THE CENTERS OF THE UNIVERSAL ENVELOPING ALGEBRAS FOR CONTRACTED LIE GROUPS

A dissertation

submitted by

Mathew Wolak

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

TUFTS UNIVERSITY

May 2015

Adviser: Professor Fulton Gonzalez

Abstract

Lie group contraction is a process by which Lie groups are "flattened out". This thesis finds the algebra of bi-invariant differential operators (identified with the center of the universal enveloping algebra) for the Galilean group (which is a contraction of the Poincaré group) and a contraction of SU(n).

To whatever institution gives me a job offer

Acknowledgments

I'd like to thank my advisor, Fulton Gonzalez, for his guidance and support.

I'd also like to thank the Tufts math department as a whole for being a friendly and welcoming place, Kim Ruane for looking out for the graduate students, and my fellow graduate students for their camaraderie.

This material is based on work supported by the National Science Foundation under Grant No. OISE-1310973.

Contents

The Centers of the Universal Enveloping Algebras for Contracted Lie Groups

Chapter 1

Introduction

1.1 Description of the problem

1.1.1 The center of the universal enveloping algebra

Given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, we define the *universal enveloping algebra* of $\mathfrak{g}$ as

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I \tag{1.1}$$

where $T(\mathfrak{g})$ is the tensor algebra over $\mathfrak{g}$ and I is the two-sided ideal generated by elements of the form $X \otimes Y - Y \otimes X - [X,Y]$. $\mathcal{U}(\mathfrak{g})$ is an associative algebra containing $\mathfrak{g}$ as the image of $T^1(\mathfrak{g})$ under the quotient [11].

If $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the inclusion map, then $\mathcal{U}(\mathfrak{g})$ satisfies

$$\iota(X)\iota(Y) - \iota(Y)\iota(X) = \iota([X,Y])$$

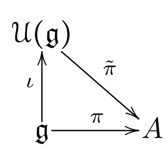
Because of this, we will identify X with $\iota(X)$.

$\mathcal{U}(\mathfrak{g})$ also satisfies a universal mapping property:

Proposition 1.1.1 *[11]: Suppose A is an associative algebra, and $\pi : \mathfrak{g} \to \mathcal{A}$ such that*

$$\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi([X,Y])$$

for all $X, Y \in A$. Then there exists a unique algebra homomorphism $\tilde{\pi}$ such that $\tilde{pi} = 1$ and the following diagram commutes:



The Poincaré-Birkhoff-Witt theorem provides a basis for $\mathcal{U}(\mathfrak{g})$:

Theorem 1.1.2 *[3] Let $\{X_1, \dots, X_n\}$ be a basis for $\mathfrak{g}$. Then the set*

$$\{X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} | m_1, \dots, m_n \in \mathbb{N} \cup \{0\}\}$$

is a basis for $\mathcal{U}(\mathfrak{g})$

When $\mathfrak{g}$ is viewed as the set of left-invariant vector fields on some Lie group G, via $\tilde{X} f(g) = \frac{d}{dt} f(g \exp tX)|_{t=0}$, Proposition 1.1.1 extends this to a map of $\mathcal{U}(\mathfrak{g})$ to the space of left-invariant differential operators on G:

Proposition 1.1.3 *[9] The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to $\mathbb{D}(G)$, the space of left-invariant differential operators on G.*

On monomials, this isomorphism is:

$$(X_1 X_2 \cdots X_k) \cdot f = \frac{\partial^k}{\partial t_1 \partial t_2 \cdots \partial t_k} f(g \exp(t_1 X_1) \exp(t_2 X_2) \cdots \exp(t_k X_k)) \bigg|_{t_1 = \cdots = t_k = 0}$$

and extends by linearity to all of $\mathcal{U}(\mathfrak{g})$. A similar isomorphism identifies $\mathcal{U}(\mathfrak{g})$ with the *right*-invariant differential operators as well.

In addition to providing some motivation for studying $U(\mathfrak{g})$, this proposition also opens the door to the use of analytic methods.

For a function $f \in C^\infty(G)$ and diffeomorphism $\phi : G \to G$, define $f^\phi = f \circ \phi^{-1}$. If D is a differential operator on G, then $D^\phi f = (D f^{\phi^{-1}})^\phi = (D(f \circ \phi)) \circ \phi^{-1}$. We say a differential operator is invariant under ϕ if $D^\phi = D$.

Example 1.1.4 For $X \in \mathfrak{g}$,

$$\tilde{X}^{L_g} f(x) = \frac{d}{dt} f(g(g^{-1}x) \exp tX) \bigg|_{t=0} = \frac{d}{dt} f(x \exp tX) \bigg|_{t=0} = \tilde{X} f(x)$$

so the left-invariant vector fields are left-invariant differential operators. $\diamond$

Example 1.1.5 Again with $X \in \mathfrak{g}$, but this time with right translation:

$$\tilde{X}^{R_g} f(x) = \frac{d}{dt} f\left(xg^{-1}\exp\left(tX\right)g\right)\Big|_{t=0} = \frac{d}{dt} f\left(x\exp\left(t\,\mathrm{Ad}(g^{-1})X\right)\right)\Big|_{t=0}$$
$$= (\mathrm{Ad}(g^{-1})X)^{\sim} f(x)$$

So if $\tilde{X}$ is right-invariant, then X is in the center of $\mathfrak{g}$ ◊

If D_1 and D_2 are differential operators, then

$$\begin{aligned}(D_1D_2)^{\phi} f &= (D_1D_2)(f \circ \phi) \circ \phi^{-1} \\ &= D_1(D_2(f \circ \phi)) \circ \phi^{-1} \\ &= D_1(D_2(f \circ \phi) \circ \phi^{-1} \circ \phi) \circ \phi^{-1} \\ &= D_1((D_2^{\phi} f) \circ \phi) \circ \phi^{-1} \\ &= (D_1^{\phi} D_2^{\phi}) f\end{aligned}$$

If we extend the adjoint action from $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$ by: $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$:

$$\mathrm{Ad}(g)(X_1X_2\cdots X_n) = (\mathrm{Ad}(g)X_1)(\mathrm{Ad}(g)X_2)\cdots(\mathrm{Ad}(g)X_n)$$

then for any left-invariant differential operator D identified with an element of $\mathcal{U}(\mathfrak{g})$, $D^{R_g} = \mathrm{Ad}(g^{-1})D$.

Again let D be a left-invariant differential operator, and consider $\mathrm{Ad}(g)D$ as a function of g. We can define $\mathrm{ad}(X)D = \frac{d}{dt}\mathrm{Ad}(\exp tX)D\big|_{t=0}$. This agrees with the definition of ad on $\mathfrak{g}$, and applying the product rule yields:

$$\mathrm{ad}(Y)X_1X_2\cdots X_n = \sum_i X_1X_2\cdots(\mathrm{ad}(Y)X_i)\cdots X_n$$

which can be simplified via the definition of the universal enveloping algebra ((1.1)) to

$$\mathrm{ad}(Y)X_1X_2\cdots X_n = YX_1X_2\cdots X_n - X_1X_2\cdots X_nY$$

If an element of $\mathcal{U}(\mathfrak{g})$ commutes with all elements of $\mathfrak{g}$, then it must be a member of the center of $\mathcal{U}(\mathfrak{g})$, denoted $Z(\mathfrak{g})$. When G is a connected Lie group, $D \in Z(\mathfrak{g})$ implies that $\mathrm{ad}(\mathfrak{g})D = 0$, and so $\mathrm{Ad}(G)D = D$. Therefore, for G connected, the center of the universal enveloping algebra can be identified with the bi-invariant differential operators.

1.1.2 Symmetrization

At this point, we've identified the center of the universal enveloping algebra with the $\mathrm{Ad}(G)$ invariant elements. $\mathcal{U}(\mathfrak{g})$ can be difficult to work with, but fortunately there are further simplifications.

$\mathcal{U}(\mathfrak{g})$ provides one way to describe the left-invariant differential operators on G, but there is another. Let $\{X_1, \ldots, X_n\}$ be a basis for $\mathfrak{g}$, and therefore a basis for the tangent space at $e \in G$. For any $g \in G$, there is a neighborhood for which $(t_1, \ldots, t_n) \mapsto g \exp(t_1 X_1 + \cdots t_n X_n)$ provides a coordinate system. This coordinate system can be used to give an alternate description of the left-invariant differential operators:

Proposition 1.1.6 *[10] Let $S(g)$ be the symmetric algebra over $\mathfrak{g}$. Then there exists a unique linear bijection*

$$\lambda : S(\mathfrak{g}) \to \mathbb{D}(G)$$

such that $\lambda(X^m) = \tilde{X}^m$. If $\{X_1, \ldots, X_n\}$ is a basis for $\mathfrak{g}$, and $P \in S(\mathfrak{g})$, then

$$\lambda(P)f(g) = P\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}\right) f(g \exp(t_1 X_1 + \cdots + t_n X_n))\Big|_{t=0}$$

Viewed as a function from $S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$,

$$\lambda(X_1 X_2 \cdots X_m) = \frac{1}{m!} \sum_{\sigma \in S_m} X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(m)}$$

where σ ranges over the permutations of m elements. It's easy to see that λ commutes with $\mathrm{Ad}(g)$:

$$
\begin{aligned}
\mathrm{Ad}(g)\,(\lambda(X_1 \cdots X_m)) &= \mathrm{Ad}(g)\left(\frac{1}{m!}\sum_{\sigma\in S_m} X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(m)}\right) \\
&= \frac{1}{m!}\sum_{\sigma\in S_m} (\mathrm{Ad}(g)X_{\sigma(1)})(\mathrm{Ad}(g)X_{\sigma(2)})\cdots(\mathrm{Ad}(g)X_{\sigma(m)}) \\
&= \lambda((\mathrm{Ad}(g)X_1)(\mathrm{Ad}(g)X_2)\cdots(\mathrm{Ad}(g)X_m)) \\
&= \lambda(\mathrm{Ad}(g)(X_1X_2\cdots X_m))
\end{aligned}
$$

which shows that the $\mathrm{Ad}(G)$-invariant polynomials in $S(\mathfrak{g})$ are mapped to $Z(\mathfrak{g})$.

Unfortunately, while λ is a linear bijection, it's not an isomorphism of algebras. In general, $\lambda(P_1P_2) \neq \lambda(P_1)\lambda(P_2)$. We do have, however, that

$$\deg(\lambda(P_1P_2) - \lambda(P_1)\lambda(P_2)) < \deg(\lambda(P_1P_2)) \tag{1.2}$$

which follows from the fact that while $\mathcal{U}(\mathfrak{g})$ isn't commutative,

$$\deg(X_1X_2\cdots X_m - X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(m)}) < m$$

for any permutation σ.

