Christiaan Huygens (14 Apr 1629 – 8 Jul 1695)

1651 ***Theoremata de quadratura hyperboles, ellipsis et circuli …***
1656 ***De Saturni lunâ observatio nova***
1657 ***Tractatus de ratiociniis in alae ludo***
1658 ***Horologium***
1659 ***Systema Saturnium***
1673 ***Horologium Oscillatorium***
1690 ***Traité de la lumière, Discours de la cause de la pesanteur***
1703 ***Christiani Hugenii Opuscula Posthuma***

Huygens on Descartes, 1693

M. des Cartes had found the way to make his conjectures and fictions pass for truths. And those who read his Principles of Philosophy experienced somewhat the same as do the readers of Romances, which give pleasure and make the same impression as veritable histories.... It seemed to me when I first read this book of the Principles that everything in it was as good as could be [*que tout alloit le mieux du monde*], and I believed, when I found some difficulty there, that it was my fault for not correctly understanding his thought. I was only 15 or 16 years old. But having since discovered there from time to time things visibly false, and others highly improbable, I have retreated far from the predilection I then had, and at the present hour I find almost nothing that I can approve as true in all his physics, or metaphysics, or meteors. [tr. Howard Stein]

[Fig. 17.]

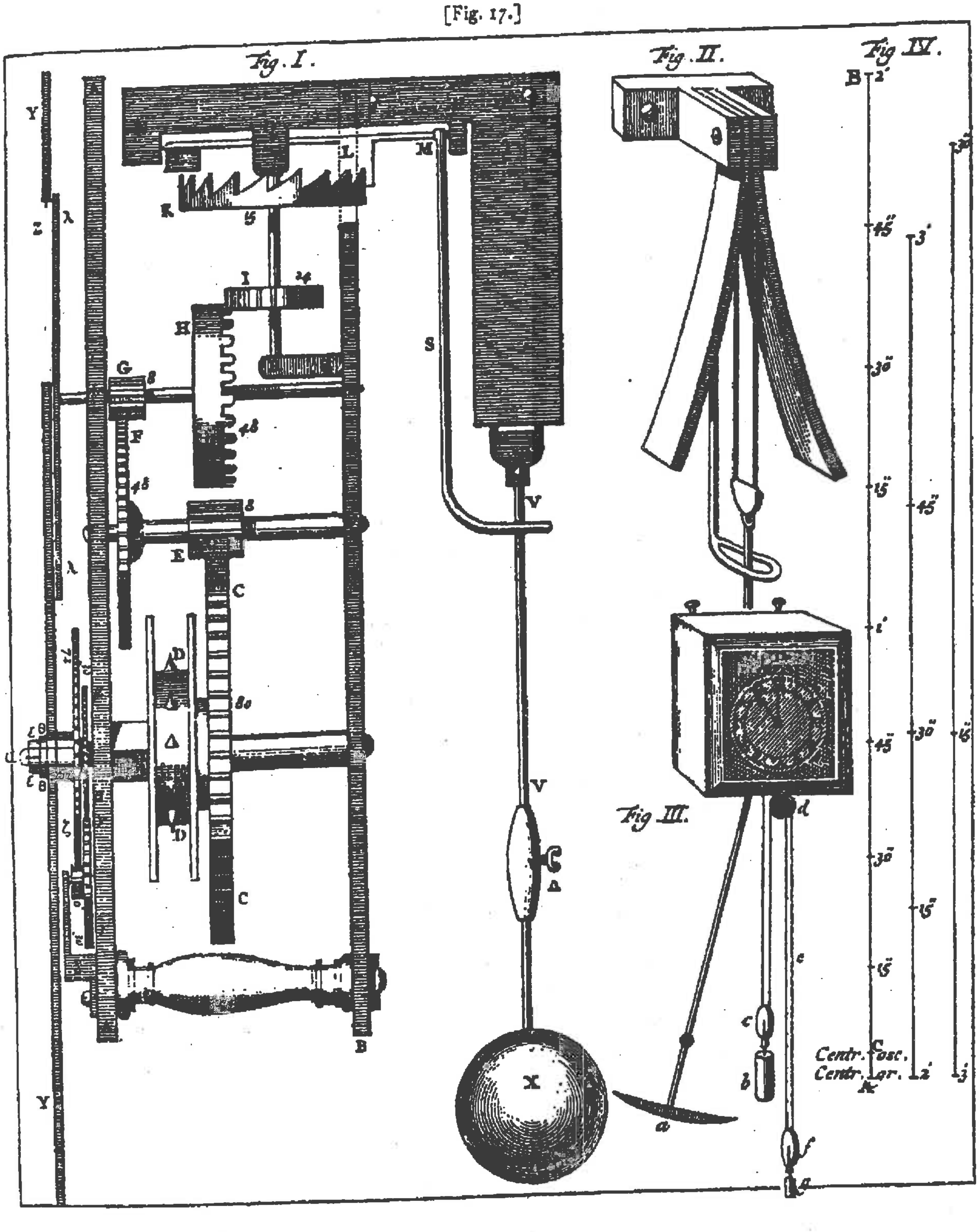

Fig. I.
Fig. II.
Fig. III.
Fig. IV.
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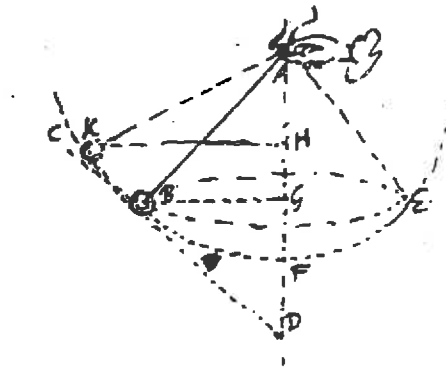
HUYGENS' METHODS OF MEASURE

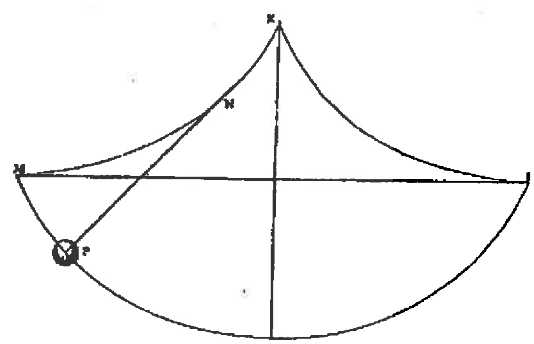
Theoretically derived law of the conical pendulum:

$$P = \pi\sqrt{2h/d_g} = 2\pi\sqrt{h/g}$$



Theoretically derived law of the cycloidal pendulum:

$$T = \pi\sqrt{\ell/2d_g} = \pi\sqrt{\ell/g}$$



Therefore, by measuring P and h, or T and ℓ, to high accuracy, can obtain a theory-mediated accurate measure of d_g, the distance of vertical fall in the first second in the absence of air resistance.

MEASURING THE STRENGTH OF GRAVITY

	DATES	METHOD	VERTICAL FALL IN FIRST SECOND	EQUIVALENT g in cm/sec^2
MERSENNE	mid-1640s	DIRECT	12 Paris ft	788.8
RICCIOLI	late 1640s	DIRECT	15 Roman ft	935.0
HUYGENS	1659	CONICAL PENDULUM	15.6 Rhen ft	979.4*
	1659	CYCLOIDAL PENDULUM	15 Rhen ft 7 1/2 in	980.9
	1660s & 70s	SECONDS-PENDULUM	15 Paris ft 1.1 in	980.7
{NEWTON	late 1660s	CONICAL PENDULUM	195 London in	989.4}**

* Without rounding, 15.625 Rhen ft and 980.9 cm/sec^2

** Apparently just to check Huygens's value

NB Modern measured g at Paris = 980.970 cm/sec^2

MEASURED LENGTHS OF THE SECONDS-PENDULUM

	LOCATION	LATITUDE	LENGTH (Paris units)	IMPLIED g (cm/sec^2)
HUYGENS	Paris	48°50′	3ft 8 1/2 lines	(980.7)*
RICHER	Paris	48°50′	3ft 8 3/5 lines	(980.9)*
	Cayenne	4°55′	$\Delta\ell = 1$ 1/4 lines	(978.1)
VARIN et al	Paris	48°50′	3ft 8 5/9 lines	(980.7)*
	Goree	14°40′	$\Delta\ell = 2$ lines	(976.4)
	Guadaloupe	15°00′	$\Delta\ell = 2$ 1/18 lines	(976.3)
PICARD	Paris	48°50′	3ft 8 1/2 lines	(980.7)*
	Uraniborg	55°54′	$\Delta\ell = 0$ lines	(980.7)
	Cape Cete	43°24′	$\Delta\ell = 0$ lines	(980.7)
MOUTON	Lyons	45°47′	3ft 6 3/10 lines	(975.8)

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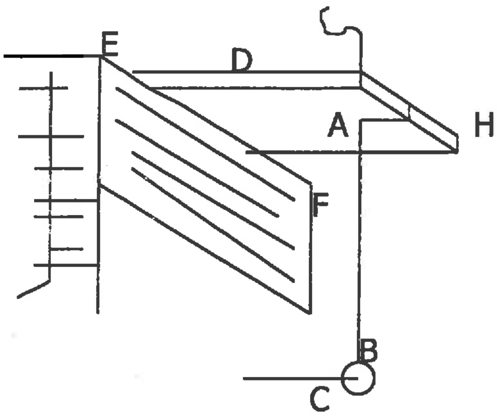
* Modern measured g at Paris = 980.970 cm/sec^2

Modern average g at Equator = 978.032 cm/sec^2

NB Latitudes vary slightly from individual to individual

Christiaan Huygens to Thomas Helder.[1]
1686.

