Numerical Notation and the Place-Value Concept in Young Children

Gabrielle A. Cayton

February 2, 2007

Qualifying Paper

MSTE Education Doctoral Program

Table of Contents

Abstract

In recent decades, we have come to accept that children's appropriation of written numbers is not automatic or simple. Various studies of children's use of notation point to several different types of notational practice possibly linked to stages in the understanding of place value in base-ten (Alvarado, 2002; Brizuela, 2004; Scheuer et al, 2000; Seron & Fayol, 1994). Scheuer et al (2000) discuss two distinct types of incorrect numerical notation strategies used by children: logogramic and compacted notation. Logogramic notation refers to children who write the entire number literally, such as 100701 for one hundred seventy-one. Compacted notation refers to children who remove some of the zeros from the logogramic coding while still not condensing the number entirely into its conventional form, such as 1071 for one hundred seventy-one. Scheuer speculates that perhaps these two types of notational strategies stem from different conceptual ideas about the numbers themselves, yet the study does not explore these ideas, focusing only on written numbers. The study described in this paper examines Scheuer's arguments by comparing numerical notation strategies with children's performance on a task of decomposing numbers created by valued tokens that will provide insights into the children's ideas.

Results show that children who have mastered numerical notation and understanding for integers 1-99 do show different strategies of numerical notation for integers 100-999, when compared to children who are still struggling with numerical notation and understanding for integers 1-99. Similarly, children who have mastered numerical notation and understanding for integers 100-999 show strategies of numerical notation for integers 1,000-9,999 that are different from those of children who are still struggling with numerical notation and understanding for integers 100-999.

The ability to produce two-digit written numerals conventionally appears to occur prior to the understanding of numbers, as over 23% of the children in the sample produced conventional written numerals without showing an understanding of decomposition while no child who showed an understanding of decomposition was unable to produce the conventional number.

Two types of numerical notation strategies; Full Literal Transcoding and Compacted Notation, are correlated with different numerical understandings. Children using Compacted Notation tend to have an understanding of numerical composition that is closer to the conventional one held by adults than children using Full Literal Transcoding.

Introduction

The intricacies of our written number system are frequently under-recognized in the teaching of elementary mathematics. Components such as place-value and zero require certain cognitive abilities such as abstract thought, logical thinking, and inference. In order to better comprehend all of the intertwined factors required to understand our written number system, it is necessary to first look at the origins of our knowledge of numbers and what characteristics comprise a number system, as well as the history of written number systems. We are then able to look deeper into the steps children take when reconstructing the system for themselves.

In this work, I distinguish between the number system and the written number system. The number system is comprised of a finite set of elements (in our Arabic number system[1], 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) and the system defines relationships among these elements, which allows us to create infinite combinations. The external representation of the number system forms a system in and of itself: the written number system. The concepts embedded in the number system are represented externally in the written number system. The separation and distinction between both systems may seem arbitrary and unnecessary; however, the learning of each in the young child involves distinct intellectual challenges.

In his discussions of figurative and operative aspects of thought, Piaget (1962, 1965, 1971, 1973; Piaget & Inhelder, 1971) insists that there is a necessary interaction between figurative aspects and operative aspects of thought. In the use of our object of analysis, the reproduction of the numbers

[1] Also known as the Hindu-Arabic system.

themselves could superficially be thought of as figurative; and the number system as a conceptual object is certainly operative. However, we cannot understand the first without the second, just as Piaget thought of both aspects of thought as wholly interwoven and interconnected.

In 1994, Zhang and Norman developed a "theory of distributed representations" to account for behavior in distributed cognitive tasks (tasks that involve both internal and external representations) and concluded the same thing that Piaget had inferred over forty years earlier about figurative and operative aspects of thought: that the representation of a cognitive task is not solely internal or external but distributed over these two indispensable parts. Thus, external representations do not necessarily need to be re-represented internally because they directly activate processes.

Previous studies of children's numerical acquisition (Fuson & Kwon, 1992; Miura & Okamoto, 1989; Power & Dal Martello, 1990; Ross, 1986; Scheuer, 1996; Seron & Fayol, 1994; Sinclair & Scheuer, 1993) have focused on place value notation, number decomposition, and oral numeration, yet there has been very little work on the correlation of any of these aspects with one another.

For example, Scheuer et al (2000) discuss, amongst others, two distinct types of incorrect numerical notation strategies used by children: logogramic and compacted notation. Logogramic notation refers to children who write the entire number literally, such as 100701 for one hundred seventy-one. Compacted notation refers to children who remove some of the zeros from the logogramic coding while still not condensing the number entirely into its conventional form, such as 1071 for one hundred seventy-one. Scheuer speculates that perhaps these two types of notational strategies stem from different working ideas about the concept of number (the operative), yet her study only focuses on written numbers (the figurative); no data on children's ideas is included. The study described in this paper examines how the strategies described by Scheuer et al (2000) and others (e.g., Power & Dal Martello, 1990; Seron & Fayol, 1994) are related to children's understanding of the structure of the number system. The core interest of the research described in this paper lies in the connections and interactions between numerical notations and numerical understandings. I seek to answer the question: How are a child's numerical notation strategies correlated in any way to his or her understanding of number decomposition?

Sources of Knowledge

Before discussing what children do or do not "know" about numbers, it is necessary to explore what it means to "know" something. Piaget (1962) distinguishes between three kinds of knowledge: physical knowledge, socio-cultural (or conventional) knowledge, and logico-mathematical knowledge. Physical knowledge is the "knowledge of objects in external reality" (Kamii, Kirkland, & Lewis, 2001, p.25) such as color, weight, and physical properties, such as 'round things roll and flat things don't.' The main source of physical knowledge is observation (Piaget, 1971).

Socio-cultural knowledge is the knowledge of conventions, created by people, such as languages and holidays. The main sources of social knowledge are convention and social interaction; this includes interaction in a school setting.

Logico-mathematical knowledge “ranges from empirical apprenticeship to experimental behavior”(Piaget, 1971, p.3). It is the knowledge of mental relationships which “adds something to its environment, namely, structure elements that were not furnished in that form by events or objects outside the organism” (Piaget, 1971, p.101). One example of logico-mathematical knowledge is spontaneous categorization, such as classifying objects as similar or different.

While the concept of number and of numerosity stems from logico-mathematical knowledge, numerical conventions grow from social knowledge, such as the words one, two, and three, and the numerals 1, 2, and 3, without which we would only have the concept of numerosity, but no words or signs[2] to organize our thoughts.

Piaget and Inhelder (1971) also conceptualized two kinds of abstraction: empirical abstraction and reflective abstraction. Empirical abstraction focuses on certain properties of an object while ignoring others. For example, in classifying by color, we ignore weight, shape, and other physical properties of an object. Reflective abstraction involves the making of mental relationships between and among objects. “Having made the theoretical distinction between empirical and constructive abstraction, Piaget went on to say that in the psychological reality of the child, one cannot take place without the other” (Kamii et al., 2001, p.26).

In the case of the written number system, we see each of these three types of knowledge (physical, socio-cultural, and logico-mathematical) and two types of abstraction (empirical and reflective) in constant interaction. For

[2] Throughout this paper, I will be using the terms “sign” and “symbol” as distinguished by Saussure (1931) and Piaget (1965). Symbols are idiosyncratic and bear a resemblance to the objects represented, such as tally marks and pictures, while signs are arbitrary and do not resemble the object represented and have their source in convention, such as letters and most numerals.

example, the shapes and meaning of numerals are acquired through social knowledge, yet the need for them would have never existed if it were not for our need to express our physical knowledge of numerosity. Finally, understanding and extrapolating knowledge of our particular number system requires using our existing logico-mathematical knowledge, empirical abstraction, and reflective abstraction.

With this perspective in mind, we can determine that in order to understand how children come to understand numerical representations we must first understand components that encompass the written number system.

Representations

Notational Systems

In 1976, Goodman noted the five principles governing all notational systems: syntactic disjointness, syntactic differentiation, unambiguity, semantic disjointness, and semantic differentiation. It is due to these five principles that someone who has deciphered the notational system will be able, upon seeing another's notation, to understand the meaning behind the representation. Further, notations can be iconic or noniconic (also referred to as symbols or signs, (see Piaget ,1965; Saussure, 1931). Drawings are usually thought of as iconic (symbols), whereas writing and numerals are considered noniconic or signs (Tolchinsky Landsmann & Karmiloff-Smith, 1992). Though historically some numerals stemmed from iconic drawings, our modern use of them does not treat them as such; thus, they have evolved into signs.

Attempts to classify differences between notations on the basis of the iconic/noniconic distinction are ultimately problematic due to the subjective nature of the categories (Goodman, 1976), especially when looking at varying human alphabets, numerals, and drawings. To aide the solution of this problem, Tolchinsky Landsmann and Karmiloff-Smith (1992) propose three distinguishing features for notational systems that transcend the realistic/arbitrary distinction at the heart of the iconic vs. noniconic and sign vs. symbol distinction: relative-closure constraint, element-string constraint, and referential-communicative constraint.

The first feature, the *relative-closure constraint*, refers to the extent of closure of the system. Drawings present a relatively open system because it is always possible to create new elements. On the other hand, writing and number notations are relatively closed systems because while new combinations of the elements are infinite, the elements within the system are nearly finite (nearly because the invention and inclusion of new elements does exist, yet is extremely rare). In addition, any element that is added does not already have an element of the same meaning and purpose as one that already exists.

The *element-string constraint* refers to the quality that strings can be parsed into discrete elements and retain separate meanings, such as in writing and numerical notation. By contrast, it is difficult to define what the individual elements are in drawings.

Finally, the *referential-communicative constraint* refers to the fact that there is a mapping from referent to notation and notation to referent. Given a referent, there are a limited number of possible notations and given a notation there are at most very few referents. This is not true in drawing. One final point to make about this constraint is that in the case of writing, for instance, the constraints refer specifically to the string of elements, not the interpretation of language. For example, a novel may be interpreted in many different ways, but there is still only one way to interpret the actual string of elements in a word or sentence.

