and lines as defined above. Thus throughout the remainder of this proof each wedge and line is meant to be defined for the particular hole under discussion.*

Often it takes more than one tile to cover a triangular hole. In these cases we may check that a tile covers one of the edges of the hole instead. Since we check strict inequalities this implies that if a tile covers an edge of a hole that it in fact covers a neighborhood of that edge.

2.6.1 Top hole in Area 1

We now complete the cover of Area 1 by adding 6 more tiles. In Area 1 there are two holes which remain to be covered. Covering the top triangular hole in Area 1 first, we focus on the tiles *Fa*, and *Ga* as shown in Figure 12.



(a) Main Picture (b) Area 1

Figure 11: Area 1 Holes

Recall that our aim is to construct a cover for all $\lambda \in \mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3$. It is sometimes useful to examine particular parts of the cover over larger or smaller intervals than this. For example we may check that a polynomial is positive on $\mathscr{I}_2 \cup \mathscr{I}_3$ by checking that it is positive on the convex hull of the two intervals. Thus we define $\mathscr{I}_r$ to be the convex hull of $\mathscr{I}_2$ and $\mathscr{I}_3$.

Claim 2.19. *If $\lambda \in \mathscr{I}$, then Fa_l is in Wedge 1 formed by H(Da,Bb,Ca).*

Proof.

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0 \tag{5}$$

(minimum estimate 0.0024163854 with mesh size 55) and

$$\lambda^6 + 2\lambda^4 - 2\lambda^3 + \lambda^2 - 2\lambda + 1 > 0$$

(minimum estimate 0.0066763173 with mesh size 20) imply that

$$Fa_l \text{ is in wedge } 1. \qquad \square$$

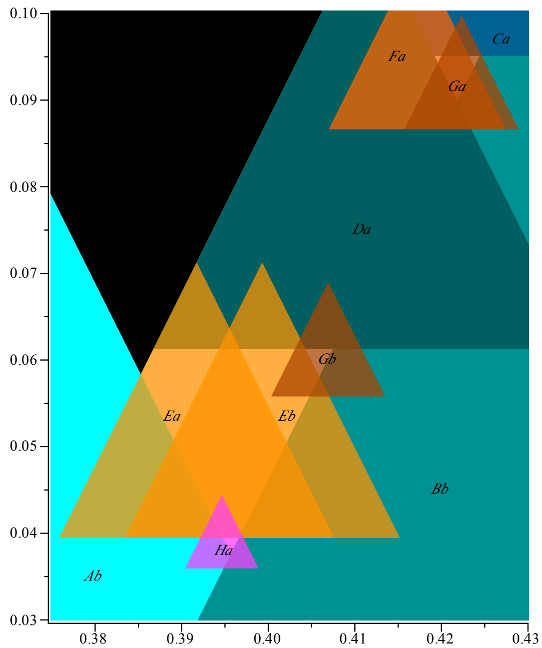


Figure 12: Area 1 Covering

Claim 2.20. *If $\lambda \in \mathscr{I}_r$, then H(Da,Bb,Ca) is completely covered by Fa.*

Proof.

$$\lambda^7 - \lambda^6 + \lambda^4 + \lambda^3 - \lambda^2 > 0$$

(minimum estimate 0.0005209584 with mesh size 108) along with Equation (5) implies that

$$Fa_r \text{ is in wedge 2.}$$

□

For $\lambda \in \mathscr{I}_1$ we need the addition of the tile *Ga* to completely cover the top hole as Fa_r is no longer contained in Wedge 2.

Claim 2.21. *If $\lambda \in \mathscr{I}_1$, then Fa and Ga cover H(Da,Bb,Ca).*

Proof.

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0032048132 with mesh size 3) and

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - 2\lambda + 1 > 0$$

(minimum estimate 0.0002031024 with mesh size 40) imply that

$$Ga_r \text{ is in wedge 2.}$$

$$\lambda^8 + \lambda^7 - 2\lambda^6 + \lambda^5 - \lambda^4 + \lambda^3 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0118443120 with mesh size 1) implies

$$\pi_x Fa_r > \pi_x Ga_l.$$

$$\lambda^8 - \lambda^6 + \lambda^4 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0033706200 with mesh size 3) implies

$$\pi_y (Fa_{rs} \cap Ga_{ls}) > \pi_y Ca_l \qquad \square$$

This concludes the proof that the top hole H(Da,Bb,Ca) contained in Area 1 is covered for all $\lambda \in \mathscr{I}$.

2.6.2 Bottom hole in Area 1

We now cover the bottom hole in Area 1. The majority of this hole is covered by two tiles *Ea*, and *Eb* which leaves uncovered two smaller holes.

- H(Eb,Bb,Da) we cover with tile *Gb*.
- H(Eb,Bb,Ab) we cover with tile *Ha*.

By Remark 2.7

$$Da = f_{1000}(I_2) \text{ and}$$
$$Ea = f_{10000}(I_2) = f_{1000}(f_0(I_2))$$

overlap since I_2 and $f_0(I_2)$ overlap for $\lambda > \frac{1}{2}$. Thus there are no holes between *Da* and the left edge of I_0.

Claim 2.22. *If $\lambda \in \mathscr{I}$, then tiles Ea and Eb cover all but two sub-holes of the bottom hole in Area 1. (See Figure 12)*

Proof.

$$2\lambda^6 - \lambda^3 + \lambda^2 - 2\lambda + 1 > 0$$

(minimum estimate 0.0003575231 with mesh size 370) implies

$$\pi_y(Ea_{rs} \cap Eb_{ls}) > \pi_y Da_l.$$

$$\frac{1}{2}\lambda^6 + \lambda^5 - \frac{3}{2}\lambda^4 + 2\lambda^3 - \lambda^2 > 0$$

(minimum estimate 0.0077964135 with mesh size 17) implies

$$\pi_x Eb_r > \pi_x(Da_{bs} \cap Bb_{ls}).$$

$$\lambda^6 - \lambda^5 + \lambda^2 + \lambda - 1 > 0$$

(minimum estimate 0.0190919770 with mesh size 7) implies

$$\pi_x(Ea_{ls} \cap Ab_{rs}) > \pi_x Ea_l. \qquad \square$$

Claim 2.23. *If $\lambda \in \mathscr{I}$, then Gb covers H(Eb,Bb,Da).*

Proof.

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^4 + 2\lambda^3 - \lambda^2 > 0 \tag{6}$$

(minimum estimate 0.0023535988 with mesh size 57) and

$$\lambda^8 - \lambda^7 + 2\lambda^5 - 2\lambda^4 + \lambda^3 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0013899660 with mesh size 95) imply

Gb_l is in wedge 1.

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0024163854 with mesh size 55) along with Equation (6) imply that

$$Gb_r \text{ is in wedge 2.} \qquad \square$$

Claim 2.24. *If $\lambda \in \mathscr{I}_r$, then H(Ab,Bb,Ea)= $\emptyset$.*

Proof.

$$\lambda^6 - \lambda^5 + \lambda^3 + \lambda^2 - \lambda > 0$$

(minimum estimate 0.0008066868 with mesh size 70) implies

$$\pi_y(Ab_{rs} \cap Bb_{ls}) > \pi_y Eb_r. \qquad \square$$

Claim 2.25. *If $\lambda \in \mathscr{I}_1$, then Ha covers H(Ab,Bb,Ea).*

Proof.

$$\lambda^9 - \lambda^8 + \lambda^7 - \lambda^6 + \lambda^3 + \lambda^2 - \lambda > 0 \tag{7}$$

(minimum estimate 0.0017708133 with mesh size 5) and

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + \lambda^2 + \lambda - 1 > 0$$

(minimum estimate 0.0027501370 with mesh size 3) imply that

$$Ha_l \text{ is in wedge 1.}$$

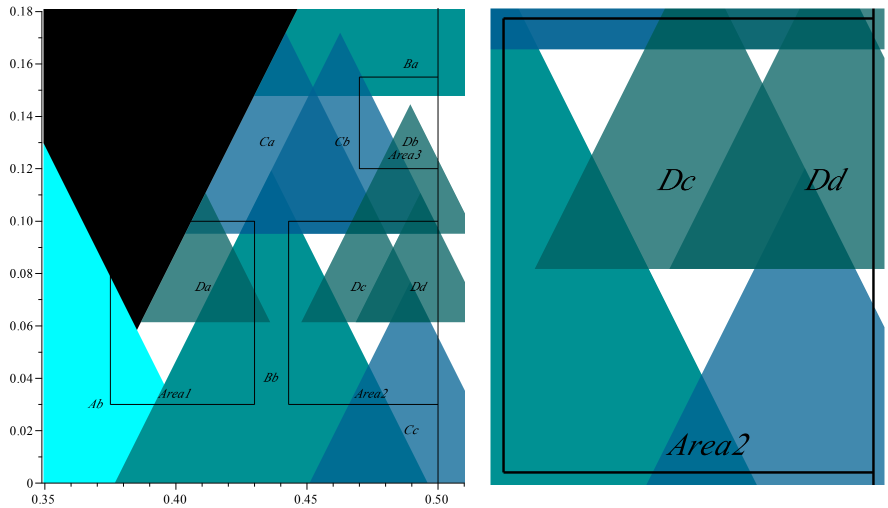
$$\lambda^9 - \lambda^8 + 2\lambda^6 - \lambda^5 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0004208982 with mesh size 20) along with Equation (7) imply that

$$Ha_r \text{ is in wedge 2.} \qquad \square$$

This concludes the proof that the bottom hole contained in Area 1 is covered for all $\lambda \in \mathscr{I}$. Therefore Area 1 is completely covered by tiles for all $\lambda \in \mathscr{I}$.

2.7 Area 2



(a) Main Picture (b) Area 2

Figure 13: Area 2 Holes

In this section we produce 18 tiles that, taken together with the previous ones, cover Area 2.

2.7.1 Bottom left hole in Area 2

We first show that the bottom left hole in Area 2, H(Bb,Cc,Dc) is covered by tiles. The tile *Ed* covers the majority of this hole leaving two smaller holes to its left and right.