Observation concerning the Length of a simple Pendulum.



XXXVI. While ashore at the Cape of Good Hope as well as especially in Batavia[2] if the voyage goes so far, or while the ship is lying very still, one will observe by using the clock how long a single pendulum must be to do each beat in a second, that is [the length] from the top end of the thread until the center of the sphere; for, here I call a single pendulum a copper or leaden little sphere of about a thumb[3] in diameter that is hanging on a thin thread.[4] With regard to the motion of the clock much depends on this experience. For, a certain Frenchman claims to have found at a location about 5 degrees north of the Equator that such a pendulum was a bit shorter there than in Paris, England, and Holland. In order then to observe this perfectly one should hang the pendulum as in the figure, in which EF is the side of a high table or windowsill; DH [is] a flat piece of wood nailed down on it and which overhangs by only 1/2 thumb; AB [is] the thread, wedged in an incision in that wood, and having a length down to the sphere C of about 3 Rhenish feet 1 1/2 thumb.

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[1] Item 2520 in <u>OCCH</u>, Vol. 9, pp. 292-293. Even though this document was found separately in the Huygens Archives, in the University Library of Leiden, the paragraph number at the beginning of it (XXXVI), makes it almost certain that it was to supplement the instructions Huygens had given Helder on the use of the clocks during the voyage (Item 2423 in <u>OCCH</u>, Vol. 9, pp. 55-76).

[2] Now Jakarta in Indonesia.

[3] A thumb is a Rhenish inch, i.e. 1/12 of a Rhenish foot, roughly 0.965 of a Paris inch, or 2.6 cm.

[4] Although we have been unable to determine what it was, the material used for the thread is of some interest. A 40 deg. F (22 deg. C) change in temperature would cause a copper or bronze thread of the length of a seconds-pendulum to increase in length by roughly 0.16 Paris lines, or 0.037 percent. The thermal expansion of a non-metallic thread could well have been as little as 1/4 of this, depending on what material was used. Huygens's drawing suggests a non-metallic thread. Cat-gut (in truth, sheep's gut), which was used for stringing musical instruments, is a good candidate. There is some evidence for silk. In his instructions to Helder, Huygens mentions that the triangular pendulum is hanging from silk threads. See item 2488, <u>OCCH</u>, Vol. 9, pp. 222-223.

One will make this pendulum[5] move very slowly, roughly just 2 or 3 thumb-widths[6] being very careful that the sphere no longer rotates, as always occurs from the start, for, through this the thread unwinds itself and becomes longer. One can impregnate it with wax, except at the top near A. Furthermore, one will observe the movements of this pendulum against one of the clocks, ensuring that one movement accords with two movements of the pendulum of the clock, and that for about a half an hour.[7] One can thus shorten or lengthen the pendulum AB, until the beats, as was said, accord perfectly. Once then this has been done, one shall measure off neatly with a straight stick, having been shortened to this measure, the true length AB from the top end of the thread until the top of the sphere, so that it just fits between the piece of wood, DH, and the sphere C. After that one can take the length of this little stick as a correct foot-measure. Adding to it half the diameter of the sphere, this yields the total length of a pendulum beating seconds, if the clock has been properly calibrated to the proper mean of the days.[8]

But as the clock usually goes several seconds too quickly or too slowly in 24 hours, so the movement of this single-pendulum will be a bit shorter or a bit longer than a second; let us suppose that a clock goes 1 minute too slow in 24 hours; then turn 24 hours into minutes, resulting in 1440, from which the previous mentioned 1 minute [should be] subtracted, resulting in 1439. Now just as the square of 1439 [is] to the square of 1440, [is] the length found for the pendulum to the correct length of a pendulum that beats in a second. For example, if the length of this little stick, including half the diameter of the sphere, is found to be 37 thumbs, 11 2/3 lines, then saying the square of 1439 [is] to the square of 1440, just as 37 thumbs 11 2/3 [lines] is to another length, the latter being very near 38 thumbs and 1/3 of a line. This is the length of a pendulum for seconds here in

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5 In Dutch the word here is *slinger*, not *pendulum*.

6 Huygens knew that a circular arc pendulum is isochronous over only infinitesimal arcs, where it truly approximates a cycloidal pendulum. He had no way, however, of calculating the departure from isochronism as a function of arc length, for the solution for the (large arc) circular pendulum had to await Euler's elliptical integrals three-quarters of a century later. The only way Huygens therefore had for determining a limit on the arc length of a simple seconds-pendulum was through trial-and-error comparison with a cycloidal pendulum. In his ***Horologium Oscillatorium*** he says, "...the pendulum should be set in motion with a small push because small oscillations, for example 5 or 6 degrees, are sufficient to give equal times, but not a large number of degrees," (OCCH, Vol. 18, p. 351; in the translations by Blackwell, p. 168). The instruction here calls for a slightly smaller arc: 2 thumbs corresponds to an arc of about 3 degrees, and 3 thumbs to one of 4.5 degrees.

The incremental error in the period of a 4.5 degree arc circular pendulum is 0.009 percent. This would require the length of a seconds-pendulum of this arc to be 0.085 Paris lines shorter than a one-second cycloidal pendulum. By contrast, the incremental error in the period of a 6 degree arc circular pendulum is 0.017 percent, which would amount to a difference in length between a circular-arc and a cycloidal seconds-pendulum of 0.151 lines. The smaller arc length that Huygens is here recommending to Helder is accordingly of some merit. (A still smaller 2 degree arc length would reduce the difference in length from 0.085 to 0.038 lines.)

7 Over the course of 30 minutes, a 0.00555 percent discrepancy between the two periods will produce a 10 percent asynchrony between their motions, which would have been quite apparent to the naked eye.

8 Huygens here ignores the correction for the center of oscillation that he so carefully takes into account in Part Four of his ***Horologium Oscillatorium***. He can safely ignore it here insofar as he is only comparing lengths, and not determining the absolute magnitude of the effective acceleration of gravity. With a 1 Paris inch diameter sphere forming the seconds-pendulum, the center of oscillation is only 0.032 lines beyond the center of the sphere.

Holland as well as in France and England. But the mentioned French observer says to have found this length 5/4 of a line less in Cayenne. When there is great calmness, it will be good to observe this on board the ship, not only at the Equator but also at several other latitudes, and to note the measures found.[9, 10]

[9] Suppose the accuracy of the pendulum clock could be determined sidereally to within at least 4 sec. per day (i.e. 1 part in 21,600); and the seconds-pendulum could be determined to be no more than 1/10 of arc out of synchrony with the pendulum clock after 30 minutes (1 part in 18,00000); and the length of the seconds-pendulum could be determined to within 0.1 Paris lines (1 part in 4405). Following the instructions Huygens gives Helder, the error in the absolute magnitude of the acceleration of gravity as determined by a seconds-pendulum would then have been less then 1 part in 2320, or 0.42 cm/sec^2. Doubling the error of the length in the pendulum to 0.2 Paris lines, to allow for thermal effects, would increase the upper bound on the error to 1 part in 1520 -- still only 0.65 cm/sec^2. Richer was claiming a difference between Paris and Cayenne amounting to 1 part in 352, or 2.78 cm/sec^2.

[10] Translation by Eric Schliesser; notes by Eric Schliesser and George E. Smith

HUYGENS'S PENDULUM MEASUREMENT OF DISTANCE OF FALL IN 1ST SECOND

Distance of fall in 1st second
$= \pi^2 \cdot$length of pendulum$/2T^2$

Length of 1 sec pendulum:

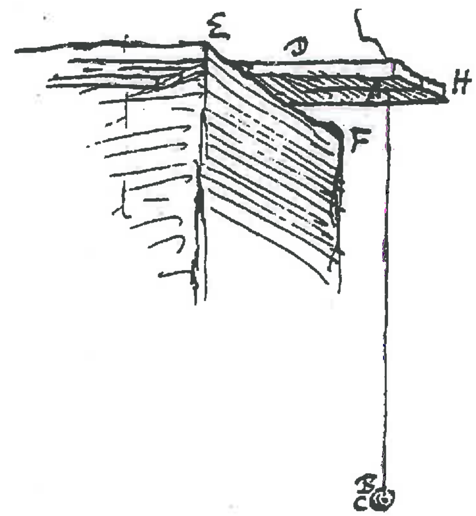
3 Paris feet 8.5 lines

i.e. 440.5 lines

Distance of fall in 1st sec:

15 Paris feet 1.1 inches

(i.e. 980.7 cm/sec/sec)



1. Adjust length until in synchrony with pendulum clock for 30 min.
2. Measure length to bob center and correct for center of oscillation
3. Correct length for any inaccuracy in clock: $(86156/\text{no. sec in day})^2$
4. Infer distance of fall in 1st sec

MEASURING GRAVITY WITH A SECONDS-PENDULUM

A MODERN ASSESSMENT OF ACCURACY

Suppose the accuracy of the pendulum clock could be determined to within 4 seconds per day : **1 part in 21600**

Suppose seconds-pendulum could be determined still to be within 1/10 of an arc in synchrony after 30 min : **1 part in 18000**

Suppose the length of the seconds-pendulum could be determined to within 0.2 Paris lines : **2 parts in 4405**

Then error in strength of gravity : **≤ 1 part in 1520**

If instead, the length of the seconds-pendulum could be determined within 0.1 Paris lines : **1 part in 4405**

Then error in strength of gravity : **≤ 1 part in 2320**

For a 2 inch diameter bob, the correction for the length of a seconds-pendulum is slightly more than 0.1 lines.