This allows us to show by induction on degree that if $\{P_1,\dots,P_m\}$ generate $S(\mathfrak{g})^G$, then $\{\lambda(P_1),\dots,\lambda(P_m)\}$ generate $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$. The degree zero case is trivial. If $D\in\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$, then we can write

$$D = \lambda(q(P_1,\dots,P_m))$$

for some polynomial q. Then

$$D - q(\lambda(P_1),\dots,\lambda(P_m))$$

is a central element whose degree is less than $\deg(D)$ by (1.2). By induction, $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ is generated by $\{\lambda(P_1),\dots,\lambda(P_m)\}$.

The fact that $\deg(\lambda(q(P_1,\ldots,P_m)-q(\lambda(P_1),\ldots,\lambda(P_m)) < \deg(\lambda(q(P_1,\ldots,P_m)))$ also implies that if $P_1,\ldots,P_m$ are algebraically independent, their images under λ will be as well.

In fact, while λ is not an algebra isomorphism, the *Duflo isomorphism* [2] is an isomorphism between $S(\mathfrak{g})^G$ and $Z(\mathfrak{g})$. In this thesis, λ is sufficient and easier to work with.

1.1.3 Coadjoint Orbits

In this thesis, $\mathfrak{g}$ will always be finite-dimensional. Therefore, by choosing a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, we can identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ via $X \mapsto \langle X,\cdot\rangle =: X^*$. For convenience, when $\mathfrak{g}$ is semisimple, we use the Killing form.

Suppose $X_1\cdots X_k$ is a monomial in $S(\mathfrak{g})$. We can apply it to an element Y^* of $\mathfrak{g}^*$ via:

$$(X_1\cdots X_k)(Y^*) = \langle X_1,Y\rangle\cdots\langle X_k,Y\rangle$$

Extending this by linearity to all of $S(\mathfrak{g})$ allows us to identify $S(\mathfrak{g})$ with the space of polynomial functions on $\mathfrak{g}^*$.

The *coadjoint representation* of G on $\mathfrak{g}^*$ is the dual of the adjoint representation:

$$(\mathrm{Ad}^*(g)X^*)(Y) = \left\langle X, \mathrm{Ad}(g^{-1})Y\right\rangle$$

With this definition, for $P \in S(\mathfrak{g})$,

$$(\mathrm{Ad}(g)P)(X^*) = P(\mathrm{Ad}^*(g^{-1})X^*)$$

This identifies $S(\mathfrak{g})^G$ with the polynomial functions on $\mathfrak{g}^*$ invariant under the coadjoint action. This allows us to use the tools of classical invariant theory to find $S(\mathfrak{g})^G$, and therefore $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$.

1.1.4 Applications

Integral transforms take a function f on a space X to a set of integrals involving f. For example, the Radon transform on $\mathbb{R}^n$ replaces a function $f(x)$ with suitable decay properties by

$$(Rf)(\xi) = \int_{\xi} f(X) d\sigma(x)$$

where ξ is a hyperplane, and σ is the Euclidean measure on ξ.

The Radon transform takes functions on $\mathbb{R}^n$ to functions on the space of hyperplanes in $\mathbb{R}^n$. Both of these spaces can be viewed as homogeneous spaces of $\mathrm{ISO}(n)$, the group of isometries of $\mathbb{R}^n$.

The Radon transform was generalized by Helgason [8] to other pairs of homogeneous spaces G/H and G/K when G, H, and K satisfy:

1. G, H, K, and $H \cap K$ are unimodular
2. For $h \in H, k \in K$, if $hK \subset KH$, then $h \in K$, and if $kH \subset HK$, then $k \in H$.
3. H, K, and HK are closed in G.

Such transforms have an invariance property: $R(g \cdot f) = g \cdot (Rf)$, where G acts by the left-regular representation $g \cdot f(x) = f(g^{-1}x)$. Taking derivatives yields actions μ, ρ of $\mathrm{Lie}(G)$ which satisfy $R(\mu(X) \cdot f) = \rho(X) \cdot (Rf)$ and can be extended to $\mathbb{D}(G)$. Finding an element $P \in \ker(\mu)$ such that $P \notin \ker(\rho)$ allows one to characterize the range of R, often completely. For an example of this technique in use, see [7]. $Z(\mathfrak{g})$ provides a good source of candidates, since every element P of $Z(\mathfrak{g})$ yields a left G-invariant operator $\rho(G)$.

Furthermore, knowledge of invariant differential operators may be used to invert Radon transforms [14]. The key is that for a Radon transform R there is a dual transform R^*, and the transform R^*R is a convolution operator.

1.2 Contractions

Imagine standing on a small but expanding sphere. As the radius of the sphere increases, the curvature decreases until it is imperceptible. As the radius goes to infinity, the curvature goes to zero: the sphere has become a plane. Applying this process to the group of symmetries of the sphere transforms SO(3) into the Euclidean motion group ISO(2). This is an example of a Lie group contraction.

Contractions are best understood at the Lie algebra level. Given a Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot])$, and a subalgebra $\mathfrak{h} = (U, [\cdot,\cdot])$, choose a complementary subspace W so that $V = U \oplus W$. For $\epsilon > 0$, let T_ϵ be the linear operator

$$T_\epsilon(u + w) = u + \epsilon w \quad (u \in U, w \in W)$$

One can define a new bracket $[\cdot,\cdot]_\epsilon$ on V by

$$[X, Y]_\epsilon = T_\epsilon^{-1}[T_\epsilon X, T_\epsilon Y]$$

Rewriting the previous equation as

$$T_\epsilon[X, Y]_\epsilon = [T_\epsilon X, T_\epsilon Y]$$

shows that T_ϵ is a Lie algebra isomorphism from $\mathfrak{g}_\epsilon := (V, [\cdot,\cdot]_\epsilon)$ to $\mathfrak{g}$.

Let us now try to take the limit as $\epsilon \to 0$. For $X \in V$, let $X = X_U + X_W$, where $X_U \in U$ and $X_W \in W$. Likewise, write $Y = Y_W + Y_U$. Then

$$\begin{aligned}
[X, Y]_\epsilon &= [X_U + X_W, Y_U, Y_W]_\epsilon \\
&= [X_U, Y_U]_\epsilon + [X_U, Y_W]_\epsilon + [X_W, Y_U]_\epsilon + [X_W, Y_W]_\epsilon \\
&= T_\epsilon^{-1}\left([X_U, Y_U] + \epsilon[X_U, Y_W] + \epsilon[X_W, Y_U] + \epsilon^2[X_W, Y_W]\right)
\end{aligned}$$

T_ϵ multiplies the W part of the result by ϵ^{-1}, and the only term in the parentheses without a factor of ϵ is $[X_U, Y_U]$. Therefore, when $\mathfrak{h} = (U, [\cdot,\cdot])$ is a subalgebra of $\mathfrak{g}$, we may define $[\cdot,\cdot]_0 := \lim_{\epsilon\to 0}[\cdot,\cdot]_\epsilon$.

Note that both the bilinearity of $[\cdot,\cdot]_\epsilon$ and the Jacobi identity hold for all $\epsilon > 0$, and so hold in the limit. The resulting Lie algebra $\mathfrak{g}_0 = (V, [\cdot,\cdot]_0)$ is in general not isomorphic to $\mathfrak{g}$, and is known as an Inönü-Wigner contraction [4]. Note that the subalgebra $\mathfrak{h}_0 = (U, [\cdot,\cdot]_0)$ is isomorphic to $\mathfrak{h}$, and that $(W, [\cdot,\cdot]_0)$ is an abelian ideal of $\mathfrak{g}_0$. The contraction process can be seen as 'flattening' W, which becomes an abelian ideal in the contracted algebra.

If H is a closed subgroup of G with Lie algebra $\mathfrak{h}$, and if W is $\mathrm{Ad}(H)$ invariant, then we can form the group $G_0 = H \ltimes W$, where H acts on W via the adjoint action. G_0 has Lie algebra $\mathfrak{g}_0$, and is referred to as the contraction of G. Often the limiting process that produced $\mathfrak{g}_0$ can be applied to objects associated with G, and yield useful information about G_0.

One particularly enlightening example comes from physics. The equations of special relativity have symmetries characterized by the Poincaré group. The Inönü-Wigner contraction of the Lorentz group preserving the rotation subgroup and time translations causes the boosts to commute with space translations, yielding the Galilean group (the symmetries of Newtonian mechanics). This is why at low velocities, special relativity looks just like Newtonian mechanics.

In this thesis, we're trying to find $Z(\mathfrak{g}_0)$ when g_0 is the contraction of some Lie algebra $\mathfrak{g}$. For all the cases we will be considering, $Z(\mathfrak{g})$ is known, so it's natural to ask what this tells us about $Z(\mathfrak{g}_0)$.

Let $P \in Z(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$. We may write P as:

$$P = \sum_i a_i X_1^{d_{i,1}} \cdots X_n^{d_{i,n}} T_1^{e_{i,1}} \cdots T_m^{e_{i,m}}$$

where $a_i \in \mathbb{R}$, $d_{i,j}, e_{i,j} \in \mathbb{Z}^+$, $X_1, \ldots, X_n$ form a basis of W, and $T_1, \ldots, T_m$ form a basis for U. For $\epsilon > 0$, $T_\epsilon : \mathfrak{g}_\epsilon \to \mathfrak{g}$ is an isomorphism, and so T_ϵ^{-1} is an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_\epsilon$.

Extending T_ϵ^{-1} to $\mathfrak{U}(\mathfrak{g})$ gives us an isomorphism $T_\epsilon^{-1} : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(g_\epsilon)$. Applying it to P gives:

$$\begin{aligned} T_\epsilon^{-1} P &= \sum_i a_i \epsilon^{-d_{i,1}} X_1^{d_{i,1}} \cdots \epsilon^{-d_{i,n}} X_n^{d_{i,n}} T_1^{e_{i,1}} \cdots T_m^{e_{i,m}} \\ &= \sum_i a_i \epsilon^{-\sum_j d_{i,j}} X_1^{d_{i,1}} \cdots X_n^{d_{i,n}} T_1^{e_{i,1}} \cdots T_m^{e_{i,m}} \end{aligned}$$

which diverges as $\epsilon \to 0$! To remedy this, let

$$M = \max_i \left\{ \sum_j d_{i,j} \right\}$$

be the maximum W-degree of terms appearing in P. ϵ^M is a scalar, and so

$$\epsilon^M T_\epsilon^{-1} P = \sum_i a_i \epsilon^{M - \sum_j d_{i,j}} X_1^{d_{i,1}} \cdots X_n^{d_{i,n}} T_1^{e_{i,1}} \cdots T_m^{e_{i,m}}$$

will still be an element of $Z(\mathfrak{g}_\epsilon)$. The exponents for ϵ are now non-negative, and zero only for terms with maximal W-degree. Since $\epsilon^M T_\epsilon^{-1}$ commutes with every element of $\mathfrak{g}_\epsilon$ for all $\epsilon > 0$, this will be true in the limit. This shows that if $P \in Z(\mathfrak{g})$, then taking only the high W-degree terms yields an element of $Z(\mathfrak{g}_0)$. We refer to such invariants as *contracted invariants*.