Claim 2.26. *For $\lambda \in \mathscr{I}$ the position of Ed is as pictured in Figure 14.*

Proof.

$$\lambda^6 - \lambda^5 + \lambda^4 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0056766960 with mesh size 24) implies

$$\pi_y(Bb_{rs} \cap Cc_{ls}) > \pi_y Ed_l.$$

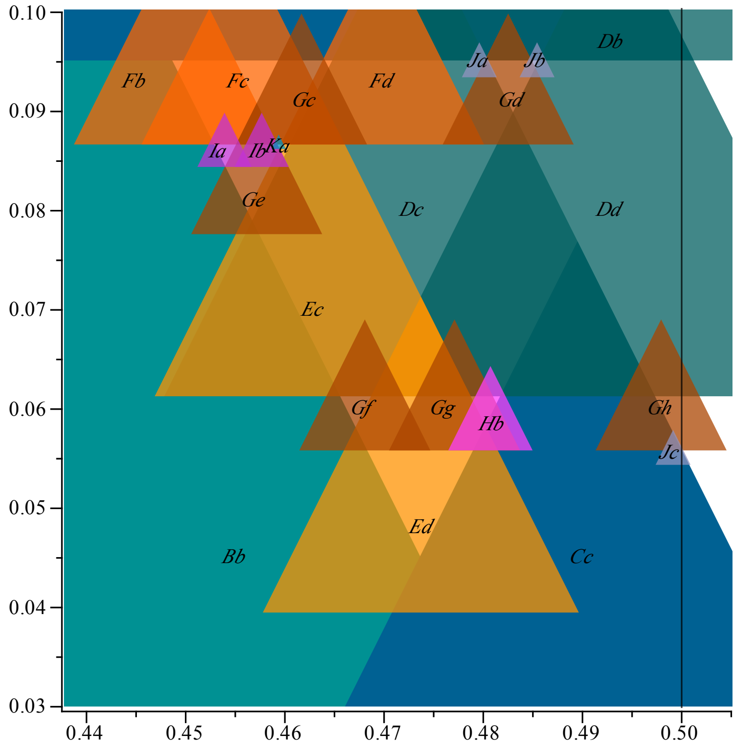


Figure 14: Area 2 Covering

$$\lambda^6 + \lambda^5 - \lambda^4 > 0$$

(minimum estimate 0.0100573390 with mesh size 14) implies

$$\pi_y Ed_t > \pi_y Dd_l.$$

$$\frac{1}{2}\lambda^6 - \frac{3}{2}\lambda^5 + \frac{3}{2}\lambda^4 - 2\lambda^3 + \lambda^2 > 0$$

(minimum estimate 0.0062788268 with mesh size 22) implies

$$\pi_x Ed_t > \pi_x (Bb_{rs} \cap Dc_{bs}).$$

$$-\frac{1}{2}\lambda^6 + \frac{1}{2}\lambda^5 - \frac{3}{2}\lambda^4 + 2\lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0046376002 with mesh size 29) implies

$$\pi_x (Dc_{bs} \cap Cc_{ls}) > \pi_x Ed_t. \qquad \square$$

Of the two smaller holes on either side of tile *Ed* we first cover the one to the left, namely H(Bb,Dc,Ed).

Claim 2.27. *For $\lambda \in \mathscr{I}$ tile Gf covers H(Bb,Dc,Ed).*

Proof.

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^4 + 2\lambda^3 - \lambda^2 > 0 \tag{8}$$

(minimum estimate 0.0023535988 with mesh size 57) and

$$\lambda^8 - \lambda^7 + 2\lambda^5 - 2\lambda^4 + \lambda^3 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0013899660 with mesh size 95) imply that

$$Gf_l \text{ is in wedge 1.}$$

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0024163854 with mesh size 55) along with Equation (8) imply that

$$Gf_r \text{ is in wedge 2.}$$ □

Next we show H(Cc,Dc,Ed) is covered.

Claim 2.28. *If $\lambda \in \mathscr{I}$, then Gg_l is contained in wedge 1 formed by H(Cc,Dc,Ed).*

Proof.

$$\lambda^8 - \lambda^7 + 2\lambda^6 - 2\lambda^5 + 2\lambda^4 - 2\lambda^3 + 2\lambda - 1 > 0 \tag{9}$$

(minimum estimate 0.0024948060 with mesh size 53) and

$$\lambda^7 + \lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0055122880 with mesh size 24) imply that

$$Gg_l \text{ is in wedge 1.}$$ □

Claim 2.29. *For $\lambda \in \mathscr{I}_r$, H(Cc,Dc,Ed) is covered by the tile Gg.*

Proof.

$$\lambda^8 - \lambda^7 + \lambda^5 + \lambda^4 - \lambda^3 > 0$$

(minimum estimate 0.0003364350 with mesh size 167) along with Equation (9) imply that

$$Gg_r \text{ is in wedge 2.} \qquad \square$$

For $\lambda \in \mathscr{I}_1$ we need the additional the tile *Hb*.

Claim 2.30. *For $\lambda \in \mathscr{I}_1$, H(Cc,Dc,Gg) is covered by the tile Hb.*

Proof.

$$2\lambda^8 - 2\lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - \lambda^3 > 0 \tag{10}$$

(minimum estimate 0.0047337820 with mesh size 2) and

$$\lambda^9 - \lambda^7 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0017063686 with mesh size 5) imply that

$$Hb_l \text{ is in wedge 1.}$$

$$\lambda^9 - \lambda^8 + \lambda^7 - \lambda^6 + \lambda^5 + \lambda^4 - 2\lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0005518500 with mesh size 15) along with Equation (10) imply that

$$Hb_r \text{ is in wedge 2.} \qquad \square$$

This concludes the proof that the bottom left hole H(Bb,Cc,Dc) in Area 2 is covered for all $\lambda \in \mathscr{I}$.

2.7.2 Bottom right hole in Area 2

We now show that the hole bordered by the line $x = \frac{1}{2}$ and the tiles *Cc* and *Dd* as pictured in Figure 13(b) is covered. The bottom vertex of this hole is defined by $Cc_{rs} \cap \{x = \frac{1}{2}\}$. As in Figure 10 we take the vertices of this hole to define three wedges and three lines.

Claim 2.31. *If $\lambda \in \mathscr{I}$, then Gh covers the top edge of the bottom right hole in Area 2.*

Proof.

$$\lambda^8 - \lambda^7 + \frac{1}{2}\lambda^5 + \frac{1}{2}\lambda^4 - \lambda^3 + \lambda - \frac{1}{2} > 0$$

(minimum estimate 0.0018850746 with mesh size 71) implies

$$\pi_x(Gh_{rs} \cap Dd_{bs}) > \frac{1}{2}.$$

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0 \tag{11}$$

(minimum estimate 0.0024163854 with mesh size 55) implies

$$\pi_y Gh_l \text{ is above Line 2.}$$ □

Claim 2.32. *If $\lambda \in \mathscr{I}_1$, then Gh completely covers the bottom right hole in Area 2.*

Proof.

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - 2\lambda + 1 > 0$$

(minimum estimate 0.0002031024 with mesh size 40) along with Equation (11) imply that

$$Gh_l \text{ is in wedge 1.}$$ □

For $\lambda \in \mathscr{I}_r$ the addition of the tile *Jc* is needed to cover the hole below *Gh*.

Claim 2.33. *If $\lambda \in \mathscr{I}_r$, then Jc covers the hole bordered by the Cc, Gh and the line $x = \frac{1}{2}$.*

Proof.

$$\lambda^{11} - \lambda^{10} + \lambda^9 - \lambda^8 + \lambda^6 - \lambda^5 + 2\lambda^4 - 2\lambda + 1 > 0 \tag{12}$$

(minimum estimate 0.0010304495 with mesh size 55) and

$$\lambda^{10} - \lambda^9 + \lambda^8 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0004453600 with mesh size 126) imply that

$$Jc_l \text{ is in wedge 1.}$$

$$\lambda^{11} - \lambda^{10} + 2\lambda^8 - 2\lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - \lambda^3 > 0$$

(minimum estimate 0.0010955991 with mesh size 52) along with Equation (12) imply that

$$Jc_r \text{ is in wedge 2.}$$ □

This concludes the proof that the bottom right hole in Area 2 is covered for all $\lambda \in \mathscr{I}$.

2.7.3 Top right hole in Area 2

In this section we cover the hole H(Db,Dc,Dd) (See Figures 13 and 14). We first show that for $\lambda \in \mathscr{I}_1$, Gd_l covers the left edge of the hole and Jb covers the right edge.

Claim 2.34. *If $\lambda \in \mathscr{I}_1$, then Gd_l is contained in Wedge 1 obtained from H(Db,Dc,Dd).*

Proof.

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0032048132 with mesh size 3) and

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - 2\lambda + 1 > 0$$

(minimum estimate 0.0002031024 with mesh size 40) imply that

$$Gd_l \text{ is in wedge 1.}$$ □

Claim 2.35. *If $\lambda \in \mathscr{I}_1$, then Jb covers H(Db,Dd,Gd).*

Proof.

$$\lambda^{10} - \lambda^9 + \lambda^8 - \lambda^6 + 2\lambda^5 - \lambda^4 > 0 \tag{13}$$

(minimum estimate 0.0009758190 with mesh size 9) and

$$\lambda^{11} - \lambda^{10} + 2\lambda^8 - 2\lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - \lambda^3 > 0$$

(minimum estimate 0.0003710732 with mesh size 22) imply that

$$Jb_l \text{ is in wedge 1.}$$

$$\lambda^{11} - \lambda^{10} + \lambda^9 - \lambda^8 + \lambda^5 + \lambda^4 - \lambda^3 > 0$$

(minimum estimate 0.0007341919 with mesh size 11) along with Equation (13) imply that

$$Jb_r \text{ is in wedge 2.}$$ □

For $\lambda \in \mathscr{I}_r$ we cover H(Db,Dc,Dd) with tiles *Gd* and *Ja*. We show that for $\lambda \in \mathscr{I}_r$, Gd_l covers the right edge of the hole and *Ja* covers the left edge.

Claim 2.36. *If $\lambda \in \mathscr{I}_r$, then Gd_l is contained in Wedge 2 obtained from the top right hole H(Db,Dc,Dd) in Area 2.*

Proof.

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0024163854 with mesh size 24) and

$$\lambda^8 - \lambda^7 + \lambda^5 + \lambda^4 - \lambda^3 > 0$$

(minimum estimate 0.0003364350 with mesh size 167) imply that

$$Gd_r \text{ is in wedge 2.}$$ □

Claim 2.37. *If $\lambda \in \mathscr{I}_r$, then Ja covers the hole H(Db,Dc,Gd).*

Proof.

$$\lambda^{10} - \lambda^9 + \lambda^8 + \lambda^3 - 2\lambda + 1 > 0 \tag{14}$$

(minimum estimate 0.0004453600 with mesh size 126) and

$$\lambda^{11} - \lambda^{10} + \lambda^9 - \lambda^8 + \lambda^6 - \lambda^5 + 2\lambda^4 - 2\lambda + 1 > 0$$

(minimum estimate 0.0010304495 with mesh size 55) imply that

$$Ja_l \text{ is in wedge 1.}$$

$$\lambda^{11} - \lambda^{10} + 2\lambda^8 - 2\lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - \lambda^3 > 0$$

(minimum estimate 0.0010955991 with mesh size 52) along with Equation (14) imply that

$$Ja_r \text{ is in wedge 2.}$$ □

This concludes the proof that the top right hole H(Db,Dc,Dd) in Area 2 is covered for all $\lambda \in \mathscr{I}$.