The solution for head-on collision of "hard" spheres (recast in symbolic form):

$$\mathbf{v_a} = \frac{B_a - B_b}{B_a + B_b}\mathbf{u_a} + \frac{2B_b}{B_a + B_b}\mathbf{u_b}$$

Four consequences of this solution:

1. *The quantity of motion which two hard bodies have may be increased or diminished by their collision, but when the quantity of motion in the opposite direction has been subtracted there remains always the same quantity of motion in the same direction.*
2. *The sum of the products made by multiplying the bulk of each hard body into the square of its velocity is always the same before and after collision.*
3. *A hard body at rest will receive more motion from another larger or smaller body if a third intermediately sized body is interposed than it would if struck directly, and most of all if this [third] is their mean proportional [i.e. their geometric mean].*

In all this I am thinking of bodies of the same material, or else I mean that their bulk can be assessed from their weight.

4. *A wonderful law of nature (which I can verify for spherical bodies, and which seems to be general for all whether the collision be direct or oblique and whether the bodies be hard or soft) is that the common center of gravity of two, three or more bodies always moves uniformly in the same direction in the same straight line, before and after their collision.* [tr. A. R. Hall, modified by GES]

Huygens, *Philosophical Transactions of the Royal Society,* 46, 12 April 1669, pp. 925-928.

NEWTON ON IMPACT OF SPHERES (ca. late 1670s)

PROBLEM XII.

Having given the Magnitudes and Motions of Spherical Bodies perfectly elastick, moving in the same right Line, and striking against one another, to determine their Motions after Reflexion.

The Resolution of this Question depends on these Conditions, that each Body will suffer as much by Re-action as the Action of each is upon the other, and that they must recede from each other after Reflexion with the same Velocity or Swiftness as they met before it. These Things being supposed, let the Velocity of the Bodies A and B, be a and b respectively; and their Motions (as being composed of their Bulk and Velocity together) will be aA and bB. And if the Bodies tend the same Way, and A moving more swiftly, follows B, make x the Decrement of the Motion aA, and the Increment of the Motion bB arising by the Percussion; and the Motions after Reflexion will be $aA-x$ and $bB+x$; and the Celerities $\frac{aA-x}{A}$ and $\frac{bB+x}{B}$, whose Difference is $=a-b$ the Difference of the Celerities before Reflexion. Therefore there arises this Equation $\frac{bB+x}{B}-\frac{aA-x}{A}=a-b$, and thence by Reduction x becomes $=\frac{2aAB-2bAB}{A+B}$, which being substituted for x in the Celerities $\frac{aA-x}{A}$, and $\frac{bB+x}{B}$, there comes out $\frac{aA-aB+2bB}{A+B}$ for the Celerity of A, and $\frac{2aA-bA+bB}{A+B}$ for the Celerity of B after Reflexion.

But if the Bodies move towards one another, then changing every where the Sign of b, the Velocities after Reflexion will be $\frac{aA-aB-2bB}{A+B}$ and $\frac{2aA+bA-bB}{A+B}$; either of which, if they come out, by Chance, Negative, it argues that Motion, after Reflexion, to tend a contrary Way to that which A tended to before Reflexion. Which is also to be understood of A's Motion in the former Case.

EXAMPLE. If the homogeneous Bodies [or Bodies of the same Sort] A of 3 Pounds with 8 Degrees of Velocity, and B a Body of 9 Pounds with 2 Degrees of Velocity, tend the same Way; then for A, a, B and b, write 3, 8, 9 and 2; and $\left(\frac{aA-aB+2bB}{A+B}\right)$ beomes -1, and $\left(\frac{2aA-bA+bB}{A+B}\right)$ becomes 5. Therefore A will return back with one Degree of Velocity after Reflexion, and B will go on with 5 Degrees.

Mr Hugens —

De motu corporum ex mutuo impulsu.

Read Jan: 7. 1668.

Hypotheses. Entd R.B. 4. 31.

1.

Corpus quodlibet semel motum, si nihil obstet, pergere moveri eadem perpetuo celeritate, et secundum lineam rectam.

2.

Cum corpora duo dura inter se aequalia aequali celeritate ac directe sibi mutuo occurrunt, resilire utrumque eadem qua advenit celeritate.

Dicuntur autem directe occurrere, cum in eadem linea recta, utriusque centra gravitatis conjungente, moventur, punctumque contactus accidit in eadem recta.

3.

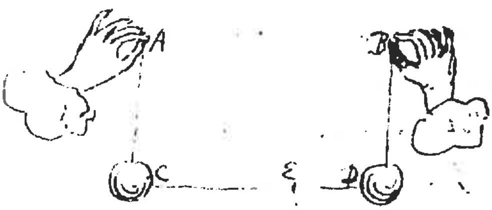
Motum corporum celeritatesque aequales aut inaequales respective intelligendas esse, facta relatione ad alia corpora quae tanquam quiescentia considerantur, etsi fortasse et haec et illa alio communi motu involvantur. ac proinde cum corpora duo sibi mutuo occurrunt, etiam si alteri praeterea motui utrumque simul obnoxium fuerit, haud aliter illa se invicem impellere respectu ejus qui eodem quoque motu defertur, ac si omnibus adventitius iste motus abesset.

Veluti siquis navi vectus, quae aequabili motu progreditur globulos duos aequales aequali celeritate in se invicem impingere faciat, suo nimirum et partium navis respectu, dicimus aequali quoque celeritate utrumque resilire debere, ejusdem vectoris respectu; plane sicut contingeret si, in navi quiescente aut in terra consistens, eodem globulos aequali celeritate collidi facere.

4.

Sive ipse corpora duo manibus meis sustinens aequalibus ea motibus concurrere faciam, sive alius illa sustinens eosdem motus mei respectu illis dederit, repercussiones eorum corporum easdem fore mei respectu.

Velut si immotus ipse consistens corpora C, D ex filis suspensa sustineam manibus A, B, easque movendo, simul corpus C transferam celeritate CE, et corpus D celeritate DE, ita ut sibi mutuo occurrant in E.



Dico eosdem motus utrique ex repercussione contingere mei respectu, sive ipse manibus meis capita filorum tenens illa ita moverim, sive alius duis manibus illa tenens eosdem motus mei respectu producat.

His positis de corporum aequalium occursu, et quibus legibus illa a se mutuo impellantur demonstrabimus, cum vero ad inaequalia ventum erit, Hypotheses quasdam necessarias jam dictis addemus.

Propositio 1.

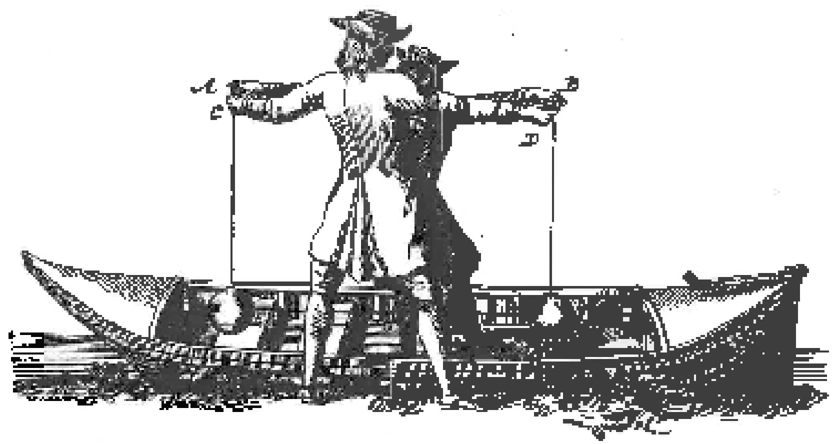
Si corpori quiescenti aliud aequale corpus directe occurrat; post contactum hoc quidem quiescet, quiescenti vero acquiretur eadem quae fuit in impellente celeritas.

GALILEAN-HUYGENSIAN RELATIVITY

"The motion of bodies and their equal and unequal speeds are to be understood, respectively, in relation to other bodies which are considered as at rest, even though perhaps both the former and the latter are involved in a common motion. And accordingly when two bodies collide with one another, even if both together are further subject to another uniform motion, they will move each other with respect to a body that is carried by the same common motion no differently than if this motion coming from outside were absent to all."