Note that in general, not all elements of $Z(\mathfrak{g})$ will be contracted ones. The simplest counterexample is to take the contraction with $U = \{0\}$. In this case, $\mathfrak{g}_0$ is abelian, and so $Z(\mathfrak{g}_0) = \mathcal{U}(\mathfrak{g}_0)$.

1.2.1 Cartan motion groups

For $\mathfrak{g}$ noncompact and semisimple, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is a compact subalgebra, with the following properties:

$$\begin{aligned} [\mathfrak{k}, \mathfrak{k}] &\subset \mathfrak{k} \\ [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] &= \mathfrak{p} \end{aligned}$$

Because $\mathfrak{k}$ is a subalgebra, we can use it to form an Inönü-Wigner contraction. Doing so produces $[\cdot,\cdot]_0$ defined by:

$$\begin{aligned}
[T_1, T_2]_0 &= [T_1, T_2] \qquad && \forall T_1, T_2 \in \mathfrak{k} \\
[T_1, X_1]_0 &= [T_1, X_1] \qquad && \forall T_1 \in \mathfrak{k}, X_1 \in \mathfrak{p} \\
[X_1, X_2]_0 &= 0 \qquad && \forall X_1, X_2 \in \mathfrak{p}
\end{aligned} \tag{1.3}$$

If G is a Lie group with Lie algebra $\mathfrak{g}$ and K a closed connected subgroup of G with Lie algebra $\mathfrak{k}$, the second relation tells us that K acts on $\mathfrak{p}$, and so we can form the contracted group $G_0 = K \ltimes \mathfrak{p}$.

The first two equations in (1.3) show that if $k \in K$, then the contracted adjoint action Ad_0 of k is the same as the adjoint action of G on $\mathfrak{g}$ restricted to K:

$$\mathrm{Ad}_0(k,0)(X+T) = \mathrm{Ad}(k)(X+T)$$

To find the adjoint action of elements $(e,Y) \in K \ltimes \mathfrak{p}$, we use the definition of Ad and its linearity:

$$\begin{aligned}
\mathrm{Ad}_0(e,Y)(X+T) &= \mathrm{Ad}_0(e,Y)X + \mathrm{Ad}_0(e,Y)T \\
&= \frac{d}{dt}(e,Y)(e,tX)(e,-Y)\Big|_{t=0} + \frac{d}{dt}(e,Y)(\exp tT,0)(e,-Y)\Big|_{t=0} \\
&= \frac{d}{dt}(e,tX)\Big|_{t=0} + \frac{d}{dt}(\exp tT, Y - \mathrm{Ad}(\exp tT)Y)\Big|_{t=0} \\
&= X + T - [T,Y]
\end{aligned}$$

Since $\mathfrak{g}$ is semisimple, the Killing form $B(\cdot,\cdot)$ provides a nondegenerate bilinear form, and therefore a pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. If $\mathfrak{g}$ is contracted, the Lie algebra is changed, but the underlying vector space is not. Thus although the Killing form on $\mathfrak{g}$ isn't invariant under the contracted group, it still provides a convenient bilinear form. We can use the invariance under G to help find the coadjoint representation Ad_0^*:

$$B(\mathrm{Ad}_0^*(k,0)(X+T), X'+T') = B(X+T, \mathrm{Ad}_0^*(k^{-1},0)(X'+T'))$$

$$= B(X + T, \mathrm{Ad}(k^{-1})(X' + T'))$$
$$= B(\mathrm{Ad}(k)(X + T), X' + T')$$

and so

$$\mathrm{Ad}_0^*(k, 0)(X + T) = \mathrm{Ad}(k)(X + T) \tag{1.4}$$

Next, we find $\mathrm{Ad}_0^*(e, Y)$:

$$\begin{aligned} B(\mathrm{Ad}_0^*(e,Y)(X,T), X' + T') &= B(X + T, \mathrm{Ad}_0^*(e, -Y)(X' + T')) \\ &= B(X + T, X' + T' + [T', Y]) \\ &= B(X + T, X' + T') + B(X + T, [T', Y]) \\ &= B(X + T, X' + T') + B([Y, X] + [Y, T], T') \\ &= B(X + T + [Y, X], X' + T') \end{aligned}$$

where in the last line we use the fact that $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal with respect to the Killing form. This gives us

$$\mathrm{Ad}_0^*(e, Y)(X + T) = X + T + [Y, X] \tag{1.5}$$

Recall that an element of $S(\mathfrak{g})^G$ can be contracted to form an invariant of $S(\mathfrak{g}_0)^{G_0}$. In [6], it was conjectured that for the Cartan motion groups formed from the classical Lie groups, *all* of the invariants are the contracted ones.

1.3 Invariant theory

1.3.1 Reduction by transverse subspaces

Suppose V is a finite dimensional vector space over $\mathbb{R}$, acted on by a Lie group G. We wish to find the algebra of polynomials on V invariant under that action; that is:

$$S(V^*)^G = \{P \in S(V^*) \mid P(g \cdot v) = P(v) \forall v \in V\}$$

To simplify this problem, we can restrict to a subgroup acting on a subspace:

Proposition 1.3.1 *With G and V as above, Let $T \subset V$ be a subspace of V, such that the set $G \cdot T$ is dense in V (with the standard topology). Let N be the subgroup of G which maps T to itself. Then the restriction map $f \mapsto f|_T$ is injective from $S(V^*)^G$ to $S(T^*)^N$*

Proof: Since N is a subgroup of G, we see that the image of the map is contained in $S(V^*)^N$. To see that the map is injective, suppose that $f|_T \equiv 0$. Since f is constant on G-orbits, and $G \cdot T$ is dense in V, this means that $f = 0$ on a dense subset of V. Since f is a polynomial, this implies that $f \equiv 0$ on V. □

Unfortunately this map is not generally surjective. However, if T is chosen so that G-orbits intersect T in isolated points, the proposition allows us to reduce the problem from finding invariants of a Lie group to finding invariants of a discrete (and usually finite!) group on a lower-dimensional vector space. The only remaining problem is to show which elements of $S(T^*)^N$ are restrictions of polynomials.

As seen in section 1.1.3, in this thesis we will be finding polynomials on $\mathfrak{g}^*$ invariant under the coadjoint action of G.

1.3.2 Polarization

Suppose we know the invariant polynomials $S(V^*)^G$ for some group G acting on a vector space V, and we would like to know the functions of more than one vector argument invariant under the simultaneous action of G on all arguments. That is, we would like to find a function f on $V \times \cdots \times V$ such that

$$f(x, y, \ldots, z) = f(g \cdot x, g \cdot y, \ldots, g \cdot z)$$

for all $g \in G$. Note that there may be more or fewer than three arguments, but for clarity they will be represented by $x, y, \ldots, z$.

Suppose that f is a homogeneous element of $S(V^*)^G$. Let $x, y \in V$, $t \in \mathbb{R}$, and consider the Taylor expansion

$$f(x+ty) = f(x) + tf_1(x,y) + \cdots \tag{1.6}$$

Since f is invariant under the simultaneous action of G on x and y, it's easy to see that each term in equation (1.6) must be too. In particular,

$$f_1(x,y) = \sum_i \frac{\partial f}{\partial x_i} y_i$$

will be invariant under this action. Define the "polarization operator" D_{xy} by

$$D_{xy} f = \sum_i \frac{\partial f}{\partial x_i} y_i$$

and note that D_{xy} reduces x degree by one, and increases y degree by one. This process can be repeated with new variables until

$$D_{xz} \cdots D_{xy} f$$

is a multilinear function in $x, y, \ldots, z$. This is called the "complete polarization" of f, and is a source of polynomials which are invariant under the simultaneous action of a group on several copies of a vector space.

To reconstruct f from $D_{xy}f$, one only has to compute $(D_{xy}f)(x,x) = (\deg_x(f))f(x)$.

In particular, the complete polarizations of the symmetric functions

$$\begin{aligned}
\phi_1(x) &= \sum_i x_i \\
\phi_2(x,y) &= \sum_{i \neq j} x_i y_j \\
&\vdots \\
\phi_n(x,y,\ldots,z) &= \sum_{i \neq j \neq \cdots \neq k} x_i y_j \cdots z_k
\end{aligned} \tag{1.7}$$

generate the polynomials invariant under the action of $\mathcal{S}_n$ on any number of n-dimensional vectors.

For a complete description of polarization, see Procesi [13] or Weyl [17].

1.3.3 Important examples

Suppose $G = \mathrm{SU}(n)$. Its Lie algebra $\mathfrak{su}(n)$ consists of skew-Hermitian matrices, and the adjoint action is $\mathrm{Ad}(g)X = gXg^{-1}$. Since $\mathfrak{su}(n)$ is semisimple, we can use the Killing form $B(X,Y) = 2n\,\mathrm{tr}(XY)$ to identify $\mathfrak{su}(n)^*$ with $\mathfrak{su}(n)$, and because the Killing form is Ad-invariant, we have:

$$\begin{aligned} B(\mathrm{Ad}^*(g)X, Y) &= B(X, \mathrm{Ad}(g^{-1})(Y)) \\ &= B(\mathrm{Ad}(g)X, Y) \end{aligned}$$

and so $\mathrm{Ad}^* = \mathrm{Ad}$.

Every element of a semisimple Lie algebra can be written as $\mathrm{Ad}(g)T$, where T is an element of a maximal abelian subalgebra. This means that the maximal torus is a transversal subspace. For $\mathfrak{su}(n)$, we can take the trace-zero imaginary diagonal matrices as our maximal torus:

$$\mathfrak{t} = \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_n \end{pmatrix}$$

where $\sum_i \lambda_i = 0$. The Weyl group of $\mathrm{SU}(n)$ is $\mathcal{S}_n$, acting by permutation of the λ_i, and so if N is the subgroup of $\mathrm{SU}(n)$ which normalizes $\mathfrak{t}$, then the $\mathrm{Ad}(N)$-invariant polynomials on $\mathfrak{t}$ are generated by the elementary symmetric polynomials in the λ_i.

The coefficients of the characteristic polynomial $\det(iX - tI)$ are invariant under conjugation, and when restricted to $X \in \mathfrak{t}$ are the elementary symmetric polynomials on the λ_i. By proposition 1.3.1, they generate the Ad-invariant polynomials on $\mathfrak{su}(n)$, and thus the Ad_0^*-invariant polynomials on $\mathfrak{su}(n)^*$.