2.7.4 Top left hole in Area 2

We complete the cover of Area 2 by showing that the hole in the top left corner is covered by 9 additional tiles. We first note some basic features of the tiles in this area that hold for $\lambda \in \mathscr{I}$. First we see that the top portion of this hole is covered by a row of 4 tiles whose bottom edges all lie on the same horizontal line $y = -\lambda^7 + \lambda^6 - \lambda^5 + \lambda^4$.

$$Fb = f_{100021}(I_2),$$

$$Fc = f_{011020}(I_1),$$

$$Fd = f_{011021}(I_2),$$

$$Gc = f_{1000212}(I_0) = f_{100021}(f_2(I_0)),$$

This follows by Remark 2.6 since the bottom edges of I_1, I_2, and $f_2(I_0)$ lie on the same horizontal line.

We also observe by Remark 2.7 that the left side of *Gc* lies on the same line as that of $Ge = f_{1000210}(I_1)$ and that the two tiles overlap each other.

We now continue by verifying the position of the remaining tiles needed to cover the top left hole in Area 2.

Claim 2.38. *For $\lambda \in \mathscr{I}$ tiles Fb, Fc and Gc cover the left side of the top edge H(Bb,Dc,Cb).*

Proof.

$$\lambda^7 - \lambda^6 + \lambda^5 - 2\lambda^4 + \lambda^3 > 0$$

(minimum estimate 0.0078147049 with mesh size 17) implies

$$\pi_y(Cb_{bs} \cap Bb_{rs}) > \pi_y Fb_l.$$

$$\lambda^6 - \lambda^5 + \lambda^4 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0056766960 with mesh size 24) implies

$$\pi_y Fb_l \text{ is above Line 2.}$$

$$2\lambda^7 - 2\lambda^6 + 2\lambda^4 - 2\lambda + 1 > 0$$

(minimum estimate 0.0046456559 with mesh size 29) implies

$$\pi_y(Fb_{rs} \cap Fc_{ls}) > \pi_y Cb_l.$$

$$\lambda^8 - \lambda^7 + 2\lambda^6 - 2\lambda^5 + 2\lambda^4 - 2\lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0024948060 with mesh size 53) implies

$$\pi_y(Fc_{rs} \cap Gc_{ls}) > \pi_y Cb_l.$$

□

Claim 2.39. *For $\lambda \in \mathscr{I}_r$ the top edge of H(Bb,Dc,Cb) is covered by the three tiles Fb, Fc and*

Gc.

Proof.

$$\lambda^8 - \lambda^7 + \lambda^5 + \lambda^4 - \lambda^3 > 0$$

(minimum estimate 0.0003364350 with mesh size 167) implies

$$\pi_y Gc_r \text{ is above Line 1.}$$

The claim then follows by Claim 2.38 since $\mathscr{I}_r \subset \mathscr{I}$. □

Claim 2.40. *For $\lambda \in \mathscr{I}_1$ the top edge of H(Bb,Dc,Cb) is covered by the four tiles Fb, Fc, Gc and Fd.*

Proof.

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + 2\lambda^4 - 2\lambda + 1 > 0$$

(minimum estimate 0.0002031024 with mesh size 40) implies

$$\pi_y(Gc_{rs} \cap Fd_{ls}) > \pi_y Cb_l.$$

$$\lambda^7 - \lambda^6 + \lambda^5 + \lambda^4 - 2\lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0110743600 with mesh size 1) implies

$$\pi_y Fd_r \text{ is above Line 1.}$$ □

We have shown that the tiles *Ge* and *Gc* overlap. Next we show that *Ge* covers the bottom vertex of H(Bb,Dc,Cb). This leaves two smaller holes on either side of *Ge*.

Claim 2.41. *If $\lambda \in \mathscr{I}$, then Ge covers the bottom vertex of the top left hole in Area 2.*

Proof.

$$\lambda^8 - \lambda^7 + 2\lambda^5 - 2\lambda^4 + \lambda^3 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0013899660 with mesh size 95) implies

$$\pi_y(Bb_{rs} \cap Dc_{ls}) > \pi_y Ge_l. \qquad \square$$

Now we cover the hole to the left of tile *Ge*.

Claim 2.42. *For $\lambda \in \mathscr{I}$ the hole H(Bb,Fc,Ge) is covered by tile Ia.*

Proof.

$$\lambda^{10} - \lambda^9 + 2\lambda^8 - 2\lambda^7 + \lambda^6 - \lambda^4 + \lambda^3 - \lambda^2 + 2\lambda - 1 > 0 \tag{15}$$

(minimum estimate 0.0000707320 with mesh size 1867) and

$$\lambda^{10} - \lambda^9 + 2\lambda^7 - 2\lambda^6 + \lambda^5 - \lambda^4 + 2\lambda^3 - \lambda^2 > 0$$

(minimum estimate 0.0005762899 with mesh size 230) imply that

$$Ia_l \text{ is in wedge 1.}$$

$$\lambda^9 + \lambda^6 - \lambda^5 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0004459880 with mesh size 296) along with Equation (15) imply that

$$Ia_r \text{ is in wedge 2.} \qquad \square$$

We now cover the hole to the right of tile *Ge*. Note that the tile bordering the right-hand side of the hole changes as λ varies.

Claim 2.43. *For $\lambda \in \mathscr{I}_1$ the tile Ec is to the left of the tile Dc.*

Proof.

$$\lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0008150606 with mesh size 10) implies

$$\pi_x Dc_l > \pi_x Ec_l. \qquad \square$$

Claim 2.44. *For $\lambda \in \mathscr{I}_r$ the tile Dc is to the left of the tile Ec.*

Proof.

$$-\lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0004424410 with mesh size 127) implies

$$\pi_x Ec_l > \pi_x Dc_l. \qquad \square$$

Thus by Claim 2.43 we aim to cover H(Ec,Gc,Ge) for $\lambda \in \mathscr{I}_1$. This is covered by the two tiles *Ib* and *Ka* as shown in Figure 14.

Claim 2.45. *For $\lambda \in \mathscr{I}_1$ the hole H(Ec,Gc,Ge) is covered by the tile Ib.*

Proof.

$$\lambda^{10} - \lambda^9 + \lambda^8 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0018466510 with mesh size 5) and

$$\frac{1}{2}(\lambda^9 - \lambda^8 + 3\lambda^7 - 3\lambda^6 + \lambda^5 + \lambda^3 - 2\lambda + 1) > 0$$

(minimum estimate 0.0015459776 with mesh size 6) imply that

$$Ib_l \text{ is in wedge } 1.$$

$$\lambda^{10} - \lambda^8 + 2\lambda^7 - \lambda^6 > 0$$

(minimum estimate 0.0032137298 with mesh size 3) implies

$$\pi_y Ib_t > \pi_y Gc_l. \qquad \square$$

Claim 2.46. *For $\lambda \in I_1$ the hole H(Gc,Ge,Ib) is covered by the tile Ka.*

Proof.

$$\lambda^{13} - \lambda^{12} + 2\lambda^{10} - 2\lambda^9 + \lambda^8 - \lambda^7 + 2\lambda^6 - \lambda^5 > 0 \tag{16}$$

(minimum estimate 0.0001538496 with mesh size 52) and

$$\lambda^{12}-\lambda^{11}+\lambda^{10}-\lambda^{8}+2\lambda^{7}-\lambda^{6}>0$$

(minimum estimate 0.0004045816 with mesh size 20) imply that

$$Ka_l \text{ is in wedge 1.}$$

$$\lambda^{13}-\lambda^{12}+\lambda^{11}-\lambda^{10}+\lambda^{7}+\lambda^{6}-\lambda^{5}>0$$

(minimum estimate 0.0003044013 with mesh size 27) along with Equation (16) imply that

$$Ka_r \text{ is in wedge 2.}$$ □

We now consider $\lambda \in \mathscr{I}_r$. For these values of λ the tile *Dc* is to the left of the tile *Ec*. Thus the hole we aim to cover is bordered by tiles *Ge, Gc* and *Dc*. For these values the hole is completely covered by the tile *Ib*. We finish the cover of Area 2 by showing the following.

Claim 2.47. *For $\lambda \in \mathscr{I}_r$ the hole H(Dc,Gc,Ge) is covered by the tile Ib.*

Proof.

$$\lambda^{10}-\lambda^{9}+\lambda^{8}-\lambda^{6}+2\lambda^{5}-\lambda^{4}>0 \tag{17}$$

(minimum estimate 0.0015119755 with mesh size 38) and

$$\frac{1}{2}(\lambda^{9}-\lambda^{8}+3\lambda^{7}-3\lambda^{6}+\lambda^{5}+\lambda^{3}-2\lambda+1)>0$$

(minimum estimate 0.0008000951 with mesh size 70) imply that

$$Ib_l \text{ is in wedge 1.}$$

$$\lambda^{10}-\lambda^{9}+\lambda^{7}+\lambda^{5}-\lambda^{4}-\lambda^{3}+2\lambda-1>0$$

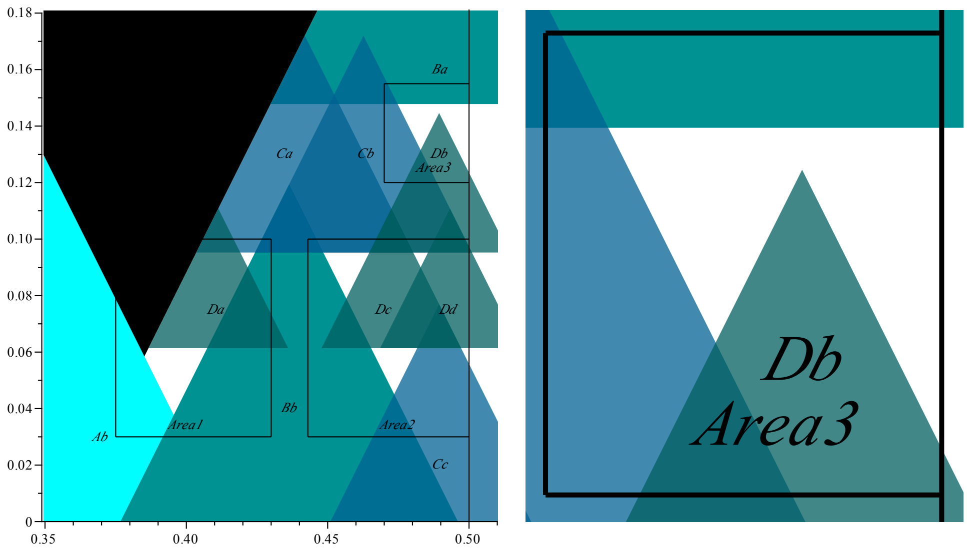
(minimum estimate 0.0005827540 with mesh size 97) along with Equation (17) imply

that

$$Ib_r \text{ is in wedge 2.} \qquad \square$$

This concludes the proof that the four holes contained in Area 2 are covered for all $\lambda \in \mathscr{I}$. Therefore Area 2 is completely covered by tiles for all $\lambda \in \mathscr{I}$.