The simplest way to represent numbers is through a one-dimensional notational system. For example, if we were to use stones, we could use one stone to represent “one,” two stones for “two,” and so on. This has a single dimension: the quantity of stones. Other one-dimensional systems would include tally marks, and even the body-counting system used by the Torres Islanders (in northern Australia), where each body part corresponds to a different number (Dehaene, 1997). The first numerical notation systems invented in nearly all ancient civilizations were one dimensional, represented by physical objects (Zhang & Norman, 1995).

In fact, looking at our own number system[3] as well as dozens of other number systems used in civilizations without any contact with one another, the numerals symbolizing “one,” “two,” and “three” are just that, symbols, consisting of either horizontal or vertical bars (Arabic “2” and “3” were at one point two and three horizontal bars that became connected through human writing). Yet at the number “four” or “five” almost all known civilizations moved from symbols to signs (Dehaene, 1997).

[3] When referring to our own number system, I am referring to the Arabic (or Hindu-Arabic) number system

The most obvious reason for a shift from symbol to sign is the noticeable fact that while one-dimensional systems may be simple and efficient for small numbers, they do not work well with large numbers since each number requires a new sign and name. Large numbers require use of another dimension, for example, a base dimension and a power dimension[4] (1X1D). A number in a 1x1D system can be represented as a polynomial: $\sum a_i x^i$. Our Arabic numeral system is one example of a 1X1D system with the base dimension represented by the shapes of the ten digits (0-9) and a power dimension represented by the position of the digits (i.e. place-value), with a base ten. Other types of base and power dimensions can be seen in Table 1.

Some numeration systems have three dimensions[5] (1x1)x1D: one main power dimension, one sub-base dimension, and one sub-power dimension. The sub-base and sub-power dimensions together form the main base dimension. Numerals in (1x1)x1D systems can be expressed as $a_i(b_{ij}y^j\ )x^i$. The Babylonian numeral system is one example of a (1x1)x1D system with the main base dimension of sixty composed of a sub-base of ten that is represented by quantity and and sub-power represented by the shape. Other known (1x1)x1D systems include the Mayan system and Roman numerals. Most of these systems use quantity to represent the sub-base dimension and shape for the sub-power dimension. The Babylonian and Mayan systems were both two of the first systems to utilize position as the main power dimension while the Roman system used shape.

[4] I will denote two-dimensional systems as (base x power) as done by Zhang and Norman (1995).
[5] I will denote three-dimensional systems as [(sub-base x sub-power)x main power] as done by Zhang and Norman (1995).

Table 1

<u>The representational structure of some 1x1D systems taken from Zhang and Norman (1995)</u>

System	Example	Base	Base dimension	Power Dimension
Abstract	$\sum a_i x^i$	X	a_i	x^i
Arabic	447 $4x10^2+4x10^1+7x10^0$	10	a_i=shape 0, 1, 2, …, 9	x^i=position . . . 10^2 10^1 10^0
Aztec	 $1x20^2+2x20^1+7x20^0$	20	a_i=quantity quantities of •'s 's and 's	x^i=shape •   20^0 20^1 20^2
Greek	νμξ $4x10^2+4x10^1+7x10^0$	10	a_i=shape α, β, γ, …, θ	x^i=shape ι κ λ … $1x10^1$ $2x10^1$ $3x10^1$ ρ σ τ … $1x10^2$ $2x10^2$ $3x10^2$

When we look at the convergence of most civilizations on the Arabic numerical system, we notice several components of the system useful to modern life. The main stepping stones of any written system are the abilities to reduce a large corpus of information into a more compact representation, to use elements and codes which are consistent across the system, and to produce notations themselves which are legible for other readers (Gardner & Wolf, 1983).

According to Zhang and Norman (1995), any distributed cognitive task can be analyzed by three aspects: formal structures, representations, and processes. "Formal structures specify the information that has to be processed in a task; representations specify how the information to be processed is represented across internal and external representations; and processes specify the actual mechanisms of information processing" (p. 290). Just as with Piaget's three kinds of knowledge (physical, socio-cultural, and logico-mathematical), these three aspects (formal structures, representations, and processes) are closely interrelated. The same formal structure can be executed by different representations, and different representations can trigger different processes. Though the main focus of this paper is the representational aspect of numerical knowledge, we must also constantly keep in mind the formal structure and the processes. For example, when viewing children's incorrect written representations, we must consider incorrect notions the child may hold about the structure of the number system. In addition, if the same child incorrectly decomposes numbers, we must look at the possibility that both of these errors are reflective of the same misunderstanding or of the same breakdown in the processing of numbers, as well as the possibility that the errors have different origins.

In Arabic numerals, separating dimensions and identifying and shifting positions are all done externally, leaving as internal the processes of retrieving base values, and multiplication tables. Some languages, such as Chinese, further relieve cognitive load as they state explicitly in the number name what the base value is. This affects the ability of children to learn the number system, as will be shown later (see Miller et al, 1995; Dehaene, 1997). At the level of bases, the Arabic system can be seen as being of a manageable size: larger bases require more symbols but less computation, smaller bases require fewer symbols, but computation becomes more burdensome (imagine the confusion that would come with only working in binary).

At the level of symbols, the Arabic system's ten symbols are easy to write and work with, thus fulfilling effectively two functions simultaneously: representation and calculation. While this dual use may seem intuitive to us, as adult users of the Arabic system, many cultures did not make use of a numerical representation system with a dual use throughout history, using objects such as abacuses and Roman counting boards to do operations while using numerals simply to notate the result. In fact, it is largely argued that Arabic numerals led to the invention of algebra due to the possibilities created when using the same system for representation and calculation (Zhang & Norman, 1994). This is one explanation for why the Greeks, though highly advanced in geometry, never developed an algebra (Zhang & Norman, 1995).

According to Strauss and Stavey (1982), in the course of mastering notational systems children often lose, at least temporarily, some of the basic intuitions underlying a domain because they are so overpowered by the implicit demands of the system itself. Similarly, Goldin (1998) claims that the interaction between internal and external representation is fundamental to effective teaching and learning. Therefore, as the internal and external systems interact, children may lose certain aspects of the domain until they realize that the particular aspect is necessary and reacquire it.

It has also been found that preliterate children are already setting personal constraints on what appears "good for reading." Strings of letters are "good" if they contain more than two and less than nine elements provided that none of the adjacent elements are identical (Ferreiro & Teberovsky, 1979). The qualifying restrictions, however, for which numbers are "good" do not have these constraints, showing that children have already separated written numbers and written language into two different systems not only containing different elements, but also governed by different rules (Tolchinsky Landsmann & Karmiloff-Smith, 1992). This further illustrates the fact that children already understand numbers as being governed by a system with its own rules, before even entering formal schooling.

Internal Representations

Numerical Systems and Internal Representation

Based on the dimensionality of numeration systems, Zhang and Norman (1995) analyzed number representation at four levels: dimensionality, dimensional representation, bases, and symbol representation. Each level has its own abstract structure despite varying representational structures within levels. For instance, in the case of dimensionality, different systems can have varying dimensions, yet each has the same abstract structure in that it represents the same entity: numbers.

The level of dimensional representations is the level whereby systems with the same dimensionality vary in the physical properties used to represent the dimensions. For example, in Table 1 we can see three different 1x1D systems that use shape x position; quantity x shape; and shape x shape. This level is critical for the representational effect of the notational system (Zhang & Norman, 1995). A system such as Arabic numerals with a position power dimension allows for simple arithmetical algorithms such as addition with carrying, subtraction with decomposition, piece-wise multiplication, and long division. Systems with other power dimensions cannot make any use of these algorithms.

At the level of bases, systems have the same dimensionality, the same dimensional representation, and different bases. This level is important for memorization and retrieval: larger bases require more symbols, and larger addition and multiplication tables.

Finally, at the level of symbol representations, the first three levels are the same, but the symbols used are different, this likely has no representational effect except in the reading and writing of the numerals (Zhang & Norman, 1995).

"In complex numerical tasks, as well as in many other tasks, people need to process the information perceived from external representations and the information retrieved from internal representations in an interwoven, integrative, and dynamic manner. External representations are the representations in the environment, as physical symbols or objects (e.g. written symbols, beads of abacuses) and external rules, constraints, or reflections embedded in physical configurations" (Zhang & Norman, 1995, p.279).

Language Effects on Internal Representations of Number

Despite the multitude of numerical notations throughout history, today almost every country and culture use the same convention and base-ten notation. However, the shape of the numbers varies slightly from culture to culture. As stated earlier, this convergence upon one system is largely credited to the efficiency of the system and its ease of use. However, this convergence has not been found in oral numeration. Some languages still have remnants of a different number system such as some Australian aborigines using numbers words in base-two and Eskimo and Yoruba concurrently use bases ten, twenty, and sixty.

Base-ten has taken over the oral representation of number in most modern languages, though to different extents in varying languages. Asian languages such as Chinese are the most simple, reflecting perfectly the decimal structure. There are only names for 1 through 9, then four multipliers for 10; 100; 1,000; and 10,000. Any number is then read through decomposition (13 is ten-three, 27 is two-ten-seven, etc). English lies in slightly more complicated form, adding names for 11-19, and decades 20-90 that do bear resemblance to 1-9 though are not predictable from the other number words. French uses almost the same system as English with a slight peculiarity: 70, 80, and 90 do not have their own number words, they are, respectively, "soixante-dix" (sixty-ten), "quatre-vingt" (four-twenty), and "quatre-vingt-dix" (four-twenty-ten). German, on the other hand, reverses the decades and units in reading numbers (432 becomes four hundred two and thirty).