A very similar calculation shows that the coefficients of the characteristic polynomial $\det(X - tI)$ generate the $\mathrm{Ad}(\mathrm{O}(n))$-invariant polynomials for $\mathfrak{o}(n)$. For this and many other examples of the technique, see [12].

1.3.4 Useful facts

When possible, we would like to find algebraically independent generators for $S(\mathfrak{g})^G$. In order to determine whether is is possible, we can use the Shepard-Todd-Chevalley theorem:

Theorem 1.3.2 *The invariant ring* $\mathbb{C}(x_1, \ldots, x_n)^G$ *for a finite matrix group* $G \subset \mathrm{GL}(n, \mathbb{C})$ *is generated by* n *algebraically independent homogeneous invariants if and only if* G *acts by reflections.*

for a proof, see [15].

Chapter 2

Computing Some Invariants

2.1 The Galilean group

2.1.1 Description of the group

The Galilean group Gal(n) is the Lie group of transformations between reference frames in Newtonian mechanics in n dimensional space. It is generated by spatial rotations, translations in space and time, and boosts, which correspond to changes in velocity (see [1]). Elements of Gal(n) can be represented as $(n+2)\times(n+2)$ matrices

$$\left(\begin{array}{ccc|cc} & & & v_1 & x_1 \\ & \rho & & \vdots & \vdots \\ & & & v_n & v_n \\ \hline 0 & \cdots & 0 & 1 & x_0 \\ 0 & \cdots & 0 & 0 & 1 \end{array}\right) \tag{2.1}$$

where ρ is an element of O(n), $\left(\begin{smallmatrix} x_1 \\ \vdots \\ x_n \end{smallmatrix}\right)$ the spatial translation, x_0 the time shift, and $\left(\begin{smallmatrix} v_1 \\ \vdots \\ v_n \end{smallmatrix}\right)$ is the boost.

To get the action on $n+1$-dimensional spacetime $\mathbb{R}^{n+1}$, we first identify affine $(n+1)$-space with the plane $x_{n+2}=1$ in $\mathbb{R}^{n+2}$. Then the action on a point $(X_1,\ldots,X_n,T)\in\mathbb{R}^{n+1}$ is:

$$\left(\begin{array}{ccc|cc} & & & v_1 & x_1 \\ & \rho & & \vdots & \vdots \\ & & & v_n & v_n \\ \hline 0 & \cdots & 0 & 1 & x_0 \\ 0 & \cdots & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ T \\ 1 \end{pmatrix} = \begin{pmatrix} X_1' \\ \vdots \\ X_n' \\ T' \\ 1 \end{pmatrix}$$

Taking derivatives, the Lie algebra $\mathfrak{gal}(n)$ of Gal(n) is identified with the Lie algebra of matrices of the form

$$\left(\begin{array}{ccc|cc} & & & v_1 & x_1 \\ & K & & \vdots & \vdots \\ & & & v_n & v_n \\ \hline 0 & \cdots & 0 & 0 & x_0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right) \tag{2.2}$$

where K is an element of $\mathfrak{o}(n)$.

The Galilean group is a contraction of the Poincaré group, and this reflects the fact that special relativity becomes Newtonian mechanics in the limit as the speed of light goes to infinity. The Poincaré group is itself a contraction of $SO(4,1)$, whose bi-invariant differential operators were found in [16] and [5].

2.1.2 The coadjoint action

In the first chapter, we saw that the algebra of bi-invariant differential operators on a group G can be identified with the polynomials on $\mathrm{Lie}(G)^*$ invariant under the coadjoint action. Thus our first step is to describe $\mathfrak{gal}(n)^*$ and the coadjoint action of $\mathrm{Gal}(n)$.

Recalling the matrix forms of the Galilean group and its Lie algebra given in (2.1) and (2.2), we must now describe the real dual space $\mathfrak{gal}^*(n)$ and the coadjoint action of $\mathrm{Gal}(n)$ on $\mathfrak{gal}^*$.

To represent $\mathfrak{gal}(n)^*$, we use the usual matrix inner product $A^*(B) = \mathrm{tr}(A^T B)$. The map $A \mapsto A^*$ identifies $\mathfrak{gal}(n)^*$ with $\mathbb{R}^{(n+2)\times(n+2)}$ modulo $\mathfrak{gal}(n)^\perp$, where

$$\mathfrak{gal}(n)^\perp = \left\{ \left(\begin{array}{c|cc} S & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline A & \begin{smallmatrix} a \\ b \end{smallmatrix} & \begin{smallmatrix} 0 \\ c \end{smallmatrix} \end{array}\right) \mid S \text{ symmetric}, A \in \mathbb{R}^{2\times n}, a, b, c \in \mathbb{R} \right\}$$

Using the fact that the trace is invariant under cyclic permutations,

$$(\mathrm{Ad}^*(g)A^*)(B) = A^*(\mathrm{Ad}(g^{-1}B))$$

$$= A^*(g^{-1}Bg)$$

$$= \mathrm{tr}(A^T g^{-1} B g)$$

$$= \mathrm{tr}(g A^T g^{-1} B)$$

$$= ((g A^T g^{-1})^T)^*(B)$$

Every element of Gal(n) can be written as $\tau\rho$, where ρ is a spatial rotation and τ is a space-time translation and boost. Because of this, we may consider the actions of rotations separately from boosts and translations.

If

$$\rho = \left(\begin{array}{c|c} \rho & \begin{smallmatrix} 0 & 0 \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \end{array}\right)$$

and

$$X = \left(\begin{array}{c|c} K^* & \begin{smallmatrix} v_1^* & x_1^* \\ \vdots & \vdots \\ v_n^* & x_n^* \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 0 & x_0^* \\ 0 & 0 \end{smallmatrix} \end{array}\right) \tag{2.3}$$

then a straightforward calculation shows:

$$\rho^{-1} = \left(\begin{array}{c|c} \rho^{-1} & \begin{smallmatrix} 0 & 0 \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \end{array}\right)$$

and

$$(\rho X^T \rho^{-1})^T = \left(\begin{array}{c|c} \rho K^* \rho^{-1} & \rho \begin{pmatrix} v_1^* & x_1^* \\ \vdots & \vdots \\ v_n^* & x_n^* \end{pmatrix} \\ \hline 0 & \begin{smallmatrix} 0 & x_0^* \\ 0 & 0 \end{smallmatrix} \end{array}\right) \tag{2.4}$$

For translations and boosts,

$$\tau = \left(\begin{array}{c|c} I & \begin{smallmatrix} v_1 & x_1 \\ \vdots & \vdots \\ v_n & x_n \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix} \end{array}\right)$$

$$\tau^{-1} = \left(\begin{array}{c|c} I & \begin{smallmatrix} -v_1 & v_1x_0-x_1 \\ \vdots & \vdots \\ -v_n & v_nx_0-x_n \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 1 & -x_0 \\ 0 & 1 \end{smallmatrix} \end{array}\right)$$

$$(\tau X\tau^{-1})^T = \left(\begin{array}{c|c} K^* + \begin{pmatrix} v_1^* & x_1^* \\ \vdots & \vdots \\ v_n^* & x_n^* \end{pmatrix} \left(\begin{smallmatrix} v_1 \dots v_n \\ x_1 \dots x_n \end{smallmatrix}\right) & \begin{smallmatrix} v_1^*+x_0x_1^* & x_1^* \\ \vdots & \vdots \\ v_n^*+x_0x_n^* & x_n^* \end{smallmatrix} \\ \hline * & \begin{smallmatrix} * & x_0^*+\sum v_ix_i^* \\ * & * \end{smallmatrix} \end{array}\right) \tag{2.5}$$

$$\equiv \left(\begin{array}{c|c} K^* + \frac{1}{2}\begin{pmatrix} v_1^* & x_1^* \\ \vdots & \vdots \\ v_n^* & x_n^* \end{pmatrix} \left(\begin{smallmatrix} v_1 \dots v_n \\ x_1 \dots x_n \end{smallmatrix}\right) - \frac{1}{2}\left(\begin{smallmatrix} v_1 \dots v_n \\ x_1 \dots x_n \end{smallmatrix}\right)^T \begin{pmatrix} v_1^* & x_1^* \\ \vdots & \vdots \\ v_n^* & x_n^* \end{pmatrix}^T & \begin{smallmatrix} v_1^*+x_0x_1^* & x_1^* \\ \vdots & \vdots \\ v_n^*+x_0x_n^* & x_n^* \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 0 & x_0^*+\sum v_ix_i^* \\ 0 & 0 \end{smallmatrix} \end{array}\right)$$

where the final congruence is modulo $\mathfrak{gal}(n)^\perp$

As described in section 1.3.1, we will reduce the problem of finding Ad^*-invariant polynomials on $\mathfrak{g}^*$ by finding a subspace transversal to the orbits. To find a transversal subspace of $\mathfrak{gal}(n)^*$, let X be a generic element written as as (2.3). We would like to find a g which can take $(g^{-1}X^Tg)^T$ to our subspace.

First, use a rotation (2.4) to put x^* in the form $(A0\dots0)^T$ and v^* in the form $(CB0\dots0)^T$. We can then use an x_0-only translation element (equation (2.5) with $x_0 = -\frac{v_1^*}{x_1^*}$) to zero out the first element of v^*. Thus we can restrict our attention to matrices of the form

$$\left(\begin{array}{c|c} K^* & \begin{smallmatrix} 0 & A \\ B & 0 \\ 0 & 0 \\ \vdots & \vdots \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 0 & x_0^* \\ 0 & 0 \end{smallmatrix} \end{array}\right)$$

where $A = \|x^*\|$ and $B = \|\mathrm{proj}_{x^{*\perp}}(v^*)\|$.

Let's now turn to the top-left quadrant. Referring again to (2.5),

$$\frac{1}{2}\begin{pmatrix} 0 & A \\ B & 0 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}\left(\begin{smallmatrix} v_1 & \dots & v_n \\ x_1 & \dots & x_n \end{smallmatrix}\right) - \frac{1}{2}\left(\begin{pmatrix} 0 & A \\ B & 0 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}\left(\begin{smallmatrix} v_1 & \dots & v_n \\ x_1 & \dots & x_n \end{smallmatrix}\right)\right)^T \tag{2.6}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & Ax_2-Bv_1 & Ax_3 & Ax_4 & \dots & Ax_n \\ Bv_1-Ax_2 & 0 & Bv_3 & Bv_4 & \dots & Bv_n \\ -Ax_3 & -Bv_3 & 0 & 0 & \dots & 0 \\ -Ax_4 & -Bv_4 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -Ax_n & -Bv_n & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.7}$$

Appropriate choices for the x_i's and v_i's allow us to zero out the topmost two rows and leftmost two columns of K^*. In particular, for $i \geq 3$, we set $x_i = -\frac{2K^*_{1i}}{A}$ and $v_i - \frac{2K^*_{2i}}{B}$. It is important to note that there is a remaining degree of freedom (from the (1,2) element in eq. (2.7)) when zeroing K^*_{12}, as this will allow us to clear x^*_0 by setting $v_1 = -\frac{x^*_0}{x^*_1}$ in (2.5).