2.8 Area 3



(a) Main Picture

(b) Area 3

Figure 15: Area 3 Hole

We finish with the cover of Area 3. A subsection of this area which we call Area 3 zoom will be covered in the next section. To cover the main part of Area 3 we need the addition of 8 more tiles. We first consider the tiles

$$Ee = f_{10020}(I_1),$$

$$Ef = f_{01120}(I_1), \text{and}$$

$$Eg = f_{10021}(I_2).$$

By Remark 2.6 we see that $\pi_y Ee_l = \pi_y Ef_l = \pi_y Eg_l$. By Remark 2.7 we also see that Ee and Eg overlap for $\lambda > \frac{1}{2}$ as this is true of $f_0(I_1)$ and $f_1(I_2)$.

To show that the tile Ee covers the left edge of the hole in Area 3 we must show that

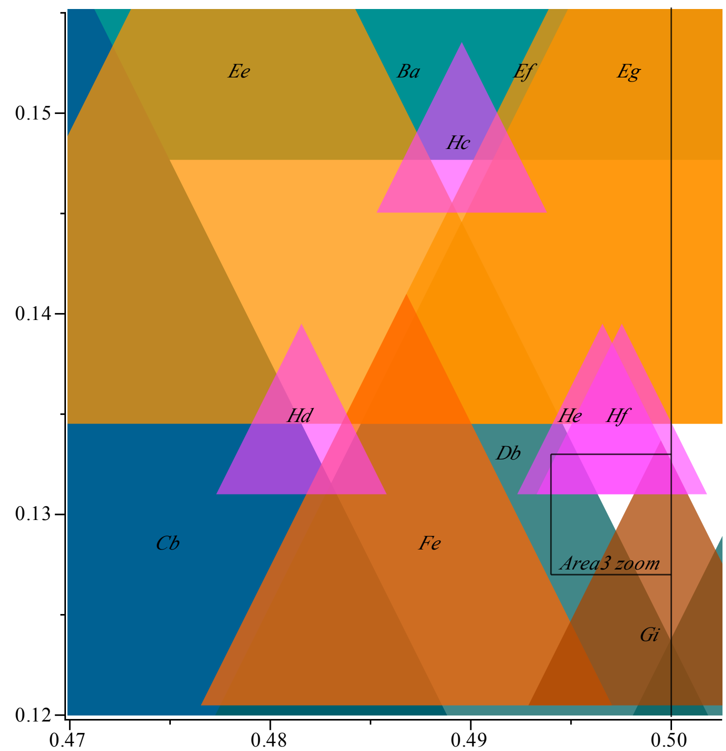


Figure 16: Area 3 Covering

$\pi_y Ee_l$ is above the line with slope 2 passing through the point defined by $Ba_{bs} \cap Cb_{rs}$. We must also show that $\pi_y Ee_t$ is above the top of the hole.

Claim 2.48. *If $\lambda \in \mathscr{I}$, then Ee covers the left edge of the hole in Area 3.*

Proof.

$$\lambda^6 - \lambda^5 + \lambda^4 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0056766960 with mesh size 24) implies

$$\pi_y Ee_l \text{ is above Line 2.}$$

$$\lambda^6 - \lambda^4 + 2\lambda^3 - \lambda^2 > 0$$

(minimum estimate 0.0186954386 with mesh size 8) implies

$$\pi_y Ee_t > \pi_y Ba_l. \qquad \square$$

We now examine the right portion of the hole in Area 3.

Claim 2.49. *If $\lambda \in \mathscr{I}$, then there is no hole in Area 3 to the right of both Ef and Eg.*

Proof.

$$\lambda^5 - \lambda^4 + \frac{1}{2}\lambda^3 - \frac{1}{2}\lambda^2 + \lambda - \frac{1}{2} > 0$$

(minimum estimate 0.0088659029 with mesh size 15) implies

$$\pi_x(Ef_{rs} \cap Ba_{bs}) > \frac{1}{2}.$$

$$\lambda^6 - \lambda^5 + \frac{3}{2}\lambda^3 - \frac{1}{2}\lambda^2 - \lambda + \frac{1}{2} > 0$$

(minimum estimate 0.0098295358 with mesh size 14) implies

$$\pi_x(Eg_{rs} \cap Ba_{bs}) > \frac{1}{2}. \qquad \square$$

We now differentiate between two cases for the relative position of tiles *Ef* and *Eg*.

Claim 2.50. *For $\lambda \in \mathscr{I}_1$ the tile Ef lies to the left of the tile Eg as pictured in Figure 16. The hole H(Ba,Ee,Ef) is covered by the tile Hc.*

Proof.

$$\lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0008150606 with mesh size 10) implies

$$\pi_x Eg_l > \pi_x Ef_l.$$

$$\lambda^9 - \lambda^8 + 2\lambda^6 - \lambda^5 + \lambda^3 - 2\lambda + 1 > 0 \qquad (18)$$

(minimum estimate 0.0004208982 with mesh size 20) and

$$\frac{1}{2}(\lambda^9 - \lambda^8 + \lambda^7 - \lambda^6 + \lambda^5 - \lambda^4 + 3\lambda^3 - \lambda^2 - 2\lambda + 1) > 0$$

(minimum estimate 0.0006841144 with mesh size 12) imply that

$$Hc_l \text{ is in wedge 1.}$$

$$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^4 + 2\lambda^3 - \lambda^2 > 0$$

(minimum estimate 0.0023535988 with mesh size 4) along with Equation (18) imply that

$$Hc_r \text{ is in wedge 2.} \qquad \square$$

Claim 2.51. *For $\lambda \in \mathscr{I}_r$ the tile Eg lies to the left of the tile Ef. The hole H(Ba,Ee,Eg) is covered by the tile Hc.*

Proof.

$$-\lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0004424410 with mesh size 127) implies

$$\pi_x Ef_l > \pi_x Eg_l.$$

$$\lambda^9 - \lambda^8 + \lambda^6 + \lambda^5 - \lambda^4 > 0 \tag{19}$$

(minimum estimate 0.0002172697 with mesh size 258) and

$$\frac{1}{2}(\lambda^9 - \lambda^8 + \lambda^7 - \lambda^6 + \lambda^5 - \lambda^4 + 3\lambda^3 - \lambda^2 - 2\lambda + 1) > 0$$

(minimum estimate 0.0006597469 with mesh size 85) imply that

$$Hc_l \text{ is in wedge 1.}$$

$$\lambda^8 - \lambda^7 + 2\lambda^5 - 2\lambda^4 + \lambda^3 - \lambda^2 + 2\lambda - 1 > 0$$

(minimum estimate 0.0040677810 with mesh size 14) along with Equation (19) imply

that

$$Hc_r \text{ is in wedge 2.} \qquad \square$$

By examining the addresses of tiles

$$Ef = f_{01120}(I_1) \text{ and}$$
$$Fe = f_{011201}(I_2) = f_{01120}(f(I_2))$$

we see that their left sides both lie along a common line because the same is true for the left side of $f_1(I_2)$ and the left side of I_1. The same is true of tiles

$$Db = f_{1002}(I_0) \text{ and}$$
$$Eg = f_{10021}(I_2) = f_{1002}(f_1(I_2))$$

as the left side of $f_1(I_2)$ also lies on the same line as the left side of I_0. Thus by Claim 2.50 we see that Fe is to the left of Db for $\lambda \in I_1$ and by Claim 2.51 Fe is to the right of Db for $\lambda \in \mathscr{I}_r$. Thus we need to cover a different hole for each of the two intervals.

Claim 2.52. *For $\lambda \in \mathscr{I}_1$ the hole H(Cb,Ee,Fe) is covered by the tile Hd.*

Proof.

$$\lambda^9 - \lambda^8 + \lambda^7 - \lambda^5 + 2\lambda^4 - \lambda^3 > 0 \tag{20}$$

(minimum estimate 0.0015154824 with mesh size 6) and

$$\lambda^8 - \lambda^7 + 2\lambda^6 - 2\lambda^5 + 2\lambda^4 - 2\lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0024948060 with mesh size 4) imply that

$$Hd_l \text{ is in wedge 1.}$$

$$\lambda^9 - \lambda^8 + 2\lambda^6 - \lambda^5 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0004208982 with mesh size 20) along with Equation (20) imply that

$$Hd_r \text{ is in wedge 2.} \qquad \square$$

Claim 2.53. *For $\lambda \in \mathscr{I}_r$ the hole H(Cb,Db,Ee) is covered by the tile Hd.*

Proof.

$$\lambda^9 - \lambda^8 + \lambda^7 - \lambda^6 + \lambda^5 + \lambda^4 - 2\lambda^3 + 2\lambda - 1 > 0 \tag{21}$$

(minimum estimate 0.0027836850 with mesh size 21) and

$$\lambda^8 - \lambda^7 + 2\lambda^6 - 2\lambda^5 + 2\lambda^4 - 2\lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0046355690 with mesh size 13) imply that

$$Hd_l \text{ is in wedge 1.}$$

$$\lambda^9 - \lambda^8 + \lambda^6 + \lambda^5 - \lambda^4 > 0$$

(minimum estimate 0.0002172697 with mesh size 258) along with Equation (21) imply that

$$Hd_r \text{ is in wedge 2.} \qquad \square$$

We now cover the hole in Area 3 below the tile *Eg* and to the right of the tile *Db*. We cover the majority of this hole with the introduction of tiles He, Hf and *Gi*. This leaves one last hole which we cover in the next section (Area 3 zoom). We now show that these tiles cover all but the hole left for Area 3 zoom.

By Remark 2.7 we see that the tiles

$$Gi = f_{0112012}(I_0) \text{ and}$$
$$He = f_{01120120}(I_1) = f_{0112012}(f_0(I_1))$$

overlap each other. We also see their right sides lie along a common line since this is true of $f_0(I_1)$ and I_0.