Does this language difference in the oral representation of number have cognitive consequences for the speakers? Are all languages the same for computation or are speakers of certain languages, by simplicity of their oral numerations, starting out with a mathematical advantage? Many studies have pointed to the superiority of Asians over English-speaking or French-speaking children in mathematics (Dehaene, 1997). However, it can be difficult to tease apart the effects of language versus schooling and home environment.

Miller, Smith, Zhu, and Zhang (1995) asked age-matched Chinese and U.S. children to recite the counting sequence as far as possible. At age four, Chinese children already counted up to about forty and U.S. children the same age recited up to about fifteen. In fact, at all tested ages, U.S. children lagged about one year behind Chinese children. Miller attributed this difference to the regularity of the Chinese language since the U.S. children were equal with the Chinese children up to the number twelve. After this, there was a sharp drop as the U.S. children struggled to remember the rest of the sequence and the Chinese children continued the sequence using the transparency of their number system.

Chinese children also show much more ease in the understanding of place-value. When asked to form numbers such as 25 from blocks, many Chinese children automatically choose two blocks of ten units and five single-unit blocks while most U.S. children of the same age will count 25 individual blocks (Dehaene, 1997).

Looking to curb this difference, at the beginning of the twentieth century the Welsh relinquished their old oral representation of numbers in favor of simplified transparent number words similar to those used in Chinese. However, this new oral numeration system has not allowed the Welsh to achieve the same mathematical competencies as the Chinese: the new number words are so long that memory suffers (Dehaene, 1997).

Cognitive Foundations of Numerical Understanding

As discussed previously, we know that Arabic numerals are more efficient than Roman and many other types of numerals for calculation, even though they all represent the same entities. Zhang and Norman (1995) point out that this "representational effect" (that different representations of the same abstract notion can lead to dramatically different cognitive behaviors) has had profound influence throughout history in the development of arithmetic, algebra, and mathematics in general. "The Arabic numeration system, remarkable as is its simplicity, has been regarded as one of the greatest inventions of the human mind" (Zhang & Norman, 1995, p. 272). That being said, how does the internal representation of numbers relate to our Arabic external representation?

Dehaene (1997) discusses one possible link from our numerical cognition to our number system: it has been shown in numerous studies that human infants distinguish between one, two, and three objects, yet not much beyond this point (Starkey & Cooper, 1980; Strauss & Curtis, 1981). While our number system was not created by infants, this innate numerical intuition in childhood does have an extension to number discrimination in adults. Bourdon (1908, as cited in Dehaene, 1997) found that the time required to name the number of dots in an array grew slightly from 1 to 3, then increased sharply beyond this point. Thus, the decision to move from bars or dots to signs after "three" was likely influenced by number discrimination abilities. Distinguishing III from IIII at a glance is difficult (Dehaene, 1997).

Models of Number Processing

What is the extent to which different numerical skills involve independent cognitive systems? One prominent theory on this topic is the modular model developed by McCloskey and colleagues (1992). McCloskey's model proposes three functionally distinct number processing systems: a numeral comprehension system that recodes stimulus numbers into an abstract semantic code, a calculation system that includes memory for number facts, and a number production system that receives output from the comprehension or calculation systems and converts it to written or spoken responses. The three systems and related subsystems are assumed to be functionally and neurologically independent.

In this model, simple number repetition engages phonological input and output as well as syntactic processing mechanisms. Individual number words or elements gain lexical access. Syntactic processing then allows the determination of the relations between elements. For instance, the comprehension of 47 entails the selection of the separate elements 4 and 7 and syntactic ordering into correct sequence.

It is proposed that in calculation, the mathematical symbols are processed first (e.g., +). Then, arithmetic facts are accessed (e.g., 12 + 9). Third, the calculation procedure is executed (e.g., 12 + 9=21). In this model, all number processing, including number repetition, must engage a component of semantic representation.

Although McCloskey's model is both simple and comprehensive, the assumptions on which it has been based have been called into question on a number of accounts and some alternative architectures have been proposed (e.g., Campbell, 1994; Campbell & Clark, 1992; Dehaene, 1992). Dehaene (1992) introduced the "triple code theory" which proposes that number processing operates on three types of codes: a visual-Arabic form, an auditory-verbal code, and an analog magnitude representation. The Arabic form mediates digital input, output, and some multi-digit operations, the analog magnitude code provides the basis for numerical size comparisons and estimation, and the auditory-verbal code mediates verbal input and output as well as counting operations and memorized multiplication and addition facts. In this model, number processing proceeds independently of the initial notation after input. This differs from McCloskey's hypothesis since in the Dehaene model differences in mathematical performance can be attributable to different pathways invoked by various notations. Thus, Arabic numerals and number words could differ in their capacity to activate certain calculation subsystems.

In McCloskey's model and its varying alternatives, verbal working memory is required during mental arithmetic to retain the numbers, operations, and intermediate results. It is easy to see, then, why a system such as Arabic numerals, that seems to balance memory load with computational difficulty so well, became the universal standard. However, if it is true that number words can invoke different methods of number processing, we need to look further at individual as well as language-specific differences in the comprehension of both written and spoken numbers.

External Representations

Theories of Place-value Acquisition

According to Dehaene (1997) "place-value coding is a must if one wants to perform calculations using simple algorithms" (p. 98). Consider calculations in the Greek or Roman numerical system. They are inconvenient for a variety of reasons; one apparent reason being that nothing indicates that N or L (50) are greater than E or V (5). For this reason, Greeks and Romans could never perform calculations without the use of the abacus. Place value allows, for us, 5, 50, 500, etc to be of transparent magnitude. We must only memorize 10 digits and products 2x2 through 9x9.

A formal theory behind the perception, production, and learning of numerical notation was put forth by Power and Longuet-Higgins (1978), which describes the learning of the number system as analogous to a computer program that learns through example. According to this model, when people translate between oral numbers and Arabic numerals they construct an intermediate semantic representation. A complex numeral such as one thousand four hundred and thirty-two has the structure ((one thousand) ((four hundred) and (thirty-two))) which is articulated when read aloud. The writer must then transfer the representation created by the oral structure into a written numeral using the known set of rules (or vice versa). Complicating this writing is also the mastery of an “over writing” (Power & Dal Martello, 1990; Seron & Fayol, 1994) operation that children must learn in order to avoid the addition of zeros. This overwriting comes hand in hand with the comprehension of a concept of place-value as the child must learn the manner in which to assure that each numeral fits into the correct “place.”

While the Power and Longuet-Higgins (1978) theory makes sense from a computational perspective, do the explanations and answers of children as they grasp this concept correspond to the model? In an attempt to see the part played by the children in the reconstruction of the system, Sinclair and Scheuer (1993) devised two tasks concerning the interpretation and understanding of written numerals with children from Argentina and Switzerland. According to Sinclair and Scheuer, "understanding of written numerical notations is a construction process that is necessary to the understanding of our numeration system, and it participates in and directly influences mathematical cognition. The grasp of numerical notation is thus deserving study in its own right, and is not to be approached exclusively as means of representing knowledge acquired in other domains (cognition, counting, computation)" (p. 203).

With this principle in mind, Sinclair and Scheuer (1993) asked first-grade children, without verbalizing (though the children were not stopped if they did read the number aloud) to complete two tasks: comparing written numerals and explaining the meaning of digits in two-digit numerals. Of sixty children studied, five could not count beyond ten, eight only went into the teens, seventeen counted to between twenty and twenty-nine, eight to between thirty and thirty-nine, eighteen continued to between forty and seventy-nine, and four children counted to one hundred or beyond. Since no difference was found between the three groups of children (upper middle class Swiss children, middle class Argentine children, and lower class Argentine children) we can assume that this is representative of children with similar oral numeration structures to Spanish and French, the languages used in the study.

Sinclair and Scheuer (1993) found four main strategies used by children to determine the greater number. The first was number of digits. All children correctly identified that numbers with more digits are greater than numbers with fewer digits (only positive integers were used); however, this strategy obviously only works with limited pairs of numbers. Children who could not write 10 and/or 15 used this approach.

The second approach involved matching the written numbers to spoken numbers and deciding which one came first/last in the counting sequence. The children that used this approach achieved correct responses when the written numbers were read aloud correctly but some children did fall victim to incorrect oral deciphering.

The third type of strategy used was face-value comparison. In this case children compared parts of the numbers independently of their place. Many of these comparisons were additive, such as saying that 19 is greater than 21 because one and nine is bigger than two and one. This argument was largely unsuccessful though some correct responses did stem from it by chance.

Quickest and most successful was the fourth type of strategy. In it, children compared cardinalities of numerals in the same position, allowing the left-most digit to override the others.

Most children did not use just one type of strategy throughout the task. Some children began with strategy one and when it was no longer useable (i.e., two numbers had the same number of digits), they used strategy three. These children clearly have not yet grasped a place-value concept. Others attempted to use all of the strategies, mostly only failing on those where strategy three was attempted. Some also used so many strategies that they would lose sight of their own approach, such as comparing left-most digits in numbers with different numbers of digits.

In explaining place-value, five children did not clearly attribute any meaning to the separate digits. They gave explanations such as “I don’t know” or evaded the question while four children wrote down the whole counting sequence to express a number (e.g., 1, 2, 3 …17 to express 17). They would not accept the individual number on its own. All of the other children (51/60) interpreted the bi-digits conventionally, that is, they maintained that the two digits written together represented the whole collection. However, their explanations of the meaning of the individual digits were not necessarily conventional. In using one-point tokens to express the numbers, some children maintained that there was one token for each digit or that both digits are for all of the tokens. However, more than half of the children interpreted digits as corresponding to their face value, that is, while recognizing that the entire number together stood for all of the tokens, the digits when taken separately each stood for the number of tokens of the single digit quantity (such as explaining that the “1” and “5” in “15” stood for 1 and 5 tokens respectfully). Some of the children knew that this explanation was not quite adequate and that somehow all of the tokens had to be accounted for, yet did not understand how, even adding extra digits for the leftover tokens despite previously stating that the original number stood for all of the tokens (such as adding a “9” onto the end of “15” in the previous example to account for the 9 tokens leftover in the explanation).