We are free to choose a rotation from the subgroup of $\mathrm{O}(n)$ which fixes the first two coordinates. We can use this freedom to conjugate the upper-left block to an element of a maximal torus in $\mathfrak{o}(n-2)$.

Our transversal subspace is thus made of matrices of the form

$$\left(\begin{array}{c|c} \begin{smallmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & K \end{smallmatrix} & \begin{smallmatrix} 0 & A \\ B & 0 \\ 0 & 0 \\ \vdots & \vdots \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{array}\right) \tag{2.8}$$

where A and B are real numbers, and K is an element of a $\mathfrak{o}(n-2)$ maximal torus, eg:

$$K = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

2.1.3 Case $n = 1$

When $n = 1$, the upper-left hand block is always zero. We use x_0 to zero out the v^*_1 and v_1 to zero out x^*_0. Thus, a generic element

$$\begin{pmatrix} 0 & v^*_1 & x^*_1 \\ 0 & 0 & x^*_0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{gal}(1)^*$ can be conjugated to

$$\begin{pmatrix} 0 & 0 & x_1^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives:

Theorem 2.1.1 *$S(\mathfrak{gal}(1))^{\mathrm{Gal}(1)}$ is generated by X_1.*

2.1.4 Case $n = 2, 3$

Theorem 2.1.2 *For $n \in \{2, 3\}$, $S(\mathfrak{gal}(n))^{\mathrm{Gal}(n)}$ is generated by $\sum_i X_i^2$ and $\left(\sum_i X_i^2\right)\left(\sum_i V_i^2\right) - \left(\sum_i X_i V_i\right)^2$, which are algebraically independent.*

Proof: When $n = 2$ or 3, K^* may not be zero, but examining (2.8) shows that we can still zero out the upper-left block entirely. If

$$\Xi = \left(\begin{array}{c|cc} K^* & v^* & x^* \\ \hline 0 & 0 & t^* \\ 0 & 0 & 0 \end{array}\right)$$

is a matrix representing an element of $\mathfrak{gal}(n)^*$, then there exists an element $g \in \mathrm{Gal}(n)$ such that

$$\mathrm{Ad}(g)^*\Xi \equiv \left(\begin{array}{c|cc} 0 & \begin{smallmatrix} 0 \\ B \end{smallmatrix} & \begin{smallmatrix} A \\ 0 \end{smallmatrix} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

where $A = \|x^*\|$ and $B = \|\mathrm{proj}_{x^{*\perp}}(v^*)\|$. Conjugating by an element of $\mathrm{O}(n)$ can switch the signs of A and B independently, so the invariant polynomials on the transversal subspace are generated by A^2 and B^2. Thus any polynomial P which is invariant under the coadjoint action of $\mathrm{Gal}(n)$ restricts to a polynomial $\overline{P} = F(A^2, B^2)$ defined on the transversal subspace.

The orbit of a generic point in $\mathfrak{gal}(n)^*$ may intersect the transversal subspace in multiple points. As we've just seen, the values of A^2 and A^2B^2 will be the same at all of these points, so we can define $Q_1 : \mathfrak{gal}(n)^* \to \mathbb{R}$ and $Q_2 : \mathfrak{gal}(n)^* \to \mathbb{R}$ to be these functions, which will be constant on the orbits. If Ξ is a matrix representing an element of $\mathfrak{gal}(n)^*$, we have:

$$Q_1(\Xi) = \|x^*\|^2 = \left(\sum_i X_i^2\right)(\Xi)$$
$$Q_2(\Xi) = \|x^*\|^2 \, \|\mathrm{proj}_{x^{*\perp}} v^*\|^2 = \left(\left(\sum_i X_i^2\right)\left(\sum_i V_i^2\right) - \left(\sum_i X_i V_i\right)^2\right)(\Xi)$$

Because the restriction is injective on the space of Ad^* invariant polynomials, we know that

$$P(\Xi) = F\left(Q_1(\Xi), \frac{Q_2(\Xi)}{Q_1(\Xi)}\right) \tag{2.9}$$
$$= F_1(Q_1(\Xi), Q_2(\Xi)) + \frac{F_2(Q_1(\Xi), Q_2(\Xi))}{Q_1(\Xi)^\ell} \tag{2.10}$$

where F_2 is not divisible by its first argument.

While we are considering only real Lie algebras, if P is a polynomial, it must extend to a polynomial on the complexification of $\mathfrak{gal}(n)^*$. With this in mind, consider

$$\Xi_0 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & i \\ 0 & 0 & z & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

for $n = 2$ or

$$\Xi_0 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & z & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

when $n = 3$. In either case, $Q_1(\Xi) = 0$, and $Q_2(\Xi) = z^2$. (2.10) then becomes

$$P(\Xi) = F_1(0, z^2) + \frac{F_2(0, z^2)}{0^\ell}$$

Since P was assumed to be a polynomial, either $\ell = 0$, or $F_2(0, z^2) = 0$ for all $z \in \mathbb{C}$. Since F_2 was assumed to be indivisible by its first element, this implies that either $\ell = 0$ or F_2 is the constant function 0. Therefore, P is a polynomial in Q_1 and Q_2 □

2.1.5 Case $n > 3$

Theorem 2.1.3 *When $n > 3$, the invariant algebra $S(\mathfrak{gal}(n))^{\mathrm{Gal}(n)}$ is generated by $\sum_i X_i^2$ and the sums of determinants of $2k \times 2k$ submatrices formed by taking the symmetric minors of*

$$\left(\begin{array}{c|cc} K^* & \begin{matrix} v_1^* \\ \vdots \\ v_n^* \end{matrix} & \begin{matrix} x_1^* \\ \vdots \\ x_n^* \end{matrix} \\ \hline \begin{matrix} -v^{*T} \\ -x^{*T} \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right)$$

which include the last two rows and columns.

Proof: When $n > 3$, the coadjoint action can no longer zero out the entire upper-left block, only the uppermost two rows and leftmost two columns. The transversal subspace S is made of matrices of the form

$$\left(\begin{array}{ccc|cc} 0 & 0 & \cdots & 0 & A \\ 0 & 0 & \cdots & B & 0 \\ \vdots & \vdots & K & 0 & 0 \\ & & & \vdots & \vdots \\ \hline & 0 & & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right)$$

where K is an $\mathfrak{o}(n-2)$ maximal torus (see (2.8)).

Conjugating by elements of $\mathrm{O}(n)$ can change the signs of A and B independently, as well as permuting and changing signs of the elements of K. The polynomials on S invariant under the subgroup of $\mathrm{Gal}(n)$ which fixes S are generated by A^2, B^2, and the coefficients of the characteristic polynomial of K [12].

By the same argument as in the $n = 2$ case, B^2 must be multiplied by A^2 to clear the denominator. This gives invariant polynomials $A^2 = \sum X_i^2$ and $A^2 B^2 = (\sum X_i^2)(\sum V_i^2) - (\sum X_i V_i)^2$

We now turn to the polynomials which depend on the K part. When conjugating to S, one step was to zero out the rows and columns corresponding to x^* and v^*. This is equivalent to pre- and post- multiplying by the matrix P, the orthogonal projection to $\mathrm{span}\{x^*, v^*\}^\perp$. The resulting element of $\mathfrak{o}(n-2)^*$ is subject to a $\mathrm{O}(n-2)$ action, and it is well-known that the resulting invariant polynomials generated by the coefficients of the characteristic polynomial. Thus we are looking for $\mathrm{charpoly}(PK^*P)$

We will also make use of the following facts about exterior algebras. Suppose that $e_1, .., e_m$ and $f_1, \dots, f_n$ are bases for vector spaces V_1 and V_2 respectively. If $A : V_1 \to V_2$ is a linear map, then define $\bigwedge^k A : \bigwedge^k V_1 \to \bigwedge^k V_2$ to be the map $x_1 \wedge x_2 \wedge \cdots \wedge x_k \mapsto Ax_1 \wedge Ax_2 \wedge \cdots \wedge Ax_k$. Then the $(e_{i_1} \wedge \cdots \wedge e_{i_k}, f_{j_1} \wedge \cdots \wedge f_{j_k})$ element of the matrix of $\bigwedge^k A$ is the determinant of the $k \times k$ matrix formed by taking elements in rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$. In particular, the coefficient of the x^{n-k} term of $\mathrm{charpoly}(A)$ is $\mathrm{tr}(\bigwedge^k A)$.

Since $\|v \wedge x\| = AB$, we can define

$$\omega = \frac{v \wedge x}{\|v \wedge x\|} = \frac{1}{AB} v \wedge x$$

(the sign of AB is ambiguous, without loss of generality we may assume it's positive), and let

$$K' = \left(\begin{array}{c|cc} K^* & \begin{matrix} v_1^* & x_1^* \\ \vdots & \vdots \\ v_n^* & x_n^* \end{matrix} \\ \hline \begin{matrix} -v^{*T} \\ -x^{*T} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array} \right)$$

Let $e_1, \dots, e_{n+2}$ be the standard column basis for $\mathbb{R}^{n+2}$, and treat x^* and v^* as elements of $\mathbb{R}^{n+2}$ by embedding $\mathbb{R}^n$ into the first n coordinates of $\mathbb{R}^{n+2}$.

If $y \in \mathrm{span}\{e_1, \dots, e_n\}$:

$$K'y = K^*y - (y \cdot v^*)e_{n+1} - (y \cdot x^*)e_{n+2} \tag{2.11}$$

And we also have the following:

$$\left(\bigwedge^2 K'\right) e_{n+1} \wedge e_{n+2} = v^* \wedge x^* = AB\omega \tag{2.12}$$

and

$$\begin{aligned}
&\left(\bigwedge^2 K'\right)\omega \\
&= \frac{1}{AB}\left(\bigwedge^2 K'\right) v^* \wedge x^* \\
&= \frac{1}{AB}(K^* v^* - (v^* \cdot v^*)e_{n+1} - (v^* \cdot x^*)e_{n+2}) \wedge (K^* x^* - (x^* \cdot v^*)e_{n+1} - (x^* \cdot x^*)e_{n+2}) \\
&= \frac{1}{AB}\left[K^* v^* \wedge (K^* x^* - (x^* \cdot v^*)e_{n+1} - (x^* \cdot x^*)e_{n+2})\right. \\
&- ((v^* \cdot v^*)e_{n+1} + (v^* \cdot x^*)e_{n+2}) \wedge K^* x^* \\
&+ \left.\left((x^* \cdot x^*)(v^* \cdot v^*) - (x^* \cdot v^*)^2\right) e_{n+1} \wedge e_{n+2}\ \right]
\end{aligned} \tag{2.13}$$

In particular, note that the $e_{n+1} \wedge e_{n+2}$ term of $(\bigwedge^2 K')\omega$ is $ABe_{n+1} \wedge e_{n+2}$.