Claim 2.54. *If $\lambda \in \mathscr{I}$, then Gi covers the bottom of the hole bordered by Ef, Db and the line $x = \frac{1}{2}$.*

Proof.

$$\lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0024163854 with mesh size 55) implies

$$\pi_y(Db_{rs} \cap \{x = 0.5\}) > \pi_y Gi_l. \qquad \square$$

To finish the cover in this area we focus on the tiles *He* and *Hf*. The position of these two tiles switch with respect to each other as λ varies. In either position we show that the hole bordered by *Db, Eg* and *Gi* is covered by *He* and *Hf*.

Claim 2.55. *For $\lambda \in \mathscr{I}_1$ the hole H(Db,Eg,Gi) is covered by He and Hf.*

Proof.

$$\lambda^8 - \lambda^7 + \frac{3}{2}\lambda^6 - \frac{3}{2}\lambda^5 + \frac{1}{2}\lambda^4 + \frac{1}{2}\lambda^3 - \lambda + \frac{1}{2} > 0$$

(minimum estimate 0.0000139420 with mesh size 574) implies

$$\pi_x(Hf_{rs} \cap Eg_{bs}) > \frac{1}{2}.$$

$$\lambda^9 - \lambda^8 + 2\lambda^6 - \lambda^5 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0004208982 with mesh size 20) implies

$$\pi_y He_l \text{ is above the Line 2.}$$

$$\lambda^9 - \lambda^7 + \lambda^6 + \lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0040274060 with mesh size 2) implies

$$\pi_y(He_{rs} \cap Hf_{ls}) > \pi_y Ef_l. \qquad \square$$

Claim 2.56. *For $\lambda \in \mathscr{I}_r$ the hole H(Db,Eg,Gi) is covered by He and Hf.*

Proof.

$$\lambda^8 - \lambda^7 + \frac{1}{2}\lambda^6 + \frac{1}{2}\lambda^5 - \frac{1}{2}\lambda^4 - \frac{1}{2}\lambda^3 + \lambda - \frac{1}{2} > 0$$

(minimum estimate 0.0000053457 with mesh size 10476) implies

$$\pi_x(He_{rs} \cap Eg_{bs}) > \frac{1}{2}.$$

$$\lambda^9 - \lambda^8 + \lambda^6 + \lambda^5 - \lambda^4 > 0$$

(minimum estimate 0.0002172697 with mesh size 258) implies

$$\pi_y Hf_l \text{ is above the Line 2.}$$

$$\frac{1}{2}(\lambda^9 - \lambda^7 + 3\lambda^6 - 3\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1) > 0$$

(minimum estimate 0.0021591542 with mesh size 26) implies

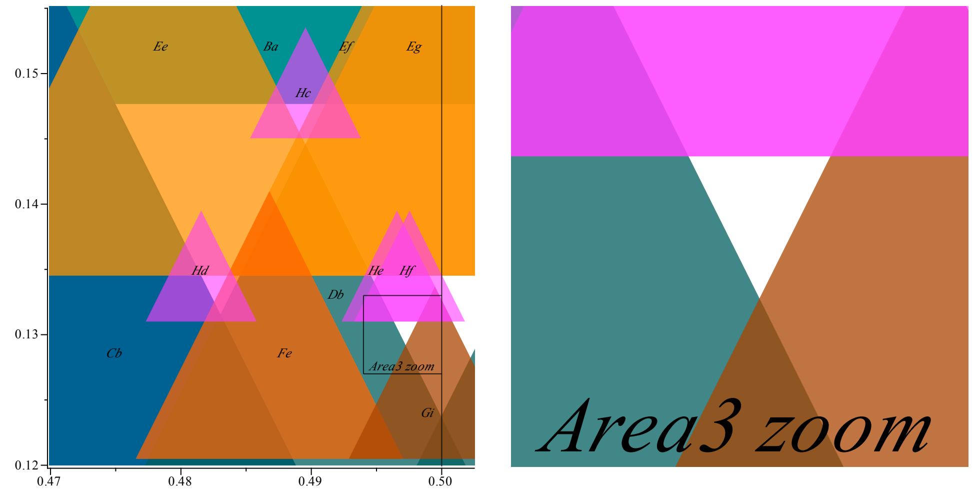
$$\pi_y(Hf_{rs} \cap He_{ls}) > \pi_y Ef_l.$$ □

2.9 Area 3 zoom

We now focus on the last remaining hole which is contained in Area 3 zoom. It is enclosed by three tiles with *He* on the top, *Db* on the left, and *Gi* on the right. In this section parts of the cover include tiles of rank up to 17. Some of the tiles are so small that the relative position of them with respect to other tiles changes dramatically as λ varies. Thus we break down the interval $\mathscr{I}$ into even finer subintervals than in the previous sections and consider the cover for Area 3 zoom on each of these subintervals.

2.9.1 Covering for $\lambda \in \mathscr{I}_1$

We first consider $\lambda \in \mathscr{I}_1$ and show that the tiles in Area 3 zoom excluding *Ma*, *Mb* and *Oa* are in the relative positions shown in Figure 18.



(a) Main Picture

(b) Area 3 zoom

Figure 17: Area 3 zoom Hole

By Remark 2.6 the bottom edges of the tiles

$$Ka = f_{10021020120}(I_1),$$

$$Kb = f_{01120120121}(I_2), \text{and}$$

$$Kc = f_{10021020121}(I_2)$$

all lie along the same horizontal line. By Remark 2.7 we see that

$$Jd = f_{0112012012}(I_0) \text{ and}$$

$$Kb = f_{01120120121}(I_2) = f_{0112012012}(f_1(I_2))$$

also overlap. Since the left side of $f_0(I_1)$ lies on the same line as the left side of I_0 we see that the left sides of Kb and Jd also lie on a common line. The same is true of the right sides of tiles

$$Je = f_{1002102012}(I_0) \text{ and}$$

$$Ka = f_{10021020120}(I_1) = f_{1002102012}(f_0(I_1))$$

since it is true of $f_0(I_1)$ and I_0.

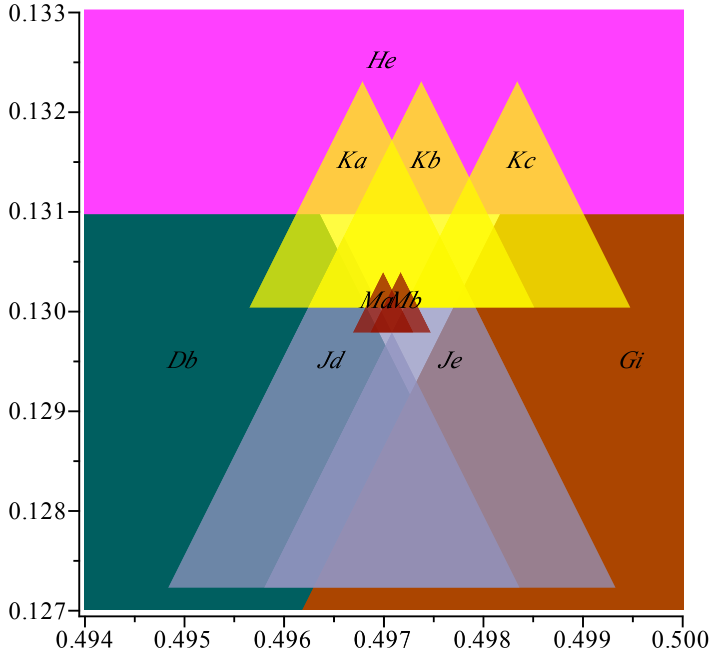


Figure 18: Area 3 zoom Covering, $\lambda = .6439 \in \mathscr{I}_1$

Claim 2.57. *If $\lambda \in \mathscr{I}_1$, then Ka is to the left of Kb which is to the left of Kc.*

Proof.

$$\lambda^{12} - 2\lambda^{11} + \lambda^{10} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0005899280 with mesh size 14) implies

$$\pi_x Kb_l > \pi_x Ka_l.$$

$$\lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0008150606 with mesh size 10) implies

$$\pi_x Kc_l > \pi_x Kb_l.$$

□

Claim 2.58. *For $\lambda \in \mathscr{I}_1$ the top edge of the hole H(Db,Gi,He) is covered by Ka, Kb and Kc.*

Proof.

$$2\lambda^{11} - 2\lambda^{10} + 2\lambda^9 - \lambda^8 + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0006047157 with mesh size 14) implies

$$\pi_y(Ka_{rs} \cap Kb_{ls}) > \pi_y He_l.$$

$$\lambda^{12} - \lambda^{10} + 2\lambda^9 - \lambda^8 - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0003688030 with mesh size 22) implies

$$\pi_y(Kb_{rs} \cap Kc_{ls}) > \pi_y He_l.$$

$$\lambda^{12} - \lambda^{11} + 2\lambda^9 - 2\lambda^8 + \lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 > 0$$

(minimum estimate 0.0002389339 with mesh size 34) implies

$$\pi_y Ka_l \text{ is above Line 2.}$$

$$\lambda^{12} - \lambda^{11} + \lambda^9 + \lambda^8 - \lambda^7 + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0007097347 with mesh size 12) implies

$$\pi_y Kc_r \text{ is above Line 1.}$$

$$\lambda^{12} - \lambda^{11} + \lambda^{10} - 2\lambda^9 + \lambda^8 > 0$$

(minimum estimate 0.0009349909 with mesh size 9) implies

$$\pi_y He_l > \pi_y Ka_l.$$

□

We next show that two tiles *Jd* and *Je* cover the bottom vertex of the hole leaving just the space between them to be covered.

Claim 2.59. *If $\lambda \in \mathscr{I}_1$, then Jd and Je cover the bottom vertex of the hole H(Db,Kb,Gi).*

Proof.

$$\lambda^{12} - \lambda^8 + \lambda^7 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0004810968 with mesh size 17) implies

$$\pi_x(Db_{rs} \cap Kb_{bs}) > \pi_x Kb_l.$$

$$\lambda^{12} - \lambda^9 + 2\lambda^8 - \lambda^7 + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0000880828 with mesh size 91) implies

$$\pi_x Ka_r > \pi_x(Gi_{ls} \cap Ka_{bs}).$$

$$\lambda^{10} - \lambda^9 + \lambda^8 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0018466510 with mesh size 5) implies

$$\pi_y(Db_{rs} \cap Gi_{ls}) > \pi_y Jd_l. \qquad \square$$

We now show that for $\lambda \in \mathscr{I}_1$ the tiles *Ma* and *Mb* cover the remaining hole.