Twelve of the sixty children in the study considered that the whole collection must be accounted for in their explanation of the numeral, yet it was not always clear how the partition should take place; noting that even once children accept that multi-digit numbers are more than individual digits, it is still not apparent what each place holder actually represents.

Also of note is Sinclair and Scheuer's (1993) finding that, though the study found children from every community were in each strategy group, on average the children from upper middle class and middle class neighborhoods performed better than the lower class children, pointing out that this is not merely a factor of natural development, but is also influenced in some way by the social environment. We could also relate this to the Power and Longuet-Higgins (1978) model if perhaps the lower class children had less numerical exposure and thus had not yet seen enough examples to generate more advanced responses.

In a related study, in individual interviews with children from widely diverse urban, rural, public and private school communities, Ross (1986) presented children with twenty-five sticks and asked them to count the sticks and write down the number. After this, Ross circled the "5" and asked, "Does this part have anything to do with how many sticks you have?" then subsequently indicated the "2" and asked the same question.

The children's responses were categorized into 4 levels; Level 1: the individual digits have no numerical meaning; Level 2: the child invents a meaning for the individual digits, unrelated to place value; level 3: the individual digits have some meaning related to place value, but it is a partial and confused idea, such as the ones place indicating tens or both places indicating ones; and Level 4: the digits represent the whole quantity partitioned into groups of tens and ones. Ross (1986) found that it was not until grade four that half of the children in the class reached level 4. By grade 5, only two-thirds of the children knew that the 5 meant five sticks and the 2 meant twenty sticks. This is surprising as children of this age are taught arithmetic algorithms based on an understanding of place-value; such as long division, multi-digit multiplication, subtraction with borrowing, and addition with carrying.

Silvern and Kamii (1988, as cited in Kamii, 1989) performed an almost identical study with sixteen tokens with very similar results, except that this time only 35% of fourth graders gave a mathematically sound answer. One possible explanation for this discrepancy is that children at this age are still using number words to guide their notational understanding so it may be easier to extrapolate that the “2” in “twenty-five” is twenty than to realize that the “1” in “sixteen” is ten. In fact, “teen” numbers have been found to be harder for children to understand in terms of correlation between written and spoken numbers (DeLoache & Willmes, 2000).

Children’s Written Numbers

It has been shown that children reconstruct our numerical written symbol system slowly over a period of many years, grasping cardinality around the age of six or seven (Sinclair & Tieche-Christinat, 1992). Thus, we may be able to better understand how children understand the system by looking at their early attempts at construction. Sinclair, Siegrist, and Sinclair (1983) interviewed 45 four, five, and six year olds in Geneva, Switzerland, who had received no formal academic mathematical instruction. The children were presented with up to eight objects and asked to “put down what is on the table.” They were later asked to write specific numerals, without any objects given. Sinclair et al (1983) found that there were six types of notation that children produced: *global* *representations of quantity* (such as lines), *representations of the object kind* (such as a B for three balls, or a house for five houses, with no attention paid to quantity), *one-to-one correspondence with symbols* (any objects at all, such as letters, of which the numerosity of the object was the same as the numerosity of the symbols), *cardinal value alone* (a numeral or number word), and *cardinal value and object-kind* (such as "4 pencils"). Sinclair and her colleagues found that global responses were mainly used among four-year-olds while cardinal values (with or with object-kind) were found mainly among children over five-and-a-half years old. The other types were found amongst children in the middle of the age range (between four and five and a half years of age).

We could use this as an example of the notational system being reconstructed as children move from one-dimensional symbolic systems towards the social convention of signs. It can be seen from this example alone how the younger children used types most similar to the earliest symbolic representations of number and the older children used those which are most similar to (or exactly the same as) the most modern forms.

If we look at children's acquisition of number signs, various studies of children's use of notation point to several different types of notational practice possibly linked to stages in the understanding of multi-digit numbers (Brizuela, 2004). In one such study, Bergeron and Herscovics (1990) studied five-year-old children grasping the convention of two-digit numerals. They found three different levels of understanding. First, in the *juxtaposition* stage, children understand that two digits side-by-side acquire a global meaning (e.g., 15 is a "fifteen" and not a "one-five"). Next, in the *chronological* stage, the child understands that when writing “fifteen” the “1” should be written first, but does not necessarily position the “5” to the right of the “1.” Finally, in the *conventional* stage, the child produces two-digit numbers in their conventional positions, regardless of the chronology of number placement. It should also be pointed out that these stages are not mutually exclusive. A child can be in the chronological stage for numbers 20 through 30 but in the conventional stage for numbers 10 through 20 and while age may contribute to the results, response levels are never very neatly correlated to age (Sinclair & Tieche-Christinat, 1992).

This phenomenon of stage co-occurrence was also noted by both Pontecorvo (1985) and Sinclair (1988, as cited in Scheuer, 1996). They noted that children, when asked to note “in another way” will change from conventional notations to one-to-one correspondences or visa versa. It has also been recognized that children who represent number in a conventional manner in a school setting may revert to an unconventional manner outside of the school environment (Bergeron, Herscovics, & Sinclair, 1992); this occurrence can also be described as one example of the trend that Piaget (1965) coined *décalage*. While we may assume that children have mastered a system when they are reliably and consistently reproducing its formal structure, it has been found that “counting and number recognition are weak evidence for the understanding of the true meaning of these systems” (Bialystok & Codd, 1996, p. 289). However, Bialystok and Codd do not refer to number production specifically. Therefore, we can still wonder, what can conventional number production inform us about children’s understanding of the meaning of the system?

Sinclair and Tieche-Christinat (1992) found that children begin by having intuitions about the precise meaning of digits in particular positions by setting up part-to-part relations in the spoken number that correspond part-to-part to the notation. This finding is backed by Fuson and Kwon (1992) who propose that children learn spoken patterns and written patterns simultaneously, by relating a particular number-word to a particular numeral. However, understanding that the “2” in “26” corresponds to the word “twenty” still occurs many years before understanding that the “2” doesn’t just signify to pronounce “twenty” but that because of its placement in the number the “2” itself actually represents the cardinal number twenty. Also of note is Brizuela’s (2004) account of a young girl trying to account for the difference in the “3” representing three and the “3” representing thirty by rotating the number to create a “capital three.” Similar accounts of number rotation were also noted by Alvarado (2002) with Spanish-speaking children. These instances of children attempting to manipulate numbers to create a new meaning point to both the fact that children are developmentally ready to learn about and understand place-value but also that they do not intuitively understand that the single-digit numeral changes its own meaning simply by its position in the number and does not need further manipulation.

The Interaction between Internal and External Representations

Dictation and Reading of Numerals by Children

A lot can be learned about children’s early understanding of place value and the number system by how children write down numbers that they hear.

Verbal numbers are used mainly in conversation while written numbers are used mainly for calculation and writing down large numbers. Thus, children do not tend to focus on writing numerals in school until first or second grade and even then only gradually increase the number of digits that are written. This being the case, Power and Dal Martello (1990) conducted a study of Italian second graders taking numerical dictation of one, two, three, and four digit numerals with and without internal zeros. Since the Italian language has a similar transparency of number words to other romance languages as well as to English, findings in this study could be considered as typical of most Western children. The second graders in this study correctly annotated all numbers below 100, demonstrating that they were both trying to succeed and also that they had mastered the number system in this range, making the findings of larger numbers quite remarkable. For numbers above 100, Power and Dal Martello found both lexical (such as using a 7 instead of an 8) and syntactical (adding extra zeros, misarranging number, etc) errors, though the syntactical errors far outweighed the lexical, indicating a misunderstanding of the system and not simply a mistake on the part of the child. There were also significantly more errors for four-digit than three-digit numbers, demonstrating that there are indeed steps in the acquisition of the system and it is not simply an “all or nothing” understanding; that is, each number range poses new problems or elicits prior problems once again.

Several trends were observed in syntactic errors. For three digit numbers without internal zeros, 14/22 errors were of type X00XX (e.g., 10025 for one hundred and twenty-five) and 6 errors were of type X0XX (e.g., 1025). This difference can be described as children who have not yet learned how to incorporate their knowledge of hundreds (100, 200, 300, etc.) with knowledge of two-digit numbers versus children who are beginning to write three-digit numbers with internal zeros (e.g., 105 as one hundred and five) without understanding place-value. Backing this interpretation is the result that 18 of 22 errors of three-digit numbers with internal zeros were of type X00X (e.g., 1005 as one hundred and five), since the second type of error found without internal zero is no longer an error with this number type.

A similar result was found in four-digit numbers, but with obviously more types of internal zero errors. For instance, the number 3194 was incorrectly transcribed by five different children as 30194, 30010094, 300010094, 30000194, and 300000100904.

All of these responses suggest that children form separate constituents for thousands, hundreds, and tens and then concatenate them. While this may not be immediately surprising, it is quite noteworthy that even after children seem to have mastered place-value and cease concatenation in three-digit numbers, they begin a new learning process with four-digit numbers, demonstrating that they do not yet truly understand the system, but are in many cases still memorizing individual number rules. This result supports the findings of Karmiloff-Smith (1979). In observing children learning both mathematical procedures and language skills, Karmiloff-Smith noted that "each time a procedure in a representational system is functioning adequately and automatically, the child steps up to a metaprocedural level and considers the procedure as a unit in its own right" (p. 91). Further, "each time children develop an adequate tool for representing their knowledge, and once the tool functions well procedurally, then the tool is considered metaprocedurally as a problem-space in its own right" (p.92). Karmiloff-Smith also noted that "young children juxtapose their procedures whereas older subjects can take their procedures as units" (p.92).