[Huygens, manuscript at Royal Society, 1669]

Huygens, 1669 Manuscript



"Thus, if someone conveyed on a boat that is moving with a uniform motion were to cause equal balls to strike one another at equal speeds with respect to himself and the parts of the boat, we say that both should rebound also at equal speeds with respect to the same passenger, just as would clearly happen if he were to cause the same balls to collide at equal speeds in a boat at rest or while standing on the ground."

Theory of Impact: Initial Fragment

Hyp. 1. Any body once moved continues to move, if nothing prevents it, at the same constant speed and along a straight line.

Hyp. 2. Whatever be the cause of the rebound of hard bodies from mutual contact when they collide with one another, we posit that when two equal bodies with equal speed collide directly with one another from opposite directions each rebounds with the same speed with which it approached.

Hyp. 3. When two bodies collide with one another, even if both together are further subject to another uniform motion, they will move each other with respect to a body that is carried by the same common motion no differently than if this motion extraneous to all were absent.

Props I and II (contrary to Descartes' Rules 6 and 3)

Hyp. 4. If a larger body meets a smaller one at rest, it will give it some of its motion and hence lose something of its own.

Prop. III. A body however large is moved by impact by a body however small and moving at any speed.

Hyp. 5. When two hard bodies meet each other, if, after impulse, one of them happens to conserve all the motion that it had, then likewise nothing will be taken from or added to the motion of the other.

Prop. IV. Whenever two bodies collide with one another, the speed of separation is the same, with respect to each other, as that of approach.

The Theory Completed

Prop. V. **If two bodies each collide again at the speed at which they rebounded from impulse, after the second impulse each will acquire the same speed at which it was moved toward the first collision.**

Prop. VI. **When two bodies collide with one another, the same quantity of motion in both taken together does not always remain after impulse what it was before, but can be either increased or decreased.**

***Stipulation (Huygens's version of Torricelli's Principle)*: For in mechanics it is a most certain axiom that the common center of gravity of bodies cannot be raised by a motion that arises from their weight.**

Prop. VIII. **If two bodies, the speeds of which correspond inversely to the magnitudes, collide with each other from opposite directions, each will rebound at the same speed at which it approached.**

Proof*: Via reductio, contradicting the stipulation in separate cases with speeds determined in accord with Galileo's sublimity principle, $v^2 \propto$ *height

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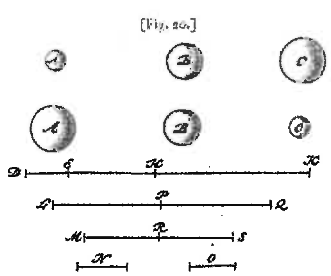
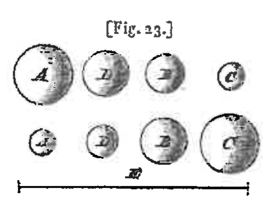
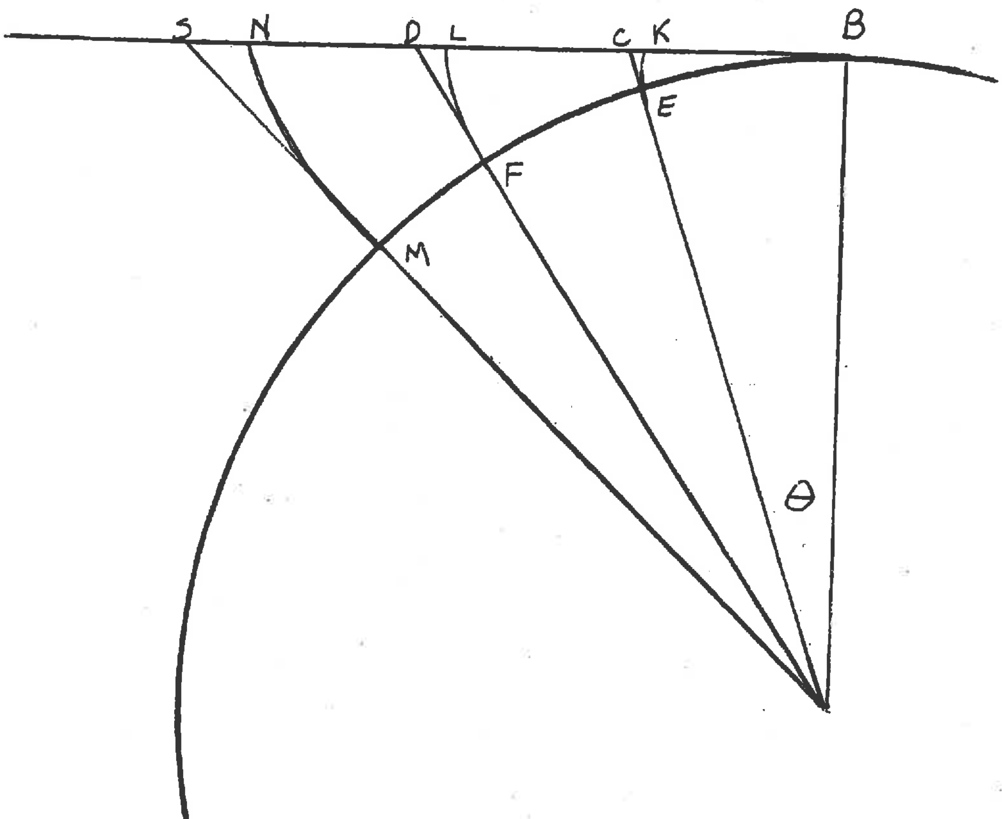




"not alien to reason and agrees above all with experiments"



for small θ

$$\left.\begin{matrix} EK \\ FL \\ MS \end{matrix}\right\} \text{ differ little from } \left\{\begin{matrix} EC \\ FD \\ MS \end{matrix}\right.$$

and

$$EC \propto \left(\frac{1}{\cos\theta} - 1\right) \Rightarrow \theta^2 \propto t^2$$

i.e.

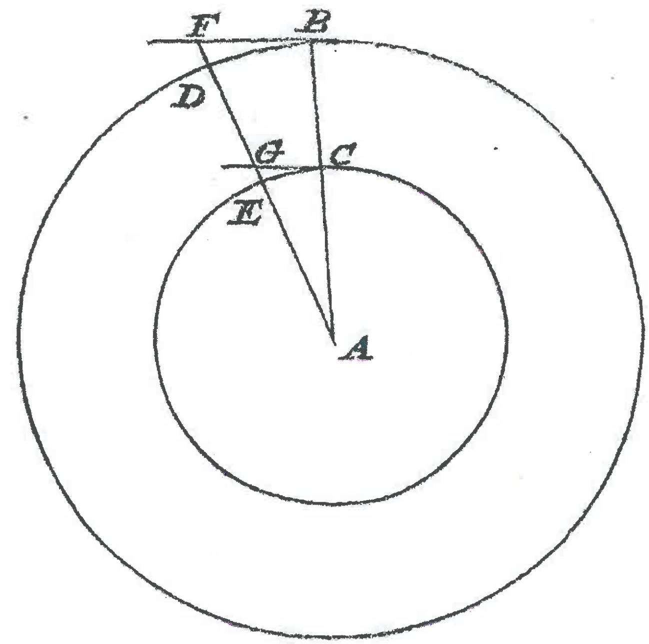
$$EC \propto t^2$$

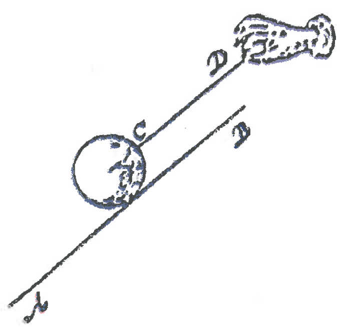
HUYGENS ON "CENTRIFUGAL FORCE"

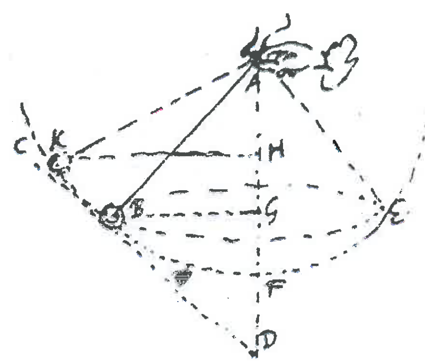
The ***tension*** in the string that retains a body in uniform circular motion varies as

$$EG/\delta t^2 \propto (GC^2/AG)/\delta t^2 \propto v^2/r \propto r/P^2$$

times the ***weight*** of the body







Euclid's Elements

Book III

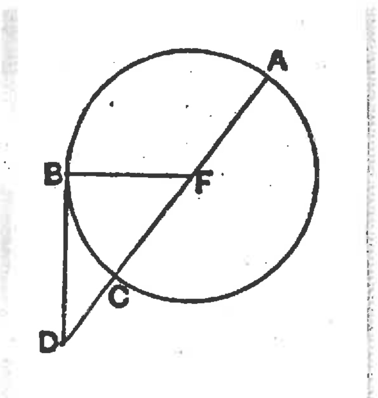
Proposition 36

If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.