Now consider any diagonal element $(\bigwedge^{k+4} K')_{(I,I)}$ for which $I = (i_1, \ldots, i_{k+2}, n+1, n+2)$. By (2.12), $(\bigwedge^{k+4} K')\alpha \wedge e_{n+1} \wedge e_{n+2} \in \bigwedge^{k+2} \mathbb{R}^{n+2} \wedge \omega$. Since we're only considering diagonal elements, this means we can restrict our attention to the subspace

$$\left(\bigwedge^k \mathbb{R}^n\right) \wedge \omega \wedge e_{n+1} \wedge e_{n+2}$$

where by $\mathbb{R}^n$ refers to $\text{span}\{e_1, \ldots, e_n\} \subset \mathbb{R}^{n+2}$. Recall that P projects to $\text{span}\{x^*, v^*\}^\perp$. The we have, for $y_1, \ldots, y_k \in \mathbb{R}^n$:

$$\begin{aligned}
&\left(\bigwedge^{(k+4)} K'\right) y_1 \wedge \cdots \wedge y_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2} \\
=&\left(\bigwedge^{(k+4)} K'\right) Py_1 \wedge \cdots \wedge Py_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2} \\
=&A^2B^2 K^* Py_1 \wedge \cdots \wedge K^* Py_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2} \\
&\quad + \text{terms without } \omega \wedge e_{n+1} \wedge e_{n+2} \\
=&A^2B^2 PK^* Py_1 \wedge \cdots \wedge PK^* Py_k \wedge \omega \wedge e_{n+1} \wedge e_{n+2} \\
&\quad + \text{terms without } \omega \wedge e_{n+1} \wedge e_{n+2}
\end{aligned} \tag{2.14}$$

By (2.14),

$$\sum_{I=(i_1,\ldots,i_k+2,n+1,n+2)} \left(\bigwedge^{k+4} K'\right)_{(I,I)} = A^2B^2 \operatorname{tr}\left(\bigwedge^{k} PK^*P\right)$$

which implies that the sum of $(k+4)\times(k+4)$ subdeterminants which include the last two rows and columns is the x^{n-k} coefficient of $A^2B^2\,\mathrm{charpoly}(PK^*P)$. Because PK^*P is skew-symmetric, the odd-k terms will be zero.

To summarize, we have the polynomials $Q_1 = x^*\cdot x^*$, $Q_2 = (v^*\cdot v^*)(x^*\cdot x^*) - (x^*\cdot v^*)^2$, and $Q_3,\ldots,Q_{2+\lfloor n/2\rfloor}$, which are Q_2 times the characteristic polynomial coefficients of PK^*P

Finally, we must show that the Q_i generate the invariant polynomials. Suppose that $F(X)$ is an invariant polynomial not generated by the Q_i. Restricting to the transversal subspace S, $\overline{F}$ is generated by the restrictions of $Q_1, \frac{Q_2}{Q_1}, \frac{Q_3}{Q_2}, \ldots, \frac{Q_{2+\lfloor n/2\rfloor}}{Q_2}$. By the injectivity of restriction to S, this means that

$$\begin{aligned} F(X) &= F'\left(Q_1(X), \frac{Q_2}{Q_1}(X), \frac{Q_3}{Q_2}(X), \ldots, \frac{Q_{2+\lfloor n/2\rfloor}}{Q_2}(X)\right) \\ &= F''(Q_1(X), Q_2(X), \ldots, Q_{2+\lfloor n/2\rfloor}(X)) + \frac{\tilde{F}(Q_1(X), Q_2(X), \ldots, Q_{2+\lfloor n/2\rfloor}(X))}{Q_1(X)^k Q_2(X)^\ell} \end{aligned} \tag{2.15}$$

where $\tilde{F}$ is assumed not to be divisible by its first or second arguments. To show that F is generated by the Q_i, we must show that $k = \ell = 0$.

While all of the preceding work was done over $\mathbb{R}$, any polynomial extends to a polynomial on $\mathbb{C}$. Consider matrices of the form

$$X_\Xi = \left(\begin{array}{cc|ccc|cc} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & i \\ \hline 0 & 0 & & & & 0 & 0 \\ \vdots & \vdots & & \Xi & & \vdots & \vdots \\ 0 & 0 & & & & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right)$$

where Ξ is an $(n-2)\times(n-2)$ skew-symmetric matrix.

For any Ξ, $Q_1(X_\Xi) = 0$ and $Q_2(X_\Xi) = 1$. For $3 \leq i \leq 2 + \lfloor n/2 \rfloor$, we get the nonconstant characteristic polynomial coefficients for Ξ, which are known to be algebraically independent. Because of this algebraic independence, and that $\tilde{F}$ was assumed not to be divisible by its first argument, there exists a Ξ such that the numerator of (2.15) is nonzero, while $Q_1(X_\Xi)$ is zero. Therefore, $k = 0$.

To show that $\ell = 0$, let

$$Y_\Xi = \left(\begin{array}{cc|ccc|cc} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ \hline 0 & 0 & & & & i & 0 \\ 0 & 0 & & & & 0 & 0 \\ \vdots & \vdots & & \Xi & & \vdots & \vdots \\ 0 & 0 & & & & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right)$$

For any $\Xi \in \mathfrak{o}(n-2)$, $Q_1(Y_\Xi) = 1$ and $Q_2(Y_\Xi) = 0$. For $j \geq 3$, $Q_j(Y_\Xi)$ is the sum of all $2j \times 2j$ subdeterminants of

$$Y'_\Xi = \left(\begin{array}{cc|ccc|cc} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ \hline 0 & 0 & & & & i & 0 \\ 0 & 0 & & & & 0 & 0 \\ \vdots & \vdots & & \Xi & & \vdots & \vdots \\ 0 & 0 & & & & 0 & 0 \\ \hline 0 & -1 & -i & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right)$$

which include the leftmost two columns and bottom two rows.

Consider a skew-symmetric submatrix of Y'_Ξ which includes the last two rows and columns. If the submatrix doesn't also include the first row and column, its leftmost column, and thus its determinant, will be zero. Similarly, any nonzero such minor must also include at least one of the second and third rows.

If the submatrix includes the second row and column, the corresponding minor is

$$\det\left(\begin{array}{cc|ccc|cc} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ \hline 0 & 0 & & & & \alpha & 0 \\ 0 & 0 & & & & 0 & 0 \\ \vdots & \vdots & & \hat{\Xi} & & \vdots & \vdots \\ 0 & 0 & & & & 0 & 0 \\ \hline 0 & -1 & -\alpha & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right) = \det(\hat{\Xi})$$

where $\hat{\Xi}$ is a $(2j-4) \times (2j-4)$ submatrix of Ξ, and α is either 0 or i depending on whether the third column is included.

If the submatrix does not include the second row and column (which can only happen for $n > 4$), it must include the third, and the minor is instead

$$\det\left(\begin{array}{cc|ccc|cc} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \xi_{1k_1} & \dots & \xi_{1k_{2k-4}} & i & 0 \\ \hline 0 & -\xi_{1k_1} & & & & 0 & 0 \\ \vdots & \vdots & & \hat{\Xi} & & \vdots & \vdots \\ 0 & -\xi_{1k_{2k-4}} & & & & 0 & 0 \\ \hline 0 & -i & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right) = -\det(\hat{\Xi})$$

where now $\hat{\Xi}$ is a $(2j-4)\times(2j-4)$ submatrix of Ξ which *does not include the first row and column*. Summing all of the $2j \times 2j$ minors of Y'_{Ξ} then yields the sum of $(2j-4)\times(2j-4)$ minors of Ξ which include the first row.

Suppose Ξ is of the form:

$$\Xi = \left(\begin{array}{c|cccc} 0 & \xi_1 & 0 & \dots & 0 \\ \hline -\xi_1 & & & & \\ 0 & & & & \\ \vdots & & & \Xi' & \\ 0 & & & & \end{array}\right)$$

Then $\text{charpoly}(\Xi) = \lambda \cdot \text{charpoly}(\Xi') - \xi_1^2 \cdot \text{charpoly}(\Xi')$ Since the nonconstant coefficients of the characteristic polynomial are the sums of the minors, and are algebraically independent on $\mathfrak{o}(m)$, this tells us that the sums of minors which include the first row and column are also algebraically independent. Putting everything together, this tells us that for any $\tilde{F}$ in (2.15), we can find a Y_{Ξ} such that $Q_2(Y_{\Xi}) = 0$ and $\tilde{F}(Y_{\Xi}) \neq 0$. Therefore, $\ell = 0$ in (2.15), and so every polynomial on $\mathfrak{gal}(n)^*$ which invariant on the coadjoint orbits is generated by our Q_j's □

2.2 Cartan motion groups

2.2.1 Prior work

Recall from section 1.2.1 that when G is a noncompact semisimple Lie group, we can contract $\mathfrak{g}$ along the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, forming the group $G_0 = K \ltimes \mathfrak{p}$, where K acts by the adjoint representation. If G is compact, we can use an involutive automorphism of G to find a non-compact dual G'. If $\mathrm{Lie}(G') = \mathfrak{k} \oplus \mathfrak{p}$, then the decomposition $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ will serve as a Cartan decomposition for G, and in fact contracting G and G' along $\mathfrak{k}$ will produce the same result.

In [5], invariants were found for the Euclidean motion group, and in [6], they were found for the groups $\mathrm{O}(n+k)/(\mathrm{O}(n) \times \mathrm{O}(k))$. In both cases, they invariants were found to be the contracted invariants described in section 1.2.

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Equations (1.5) and (1.4) in section 1.2 gave us the coadjoint action for the Cartan motion group $G_0 = K \ltimes \mathfrak{p}$:

$$\mathrm{Ad}_0^*(k,0)(X+T) = \mathrm{Ad}(k)(X+T)$$
$$\mathrm{Ad}_0^*(e,Y)(X+T) = X + T + [Y,X]$$

where the unsubscripted Ad and $[\cdot,\cdot]$ refer to the uncontracted versions, $X \in \mathfrak{p}$, and $T \in \mathfrak{k}$.

Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and let $\mathfrak{m} \subset \mathfrak{k}$ be centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Finally, choose $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{m}$ so that $\mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$. Note that this subspace is transversal to the coadjoint action of G on $\mathfrak{g}^*$. In fact, it will still be a transversal subspace for G_0.