As stated on page 2 of this paper, Scheuer et al (2000) conducted a similar dictation study with Spanish-speaking children and discuss, among others, two distinct types of incorrect numerical notation strategies used by children: logogramic and compacted notation. Logogramic notation refers to children who write the entire number literally, such as 100701 for one hundred seventy-one while compacted notation refers to children who remove some of the zeros from the logogramic coding while still not condensing the number entirely into its conventional form, such as 1071 for one hundred seventy-one. These two error types are quite similar to the errors noted by Power and Dal Martello (1990). Scheuer speculates that perhaps these two types of notational strategies stem from different ideas about the numbers themselves, yet Scheuer's study only focuses on written numbers and she never goes on to test this speculation.

As Seron and Fayol (1994) pointed out, it remains to be determined where in the functional architecture of number processing the children's transcoding errors originate. Is it the result of inadequate comprehension of the verbal number forms, difficulties located at the production stage of Arabic forms, inadequate comprehension of the number system itself, or a conjunction of two or three of those possibilities?

Seron and Fayol (1994) devised a follow-up experiment aimed at answering whether the children in the Power and Dal Martello (1990) study comprehended correctly the verbal number forms and to further understand what role language might have in this process.

As mentioned earlier, the French language has a very similar numeration system to English and Italian, aside from the forms for 70, 80, and 90 which translate to sixty-ten, four-twenty, and four-twenty-ten respectively. However, in Wallonia, a region of Belgium where French is spoken, the words for 70 and 90 mirror those used in English and most romance languages. Due to this small difference in the languages, Seron and Fayol (1994) conducted a similar transcoding experiment to Power and Dal Martello (1990) to see if the French and Walloon children would have any differences in numerical dictation or understanding of the written number system caused by the differences in the oral numeration of each language.

Seron and Fayol (1994) adopted a longitudinal approach by interviewing children at three sessions, each distant by a three month interval, and coding for the types of errors present in Power and Dal Martello (1990) in each separate type of number, which they termed Unit-Hundred (UH, this would include numbers such as two hundred); Hundred-Unit (HU, such as one hundred and three); Hundred-Decade (HD, such as one hundred and thirty); and Hundred-Teen (HT, such as one hundred and thirteen); as well as each of these types following a thousand (MU, UM, MD, MT, DM, TM). For example, 1,001; 2,000; 1,010; 1,013; 10,000; and 13,000, respectively.

Five different tasks were then administered: In Task 1, children were asked to write ninety-five orally presented numbers that covered all of the categories listed above. Task 2 was a number magnitude comparison task where children indicated which of two orally presented numbers was the larger one. Task 3 was a grammaticality judgment task whereby each child was asked to determine whether a sequence of number words was legal or not. The aim of Task 4 was to evaluate the semantic component of the McCloskey (1992) model (see description on page 18). Children were presented with oral numbers and asked to represent the quantity using tokens of three different colors representing one franc, ten francs, and one hundred francs. Finally, in Task 5, children were asked to write down Arabic numerals corresponding to tokens presented to them.

As opposed to the Power and Dal Martello (1990) task and the Scheuer et al (2000) task, which interviewed children only once, Seron and Fayol were able to explore order of acquisition of the various number forms using their set-up of three interviews over six months. In the case of Task 1, there was a significant order of acquisition of UH then HU (e.g., 200 precedes 104), followed by HT/HD (e.g. 113 or 120 would be acquired next). For four-digit numbers, the order of MU, MT, then MD was also significant (e.g., 1,002 precedes 1,014, followed by 1,040).

Seron and Fayol found very similar lexical and syntactical errors to Power and Dal Martello (1990) and Scheuer et al (2000). Not surprisingly, certain incorrect syntactical responses were seen only by French children such as 6018 for “soixante-dix-huit” (seventy-eight) and 42017 for “quatre-vingt dix-sept” (ninety-seven). Interestingly, these errors are very similar to those produced by French adult aphasics (DeLoache & Seron, 1982), indicating that this is likely an error at the level of processing or comprehension, not an error at the stage of production (though we cannot be certain: it is always possible that two different sources of error produce the same outcome as was discussed on page 12 of this paper).

In the number comparison task (Task 2), most children did too well to allow for an analysis due to a ceiling effect, indicating children are able to elaborate from an oral number to a corresponding quantity without first transcribing the number. This also indicated a lack of place-value concept, since the same numbers that were correctly judged to be greater than others when presented orally were then transcribed using fewer digits. The grammaticality judgment task (Task 3) also had a ceiling effect.

Task 4 (semantic representation) was the first task to show a marked difference between the numerical structures of the two languages. Walloon children made less than 1 percent errors (8 out of 900 responses created over 10 children) while French children made 53 errors (also out of 900 responses over 10 children). Twenty-two of these errors were semantic errors in the representation of complex tens (for instance, 79 was represented by 7 tens tokens plus 19 units tokens or 6 tens tokens and nine units tokens[6]). These types of errors were seen both in numbers in the 70s and in the 90s[7].

Since Task 5 was intended to test the production component of McCloskey (1992), the same numbers were used as in the Task 4. This time, out of the 1800 responses, there were 304 errors, 196 (64.5%) of which came from the French children. This language difference was not significant, but there was

[6] Since 6 tens tokens and 19 units tokens would be considered a correct response, Seron and Fayol (1994) did not list how many children made this coding as opposed to the expected coding of 7 tens tokens and 9 units tokens.
[7] Both the French and Walloon numeration systems use quatre-vingt (four-twenty) to represent 80, thus this number was not explored in the study which was focusing on the language differences in the oral representations of numbers, specifically 70-79 and 90-99.

a significant effect of session, showing steady improvement over the six months that the study lasted. In addition, the French children made more than twice as many errors than the Walloon children with complex tens.

The types of tasks at which the children from Wallonia excelled versus erred demonstrate that their difficulties in transcoding from oral numbers to Arabic numerals were mainly due to the production of the numerals themselves and not a difficulty in understanding what numerosity the spoken number referred to. The same cannot be said for French children, who produced token arrangements very similar to their transcoding errors (from oral numbers to Arabic numerals).

That being said, both groups performed better than the U.S. children in comprehension of the base-ten structure of written word numerals reviewed by Fuson (1990). This cannot be explained linguistically as was done with the French/Walloon difference since English is more transparent than French in higher numbers and contains similar irregularities to the Walloon number system. However, we must keep in mind that Fuson (1990) reviewed many studies of unselected samples of children at varying tasks and Seron and Fayol (1994) interviewed only children who presented no mathematical difficulties.

With these differences amongst children of different languages that have such similar numerical structures, we would expect to find that children with a greater difference in their oral numerical systems would perform at an even greater disparity. This is exactly what Miura and Okamoto (1989) showed. They interviewed 24 U.S. first graders and 24 Japanese first graders using base-ten blocks. The equivalence between ten unit-blocks and one ten-block was explained explicitly to the children. Children were then asked to read a number written on a card (11, 13, 28, 30, and 42 in random order) and show that number by using the blocks. These representations were then scored as one of three types: a) one-to-one (i.e., the child only used unit blocks), b) canonical base-ten representation (i.e., the child incorporated both ten-blocks and unit blocks with no more than 9 unit blocks), or c) non-canonical base-ten representation (i.e., the child incorporated both ten-blocks and unit blocks, allowing there to be more than 9 unit blocks, such as 1 ten-block and 18 unit blocks for 28).

The children also performed more explicit place-value tasks such as being shown the number 32 and being asked to point to the number in the ones place and in the tens place. They then were told to show the 3 and the 2 using base-ten blocks. After, they were shown a 44 along with its construction and asked which set of 4 blocks showed the first 4 and which showed the second 4. The children were also showed non-canonical construction of 3 ten-blocks and 12 unit-blocks and asked to represent the number. If the number was correct, they were asked if the 4 and the 2 had anything to do with the number of blocks. On a fifth task, the children were given 13 unit blocks and asked to put 4 blocks each into plastic cups and then give the total. Children were then shown the number 13 and asked if the 1 and the 3 had anything to do with how many blocks there were. This was then repeated with 26 blocks.

The results of the Miura and Okamoto (1989) study were as overwhelming as expected: 67% of Japanese children used a canonical base-ten representation for all five numbers compared to only 8% of U.S. children. In fact, 50% of the U.S. children did not use a canonical representation for any number.

When asked to create another representation of the same number, 79% of the Japanese children were able to think of a different method of representation while only 13% of U.S. children did so. This last difference should be taken with a grain of salt, however, since children who used a canonical representation first could be assumed to understand the equivalency with unit blocks only, thus the second result is merely repeating the first.

The explicit place-value tasks were also quite telling. 42% of Japanese children answered all of the questions correctly compared to 16% of U.S. children. In fact, every Japanese child got at least one place-value question correct while half of the U.S. children did not get a single correct answer. Surprisingly, and contrary to popular belief, the U.S. children in this study had been taught about place-value using manipulative materials, pictorial arrays, and expanded notation; yet the Japanese children had only been explicitly working with numbers 1-9 in school at the point of the testing. The reality that the U.S. children had not understood the place-value lessons while the Japanese children found it intuitive seems to only be explained by the explicitness of the canonical forms in the Japanese oral numeration. When numbers have names such as ten-one and two-ten eight (the English translation of two Japanese numbers used in this experiment) it seems only natural to construct the number exactly as it is told, with the number of ten-blocks and unit blocks explicit. Lacking a system for producing oral numeration that explicitly incorporates place-value, English-speaking children must develop an understanding of the concept in some other way (Miura & Okamoto, 1989). While the use of manipulatives has been explored, it continues to appear as though U.S. children do not understand place-value until past the primary grades (Fuson, 1986).

