b. Only a line of apsides through the true sun has its end points maximally near and far from the true sun, and a line of apsides through the mean sun does not have its endpoints thus maximally near and far from it

c. Confirmation of the location of the aphelion, with aphelial distance = 166510, perihelial distance = 138173, eccentricity near 14169 (0.093 vs earlier 0.09265) for earth-sun orbit radius = 100000

7. Finally, notice that the combination of using true sun as a reference point and the area rule has eliminated almost all the error from a classic eccentric circular model for Mars! (see table)

a. Why then is Kepler more famous for the ellipse than for these two steps?

b. Suspect the answer is twofold:

(1) The shift from using the circular geometries of the 2000 years of previous astronomy

(2) No one worked through *Astronomia Nova* sufficiently to realize how little work the ellipse was doing for Mars

H. "Phase 5": Alternative Oval Trajectories

1. One might naturally expect Kepler at this point to adopt the classical oval from geometry, the ellipse, and see how it does in conjunction with the area rule in yielding heliocentric longitudes

a. But not what he does, undoubtedly because saw no physical reason why orbit might be elliptical

b. Not engaged in just finding a geometrically familiar trajectory that agrees within observational limits, but wants one with some physical basis

2. Thus tries same sort of epicyclet model as before, but now with uniform circular motion on epicycle

a. This together with area rule yields a slightly egg-shaped oval, with bulge end at perihelion

b. Uniform motion on epicycle less physically objectionable than former non-uniform motion

3. Difficulties in carrying through the area calculations led him to approximate this oval by an ellipse (his "auxiliary ellipse", as shown in the figure from Gingerich in Appendix)

a. Result: good agreement again in apsides and quadrants, but error in octants -- errors essentially the reverse of before

b. -8 min for first octant, +7 1/2 min for third (Fig 9) -- i.e. planet moving too slowly in apsides, too fast in quadrants

4. Upshot: auxiliary ellipse yields excessive correction versus circle, roughly twice the amount of correction needed; proceeds to check, for several chapters, whether approximations, including auxiliary ellipse, might be responsible for the excessive discrepancies

5. The auxiliary ellipse an idealization, but one not prompted by idea that nature will conform with mathematics; rather, purely to ease computation, which is recognized throughout to involve uncertainty owing to observational inaccuracies, and hence is at best only approximate to begin with

I. "Phase 6": The Elliptical Trajectory

1. Working under the assumption of the area rule, the discrepancies at the octants between the circle of "Phase 4" and the "auxiliary ellipse" of "Phase 5" are opposite, and almost exactly equal

a. Indeed, equal to within bounds of uncertainty from multiple sources

b. But then nothing empirical standing in the way of simply saying that they are equal and opposite

c. I.e. an idealization of the data, but one within bounds of uncertainty

2. Once this move is made, then orbit must be precisely midway between the circle of "Phase 4" and the "auxiliary ellipse" of "Phase 5"

a. That is, it must be an ellipse!, ingressing by 429 out of 100,000 parts (the value I gave above)

b. Notice here how the combination of the area rule and Tycho's observations is entailing that the trajectory is an ellipse, at least to very high approximation

> In effect the area law controls the shape of the orbit. The areas swept out around the sun are assumed to be proportional to the times. Variously shaped orbits distribute the total area of the orbit in different ways; only one shape of orbit will get the planet to the right place at the right time. On the assumption of the area law the right orbit can differ only negligibly from the intermediate ellipse. (Wilson, p. 102)