To see this, first note that $\mathrm{Ad}(K)(\mathfrak{a}) = \mathfrak{p}$, so there exists a $k \in K$ such that

$$\mathrm{Ad}_0^*(k,0)(X+T) = (H+T') \in \mathfrak{a} + \mathfrak{k}$$

Because the set of regular elements in $\mathfrak{a}$ is dense, we may assume H is regular, that is $\alpha(H) \neq 0$ for all α in the restricted root system Σ.

Now consider $\mathrm{ad}(H)^2$. This is a linear operator which preserves both $\mathfrak{p}$ and $\mathfrak{k}$. Because H was assumed to be regular, we can take eigenspace decompositions

$$\mathfrak{k} = \mathfrak{m} + \sum_{\alpha \in \Sigma^+} \mathfrak{k}_\alpha$$
$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha$$

where

$$\mathfrak{k}_\alpha = \left\{ T \in \mathfrak{k} \mid \mathrm{ad}(H)^2(T) = \alpha(H)^2 T \right\}$$
$$\mathfrak{p}_\alpha = \left\{ X \in \mathfrak{p} \mid \mathrm{ad}(H)^2(X) = \alpha(H)^2 X \right\}$$

and $\mathrm{ad}(H)$ maps $\mathfrak{k}_\alpha$ to $\mathfrak{p}_\alpha$ bijectively. Writing

$$T' = T'_{\mathfrak{m}} + \sum_{\alpha \in \Sigma^+} T'_\alpha$$

with $T'_\alpha \in \mathfrak{k}_\alpha$, let

$$Y = \sum_{\alpha \in \Sigma^+} \frac{1}{\alpha(H)^2} [H, T'_\alpha]$$

so that

$$\begin{aligned} \mathrm{Ad}_0^*(e, Y)(H + T') &= H + T' + [Y, H] \\ &= H + T' - \sum_{\alpha \in \Sigma^+} T'_\alpha \\ &= H + T'_{\mathfrak{m}} \end{aligned}$$

Finally, we can use an element $k' \in K$ to conjugate $T'_{\mathfrak{m}}$ to $\mathfrak{h}_{\mathfrak{k}}$, without perturbing H. Thus we have:

$$\mathrm{Ad}_0^* \left((k', 0) \cdot (e, Y) \cdot (k, 0) \right) (X + T) \in \mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$$

Now we would like to know the normalizer of $\mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$. Because H doesn't commute with any nonzero element of $[H, \mathfrak{p}]$, we know that the $\mathfrak{p}$-part of any normalizer must be in $\mathfrak{a}$. Since $\mathrm{Ad}_0^*(k,e) = \mathrm{Ad}(k)$, the k part must normalize $\mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$ in G as well. Since $\mathrm{Ad}(k)$ leaves both $\mathfrak{k}$ and $\mathfrak{p}$ invariant, the image of this action is the subset of the Weyl group which leaves $\mathfrak{a}$ invariant, denoted W_θ [9].

Compare this with the uncontracted case, where we have the same transversal subspace, but with the full Weyl group acting on it. Since we have a smaller group in the contracted case, one may expect new invariants.

2.2.2 Normal real forms

If $\mathfrak{g}$ is a normal real form, that is, the maximal abelian subalgebra is contained entirely in $\mathfrak{p}$. In this case $W_\theta = W$.

Suppose P is a homogeneous invariant polynomial on $\mathfrak{g}^*$. Since $\mathfrak{g}$ is semisimple, the restriction of $S(\mathfrak{g})^G$ to $S(\mathfrak{a})^W$ is both injective and surjective. Let P_0 be the contracted invariant corresponding to P. Any terms in P which have nonzero $\mathfrak{k}$-degree will be zero on $\mathfrak{a}$, and thus $P|_{\mathfrak{a}} \equiv P_0|_{\mathfrak{a}}$. Since the $\mathrm{Ad}^*(G)$-invariant polynomials map surjectively onto the W-invariant polynomials on $\mathfrak{a}$, so too will the contracted polynomials. Therefore, when $\mathfrak{g}$ is a normal real form, the contracted invariants are the only elements of $S(\mathfrak{g}_0)^{G_0}$.

2.2.3 $S(p+q)/S(U(p)\times U(q))$

Consider $\mathrm{SU}(p+q)$, assuming without loss of generality that $p \geq q$. Its Lie algebra, $\mathfrak{su}(p+q)$, is the set of $(p+q)\times(p+q)$ skew-Hermitian matrices. The killing form is $B(X,Y) = 2(p+q)\,\mathrm{tr}(XY)$, which is proportional to the usual matrix inner product. It's important to remember that although $\mathfrak{su}(p+q)$ is defined using complex numbers, it is a *real* Lie algebra.

Let I_p be the $p\times p$ identity matrix, and $I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$. Then $\theta(X) = I_{p,q} X I_{p,q}$ is a Cartan involution, with

$$\mathfrak{k} = \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}$$

$$\mathfrak{p} = \begin{pmatrix} 0 & X \\ -\overline{X}^T & 0 \end{pmatrix}$$

where T_p and T_q are elements of $\mathfrak{u}(p)$ and $\mathfrak{u}(q)$ respectively, subject to $\mathrm{tr}(T_p) + \mathrm{tr}(T_q) = 0$; and X is an arbitrary $p \times q$ complex matrix.

A routine calculation shows that

$$\mathfrak{a} = \left\{ \left(\begin{array}{cc|c} 0 & 0 & S \\ 0 & 0 & 0 \\ \hline -S & 0 & 0 \end{array} \right) \middle| S \text{ is a } q \times q \text{ real diagonal matrix} \right\}$$

is a maximal abelian subspace of $\mathfrak{p}$.

To extend $\mathfrak{a}$ to a maximal torus in $\mathfrak{g}$, let $\begin{pmatrix} A & B & 0 \\ -\overline{B}^T & C & 0 \\ 0 & 0 & D \end{pmatrix}$ be an element of $\mathfrak{k}$, and compute:

$$\begin{aligned} 0 &= \begin{pmatrix} A & B & 0 \\ -\overline{B}^T & C & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & 0 \\ -S & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & 0 \\ -S & 0 & 0 \end{pmatrix} \begin{pmatrix} A & B & 0 \\ -\overline{B}^T & C & 0 \\ 0 & 0 & D \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & AS \\ 0 & 0 & -\overline{B}^T S \\ -DS & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & SD \\ 0 & 0 & 0 \\ -SA & -SB & 0 \end{pmatrix} \\ &\Rightarrow AS = SD \\ &\quad \overline{B}^T S = 0 \end{aligned}$$

Since this must hold for all S, this shows that $A = D$ is diagonal, and $B = 0$. Thus the subalgebra of $\mathfrak{k}$ which commutes with all of $\mathfrak{a}$ is:

$$\mathfrak{m} = \left\{ \left(\begin{array}{cc|c} iD & 0 & 0 \\ 0 & T & 0 \\ \hline 0 & 0 & iD \end{array} \right) \middle| \begin{array}{l} D \text{ is a } q \times q \text{ real diagonal matrix,} \\ T \in \mathfrak{u}(p-q), \\ \mathrm{tr}(T) + 2i \,\mathrm{tr}\, D = 0 \end{array} \right\}$$

and for a maximal abelian part ($\mathfrak{h}_{\mathfrak{k}}$), we can take T to be a (pure imaginary) diagonal matrix. This gives our maximal torus for $\mathfrak{su}(p+q)$:

$$\mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{a} = \left\{ \left(\begin{array}{cc|c} iD & 0 & S \\ 0 & iT & 0 \\ \hline -S & 0 & iD \end{array} \right) \middle| \begin{array}{l} D, S \ q \times q \text{ real diagonal} \\ T \ (p-q) \times (p-q) \text{ real diagonal} \\ \operatorname{tr}(T) + 2\operatorname{tr}(D) = 0 \end{array} \right\} \tag{2.16}$$

Conjugating an element $X \in \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{a}$ by

$$g = \begin{pmatrix} \frac{\sqrt{2}}{2} I_q & 0 & i\frac{\sqrt{2}}{2} I_q \\ 0 & I_{p-q} & 0 \\ i\frac{\sqrt{2}}{2} I_q & 0 & \frac{\sqrt{2}}{2} I_q \end{pmatrix} \in \mathrm{SU}(p+q)$$

diagonalizes it. Taking the diagonal matrices as a maximal torus makes it easy to see that the Weyl group $W \cong S_{p+q}$.

If

$$X = \left(\begin{array}{ccc|ccc|ccc} ia_1 & & & & & & s_1 & & \\ & \ddots & & & & & & \ddots & \\ & & ia_q & & & & & & s_q \\ \hline & & & i\lambda_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & i\lambda_{p-q} & & & \\ \hline -s_1 & & & & & & ia_1 & & \\ & \ddots & & & & & & \ddots & \\ & & -s_q & & & & & & ia_q \end{array} \right)$$

then

$$gXg^{-1} = \left(\begin{array}{ccc|ccc|ccc} ia_1 - is_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & ia_q - is_q & & & & & & \\ \hline & & & i\lambda_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & i\lambda_{p-q} & & & \\ \hline & & & & & & ia_1 + is_1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & ia_q + is_q \end{array}\right)$$

Let W_θ be the subgroup of W which leaves $\mathfrak{a}$ invariant. From the form of gXg^{-1}, we can see that an element of W_θ can permute the λ_i, permute the a_i and s_i simultaneously, and can switch the signs of the s_i. Any other permutation would mix $\mathfrak{a}$ and $\mathfrak{h}_\mathfrak{k}$. Thus

$$W_\theta \cong S_{p-q} \times \left(S_q \ltimes \mathbb{Z}_2^q\right)$$

Since the first factor acts on the λ_i only, and is the only factor to act on them, the invariant polynomials involving the λ_i are generated by the symmetric polynomials on them.

The second factor acts on the s_i and a_i simultaneously. The $\mathbb{Z}_2^q$ part acting on the s_i means we must square them. Finally, when $\mathcal{S}_n$ acts on $2n$ elements, the invariants are generated by the polarizations of the symmetric polynomials [17].

Putting these together, the W_θ-invariant polynomials on $\mathfrak{h}_\mathfrak{k} \oplus \mathfrak{a}$ are generated by:

$$\sigma_1(\lambda_1, \ldots, \lambda_{p-q})$$
$$\vdots$$
$$\sigma_{p-q}(\lambda_1, \ldots, \lambda_{p-q})$$

where the σ_i are the elementary symmetric polynomials on $p-q$ arguments, and

$$\begin{gathered}\phi_1(v_1)\\ \vdots\\ \phi_q(v_1,\dots,v_q)\end{gathered}$$

where the ϕ_i are the completely polarized elementary symmetric polynomials on q elements, and each v_i is either $(a_1,\dots,a_q)$ or $(s_1^2,\dots,s_q^2)$.