Studies such as Miura and Okamoto (1989) and Seron and Fayol (1994) demonstrate the varying difficulties that children may have in interpreting oral numbers into canonical base-ten forms and vise versa. In addition, Scheuer (1996) and Power and Dal Martello (1990) indicate similar difficulties in interpreting spoken numbers into written numerals and the reverse. However, there has not yet been any analysis comparing how these two relationships (i.e., oral numbers with canonical base-ten and oral numbers with written numerals) are interrelated. If we look at the heart of these authors' hypothesis and conclusions, there are many suppositions of numerical representations through tokens as being indicative of a child's internal representation and understanding of the number. By definition, numerical notation is one type external representation of a number. Numerical notations, as discussed earlier, are also our modern numerical tool and contain within them a great deal of information about the number. We must then question, how are these two types of representation (internal and external) related? Do children who make a particular error in one type of representation (i.e., oral, written, canonical) make similar errors to one another in the other type of representation?

The Study

Objective

The study reported in this paper examines young children's understanding of place value in the base-ten number system through their use of numerical notation and strategies in object manipulation tasks. Various studies of children's use of notation point to several different types of notational practices possibly linked to stages in the understanding of place value in base-ten (Alvarado, 2002; Brizuela, 2004; Scheuer et al, 2000; Seron & Fayol, 1994, among others). As discussed earlier, many children use similar incorrect strategies to write numbers such as 10071 or 1071 for one hundred seventy-one. I seek to examine what correlations, if any, exist between children's techniques of representing multi-digit numbers and their knowledge of place value and number composition. The particular analysis presented in this paper focuses on two particular incorrect notation strategies termed Full Literal Transcoding (Seron & Fayol, 1994) and Compacted Notation (Scheuer et al, 2000) and their correlation with correct and incorrect decomposition strategies. I seek to answer the question: How is a child's numerical notation strategy correlated in any way to his or her understanding of number decomposition? Broader questions that are also addressed include: What can we learn from looking at children's numerical notation? Are children's numerical notations correlated to their nonverbal representations of number? Do children's incorrect notations help us to understand incorrect concepts of the number system?

Methods

Sample

Participants were 45 kindergarten and first grade children (21 male, 24 female; ages ranging from 4;10 to 7;11, mean age 6;2) from a public school within the greater Boston area in a community that is racially, ethnically, and culturally diverse. Each child was individually interviewed in the reading, writing, and decomposing of 2-digit, 3-digit, and 4-digit numbers as shown in Table 2. Two kindergarten children were not included in the analysis due to either not yet conserving number (a necessity for the decomposition task) or refusal to answer enough questions to allow for categorization of strategies used. For this same reason, ten children are not included in the analysis of four-digit numbers as they did not answer enough questions.

Table 2

Numbers used in Tasks 1 through 3

Series 1	2 digit – "transparent"[8] with final 0	40	60	70	80	90
Series 2	2 digit –nontransparent with final 0	50	30	20		
Series 3	2 digit – transparent without 0	43	64	79	88	91
Series 4	2 digit – nontransparent without 0	53	21	19	35	17
Series 5	3 digit – without 0	127	143	324	465	132
Series 6	3 digit – internal 0	101	207	301	401	504
Series 7	3 digit – final 0	300	760	640	430	910
Series 8	4 digit – without 0	1127	3143	4324	5465	7132
Series 9	4 digit – X0XX	3064	2053	1019	4035	5091
Series 10	4 digit – XX0X	2101	3504	1401	4207	1706
Series 11	4 digit – XXX0	1300	3760	2640	1430	1910

[8] The transparency of a number refers to the degree to which a number sounds like its written form when spoken aloud. For example, in English, 60 is more transparent than 30 since the "6" is clearly heard in the pronunciation of the number. We cannot say that 60 is completely transparent since the "ty" does not sound exactly like "ten" or "zero." For more information on transparency, see Alvarado & Ferreiro (2002).

Task Details

Each child engaged in an individual interview of approximately 30 minutes in the style of a Piagetian Clinical Interview (Piaget, 1965), where the child was asked to write and read numbers as well as name the total value of a group of tokens, of which each color was assigned a value, to pinpoint his or her understanding of our base-ten number system. Each interview was videotaped and later transcribed to look for any relevant statements or actions made by the child that could be overlooked while the interview was occurring.

The three tasks focused on different aspects of the child's understanding of the number system: written numerals and the consistency of words with symbols (Tasks 1 and 3) and the correspondence between words and object numeracy (Task 2). While the focus of the study is on multi-digit numbers, it was also necessary to include numbers less than ten in every task to understand the child's representation of those numbers and its correspondence to greater numbers. For example, Alvarado (2002) and Alvarado and Ferreiro (2002) noted some interesting connections and differences between a child's notation for single-digit numbers and the same numbers in a different place-value, such as differing orientations of the numbers. Brizuela (2004) also reported on one child discussing "capital numbers" and how numbers change when in different positions just as letters do. Questions in the interview were about numbers ranging from 10-9,999; however, numbers 1,000-9,999 were not included for ten children that expressed the desire to end the interview and were already demonstrating a guessing strategy with smaller numbers.

Task 0: Introduction

Each child was given a pencil and paper and told that we would be talking about numbers. The child was asked introductory questions such as: "What's the biggest number you know?"; "Do you know how to count from 1 to X?"; "Could you put down those numbers on the paper?" The introduction task ended once I was familiar and comfortable with the general knowledge, vocabulary, and number fluency of the child.

Task 1: Written numerals

The written numerals task is based on the work of Brizuela (2004). In this task, I aimed to understand the child's fluency with written numbers. Every child was asked to write at least two numbers from each structural type such as X0, XX, XX0, etc. Children who wrote the first two numbers in a series (see each row of Table 2) conventionally were presented with the next category. Children who wrote at least one number incorrectly received 1-3 more numbers until the child's strategy within that series was apparent (see Table 2 for a complete list of numbers presented).

Task 2: Object numeracy in the base-ten system

This task was designed for the purpose of understanding the consistencies/inconsistencies in the child's understanding of our number system without the use of notation. In the object numeracy tasks, children were presented with a number of tasks involving the use of counters (poker chips) of different colors. Counters were chosen based on the work of Nunes Carraher (1985) performing similar tasks in the understanding of place value in young children and illiterate adults. The child was told that red counters are worth 1 point, blue counters are worth 10 points, white counters are worth 100 points, and brown counters are worth 1,000 points. The child was presented with the same numbers as in Task 1 but in token-form and asked how many points he/she has. Once again, children who correctly named the first two numbers within a series were presented with the next series. Children who named at least one number incorrectly received 1-3 more numbers until the child’s strategy with that series was apparent.

Task 3: Reading Numerals

In the reading numerals task, children were asked to read from a piece of paper the same numbers that they were asked to write in Task 1. Once again, children who correctly read the first two numbers within a series were presented with the next series. Children who read at least one number incorrectly received 1-3 more numbers until the child’s strategy with that category was apparent. Only Tasks 1 and 2 are analyzed in this paper. Results from Task 3 will not be discussed in this paper.

Analysis

Transcripts of the interviews were reviewed along with any notes made during or after the interview, the written work of the children, and the physical manipulations of the children during the object numeracy tasks. These pieces constituted the data for the study.

Data was arranged into categories for different types of strategies. In Task 1, children were classified by the strategy used to produce written numerals. Written numeral strategies were coded separately for each type of number (for a complete list of numbers and number types, see Table 2). There were seven categories coded:

A) *Idiosyncratic, various strategies* – Child has no consistent strategy for writing numerals in the given number range (i.e., 10-99, 100-999, 1,000-9,999).

B) *Other* – The child has a consistent strategy not used by any other children.

C) *Full Literal Transcoding (FLT)* – Child writes out number literally, for example, 100701 or 10071 for one hundred seventy-one. This category is taken from Seron and Fayol (1994). This is similar to the Scheuer et al (2000) category of logogramic notation except that Scheuer would only allow for 100701 to be considered in this category. I allow for both types of literal transcoding (100701 and 10071) as FLT since for children who are conventionally writing 2-digit numbers, "71" has become the literal writing of seventy-one.

D) *Compacted Notation (CN)* – Child writes extra zeros in numbers but fewer than the FLT notation, for example, 1071 for one hundred and seventy-one.

E) *Correct Strategy, some errors* – The child correctly notates most numbers, but makes at least two errors within the given number range, for example, making one error in series 5 and one error in series 6 to total two errors in 3-digit numbers.

F) *Error due to incorrect use of comma* – This category only pertains to numbers over 999. The child uses a comma in notation and leaves out zeros. For example, one thousand seventy-one would be written 1,71.

G) *Zero or one error* - Children making only one error in a given number range were considered to be demonstrating proficiency with numerical notation in the range. This was to account for natural human error such as forgetting a number requested or mishearing the interviewer. For example, a child whose only error was writing 137 for one hundred and twenty-seven would be considered proficient in 3-digit numbers.

As can be seen by the descriptions, categories A, B, and C, and D all produce incorrectly written numbers while E and F produce many correct numbers with a few mistakes. Category G produced at most 1 incorrect number in the number range.

In Task 2, children were classified by the strategy used to decipher the point value of the tokens. Decomposition number strategies were coded separately for each type of number. The same six categories were used for each type of number:

A) *Incorrect Understanding* – Child fails to understand the multiplicative nature of the tokens. The most common example of this is counting every token as one point regardless of its color.

B) *Varied Strategies* – Child shows some understanding of token value, but does not have a consistent strategy for naming the number represented by the tokens, resulting in at least two errors.

C) *Counting by ones* – Child does show some understanding of token value, but can only add up the points by counting by ones (i.e., pointing to a 10-point token and counting (1, 2, 3, …, 10), leading to at least two errors with larger numbers.

D) *Ordering/Combining Difficulty* – Child correctly adds each individual color token correctly, but makes several errors in adding the colors together to form a single number.

E) *Correct Strategy, some errors* – The child correctly adds the tokens in a conventional manner, but makes at least two errors within the given number range.