Let a point *D* be taken outside the circle *ABC,* and from *D* let the two straight lines *DCA* and *DB* fall on the circle *ABC.* Let *DCA* cut the circle *ABC,* and let *BD* touch it.

I say that the rectangle *AD* by *DC* equals the square on *DB.*

Then *DCA* is either through the center or not through the center.



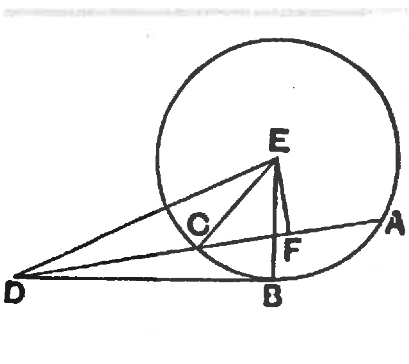
First let it be through the center, and let *F* be the center of the circle *ABC.* Join *FB.* Therefore the angle *FBD* is right. III.18

And, since *AC* has been bisected at *F,* and *CD* is added to it, the rectangle *AD* by *DC* plus the square on *FC* equals the square on *FD.* II.6

But *FC* equals *FB,* therefore the rectangle *AD* by *DC* plus the square on *FB* equals the square on *FD.*

And the sum of the squares on *FB* and *BD* equals the square on *FD,* therefore the rectangle *AD* by *DC* plus the square on *FB* equals the sum of the squares on *FB* and *BD.* I.47

Subtract the square on *FB* from each. Therefore the remaining rectangle *AD* by *DC* equals the square on the tangent *DB.*



Again, let *DCA* not be through the center of the circle *ABC.* Take the center *E,* and draw *EF* from *E* perpendicular to *AC.* Join *EB, EC,* and *ED.* III.1

Then the angle *EBD* is right. III.18

And, since a straight line *EF* through the center cuts a straight line *AC* not through the center at right angles, it also bisects it, therefore *AF* equals *FC.* III.3

Now, since the straight line *AC* has been bisected at the point *F,* and *CD* is added to it, the rectangle *AD* by *DC* plus the square on *FC* equals the square on *FD.* II.6

Add the square on *FE* to each. Therefore the rectangle *AD* by *DC* plus the sum of the squares on *CF* and *FE* equals the sum of the squares on *FD* and *FE.*

But the square on *EC* equals the sum of the squares on *CF* and *FE,* for the angle *EFC* is right, and the square on *ED* equals the sum of the squares on *DF* and *FE,* therefore the rectangle *AD* by *DC* plus the square on *EC* equals the square on *ED*. I.47

And *EC* equals *EB,* therefore the rectangle *AD* by *DC* plus the square on *EB* equals the square on *ED.*

But the sum of the squares on *EB* and *BD* equals the square on *ED,* for the angle *EBD* is right, therefore the rectangle *AD* by *DC* plus the square on *EB* equals the sum of the squares on *EB* and *BD.* I.47

Subtract the square on *EB* from each. Therefore the remaining rectangle *AD* by *DC* equals the square on *DB.*

Therefore *if a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.*

Q.E.D.

Guide

This proposition is used in the next one.

Next proposition: III.37

Previous: III.35

Book III introduction

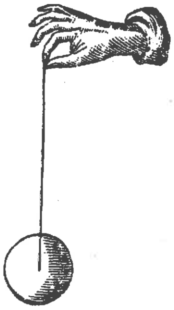
Select from Book III

Select book

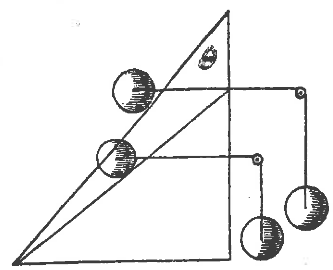
Select topic

Some Pertinent Principles from Statics

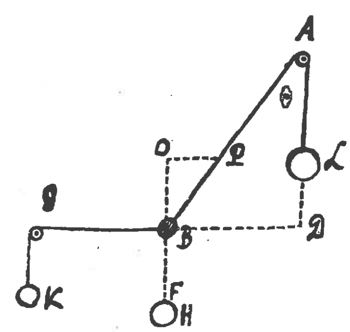
The tension in a vertically suspended string produced by a body hanging from it varies as the density and volume of the body and the strength of the tendency bodies at the location in question have to descend.



The tension in a string retaining two bodies in static equilibrium, one vertically and the other on an inclined plane, varies as the cosine of the angle of the plane θ and the tension in a vertically suspended string that would be produced were the body on the inclined plane hanging from it.



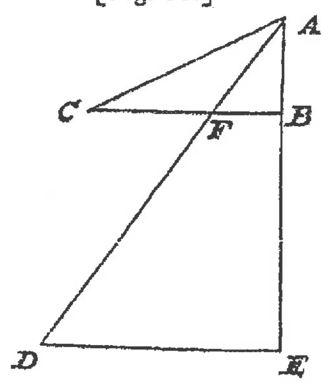
The tension in a string BK required to maintain a body in equilibrium at an angle θ varies as the tangent of that angle and the tension in string BH when that body is hanging vertically at its end.



HUYGENS'S THREE WAYS OF TESTING HIS THEORY OF CENTRIFUGAL FORCE

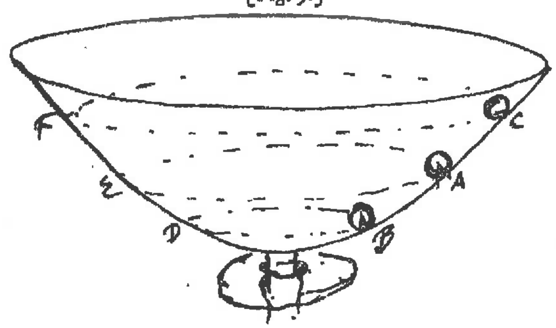
Periods of conical pendulums with strings of different lengths vary as the square root of their heights AB, AE

[Fig. 18.]



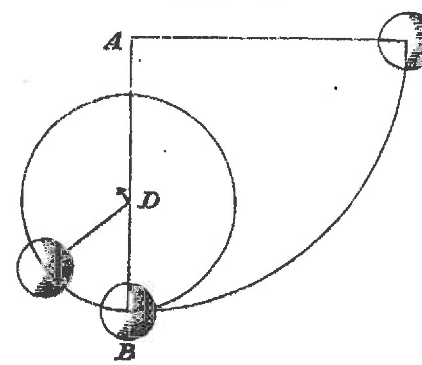
There is one rotational speed of a paraboloid at which a loose ball remains in equilibrium regardless of where on the surface it is placed.

[Fig. 9.]



A 90 deg arc circular pendulum will ascend with a taut string and complete a full circle if it is intercepted at DB = 2/5AB

[Fig. 24.]



HUYGENS'S CONSTANT-HEIGHT CONICAL PENDULUM MEASUREMENT OF DISTANCE OF FALL IN 1^{ST} SECOND (1659)

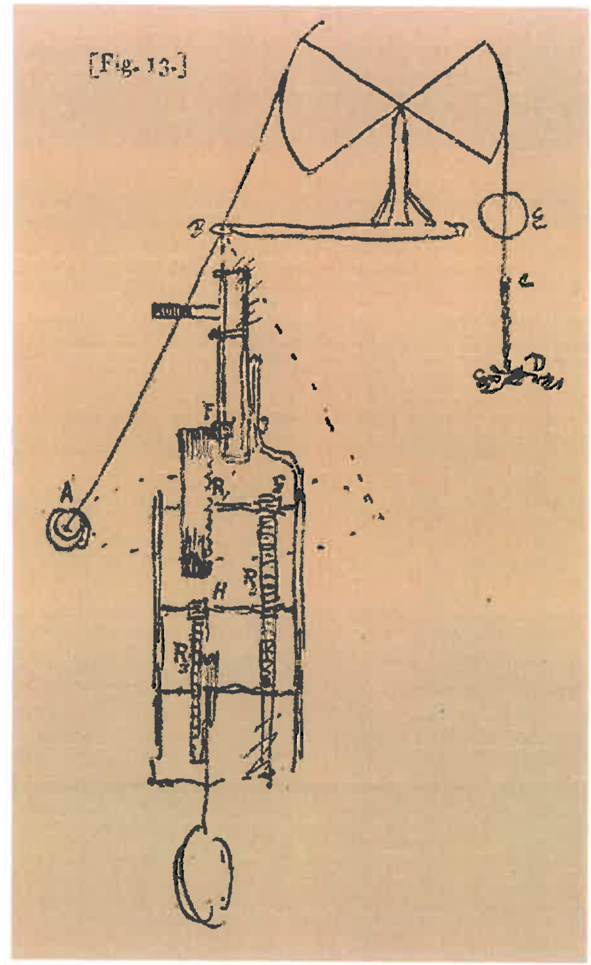
Distance of fall in 1^{st} second

$= 2\pi^2 \cdot$height of pendulum$/P^2$

Height of a three-quarter sec constant-height conical-pendulum clock: 5 inches 1.9 lines

Distance of fall in 1^{st} second:

15 Paris feet 1.1 inches



HUYGENS'S PARABOLOIDAL CONICAL PENDULUM CLOCK MEASUREMENT OF DISTANCE OF FALL IN 1ST SECOND (1673)

Distance of fall in 1st second

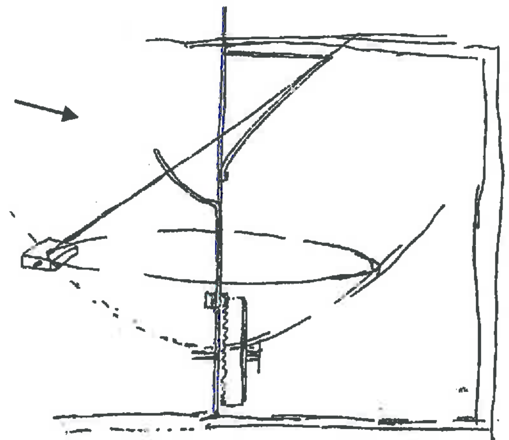
$= \pi^2 \cdot \text{latus rectum}/P^2$

Latus rectum of a one-half sec paraboloidal-conical-pendulum clock: 4 inches 7.1 lines

Distance of fall in 1st second:

15 Paris feet 1.1 inches

Original sketch



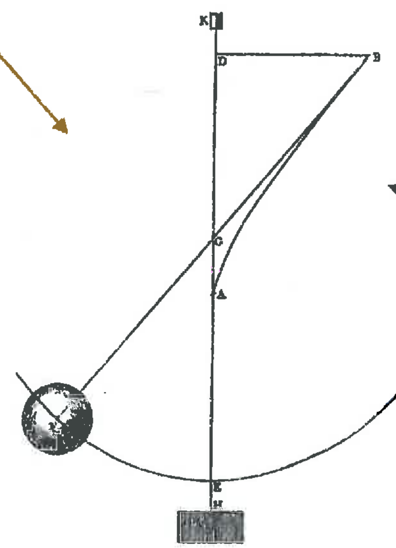


Diagram in *Horologium Osclatorium*

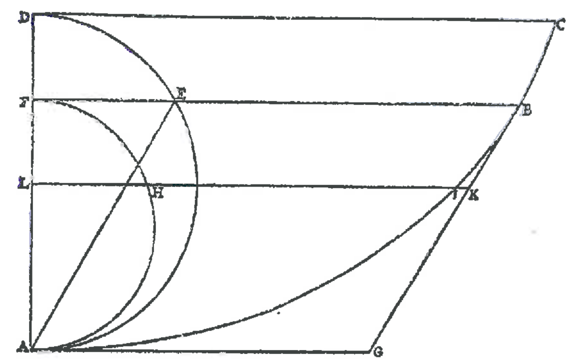
HUYGENS'S THEORY OF THE PENDULUM

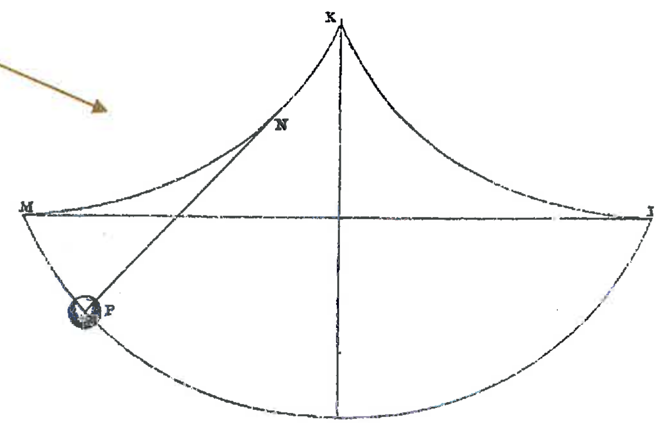
From four Galilean principles, using height as proxy for v^2, deduced:

1. Cycloid is the isochronal path.
2. Cycloidal cheeks make the pendulum path same cycloid.
3. Law of cycloidal pendulum:

   $T = \pi\sqrt{(\text{length}/2\cdot\text{fall in } 1^{st}\text{ sec})}$

4. Same law holds for small-arc circular pendulum.
5. Bound on small-arc: ≤ 4.5 deg





Arc **MP** = Arc **KN**

QUESTION-ANSWERING EXPERIMENTS

IF

Galilean principles of pathwise independence, return to height, and no effect of weight

THEN

Uniform acceleration

IF AND ONLY IF

The cycloidal and small-arc-circular pendulum measures of distance of fall in 1st second

STABLE

IF

Force $\propto$ weight·(distance of departure from uniform straight line motion)/δt^2

THEN

Uniform acceleration

IF AND ONLY IF

The conical and paraboloidal pendulum measures of distance of fall in 1st second

STABLE

CONVERGENT

HYPOTHESES

I

If there were no gravity, and if the air did not impede the motion of bodies, then any body will continue its given motion with uniform velocity in a straight line.

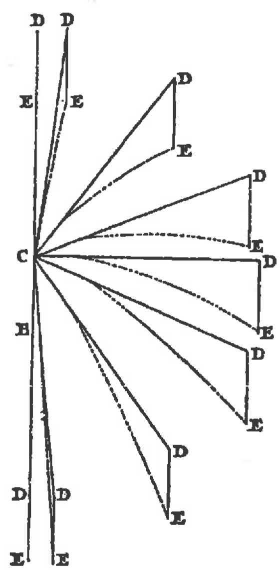
II

By the action of gravity, whatever its sources,[1] *it happens that bodies are moved by a motion composed both of a uniform motion in one direction or another and of a motion downward due to gravity.*

III

These two motions can be considered separately, with neither being impeded by the other.

LET C be a heavy body which, starting from rest, crosses the distance CB by the force of gravity in a certain time F [Fig. 7]. And let the same body be imagined to undergo another motion by which, assuming that gravity does not exist, it crosses the straight line CD with a uniform motion in the same time F. When the force of gravity is added, the body will not move from C to D in the time F



[Fig. 7]

but rather to some point E vertically below D such that the distance DE equals the distance CB. And thus the uniform motion and the motion due to gravity each make a contribution, and neither impedes the other. In what follows later we will define the line in which the body moves with this composite motion when the uniform motion is neither straight up or down but in an oblique direction. But when the uniform motion CD occurs downward on the perpendicular, it is obvious that the line CD is increased by the straight line DE when the motion due to gravity is added. Likewise, when the uniform motion CD is directed upward, CD is decreased by the straight line DE, so that, for example, after the time F the body will always be found at the point E. Thus, if we consider the two motions separately in each case, as we said, and if we recognize that neither motion is in any way impeded by the other, then from this we can discover the cause and the laws of acceleration of heavy falling bodies. And first we will show the following two things.

PROPOSITION I

In equal times equal amounts of velocity are added to a falling body, and in equal times the distances crossed by a body falling from rest are successively increased by an equal amount.

[Fig. 8]

Let there be a heavy body at rest at A [Fig. 8]. In the first unit of time it falls through the distance AB; and when it has arrived at B, it has acquired a velocity by which it next could cross the distance BD with a uniform velocity in the second unit of time. But we know that in the second unit of time it will cross a distance greater than BD because it would travel the distance BD only if all the action of gravity had ceased at B. Actually it moves with a motion composed of the uniform motion by which it would have crossed the distance BD and of a motion characteristic of falling bodies by which it necessarily falls through a distance equal to AB. Hence by adding DE, which is equal to AB, to BD, we know that in the second unit of time the body will arrive at E.

But if we inquire what velocity the body has at E at the end of the second unit of time, we find that this ought to be double the velocity which it had at B at the end of the first unit of time. For we said that it is moved by a motion composed of a uniform motion equal to the velocity acquired at B and of a motion due to gravity, which clearly is the same in the second unit of time as in the first. Hence a velocity ought to be added to the falling body in the second unit of time which is equal to the velocity added in the first unit of time. Thus, since it conserves the whole velocity acquired at the end of the first unit of time, it is clear that at the end of the second unit of time it has twice, or double, the velocity which it acquired at the end of the first unit of time.

Now if, after having arrived at E, the body were to be moved with a uniform velocity equal to what it has acquired at E, it is clear that in a third unit of time equal to each of the first two it would cross the distance EF, which is double the distance BD. For we said that the latter was crossed with half this velocity by a uniform motion in an equal time. But by adding again the action of gravity, in the third unit of time the body will cross the distance EF and also the distance FG, which is equal to AB or to DE. And thus at the end of the third unit of time the body will be found at G. It will have here a velocity which is triple that which it had at B at the end of the first unit of time. For in addition to the velocity acquired at E, which we said was double that acquired at B, in the third unit of time of the fall a velocity is added which again is equal to the velocity at the end of the first unit of time. Hence at the end of the third unit of time both velocities add up to triple the velocity found at the end of the first unit of time.

In the same way it can be shown that in the fourth unit of time the body ought to cross both the distance GH, which is triple BD, and the distance HK, which is equal to AB; and the velocity at K, at the end of the fourth unit of time, will be quadruple what it was at B at the end of the first unit of time. Therefore, it is clear that whatever distances we take to be crossed successively in equal times, these distances will each increase by an amount equal to BD, and simultaneously the velocities will also be increased equally in equal times.