The Cartan motion group associated with $\mathrm{SU}(p+1)/S(U(p)\times U(q))$ is $S(U(p)\times U(q))\ltimes\mathbb{C}^{p\times q}$, where the action is

$$(g_p,g_q)\cdot\begin{pmatrix} z_{11} & \cdots & z_{1q}\\ \vdots & \ddots & \vdots\\ z_{p1} & \cdots & z_{pq}\end{pmatrix} = g_p\begin{pmatrix} z_{11} & \cdots & z_{1q}\\ \vdots & \ddots & \vdots\\ z_{p1} & \cdots & z_{pq}\end{pmatrix} g_q^{-1}$$

2.2.3.1 $q = 1$

When $q=1$, every element is of the form $(g,\det(g)^{-1})$, so $S(U(p)\times U(1))\cong U(p)$. The result is a "twisted" version of the usual complex motion group, where the action of $U(p)$ on $\mathbb{C}^p$ is:

$$k\cdot\begin{pmatrix} z_1\\ \vdots\\ z_p\end{pmatrix} = \det(k)g\begin{pmatrix} z_1\\ \vdots\\ z_p\end{pmatrix}$$

Suppose $B(X,\cdot)$ is a generic element of $\mathfrak{g}_0^*$. We can write X as:

$$X = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1p} & z_1\\ \vdots & & \vdots & \vdots\\ a_{p1} & \cdots & a_{pp} & z_p\\ \hline -\overline{z}_1 & \cdots & -\overline{z}_p & b\end{array}\right) \tag{2.17}$$

where $a_{ij} = -\overline{a_{ji}}$.

Let Z be the column vector $\begin{pmatrix} z_1\\ \vdots\\ z_p\end{pmatrix}$. The coadjoint action allows us to conjugate by an element $k\in U(p)$ until $Z = \begin{pmatrix} \|Z\|\\ 0\\ \vdots\\ 0\end{pmatrix}$. This puts the $\mathfrak{p}$ part of X in $\mathfrak{a}$, and henceforth we will assume X is in this form. Define the matrix $(a'_{ij}) = k(a_{ij})k^{-1}$.

Note that

$$\left[\begin{pmatrix} 0 & \cdots & 0 & \|Z\| \\ \vdots & & \vdots & 0 \\ 0 & \cdots & 0 & \vdots \\ -\|Z\| & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & & 0 & y_1 \\ \vdots & & & \vdots & y_2 \\ & & & & \vdots \\ 0 & \cdots & & 0 & y_q \\ -\overline{y_1} & -\overline{y_2} & \cdots & -\overline{y_q} & 0 \end{pmatrix}\right] = \|Z\| \begin{pmatrix} 2i\,\mathrm{Im}(y_1) & -\overline{y_2} & \cdots & -\overline{y_q} & 0 \\ y_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ y_q & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -2i\,\mathrm{Im}(y_1)\|Z\| \end{pmatrix}$$

Proper choice of y allows us to use the coadjoint action to put X in the form

$$\mathrm{ad}^*(k,y)X = \left(\begin{array}{c|ccc|c} \frac{a'_{11}+b}{2} & 0 & \cdots & 0 & \|Z\| \\ \hline 0 & a'_{22} & \cdots & a'_{2q} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a'_{q2} & \cdots & a'_{qq} & 0 \\ \hline -\|Z\| & 0 & \cdots & 0 & \frac{a'_{11}+b}{2} \end{array}\right)$$

where $a'_{11} = \frac{\overline{Z}^T AZ}{\|Z\|^2}$ with $A = (a_{ij})$. If P is the operator which projects onto the orthogonal complement of the column space of X, then:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a'_{22} & \cdots & a'_{2q} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{q2} & \cdots & a'_{qq} \end{pmatrix} = kPAPk^{-1}$$

which we can diagonalize to find the λ_i.

We are now in a position to calculate the generators of the invariant polynomials. When $q = 1$, the invariant polynomials on $\mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{a}$ from the previous subsection become:

$$\sigma_1(\lambda_1, \ldots, \lambda_{p-1})$$
$$\vdots$$
$$\sigma_{p-1}(\lambda_1, \ldots, \lambda_{p-1})$$
$$\|Z\|^2$$
$$\frac{\overline{Z}^T AZ + b\|Z\|^2}{2\|Z\|^2}$$

where as before, the σ_i are the elementary symmetric polynomials on $p-q$ elements. The values of these polynomials can be found as the coefficients of the characteristic polynomial of PAP. Because tr = 0 for all elements of $\mathfrak{g}_0$, the first and the last polynomials are the same!

A calculation similar to that for the Galilean group shows that the sums of $m \times m$ subdeterminants of

$$i\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1p} & z_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{p1} & \cdots & a_{pp} & z_p \\ \hline -\overline{z_1} & \cdots & -\overline{z_p} & 0 \end{array}\right) \tag{2.18}$$

which include the last row and column result in $\|Z\|^2 \sigma_{m-2}(\lambda_1, \ldots, \lambda_{p-1})$.

Theorem 2.2.1 *With notation as in equation* (2.17)*; when G_0 is the Cartan motion group $U(p) \ltimes \mathbb{C}^p$, $S(\mathfrak{g}_0)^{G_0}$ is generated by the sums of $m \times m$ subdeterminants of* (2.18) *which include the last row and column, with $m \geq 2$.*

Proof: When $m = 2$, the sum of subdeterminants of (2.18) which include the last row and column is:

$$\sum_i i\begin{pmatrix} a_{ii} & z_i \\ -\overline{z}_i & 0 \end{pmatrix} = \|Z\|^2$$

Similarly to the Galilean group, we know that the restriction of any invariant polynomial to the transversal subspace results in a polynomial of the form

$$P(\|Z\|^2, \sigma_1(\lambda_1, \ldots, \lambda_{p-1}), \ldots, \sigma_{p-1}(\lambda_1, \ldots, \lambda_{n-1})) \qquad \square$$

Defining Q_i to be the sum of the $(i+2) \times (i+2)$ symmetric subdeterminants, we can use the injectivity of restriction to re-write this as:

$$P\left(Q_0(X), \frac{Q_1(X)}{Q_0(X)}, \ldots, \frac{Q_{p-1}(X)}{Q_0(X)}\right)$$

and split P:

$$P\left(Q_0(X), \frac{Q_1(X)}{Q_0(X)}, \ldots, \frac{Q_{p-1}(X)}{Q_0(X)}\right) = \tag{2.19}$$

$$P'(Q_0(X), Q_1(X), \ldots, Q_{p-1}(X)) + \frac{P''(Q_0(X), Q_1(X), \ldots, Q_{p-1}(X))}{Q_0(X)^k} \tag{2.20}$$

where P' and P'' are polynomials, and P'' isn't divisible by its first argument.

As with the Galilean group, we complexify. If P is a polynomial, there cannot be an element of $\mathfrak{g}_0^C$ for which P is undefined. Consider

$$X = \left(\begin{array}{cc|ccc|c} 0 & -i & 0 & \cdots & 0 & -\frac{i}{2} \\ -i & 0 & 0 & \cdots & 0 & \frac{i}{2} \\ \hline 0 & 0 & -i\lambda_1 & & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & & -i\lambda_{p-2} & 0 \\ \hline -\frac{i}{2} & \frac{i}{2} & 0 & \cdots & 0 & -ib \end{array}\right) + i \left(\begin{array}{cc|ccc|c} 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|ccc|c} 0 & -i & 0 & \cdots & 0 & -i \\ -i & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & -i\lambda_1 & & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & & -i\lambda_{p-2} & 0 \\ \hline 0 & -i & 0 & \cdots & 0 & -ib \end{array}\right)$$

With this X, (2.18) becomes

$$\left(\begin{array}{cc|ccc|c} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & & \lambda_{p-2} & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 & 0 \end{array}\right)$$

With this, the 2×2 subdeterminants which include the last row and column are all zero. In fact, the only way for a subdeterminant to be nonzero is to also include the first two rows and columns as well. Thus

$$Q_1 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

For $j > 1$,

$$Q_j = \sum_{1 \le k_1 < \cdots < k_j \le p-1} \left|\begin{array}{cc|ccc|c} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \lambda_{k_1} & & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & & \lambda_{k_j} & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 & 0 \end{array}\right| = \sum_{1 \le k_1 < \cdots < k_j \le p-1} \begin{vmatrix} \lambda_{k_1} & & \\ & \ddots & \\ & & \lambda_{k_j} \end{vmatrix}$$

$$= \sigma_j(\lambda_1, \dots, \lambda_{p-1})$$

Referring back to (2.20), we see that either $k = 0$, or $P''(Q_0(X), Q_1(X), \dots, Q_{p-1}(X)$ must be zero. Since the symmetric polynomials σ_i are algebraically independent, and P'' was chosen not to be divisible by its first argument, we can choose $\lambda_1, \dots, \lambda_{p-1}$ to force $P'' \equiv 0$ if $k \neq 0$. Therefore, $Q_0, \dots, Q_{p-1}$ generate the invariant algebra $S(\mathfrak{g}_0)_0^G$.

Bibliography

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1989.

[2] D. Calaque and C Rossi, *Lectures on Duflo isomorphisms in Lie algebra and complex geometry*, European Mathematical Society, 2011.

[3] J. Dixmier, *Enveloping Algebras*, North-Holland Publishing Company, 1977.

[4] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Dover Publications, Inc., 2002.

[5] F. Gonzalez, *Bi-invariant differential operators on the Euclidean motion group and applications to generalized Radon transforms*, Arkiv för Mathematik, 26 (1988), pp. 191–204.

[6] ——, *Invariant differential operators on matrix motion groups and applications to the matrix radon transform*, Contemporary Mathematics, 464 (2008), pp. 107–127.

[7] F. Gonzalez and T. Kakehi, *Pfaffian systems and Radon transforms on affine Grassmann manifolds*, Math. Ann., 326 (2003), pp. 237–273.

[8] S. Helgason, *A duality in integral geometry on symmetric spaces*, in Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto 1965, Nippon Hyronsha, Tokyo, 1966.

[9] ——, *Differential Geometry, Lie Groups, and Symmetric Spaces*, American Mathematical Society, 1978.

[10] ——, *Groups and Geometric Analysis*, American Mathematical Society, 1984.

[11] A. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, 1996.

[12] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Volume II*, John Wiley & Sons, Inc., 1986.

[13] C. Procesi, *Lie Groups: An Approach through Invariants and Representations*, Springer, 2007.

[14] F. Rouvière, *Inverting Radon transforms: the group-theoretic approach*, Enseign. Math., 47 (2001), pp. 205–252.

[15] B Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag/Wien, 2008.

[16] S. Takiff, *Invariant polynomials on Lie algebras of inhomogeneous unitary and special orthogonal groups*, Trans. Amer. Math. Soc., 170 (1972), pp. 221–230.

[17] H. Weyl, *The Classical Groups*, Princeton University Press, 1946.