Claim 2.60. *For $\lambda \in \mathscr{I}_1$ the hole H(Jd,Je,Kb) is covered by Ma and Mb.*

Proof.

$$\lambda^{15} - \lambda^{14} + \lambda^{13} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0 \tag{22}$$

(minimum estimate 0.0000035520 with mesh size 2253) and

$$\lambda^{14} - \lambda^{13} + 2\lambda^{12} - \lambda^{11} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0001420280 with mesh size 57) imply that

Ma_l is in Wedge 1.

$$\lambda^{14} - \lambda^{13} + 2\lambda^{12} - \lambda^{11} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0001420280 with mesh size 57) along with Equation (22) imply that

$$Mb_r \text{ is in Wedge 2.}$$

$$2\lambda^{15} - 2\lambda^{14} + \lambda^{12} + \lambda^{11} - \lambda^{10} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0000341254 with mesh size 235) implies

$$\pi_y(Ma_{rs} \cap Mb_{ls}) > \pi_y Ka_l.$$ □

We have now completed the cover of the left half of I_0 for $\lambda \in \mathscr{I}_1$.

2.9.2 Common features in covering for $\lambda \in \mathscr{I}_r$

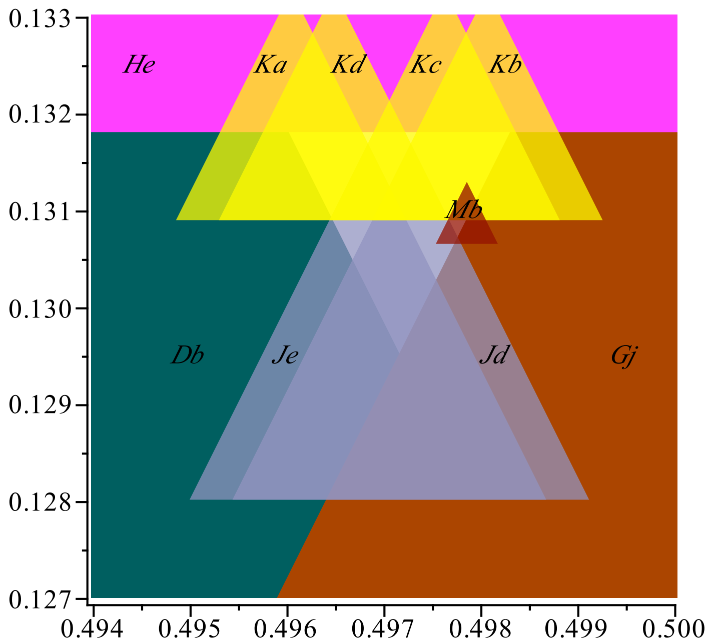


Figure 19: Area 3 zoom Covering, $\lambda = .6458 \in \mathscr{I}_{2a}$

To cover Area 3 zoom for $\lambda \in \mathscr{I}_r$ we start by showing the relative positions of the tiles *Ka*, *Kb*, *Kc* and *Kd* is as pictured in Figure 19.

Claim 2.61. *For $\lambda \in \mathscr{I}_r$ the tiles of rank 11 appear from left to right in the order Ka, Kd, Kc, and Kb as in Figure 19.*

Proof.

$$-\lambda^6+2\lambda^5-\lambda^4-\lambda^3+2\lambda-1>0$$

(minimum estimate 0.0004424410 with mesh size 127) implies

$$\pi_x Kd_l > \pi_x Ka_l.$$

$$\lambda^{12}-2\lambda^{11}+\lambda^{10}+\lambda^6-2\lambda^5+\lambda^4+\lambda^3-2\lambda+1>0$$

(minimum estimate 0.0001340605 with mesh size 418) implies

$$\pi_x Kc_l > \pi_x Kd_l.$$

$$-\lambda^6+2\lambda^5-\lambda^4-\lambda^3+2\lambda-1>0$$

(minimum estimate 0.0004424410 with mesh size 127) implies

$$\pi_x Kb_l > \pi_x Kc_l. \qquad \square$$

Claim 2.62. *For $\lambda \in \mathscr{I}_r$ the top edge of the hole H(Db,Gj,He) is covered by Ka, Kb, Kc and Kd.*

Proof.

$$\lambda^{12}-\lambda^{10}+2\lambda^9-\lambda^8+\lambda^6-2\lambda^5+\lambda^4+\lambda^3-2\lambda+1>0$$

(minimum estimate 0.0000985634 with mesh size 569) implies

$$\pi_y(Ka_{rs}\cap Kd_{ls}) > \pi_y He_l.$$

$$2\lambda^{11}-2\lambda^{10}+2\lambda^9-\lambda^8-\lambda^6+2\lambda^5-\lambda^4-\lambda^3+2\lambda-1>0$$

(minimum estimate 0.0003261560 with mesh size 172) implies

$$\pi_y(Kd_{rs}\cap Kc_{ls}) > \pi_y He_l.$$

$$\lambda^{12}-\lambda^{10}+2\lambda^9-\lambda^8+\lambda^6-2\lambda^5+\lambda^4+\lambda^3-2\lambda+1>0$$

(minimum estimate 0.0000985634 with mesh size 569) implies

$$\pi_y(Kc_{rs}\cap Kb_{ls})>\pi_y He_l.$$

$$\lambda^{12}-\lambda^{11}+2\lambda^9-2\lambda^8+\lambda^7-\lambda^6+2\lambda^5-\lambda^4>0$$

(minimum estimate 0.0007075379 with mesh size 80) implies

$$\pi_y Ka_l \text{ is above Line 2.}$$

$$\lambda^{12}-\lambda^{11}+\lambda^9+\lambda^8-\lambda^7-\lambda^6+2\lambda^5-\lambda^4-\lambda^3+2\lambda-1>0$$

(minimum estimate 0.0005009600 with mesh size 112) implies

$$\pi_y Kb_r \text{ is above Line 1.}$$

$$\lambda^{12}-\lambda^{11}+\lambda^{10}-2\lambda^9+\lambda^8>0$$

(minimum estimate 0.0008873708 with mesh size 64) implies

$$\pi_y He_l>\pi_y Ka_l.$$

□

Claim 2.63. *If $\lambda\in\mathscr{I}_r$, then Je covers the left edge of the hole H(Db,Gj,Kc).*

Proof.

$$\lambda^{12}-\lambda^8+\lambda^7-\lambda^6+2\lambda^5-\lambda^4>0$$

(minimum estimate 0.0000339551 with mesh size 1650) implies

$$\pi_x(Db_{rs}\cap Kc_{bs})>\pi_x Kc_l.$$

$$\lambda^{10}-\lambda^9+\lambda^8-\lambda^6+2\lambda^5-\lambda^4>0$$

(minimum estimate 0.0015119755 with mesh size 38) implies

$$\pi_y(Db_{rs} \cap Gj_{ls}) > \pi_y Je_l. \qquad \square$$

2.9.3 Covering for $\lambda \in \mathscr{I}_2$

For λ in this range we split the interval $\mathscr{I}_2$ into three subintervals. Let

$$\mathscr{I}_{2a} := [.6458, .64605],$$

$$\mathscr{I}_{2b} := [.64605, .64625], \text{ and}$$

$$\mathscr{I}_{2c} := [.64625, .6466].$$

Thus $\mathscr{I}_2 = \mathscr{I}_{2a} \cup \mathscr{I}_{2b} \cup \mathscr{I}_{2c}$.

For $\lambda \in \mathscr{I}_{2b}$ the bottom vertex of the hole is covered by the tiles *Jd* and *Je*. We first show that the right edge of the hole in Area 3 zoom is covered by *Jd* for $\lambda \in \mathscr{I}_r \setminus \mathscr{I}_{2a}$.

Claim 2.64. *If $\lambda \in \mathscr{I}_r \setminus \mathscr{I}_{2a}$, then Jd covers the right edge of the hole H(Db,Gj,Kd).*

Proof.

$$\lambda^{12} - \lambda^9 + 2\lambda^8 - \lambda^7 - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0000283630 with mesh size 1622) implies

$$\pi_x Kd_r > \pi_x(Gj_{ls} \cap Kd_{bs}).$$

Then the claim follows by Claim 2.63. $\square$

Claim 2.65. *If $\lambda \in \mathscr{I}_{2b}$, then Je and Jd cover the hole H(Db,Gj,Kc).*

Proof.

$$\lambda^{12} + \lambda^{11} - \lambda^{10} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0000380663 with mesh size 211) implies

$$\pi_y(Je_{rs} \cap Jd_{ls}) > \pi_y Kc_l.$$

Then the claim follows by Claim 2.63 and Claim 2.64. □

Therefore we have shown Area 3 zoom is covered by tiles for all $\lambda \in \mathscr{I}_{2b}$. Next we show Area 3 zoom is covered for all $\lambda \in \mathscr{I}_{2a}$ by including the additional tile *Mb*.

Claim 2.66. *If $\lambda \in \mathscr{I}_{2a}$, then Mb covers the hole H(Gj,Jd,Kb).*

Proof.

$$\lambda^{15} - \lambda^{14} + \lambda^{13} - \lambda^{12} + \lambda^{11} - \lambda^{9} + 2\lambda^{8} - \lambda^{7} - \lambda^{6} + 2\lambda^{5} - \lambda^{4} - \lambda^{3} + 2\lambda - 1 > 0 \tag{23}$$

(minimum estimate 0.0000723120 with mesh size 139) and

$$\lambda^{15} - \lambda^{14} + \lambda^{12} + \lambda^{11} - \lambda^{10} > 0$$

(minimum estimate 0.0000157611 with mesh size 635) along with Claim 2.63 imply that

$$Mb_l \text{ is in wedge 1.}$$

$$\lambda^{14} - \lambda^{13} + 2\lambda^{12} - 2\lambda^{11} + \lambda^{10} - \lambda^{9} + 2\lambda^{8} - \lambda^{7} - \lambda^{6} + 2\lambda^{5} - \lambda^{4} - \lambda^{3} + 2\lambda - 1 > 0$$

(minimum estimate 0.0002066510 with mesh size 49) along with Equation (23) imply that

$$Mb_r \text{ is in wedge 2.}$$ □

Therefore we have shown Area 3 zoom is covered by tiles for all $\lambda \in \mathscr{I}_{2a}$. Next we show Area 3 zoom is covered for all $\lambda \in \mathscr{I}_{2c}$. By Claims 2.63 and 2.64 for $\lambda \in \mathscr{I}_{2c}$ we must cover the hole bordered by *Je*, *Jd* and *Kc*. We do this with the addition of the two tiles *Mc* and *Md*.