3. Still using bisected eccentricity in a circular earth-sun orbit, predicted observed positions of Mars from the Earth for 28 of Tycho's observations bracketing seven oppositions from 1583 to 1595

a. Elliptical theory of Mars gives heliocentric longitude and distance of Mars from sun versus time; bisected circle gives earth-sun distance versus time: two sides and angle determine triangle

b. As Table (p. 341) displayed in the Appendix shows, average discrepancy less than 3 min and maximum less than 6 min

c. 20 of the 28 points within 3 min, 15 within 2 min

4. Of the four worst residual discrepancies in this table, three are primarily from bad observations, according to Gingerich's computer program, as used by Harper and Smith:

a. 30 Dec 1582: correct locus 16.4.14 and discrepancy 2 min 6 sec

b. 13 May 1591: correct locus 2.17.53 and discrepancy 2 min 17 sec

c. 10 Jun 1591: correct locus 26.0.47 and discrepancy 2 min 50 sec

d. 28 Jun 1591: correct locus 21.9.39 and discrepancy 5 min 18 sec

5. This is more than an order of magnitude improvement on all prior accounts of the orbit; indeed, more than a factor of 20 improvement

a. The first real advance in modeling planetary motion since Ptolemy -- 1450 years earlier

b. A totally new standard set by this table, replacing the one that had held sway since Ptolemy

c. Any alternative theory of Mars was going to have to do at least this well versus Tycho's data

J. The Elliptical Trajectory and Diametral Distances

1. Kepler now knew that an ellipse midway between his circle and his auxiliary ellipse would yield heliocentric (and geocentric) longitudes within more or less his bounds of uncertainty

a. But so too, undoubtedly with some other trajectories approximating it

b. Moreover, the reasoning to the ellipse was predicated on the area rule, and it had been introduced as merely an approximation to a preferred rule, giving added reason to question the ellipse

2. The comparison between the circle and the auxiliary ellipse, both of which were within the bounds of uncertainty at apsides and quadrants, indicated that the 100000 circular radius of the orbit at the quadrants needed to be decreased from 858 units to half that value, 429
    a. He says he then stumbled on the fact that the secant of the angle (in the circle), CQS, = 1.00429
    b. That is, get the desired distance CM at the quadrants by setting SM to 100000 there
3. Now generalizes this: the right way to measure the required distances from the Sun to Mars is by taking the projection on the diameter, TP, of the distances to points on the circle
    a. I.e. the SM distances everywhere are given by the rule, $r(1+e\cos x)$, where $r$ is radius of circle and $x$ called the "eccentric anomaly" -- see figure from Wilson in Appendix
    b. Kepler calls these the diametral distances: the "diametral distance" rule
4. Checks this rule against the triangulated distances he had derived from Tycho's observations before
    a. Table (in Appendix) shows that rule gives correct distances within the bounds of uncertainty
    b. Notice that this rule and its confirmation do not presuppose either the area rule or the ellipse; the evidence presupposes only the heliocentric longitude curve-fit of the vicarious theory and Kepler's earth-sun theory with its bisected eccentricity
5. Now asks how the distances in question should be laid off from S, that is, as radius vectors extending from S, under the requirement that the resulting heliocentric longitudes vs. time be correct
    a. First try, consistent with the 429 shortening at the quadrant: intersection between radius vector of appropriate length and radius from center of circle for each value of the angle $x$
    b. Result: a "puff-cheeked" orbit not symmetric about the diameter at the quadrants, with calculated heliocentric longitudes (using the area rule) outside bounds of vicarious theory uncertainty
    c. Second try, consistent with the 429 shortening at the quadrant: intersection between radius vector of appropriate length and a perpendicular to the line of apsides
    d. Result: an exact ellipse, with the 429 shortening at the quadrant, and one for which the area rule holds exactly; by earlier comparison of discrepancies for circle and auxiliary ellipse, eliminates the discrepancies (vis-à-vis the vicarious theory) in the octants and everywhere else
6. On this construal of Kepler's reasoning, played off triangulated distances against heliocentric longitudes: any two of the three rules -- ellipse, area rule, and diametral distance rule -- if taken to hold exactly, has the vicarious theory entailing that the third holds as exactly as that theory holds!

K. "Phase 7": A Physical Explanation

1. Kepler now had a cinematic model of the Mars orbit accurate to within observational accuracy, give or take a little, but he still had no physical basis supporting the claim that the model gives the true motion, nor for that matter much of any basis for insisting that the true path is exactly an ellipse rather than something closely approximating it
    a. Still no physical reason why the orbit might be an ellipse instead of a circle
    b. Given that the area rule only approximates the inverse distance rule, no physical reason for it