F) *Zero or 1 error* - Children making only one error in a given number range were considered to be demonstrating proficiency with numerical decomposition in the range. This was to account for natural human error such as miscounting or forgetting the number counted to.

Results

In the analysis presented here, I sought to compare and contrast children's responses on Tasks 1 and 2 (notation of number and decomposition of number). Table 3 relates the results in Tasks 1 and 2 for the case of two-digit numbers. It shows that 40 of 43 children (93%), fell into one of three cases: *Incorrect understanding of tokens* in Task 2 with *idiosyncratic notation* in Task 1 (9 of 43 children, 21%); *Incorrect understanding of tokens* in Task 2 with *correct notation in Task 1* (10 of 43 children, 23%); and *Correct decomposing* with tokens in Task 2 with *correct notation* in Task 1 (21 of 43 children, 49%). *No* *child* displayed *correct decomposing with tokens* in Task 2 and *incorrect notation* in Task 1. It is interesting to note that half of the children (10/19) who did not understand how to use the tokens still produced correct notation while all of the children who were not able to produce correct notation were also unable to understand the tokens. One possible explanation for this finding is that the children who were producing correct notations with incorrect tokens had simply memorized two-digit numbers each as their own entity, but still did not understand how the numbers were constructed.

Table 3

Two-digit number decomposition and notation strategies

Decomposing / Notation	Incorrect understanding	Varied strategies	Counting by 1s	Ordering difficulty	Correct strategy, some errors	Zero or 1 error
Idiosyncratic, various strategies	9	0	0	0	1	0
Other	0	0	0	0	0	0
FLT	0	0	0	0	0	0
Compacted	0	0	0	0	0	0
Correct strategy, some errors	0	0	0	0	0	0
Zero or 1 error	10	0	0	0	2	21

Note: N=43; cells containing zero responses are shaded for clarity

In the case of three- and four-digit numbers (Table 4), we do see children spread out over more of the cases. This time, the table also included information on children's performance in previous number groups. When analyzing three-digit performance, children who performed in the zero or 1 error group with both notation and the tokens tasks with two-digit numbers were compared with children who were in an error group in at least one of those tasks. This was done to see if children who had performed perfectly in the previous number range would go on to have different types of problems with the next number range than children who had difficulty in the pervious range.

Table 4

Three-digit number decomposition and notation strategies

Decomposing / Notation	Incorrect understanding	Varied strategies	Counting by 1s	Ordering/ combining difficulty	Correct strategy, some errors	Zero or 1 error
Idiosyncratic, various strategies	4	1	0	1	0	0
Other	1	0	0	0	0	0
FLT	3	0	0	1	1	1
Compacted	1	0	0	1	1	3
Correct strategy, some errors	0	0	0	0	1	0
Zero or 1 error	0	0	0	0	1	12

Note: N=43;

Light gray = Correct with Tasks 1 and 2 in 2-digit numbers;

Dark gray = Incorrect with at least one task in 2-digit numbers

Similarly, when analyzing four-digit number performance, Table 5 included information on how children who performed in the zero or 1 error group with both notation and the tokens tasks with three-digit numbers compare with children who were in an error group for at least one task. Surprisingly, *every combination of notation strategy with decomposing strategy was utilized solely by children who performed at ceiling with the previous number group or by children who did not perform at ceiling with the previous number group*. In other words, children who have mastered numerical notation and understanding (an indicated by the token task) for integers 1-99 show different strategies of numerical notation for integers 100-999 than children who are still struggling with 1-99. Similarly, children who have mastered numerical notation and understanding for integers 100-999 show different strategies of numerical notation for integers 1,000-9,999 than children who are still struggling with 100-999.

Table 5

Four-digit number decomposition and notation strategies

Decomposing / **Notation**	Incorrect understanding	Varied strategies	Counting by 1s	Ordering/ combining difficulty	Correct strategy, some errors	Zero or 1 error
Idiosyncratic, various strategies	11	0	0	1	0	0
Other	6	0	0	2	0	0
FLT	0	0	0	0	0	0
Compacted	0	0	0	1	0	3
Correct strategy, some errors	0	0	0	0	1	0
Error due to use of comma	0	0	0	1	1	1
Zero or 1 error	0	0	0	0	0	5

Note: N=33;

Light gray = Correct with Tasks 1 and 2 in 3-digit numbers;

Dark gray = Incorrect with at least one task in 3-digit numbers

For the next analysis, I isolated the Full Literal Transcoding (FLT) and Compacted Notation (CN) categories among the children to explore Scheuer's hypothesis that perhaps the children using these notational strategies have different numerical concepts. The results support that these two incorrect strategies of numerical notation do appear to be correlated with different numerical understanding. Of 6 children that used FLT to notate 3-digit numbers, only 2 children (33%) had correctly decomposed and notated 2-digit numbers. Meanwhile, of 6 children that used CN to transcribe 3-digit numbers, 5 (83%) had correctly decomposed and notated 2-digit numbers (see Table 6).

Table 6

Two- and Three- digit interaction amongst children using FLT and CN

	Incorrect with at least one task on 2-digit numbers	Correct with both tasks on 2-digit numbers
FLT with 3-digit numbers	4	2
CN with 3-digit numbers	1	5

Note: N = 12

Next, I looked at these two types of notation in three-digit numbers compared not to whether children performed perfectly with two-digit numbers, but whether they demonstrated that they understood that the various tokens were worth different values, even if unable to correctly utilize this fact. Three out of 6 (50%) of those using FLT DO NOT understand the value of individual tokens, while 5 out of 6 (83%) of those using CN DO understand the symbolic value of the tokens (see Table 7). Though the sample was too small to perform any statistical tests, these results do indicate that this topic is worth further exploration in a larger-scale study. Together, Tables 6 and 7 both point to the possibility that children using CN have a more advanced understanding of the number system than those using FLT. This is a surprising result seeing that both of these strategies are incorrect, never taught, and both formed by the over-use of zero in numerical notation.

Table 7

Three-digit number decomposition amongst children using FLT and CN

	Incorrect Understanding of tokens	Correct Understanding of tokens
FLT	3	3
CN	1	5

Note: N = 12

Discussion

These results lead us to wonder: why do FLT and CN repeatedly come up as strategies and what causes them to be related to different numerical understandings? They are certainly strategies never taught in school such that children would not be repeating from example. In the case of FLT, one could call it the numerical equivalent of sounding out the spelling of a word, which children are taught to do in written language. If this is the case, it still does not explain the prevalence of CN. Why do children choose to eliminate some but not all zeros in a number? Moreover, why do children who are performing this way appear to be closer to a conventional numerical understanding as demonstrated in their token interpretation? Seron and Fayol (1994) posit that perhaps children are absorbing some rules of notation, such as the “overwriting” of zeros, but are not yet grasping the entire concept of the place-value system. The data described here does seem to support this possibility, as even children who understood the concept of the tokens still did not seem to realize that they could use each color token to represent each digit of the number.

This leads us to the next obvious question, which is: are numerical notation strategies shaped by numerical understanding or does the notation influence what children think about numbers (i.e., their numerical understanding)? The lack of a single child who displayed a conventional understanding of the tokens task yet produced idiosyncratic numerical notation seems to point in the direction of the notation as a reflection of the child's understanding. In the case of children with an idiosyncratic understanding of the tokens task yet conventional notation, one potential explanation is simply memorization. First grade children have had many classroom experiences writing numbers 1-100. For the child who does not yet have the proper number concept to allow for a place-value understanding, memorization is the key to producing numbers within this range. This would also help to explain why this category only showed up for numbers less than 100. Above 100, the children have not yet received formal instruction, so the memorization factor does not come into play and thus there were no children who did not understand the tokens task that produced correct notation for numbers above 100.

Noticing that performance in one number range is correlated to the next range (i.e., performance with two digits predicts a range of performance with three digits), we must then ask if there is a critical number of digits after which, when children have mastered both writing and decomposing, they could begin to produce numbers with any number of digits. This question and the preceding ones will be addressed in further studies towards my dissertation. The next study conducted will include numbers beyond 9,999 and also a fourth task. In Task 4, students will be orally presented with a number and then asked to write the number and compose it using the tokens. In this task, we will be able to see if the students' notational strategy influences how they choose to compose the number. In addition, some students may revise the notation after noticing correlations with the tokens that they were unaware of previously. Finally, the need for a larger sample size is apparent. The difference in performance between children using FLT and CN strategies could be quite a remarkable finding, but it is evident that we first need to reproduce the study on a larger scale. The plan for this larger study is currently underway.

Conclusions

Though many numerical systems have existed throughout time, the world has converged on one system in a manner unprecedented by any other form of communication. This is likely due to the manner in which the Arabic number system creates an ideal balance between memory usage, ease of manipulation, and simplicity of understanding. However, when expecting children to converge upon the same understanding, it is important to keep in mind the thousands of years that it took for this system to develop and become the world's norm. In Psychogenesis and the History of Science, Piaget and Garcia (1989) demonstrate parallels between individual development of children's understanding of physical phenomenon and the history of the field of mechanics. Similar parallels can also be drawn in the field of mathematics and the acquisition of the number system (Safuanov, 2004).

There are several theories about how children come to understand the number system using various metaphors such as computer programs that learn through experience and the reconstruction of history over a period of years (Bergeron & Herscovics, 1990; Bergeron, Herscovics, & Sinclair, 1992; DeLoache & Seron, 1982; McCloskey, 1992; Piaget & Garcia, 1989; Sinclair, & Tieche-Christinat, 1992; Power & Longuet-Higgins, 1978; and others).

Regardless of the process that seems to takes place while children are learning the number system, it is clear that children do go through various stages of both internal and external ways of representing numbers.

By looking at children's external representations and explanations of numerical notation we can learn a plethora of information about what children are grasping as we teach them about numbers both in school and in daily life. There are also noticeable differences in children's understanding of place-value correlated to their native language (Dehaene, 1997; Miller, Smith, Zhu, & Zhang, 1995; Miura & Okamoto, 1989; Power & Dal Martello, 1990). Asian languages that act as transparent clarifiers of numerical quantity help children, at least in primary grades, a great deal with learning to write, read, and manipulate numbers.