Claim 2.67. *If $\lambda \in \mathscr{I}_{2c}$, then Mc and Md cover the hole H(Jd,Je,Kc).*

Proof.

$$-\lambda^{15} + 2\lambda^{14} - \lambda^{13} + \lambda^{12} - 2\lambda^{11} + \lambda^{10} + \lambda^{6} - 2\lambda^{5} + \lambda^{4} + \lambda^{3} - 2\lambda + 1 > 0$$

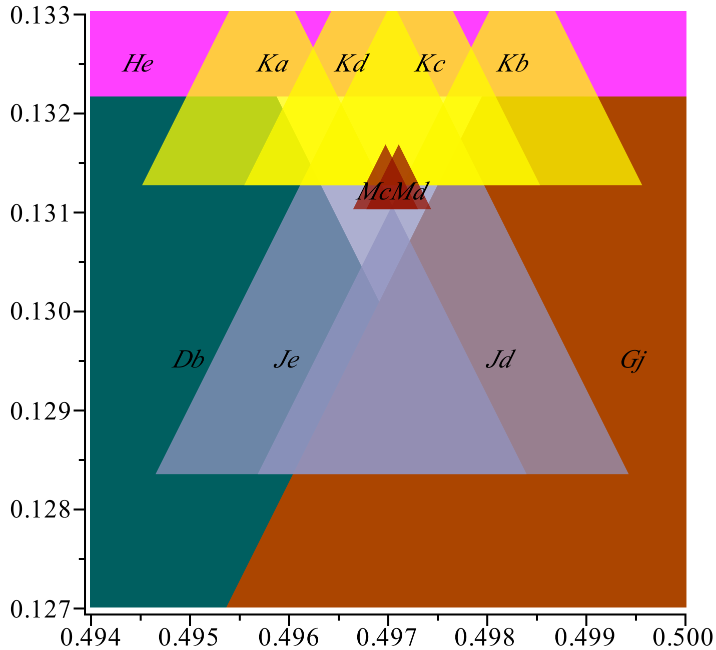


Figure 20: Area 3 zoom Covering, $\lambda = .6463 \in \mathcal{I}_{2c}$

(minimum estimate 0.0001332740 with mesh size 106) implies

$$\pi_x Md_l > \pi_x Mc_l.$$

$$\lambda^{15} - \lambda^{14} + \lambda^{13} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0 \tag{24}$$

(minimum estimate 0.0000380156 with mesh size 396) and

$$\lambda^{14} - \lambda^{13} + 2\lambda^{12} - \lambda^{11} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0001705389 with mesh size 83) imply that

$$Mc_l \text{ is in wedge 1.}$$

$$\lambda^{14} - \lambda^{13} + 2\lambda^{12} - \lambda^{11} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0001705389 with mesh size 83) along with Equation (24) imply

that

$$Md_r \text{ is in wedge } 2.$$

$$2\lambda^{15} - 2\lambda^{14} + \lambda^{12} + \lambda^{11} - \lambda^{10} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0000175130 with mesh size 800) implies

$$\pi_y(Mc_{rs} \cap Md_{ls}) > \pi_y Kc_l. \qquad \square$$

Therefore we have shown that the left half of I_0 is covered by tiles for all $\lambda \in \mathscr{I}_2$.

2.9.4 Covering for $\lambda \in \mathscr{I}_3$

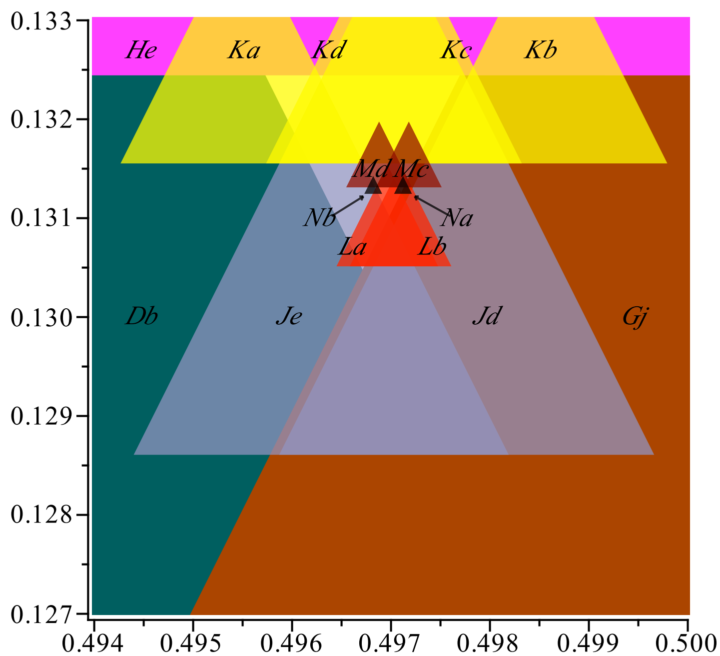


Figure 21: Area 3 zoom covering, $\lambda = .6472 \in \mathscr{I}_3$

By Claims 2.63 and 2.64 to show that Area 3 zoom is covered by tiles for $\lambda \in \mathscr{I}_3$ we must cover the hole H(Jd,Je,Kc). We again cover a portion of this hole with tiles *Mc* and *Md* although the position of the two tiles have been interchanged. This is shown by the claim below.

Claim 2.68. *If $\lambda \in \mathscr{I}_3$, then Md is to the left of Mc.*

Proof.

$$\lambda^{15} - 2\lambda^{14} + \lambda^{13} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} - \lambda^{6} + 2\lambda^{5} - \lambda^{4} - \lambda^{3} + 2\lambda - 1 > 0$$

(minimum estimate 0.0001563810 with mesh size 52) implies

$$\pi_x Mc_l > \pi_x Md_l. \qquad \square$$

Claim 2.69. *If $\lambda \in \mathscr{I}_3$, then Mc and Md cover the top edge of the hole H(Jd,Je,Kc).*

Proof.

$$\lambda^{15} - \lambda^{14} + \lambda^{12} + \lambda^{11} - \lambda^{10} > 0$$

(minimum estimate 0.0000380156 with mesh size 166) implies

$$\pi_y Md_l \text{ is above line 2.}$$

$$\lambda^{15} - \lambda^{14} + \lambda^{12} + \lambda^{11} - \lambda^{10} > 0$$

(minimum estimate 0.0000380156 with mesh size 166) implies

$$\pi_y Mc_r \text{ is above line 1.}$$

$$\lambda^{15} - \lambda^{14} + \lambda^{13} - 2\lambda^{12} + \lambda^{11} > 0$$

(minimum estimate 0.0002405585 with mesh size 34) implies

$$\pi_y Kc_l > \pi_y Md_l.$$

$$\frac{1}{2}(2\lambda^{14} - 2\lambda^{13} + 3\lambda^{12} - 3\lambda^{11} + \lambda^{10} + \lambda^{6} - 2\lambda^{5} + \lambda^{4} + \lambda^{3} - 2\lambda + 1) > 0$$

(minimum estimate 0.0000622188 with mesh size 129) implies

$$\pi_y(Md_{rs} \cap Mc_{ls}) > \pi_y Kc_l. \qquad \square$$

To show that the hole H(Jd,Je,Md) is covered, we need to write $\mathscr{I}_3 = \mathscr{I}_{3a} \cup \mathscr{I}_{3b} \cup \mathscr{I}_{3c}$ where

$$\mathscr{I}_{3a} := [.647, .64707],$$

$$\mathscr{I}_{3b} := [.64707, .64718], \text{ and}$$

$$\mathscr{I}_{3c} := [.64718, .6472].$$

Claims 2.70 and 2.71 below show that for $\lambda \in \mathscr{I}_{2b}$ the two tiles *La* and *Lb* cover this last remaining hole.

Claim 2.70. *If $\lambda \in \mathscr{I}_{3a} \cup \mathscr{I}_{3b}$, then La and Lb cover the bottom vertex of the hole H(Jd,Je,Md).*

Proof.

$$\lambda^{13} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0 \tag{25}$$

(minimum estimate 0.0004332883 with mesh size 17) and

$$\lambda^{15} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0000072718 with mesh size 991) imply that

$$La_l \text{ is in Wedge 1.}$$

$$\lambda^{15} + \lambda^6 - 2\lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1 > 0$$

(minimum estimate 0.0000072718 with mesh size 991) along with Equation (25) imply that

$$Lb_r \text{ is in Wedge 2.}$$ □

Claim 2.71. *If $\lambda \in \mathscr{I}_{3b}$, then La and Lb cover the hole H(Jd,Je,Md).*

Proof. This follows from Claim 2.70 and the fact that

$$\lambda^{15} + \lambda^{14} - \lambda^{13} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0000020560 with mesh size 2530) implies

$$\pi_y(La_{rs} \cap Lb_{ls}) > \pi_y Mc_l. \qquad \square$$

Therefore we have shown Area 3 zoom is covered by tiles for all $\lambda \in \mathscr{I}_{3b}$. To show it is covered by tiles for $\lambda \in \mathscr{I}_{3a}$ we need the addition of the tile *Na*. This covers the hole bordered by the tiles *La*, *Lb* and *Mc*.

Claim 2.72. *If $\lambda \in \mathscr{I}_{3a}$, then Na covers the hole H(La,Lb,Mc).*

Proof.

$$\lambda^{18} - \lambda^{17} + \lambda^{16} - \lambda^{15} + 2\lambda^{14} - \lambda^{13} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0 \quad (26)$$

(minimum estimate 0.0000163050 with mesh size 172) and

$$\lambda^{18} - \lambda^{17} + \lambda^{15} + \lambda^{14} - \lambda^{13} > 0$$

(minimum estimate 0.0000130608 with mesh size 215) imply that

$$Na_l \text{ is in Wedge 1.}$$

$$\lambda^{17} - \lambda^{16} + 2\lambda^{15} - \lambda^{14} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} - \lambda^6 + 2\lambda^5 - \lambda^4 - \lambda^3 + 2\lambda - 1 > 0$$

(minimum estimate 0.0000519470 with mesh size 54) along with Equation (26) imply that

$$Na_r \text{ is in Wedge 2.} \qquad \square$$

Therefore by Claim 2.70 and Claim 2.72, Area 3 zoom is covered by tiles for all $\lambda \in \mathscr{I}_{3a}$. For $\lambda \in \mathscr{I}_{3c}$ the left side of tile *Gj* is to the left of the left side of tile *Jd*. Therefore on this interval the hole we aim to cover is bounded by *Je*, *Gj* and *Mc*.

Claim 2.73. *For $\lambda \in \mathscr{I}_{3c}$ the left side of Gj is to the left of the left side of Jd.*

Proof.

$$-\lambda^{11}+\frac{1}{2}\lambda^{10}-\frac{1}{2}\lambda^{9}+2\lambda^{8}-\lambda^{7}-\lambda^{6}+2\lambda^{5}-\lambda^{4}-\lambda^{3}+2\lambda-1>0$$

(minimum estimate 0.0035981160 with mesh size 1) implies

$$\pi_x Jd_l > \pi_x Gj_l. \qquad \square$$

To cover this hole we include the tile *Nb*. This combined with *La* and *Lb* covers the hole bounded by *Je*, *Gj* and *Mc*.