From this it will not be difficult to prove the following proposition which Galileo asked that we accept as in a sense being self-evident.[4] For the demonstration which he tried to give later and which appears in the later edition of his works does not seem to me to be too strong. The proposition is the following.

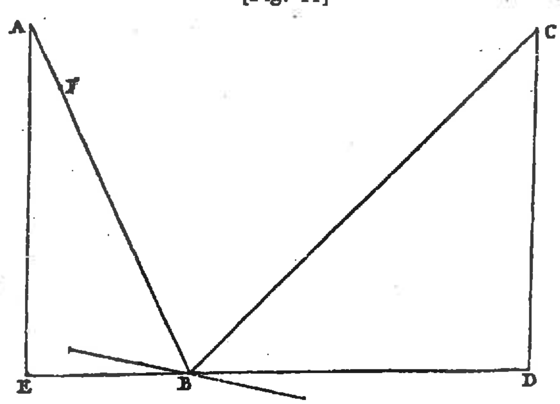
PROPOSITION VI

The velocities acquired by bodies falling through variably inclined planes are equal if the elevations of the planes are equal.

The elevation of the plane will be called its height on the perpendicular.

Let AB and CB [Fig. 11] be sections of inclined planes extended to the horizontal plane, and let their heights AE and CD be equal. Then let a body fall from A through the plane AB, and a body fall from C through the plane CB. Now I say that in each case the same degree of velocity will be acquired at the point B.

[Fig. 11]



For if we were to assume that the body falling through CB were to acquire less velocity than the body falling through AB, it would follow that the body falling through CB would acquire exactly the same velocity as a body falling only through FB, where FB is less than AB. But the body falling through CB acquires a velocity by which it could ascend again through the whole of BC [Proposition 4]. Therefore, if the body falling

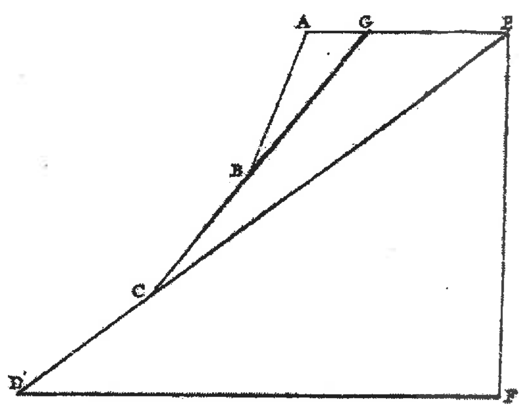
4. Galileo, *Discourses Concerning Two New Sciences.* Third Day. (*Le Opere di Galileo Galilei* 8:205.) ". . . the Author requires and takes as true one single assumption; that is, [Postulate] I assume that the degrees of speed acquired by the same moveable over different inclinations of planes are equal whenever the heights of those planes are equal." [S. Drake's translation, 162.]

PROPOSITION VIII

If from the same height a body descends by a continuous motion through any number of contiguous planes having any inclinations whatsoever, it will always acquire at the end the same velocity; namely, a velocity equal to that which would be acquired by falling perpendicularly from the same height.

Let AB, BC, and CD be contiguous planes [Fig. 13] whose terminus A has a height above the horizontal line DF, drawn through the lower terminus D, equal to the perpendicular EF. And let a body descend through these planes from A to D. Now I say that at D it will have the same velocity which it would have at F by falling from E.

[Fig. 13]

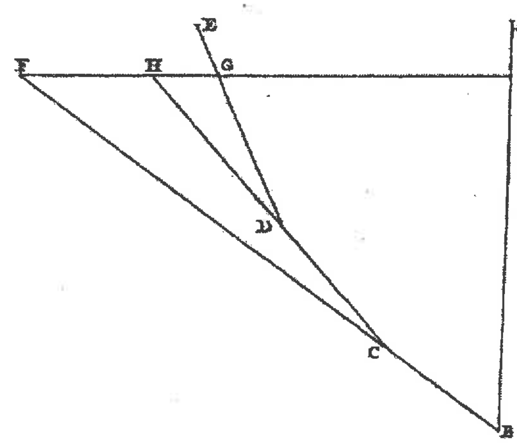


PROPOSITION IX

If, after falling, a body converts its motion upwards, it will rise to the same height from which it came, no matter how many contiguous plane surfaces it may have crossed, and no matter what their inclinations are.

Let a body fall from the height AB [Fig. 14]. From the point B let the planes BC, CD, and DE be inclined upwards such that their extremity E has the same height as the point A. Now I say that if a body, after falling through AB, converts its motion so that it continues to be moved through these inclined planes, it will rise up to the point E.

[Fig. 14]

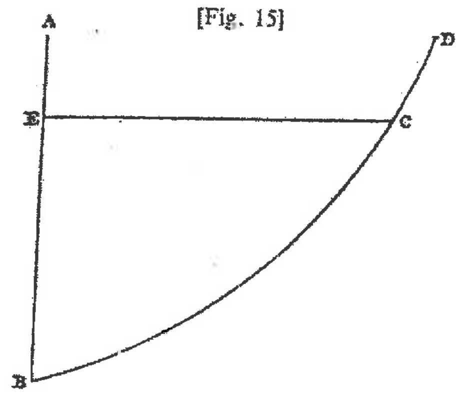


PROPOSITION X

If a body falls perpendicularly or through any surface, and if it later moves upwards by the acquired impetus through any other surface, then it will always have the same velocity at points of equal height in its descent and ascent.

Let a body fall from the height AB [Fig. 15] and then continue its motion through the surface BCD, in which the point C has the same height as the point E in AB. Now I say that the same velocity is present in the body at C as was present in it at E.

[Fig. 15]

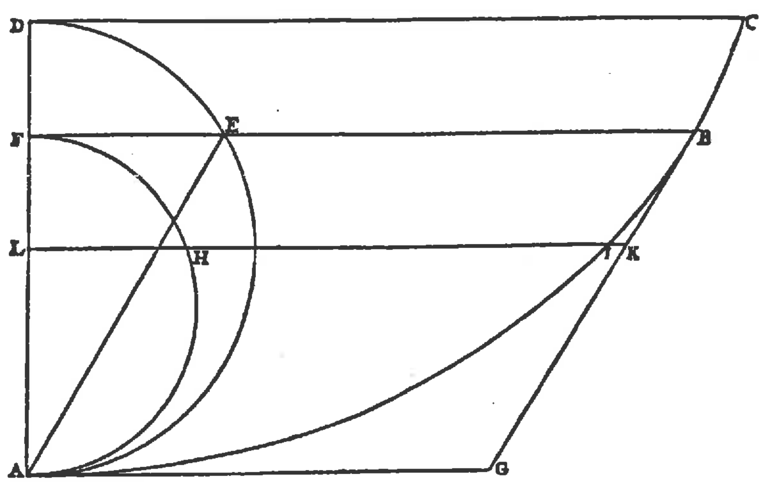


PROPOSITION XXV

On a cycloid whose axis is erected on the perpendicular and whose vertex is located at the bottom, the times of descent, in which a body arrives at the lowest point at the vertex after having departed from any point on the cycloid, are equal to each other; and these times are related to the time of a perpendicular fall through the whole axis of the cycloid with the same ratio by which the semicircumference of a circle is related to its diameter.

Let ABC [Fig. 35] be a cycloid whose vertex A is located at the bottom and whose axis AD is erected on the perpendicular. Select any point on the cycloid, for example B, and let a body descend by its natural impetus through the arc BA, or through a surface so curved. Now I say that the time of this descent is related to the time of a fall through the axis DA as the semicircumference of a circle is related to its diameter. When this has been demonstrated, it will also be established that the times of descent through all arcs of the cycloid terminating at A are equal to each other.

[Fig. 35]

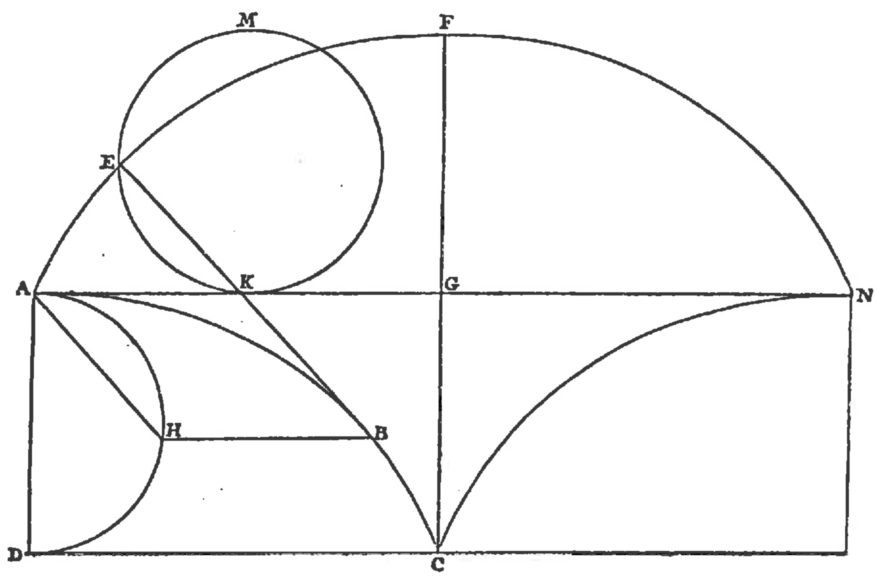


PROPOSITION V

If a straight line is tangent to a cycloid at its apex, and if on that line as a base another cycloid similar and equal to the first is constructed starting from the point of the apex just mentioned, then any straight line which is tangent to the lower cycloid will meet the arc of the higher cycloid at right angles.