We can see through Brizuela's (2004) and Alvarado's (2002) accounts of children attempting to rotate or otherwise alter digits to change their meaning as a sign that children are attempting, yet with difficulty, to make sense of the number system. These accounts remind us that while we take the use of place-value for granted, it is not obvious, as can be seen by the thousands of years that it took to develop.

That being said, we still have much to learn about how children understand this unique and global system. The data presented in this paper show that we may be able to learn quite a bit about how children understand the number system by looking at their notational strategies. One further point of exploration is the number zero. What do children understand is the role of this object in numbers? How is it related to place-value? Having taken thousands of years to develop this symbol of null quantity (and even longer to incorporate it as a place holder within written numeral), we cannot assume that its meaning is intuitive to children. As a follow-up to this study, plans for a larger, longitudinal study have begun tracing the same children from kindergarten until third grade with their oral, nonverbal, and written representations of number. Using this longitudinal approach, we will be able to see not only the correlations between one representation and another, but also the order in which certain types of representations develop. .Until we answer more of these questions, we cannot begin to state that we understand how children represent numbers, internally or externally.

References

Alvarado, M. (2002). *La Construcción del Sistema Gráfico Numérico en los mometos iniciales de la adquisición del Sistema Gráfico Alfabético.* Unpublished doctoral dissertation, CINVESTAV, IPN, México.

Alvarado, M., & Ferreiro, E. (2002). Four- and five-year old children writing two-digit numbers. In M. A. Pinto (Ed.), *Rivista di Psicolinguistica* (Vol. II-3, pp. 23-38). Pisa: Istituti Editoriali e Poligrafici Internazionali.

Bergeron, A., Herscovics, N., & Sinclair, H. (1992). Contribution a la genese du nombre. *Archives de Psychologie, 60*, 147-170.

Bergeron, J., & Herscovics, N. (1990). Psychological aspects of learning early arithmetic. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge, England: Cambridge University Press.

Bialystok, E., & Codd, J. (1996). Developing representations of quantity. *Canadian Journal of Behavioral Science, 28*, 281-291.

Brizuela, B. M. (2004). *Mathematical development in young children: exploring notations*. New York: Teachers College Press.

Campbell, J. I. D. (1994). Architectures for numerical cognition. *Cognition, 53*, 1-44.

Campbell, J. I. D., & Clark, J. M. (1992). The nature and origins of mathematical skills. *Advances in psychology, 91*(3), 457-491.

Dehaene, S. (1992). Varieties of numerical abilities. *Cognition, 44*(1-2), 1-42.

Dehaene, S. (1997). *The number sense: how the mind creates mathematics*. Oxford: Oxford University Press.

DeLoache, G., & Seron, X. (1982). From one to 1: An analysis of a transcoding process by means of neuropsychological data. *Cognition, 12*, 119-149.

DeLoache, G. & Willmes, K. (2000). Cognitive neuropsychological models of adult calculation and number processing: The role of the surface format of numbers. *European Child & Adolescent Psychiatry*, 9 (2), 27-40.

Ferreiro, E. and Teberovsky, A., (1979) *Los sistemas de escritura en el desarrollo del niño [Writing systems in the development of the child]*. Ciudad de México. Siglo XXI Editores.

Fuson, K. (1986). Roles of representation and verbalization in the teaching of multi-digit addition and subtraction. *European Journal of Psychology of Education, 1*, 35-36.

Fuson, K. (1990). Conceptual structures for multiunit numbers: implications for learning and teaching multidigit addition, subtraction, and place-value. *Cognition and Instruction, 7*(4), 343-403.

Fuson, K. C., & Kwon, Y. (1992). Learning addition and subtraction: effects of number words and other cultural tools. In J. Bideaud, C. Meljac & J. P. Fischer (Eds.), *Pathways to number: Children's developing numerical abilities*. Hillsdale, NJ: Erlbaum.

Gardner, H., & Wolf, D. (1983). Waves and streams of symbolization: Notes on the development of symbolic capacities in young children. In J. Sloboda & D. Rogers (Eds.), *The Acquisition of Symbolic Skills* (pp. 19-42). New York: Plenum Press.

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior, 17*(2), 137-165.

Goodman, N. (1976). *Languages of art: an approach to a theory of symbols*. Indianapolis, IN: Bobbs-Merrill Co.

Kamii, C. (1989). *Young children continue to reinvent arithmetic - 2nd grade: Implications of Piaget's theory*. New York: Teachers College Press.

Kamii, C., Kirkland, L., & Lewis, B. A. (2001). Representation and abstraction in young children's numerical reasoning. In *The roles of representation in school mathematics*: National council of teachers of mathematics.

Karmiloff-Smith, A. (1979). Micro- and macrodevelopmental changes in language acquisition and other representational systems. *Cognitive Science, 3*, 91-118.

McCloskey, M. (1992). Cognititve mechanisms in numerical processing: Evidence from acquired dyscalculia. *Cognition, 44*, 107-157.

Miller, K., Smith, C. M., Zhu, J., & Zhang, H. (1995). Pre-school origins of cross-national differences in mathematical competence: The role of number-naming systems. *Psychological Science, 6*, 56-60.

Miura, I. T., & Okamoto. (1989). Comparisons of U.S. and Japanese first graders' cognitive representation of number and understanding of place value. *Journal of Educational Psychology, 81*(1), 109-113.

Nunes Carraher, T. (1985, July 22-26). *The decimal system: Understanding and notation.* Paper presented at the Ninth International Conference for the Psychology of Mathematics Education, Noordwijkerhout.

Piaget, J. (1962). *Play, dreams and imitation in childhood* (C. Gattegno & F. M. Hodgson, Trans.). New York: W. W. Norton & Company.

Piaget, J. (1965). *The Child's Conception of the World* (J. Tomlinson & A. Tomlinson, Trans.). Totowa, NJ: Littlefield, Adams, & Co.

Piaget, J. (1971). *Biology and Knowledge*. Chicago: University of Chicago Press.

Piaget, J., & Inhelder, B. (1973). *Memory and intelligence* (A. J. Pomerans, Trans.). New York: Basic Books.

Piaget, J., & Inhelder, B. (1971). *Mental imagery in the child* (P. Chilton, Trans.). New York: Basic Books.

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the History of Science*. New York: Columbia University Press.

Pontecorvo, C. (1985). Figure, parole, numeri: Un problema di simbolizzazione [Pictures,words, numbers: A problem of symbolization]. *Età Evolutiva, 22*, 5-33.

Power, R. J. D., & Dal Martello, M. F. (1990). The dictation of Italian Numerals. *Language and Cognitive Processes, 5*(3), 237-254.

Power, R. J. D., & Longuet-Higgins, H. C. (1978). Learning to count: A computational model of language acquisition. *Proceedings of the Royal Society of London, B200*, 391-417.

Ross, S. (1986). *The development of children's place-value numeration concepts in grades two through five.* Paper presented at the Annual meeting of the American Educational Research Association,, San Francisco.

Safuanov, I. S. (2004, July 22-26). *Psychological aspects of genetic approach to teaching mathematics.* Paper presented at the Twenty-eighth International

Conference for the Psychology of Mathematics Education, Bergen, Norway.

Saussure, F. (1931). *Course in general linguistics*. New York: McGraw-Hill.

Scheuer, N. I. (1996). *La construction du systeme de notation numerique chez l'enfant.* l'Universite de Geneve, Geneva, Switzerland.

Scheuer, N., Sinclair, A., Merlo de Rivas, S., & Tieche-Christinat, C. (2000). Cuando ciento setenta y uno se escribe 10071: niños de 5 a 8 años produciendo numerales [When a hundred and seventy one is written 10071: Five to eight-year-olds' production of written numerals]. *Infancia y Aprendizaje, 90*, 31-50.

Seron, X., & Fayol, M. (1994). Number transcoding in children: A functional analysis. *British Journal of Developmental Psychology, 12*, 281-300.

Sinclair, A., & Scheuer, N. I. (1993). Understanding the written number system: 6 year-olds in Argentina and Switzerland. *Educational Studies in Mathematics, 24*, 199-221.

Sinclair, A., Siegrist, F., & Sinclair, H. (1983). Young children's ideas about the written number system. In D. Rogers & J. A. Sloboda (Eds.), *The acquisition of symbolic skills* (pp. 535-541). New York: Plenum.

Sinclair, A., & Tieche-Christinat, C. (1992). Constructing and understanding of place value in numerical notation. *European Journal of Psychology of Education, 7*(3), 191-207.

Starkey, P., & Cooper, R. G., jr. (1980). Perception of numbers by human infants. *Science, 210*, 1033-1035.

Strauss, M. S., & Curtis, L. E. (1981). Infant perception of numerosity. *Child Development, 52*, 1146-1152.

Strauss, S., & Stavy, R. (1982). U-shaped behavioral growth: Implications for theories of development. In W. W. Hartup (Ed.), *Review of child development research* (Vol. 6). Chicago: University of Chicago Press.

Tolchinsky Landsmann, L., & Karmiloff-Smith, A. (1992). Children's understanding of notations as domains of knowledge versus referential-communicative tools. *Cognitive Development, 7*, 287-300.

Zhang, J., & Norman, D. A. (1994). Representations in distributed cognitive tasks. *Cognitive Science, 18*, 87-122.

Zhang, J., & Norman, D. A. (1995). A representational analysis of numeration systems. *Cognition, 57*, 271-295.

Acknowledgements

I'd like to thank Bárbara Brizuela for her countless hours of assistance with data collection. Thank you also to Bárbara Brizuela and Analúcia Schliemann for their invaluable suggestions on data interpretation and feedback during the writing of this paper.