Claim 2.74. *For $\lambda \in \mathscr{I}_{3c}$ the right edge of the hole H(Gj,Je,Mc) is covered by La and Lb.*

Proof.

$$\lambda^{13}-\lambda^{12}+\lambda^{11}-\lambda^{9}+2\lambda^{8}-\lambda^{7}>0$$

(minimum estimate 0.0005178308 with mesh size 2) and

$$\lambda^{15}-\lambda^{11}+\lambda^{10}-\lambda^{9}+2\lambda^{8}-\lambda^{7}>0$$

(minimum estimate 0.0000918143 with mesh size 9) imply that

$$Lb_r \text{ is in Wedge 1.}$$

$$\lambda^{15}+\lambda^{14}-\lambda^{13}-\lambda^{12}+2\lambda^{11}-\lambda^{10}-\lambda^{6}+2\lambda^{5}-\lambda^{4}-\lambda^{3}+2\lambda-1>0$$

(minimum estimate 0.0000822900 with mesh size 10) implies

$$\pi_y(La_{rs}\cap Lb_{ls}) > \pi_y Mc_l. \qquad \square$$

To show that Area 3 zoom is covered by tiles for all $\lambda \in \mathscr{I}_{3c}$ we cover the hole H(Je,La,Mc) with the tile *Nb*.

Claim 2.75. *If $\lambda \in \mathscr{I}_{3c}$, then Nb covers the hole H(Je,La,Mc).*

Proof.

$$\lambda^{18}-\lambda^{17}+\lambda^{16}-\lambda^{15}+\lambda^{14}+\lambda^{6}-2\lambda^{5}+\lambda^{4}+\lambda^{3}-2\lambda+1>0 \tag{27}$$

(minimum estimate 0.0000585182 with mesh size 14) and

$$\lambda^{18} - \lambda^{17} + 2\lambda^{15} - 2\lambda^{14} + \lambda^{13} - \lambda^{12} + 2\lambda^{11} - \lambda^{10} > 0$$

(minimum estimate 0.0000775613 with mesh size 11) imply that

$$Nb_l \text{ is in Wedge 1.}$$

$$\lambda^{17} - \lambda^{16} + \lambda^{15} + \lambda^{14} - \lambda^{13} + \lambda^{12} - 2\lambda^{11} + \lambda^{10} + \lambda^{6} - 2\lambda^{5} + \lambda^{4} + \lambda^{3} - 2\lambda + 1 > 0$$

(minimum estimate 0.0000306896 with mesh size 27) along with Equation (27) imply that

$$Nb_r \text{ is in Wedge 2.} \qquad \square$$

Therefore Area 3 zoom is covered by tiles for all $\lambda \in \mathscr{I}_{3c}$. Thus the left half of the primary overlap I_0 is covered by tiles for all $\lambda \in \mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3$. This completes the proof of the Main Theorem. $\square$

3 Open questions

The study of "fat" Sierpinski triangles in this thesis has lead us to consider a few open questions we hope to answer in the future. The method we used to prove Λ_λ has nonempty interior involved the creation of a set N which satisfies Criterion 1.17. Since any set N satisfying this criterion is a subset of the attractor we chose to show Λ_λ is generalized radial for $\lambda \in \mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3$. We have also shown a lower bound for the parameter λ, given by λ_*, for which the attractor can be generalized radial. However, this does not imply the attractor has empty interior below this lower bound for the generalized radial case. A natural continuation of our work is to determine if Λ_λ has empty interior below this parameter. In particular, can another set N with nonempty interior be defined that satisfies Criterion 1.17 for $\lambda < \lambda_*$?

Another question arising from the proof of our Main Theorem involves the attractor for parameter values in the interval between $\mathscr{I}_1$ and $\mathscr{I}_2$. In Conjecture 1.15 we state our belief that the attractor is not generalized radial for any $\lambda \in (.64415, .64578)$. Can this conjecture be shown to be true? If so, can it still be shown the attractor has nonempty interior for λ in this interval?

There is one question which would simplify the proof of our Main Theorem greatly if the answer to it is positive. This question asks if the property of nonempty interior is "monotone" as λ increases. That is, given $\Lambda_{\mathring{\lambda}}$ has nonempty interior does Λ_λ have nonempty interior for all $\lambda \geq \mathring{\lambda}$? If the answer to this question is positive then finding a single set N satisfying Criterion 1.17 for a given value of λ would imply the attractor has nonempty interior for all λ greater than or equal to that value. Thus checking our cover at the left endpoint of $\mathscr{I}_1$ would be sufficient to prove our Main Theorem as well as answer the second open question. On the other hand if it can be shown Λ_λ has empty interior for any value of λ between the intervals $\mathscr{I}_1$ and $\mathscr{I}_2$ then it would be shown the property of nonempty interior is not monotone.

We have shown that finding a set N with nonempty interior satisfying Criterion 1.17 is sufficient to show that the attractor of an iterated function system has nonempty interior. We would like to know if this property of the attractor is also necessary for the attractor to have nonempty interior. That is, if an attractor has nonempty interior is there a subset of the attractor with nonempty interior that maps onto itself under the iterated function system?

A somewhat less related question asks if there are examples of self-similar sets with positive measure but empty interior in $\mathbb{R}$. However, what we have accomplished in this thesis could easily be applied to iterated function systems in higher dimensions. For example our proof could be generalized to apply to a “fat” Sierpinski pyramid.

We have also examined the feasibility of extending our ideas to other iterated function systems in the plane. One iterated function system we have considered is the “fat” Sierpinski carpet.

The standard contraction rate for the Sierpinski carpet is $\frac{1}{3}$. As we increase the contraction rate for this iterated function system from $\frac{1}{3}$, the corresponding attractor again becomes “fat”. For contraction rates between .48 and $\frac{1}{2}$ we noticed that the attractor ap-

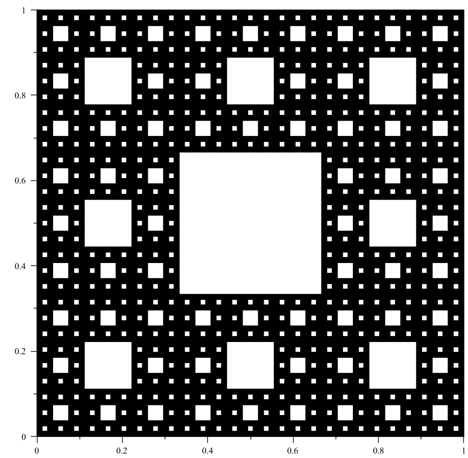


Figure 22: Standard Sierpinski Carpet

pears to have holes in a pattern analogous to that of the radial case for the "fat" Sierpinski triangle.

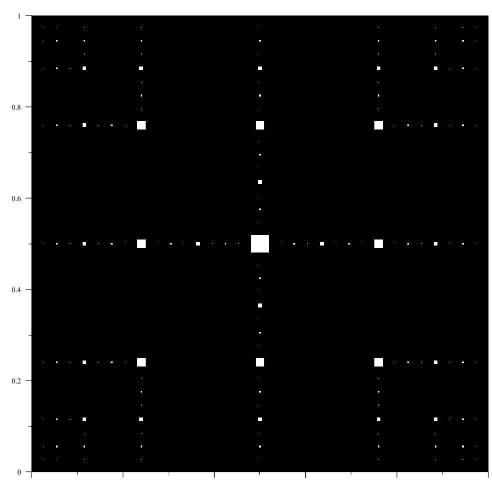


Figure 23: "Fat" Sierpinski Carpet

The "fat" Sierpinski carpet is the resulting attractor of an iterated function system with 8 maps. Unlike the "fat" Sierpinski triangle the individual components of the Sierpinski carpet do not pairwise overlap. Thus it is not clear what subset of the plane should be chosen to construct a covering set N as we have done with the "fat" Sierpinski triangle.

In the future we would like to continue this work and generalize our methods to easily distinguish between empty and nonempty interior for the attractor associated to any iterated function system.

4 Labels

Label	Subset
Aa	$f_0(I_1)$
Ab	$f_0(I_0)$
Ba	$f_{10}(I_2)$
Bb	$f_{01}(I_0)$
Ca	$f_{100}(I_2)$
Cb	$f_{011}(I_2)$
Cc	$f_{100}(I_0)$
Da	$f_{1000}(I_2)$
Db	$f_{1002}(I_0)$
Dc	$f_{1000}(I_1)$
Dd	$f_{0110}(I_1)$
Ea	$f_{10000}(I_2)$
Eb	$f_{01010}(I_1)$
Ec	$f_{01102}(I_0)$
Ed	$f_{01101}(I_2)$
Ee	$f_{10020}(I_1)$
Ef	$f_{01120}(I_1)$
Eg	$f_{10021}(I_2)$
Fa	$f_{010121}(I_2)$
Fb	$f_{100021}(I_2)$
Fc	$f_{011020}(I_1)$
Fd	$f_{011021}(I_2)$
Fe	$f_{011201}(I_2)$
Ga	$f_{1000020}(I_1)$
Gb	$f_{1000202}(I_0)$
Gc	$f_{1000212}(I_0)$
Gd	$f_{0110212}(I_0)$
Ge	$f_{1000210}(I_1)$
Gf	$f_{1000120}(I_1)$

Label	Subset
Gg	$f_{1000121}(I_2)$
Gi	$f_{0112012}(I_0)$
Gj	$f_{1002102}(I_0)$
Ha	$f_{10000021}(I_2)$
Hb	$f_{01101202}(I_0)$
Hc	$f_{01120210}(I_1)$
Hd	$f_{10020121}(I_2)$
He	$f_{01120120}(I_1)$
Hf	$f_{10021020}(I_1)$
Ia	$f_{011020120}(I_1)$
Ib	$f_{011020121}(I_2)$
Ja	$f_{0110212012}(I_0)$
Jb	$f_{0110212102}(I_0)$
Jc	$f_{0110121021}(I_2)$
Jd	$f_{0112012012}(I_0)$
Je	$f_{1002102012}(I_0)$
Ka	$f_{10021020120}(I_1)$
Kb	$f_{01120120121}(I_2)$
Kc	$f_{10021020121}(I_2)$
Kd	$f_{01120120120}(I_1)$
La	$f_{0112012012012}(I_0)$
Lb	$f_{0112102012102}(I_0)$
Ma	$f_{10021020120121}(I_2)$
Mb	$f_{01120120121020}(I_1)$
Mc	$f_{01120120120121}(I_2)$
Md	$f_{10021020121020}(I_1)$
Na	$f_{01120120120121020}(I_1)$
Nb	$f_{10021020121020120}(I_1)$

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