Let the straight line AG be a tangent to the cycloid ABC at the apex A [Fig. 42]. On AG as a base construct another similar cycloid AEF whose apex is F. Moreover let the line BK be a tangent to the cycloid ABC at B. Now I say that the extension of BK will meet the cycloid AEF at right angles.

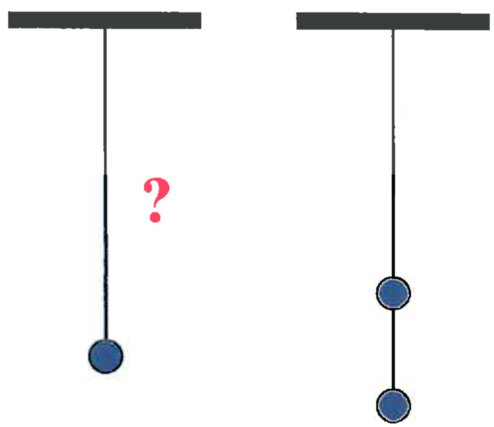
[Fig. 42]



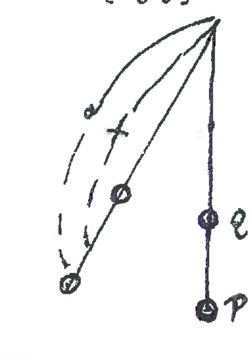
Let the generative circle AHD be drawn around AD, which is the axis of the cycloid ABC. Let BH, which is parallel to the base, meet the circle at H, and draw HA. Since BK is tangent to the cycloid at B, it follows that it is parallel to HA [Proposition 15, Part II]. Thus AHBK is a parallelogram, and so AK is equal to HB, i.e., to the arc AH [Proposition 14, Part II]. Next construct the circle KM equal to the generative circle AHD, making it tangent to the base AG at K, and extend BK to the point E. Now since AH is parallel to BKE, and since the angle EKA thus equals KAH, it is clear that the extension of BK cuts an arc from the circle KM equal to the arc which AH cuts from the circle AHD. Thus the arc KE is equal to the arc AH, that is, to the line HB and to the line KA. From this it follows, according to the properties of a cycloid, that since the generative circle touches the base at K, the point describing the cycloid would be at E. Thus the line KE meets the cycloid at E at right angles [Proposition 15, Part II]. But KE is the extension of BK. Therefore it is clear that BK when extended meets the cycloid at right angles. Q.E.D.

WHAT IS THE LENGTH OF A PENDULUM?

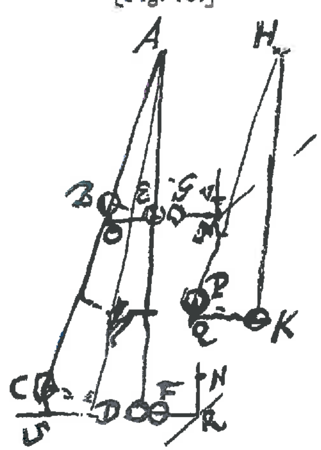
Huygens's Solution for the Center of Oscillation



[Fig. 3.]



[Fig. 16.]



***Generalized Galilean Principles*:**

1. "If any number of weights begin to move by the force of their own gravity, their center of gravity cannot rise higher than its location at the beginning of the motion."

2. "Abstracting from the air and every other impediment, the center of gravity of a pendulum crosses through equal arcs in descending and ascending."

Length of a one-second pendulum with a 2 inch diameter spherical bob is 0.1 lines longer than the distance to the center of the bob.

Center of Oscillation

Prop. III. **If any magnitudes all descend or ascend, albeit through unequal intervals, the heights of descent or ascent of each, multiplied by the magnitude of itself, yield a sum of products equal to that which results from the multiplication of the height of descent or ascent of the center of gravity of all the magnitudes times all the magnitudes.**

Prop. IV. **Assume that a pendulum is composed of many weights, and beginning from rest, has completed any part of its whole oscillation. Imagine next that the common bond between the weights has been broken and that each weight converts its acquired velocity upwards and rises as high as it can. Granting all this, the common center of gravity will return to the same height which it had before the oscillation.**

***Solution for the length of a compound pendulum*:**

$$\ell = \frac{\sum m_i \ell_i^2}{\sum m_i \ell_i} = \frac{\sum m_i \ell_i^2}{\ell_{cg} \sum m_i} \quad \text{or} \quad = \frac{\int y^2 dm}{\int y dm}$$

***Leibniz (1686)*:** **Forces are proportional, jointly, to bodies (of the same specific gravity or solidity) and to the heights which produce their velocity or from which their velocities can be acquired. More generally, since no velocities may actually be produced, the forces are proportional to the heights which might be produced by these velocities. They are not generally proportional to their own velocities.... Many errors have arisen from this latter view.... I believe this error is also the reason why a number of scholars have recently questioned Huygens's law for the center of oscillation of a pendulum, which is completely true.**

Prop. XXIV. **It is not possible to determine the center of oscillation for pendula suspended between cycloids, and how to overcome the difficulty which this causes.**

If one very closely compares what we have demonstrated above concerning pendula suspended between cycloids with our discussion of the center of oscillation, he will see that these oscillations fall short of the perfect equality which we would prefer. First he will have doubts, in determining the generating circle of a cycloid, as to whether the length of the pendulum should be measured from the point of suspension to the center of gravity of the attached lead weight or to the center of oscillation.... If we say that that length should be measured to the center of oscillation, then it will not be clear how what was proven about the center of oscillation applies to a pendulum which is continually changing its length, as is the case for a pendulum which moves between cycloids. For it would seem that its center of oscillation changes for each different length....

However, if we wish to escape these problems completely, we can succeed if we make the sphere or lentil of the pendulum move around its own horizontal axis. This is done by inserting both ends of that axis into the bottom of the rod of the pendulum; the rod having been split in half for the purpose. For in this way the nature of motion is such that the sphere of the pendulum will maintain perpetually the same position in respect to the horizontal plane; and any point in it, as well as its center, will cross the same cycloids. Hence a consideration of the centers of oscillation is no longer relevant. Such a pendulum will maintain an equality of times which is no less perfect than if all of its weight were contained in one point.

Questions Answered by Huygens: 1652-1673

1. What is the distance of vertical fall in the first second in the absence of a resisting medium -- i.e. the proportionality constant in Galileo's "law" of free-fall?

2. What rules of impact for perfectly "hard" spheres in head-on collision, in contrast to those proposed by Descartes, agree with experience?

3. Is Descartes' quantity of motion conserved in head-on impact of perfectly "hard" spheres, and, if not, what quantity is (or quantities are) conserved?

4. What is the strength of the tendency (*conatus*) to recede from the center in uniform circular motion?

5. What is the tension in the string retaining a body moving in uniform circular motion?

6. What is the "law" fully characterizing the relationship between the dimensions of conical pendulums and their periods?

7. Where must a 90-degree circular-arc pendulum be intercepted for its bob to reach the vertical with its string remaining taut in ascent?

8. From what principles can Galileo's claim of pathwise-independence of speed acquired in the absence of a resisting medium be derived, and does it hold regardless of the trajectory of descent?

9. Given that the circular arc is not the answer, what is the isochronous path in descent, assuming uniform gravity acting along parallel lines?

10. How can an isochronous pendulum be constructed -- i.e., one that maintains the same time to complete a full arc regardless of the arc-length of descent?

11. With gravity as in (9), what is the "law" fully characterizing the relationship between the dimensions of simple isochronous pendulums and their periods?

12. Where is the "center of oscillation" of a circular-arc pendulum with multiple small bobs or a physically large bob, and how can the solution for this center be used to tune pendulum clocks?

Beyond Galileo

- **The range of topics covered by Huygens under largely the same basic hypotheses as Galileo (+ Torricelli and Cartesian inertia) reaches far beyond the theory in *Two New Sciences*: the *Galilean-Huygensian theory* of motion under uniform (parallel) acceleration.**

- **Huygens introduced multiple theory-mediated means for measuring the fundamental quantities – i.e. the constants of proportionality – of that theory, opening the way to a new form of evidence for it from agreeing measurements that presuppose different hypotheses.**

- **In extending the theory beyond its initial idealizations Huygens opened the way to another new form of evidence by showing that the initial theory is a first approximation that can be extended to cover deviations from it without requiring new basic hypotheses.**

***Theoretical physics*: theoretical solutions to new problems, proceeding as much as possible from principles – or at least direct generalizations of them – that yielded empirically supported solutions to previous problems**

The Development of "Newtonian" Mechanics