A Study of Normality
By Cayley Graphs and Their Quotients

An honors thesis for the Department of Mathematics

Devin Ivy

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Chapter 0

Introduction and Conventions

This thesis grew out of a collaboration with Dr. Kim Ruane which began some months ago. Our goal was to prove geometrically that the alternating groups on ≥ 5 letters are all simple. Whereas algebraic proofs of this fact tend to be dry exercises in symbolic manipulation, we were motivated by finding a proof that caters to visual, geometric, and concrete intuition. The symmetric groups can be realized as symmetries of simplices, so the original thought was to study the alternating groups as they acted faithfully on these simplices. We hoped to “see” simplicity via geometric actions. In the process, I headed down a much more combinatorial path and ended up stumbling upon a more general result which characterizes normal subgroups as they live within Cayley graphs.

The first result in Theorem 1.4, which is very much the motivation of the thesis as a whole, links the property of a subset of a group being normal with the ability to color any Cayley graph for the parent group in a certain fashion. By showing that some Cayley graph for A_5 does not permit certain colorings, we were able to show visually why the group is simple. Finding these normal subgroups within Cayley graphs via colorings lead to some natural questions and concerns. Firstly, using the colorings we obtain an embedding of the normal subgroup in the parent group, but not any sort of general presentation for it. So at first the groups were not necessarily recognizable. Secondly, it was unclear how useful this new way of seeing normality would be in the context of infinite groups. And so began the foray into treating Cayley graphs as covering spaces.

Of course all graphs have free fundamental groups, so it is often the case that when graphs and covering space theory mix, facts about free groups result. Many of these proofs begin by covering a wedge of circles by some nicely-constructed graph, thereby realizing a subgroup of a free group as the fundamental group of one of its covers. But Cayley graphs provide a new level of structure which actually allows us to realize subgroups as deck transformation groups of covers rather than fundamental groups. This, I think, adds lots of flexibility and latitude in dealing with purely algebraic questions by graph coverings. Moreover, graph coverings are often convenient to deal with because they are so combinatorally grounded and easily described. While the first chapter of the thesis is solely combinatorial and elementary, the second half of the thesis takes advantage of these topological ideas to fully address the concern about presenting normal subgroups found by coloring. And we will end with a result about infinite groups requiring most of the theory from both chapters.

Preliminaries and Conventions

A Cayley graph is an edge-labeled digraph whose vertices can be thought of as elements of some finitely generated group G. If S is any generating set for G then the *Cayley graph of G with respect to S*, which we will denote by $\Gamma_{G,S}$, has a directed edge labeled s from initial vertex g to terminal vertex gs for all $g \in G$ and for all $s \in S$. Edges corresponding to generators that are involutions will often be depicted without direction. Some crucial properties of Cayley graphs are that they are connected, regular with respect to edge labels and direction, vertex-transitive, and the automorphism group of $\Gamma_{G,S}$ respecting edge labels and directions is isomorphic to G. Credit is due to [W] for the Cayley graph skeletons found in all the examples within this thesis.

We are concerned primarily with finitely generated groups, i.e., groups with finite generating sets. This bounds the vertex degrees within the Cayley graph for G, which is an effort to keep the structure of the graphs tame enough to work with. We will later find that these graphs are also much more reasonable topologically. However, the groups we work with may be infinite groups.

By convention we will distinguish group elements from corresponding vertices in a Cayley graph by writing $\bullet_g$ for the vertex associated with group element g. Additionally $\bullet_J$ may be used as the collection of vertices associated with elements of $J \subseteq G$. The vertices of certain types of quotient graphs may be shown as $\circ_H$ where H will be some coset of a larger group. The full collection of vertices in any graph Γ will be denoted $V(\Gamma)$ and the edgeset as $E(\Gamma)$. Lastly, $\mathrm{Aut}(\Gamma)$ will always refer to the automorphism group of Γ respecting any associated edge labels and directions unless otherwise noted.

Chapter 1

Coloring and the Graph Quotient

1.1 Coloring

Let $\Gamma_{G,S}$ be a Cayley graph of group G generated by S and $\mathcal{C}$ be a set of colors, not necessarily finite. Then we may assign to each vertex in $\Gamma_{G,S}$ a color in $\mathcal{C}$ via some surjection $c : V(\Gamma_{G,S}) \twoheadrightarrow \mathcal{C}$. This function c will denote a *coloring* of the Cayley graph where we may pay attention to the partitioning of $V(\Gamma_{G,S})$ into *color-equivalence classes* of monocolored subsets.

Colorings arise naturally from any choice of subgroup J. Recall that by the construction of $\Gamma_{G,S}$ its vertices are in one-to-one correspondence with elements of G. Let $J \leq G$ and define c_J, the *coset coloring of* $\Gamma_{G,S}$ *with respect to subgroup* J, so that each left coset of J in $\Gamma_{G,S}$ has a distinct color. Notice this will mean $c_J(\bullet_{g_1}) = c_J(\bullet_{g_2})$ if and only if there exists $g_0 \in G$ such that $g_1, g_2 \in g_0 J$. And relatedly, given $k \in \mathcal{C}$, $c_J^{-1}(k) = \bullet_{g_0 J}$ for some $g_0 \in G$. To rephrase what's happening, c_J assigns each left coset of J a distinct color-equivalence class.

Now we may characterize normal subgroups in terms of their coset colorings.

Lemma 1.1. Let $J \leq G$ and $c = c_J$. Then $J \trianglelefteq G$ if and only if given any color $k \in \mathcal{C}$, for each $s \in S \cup S^{-1}$ there is a $k_0 \in \mathcal{C}$ such that $c(\bullet_{vs}) = k_0$ for every $\bullet_v$ colored k.

Proof. Assume $J \trianglelefteq G$. Fix $k \in \mathcal{C}$ and $s \in S \cup S^{-1}$. Take any two $\bullet_{v_1}, \bullet_{v_2} \in c^{-1}(k)$. $\bullet_{v_1}, \bullet_{v_2}$ have the same color, so $\bullet_{v_1}, \bullet_{v_2} \in \bullet_{g_0 J}$ for some $g_0 \in G$. Choose k_0 to be the color of the left coset $\bullet_{g_0 sJ}$. Since J is normal we know $sJ = Js$, so the following is true,

$$k_0 = c(\bullet_{g_0 sJ}) = c(\bullet_{g_0 Js}) = c(\bullet_{v_1 s})$$

By the same reasoning $k_0 = c(\bullet_{v_2 s})$.

$\therefore\ c(\bullet_{vs}) = k_0$ for every $\bullet_v \in c^{-1}(k)$.

Conversely, assume given any color $k \in \mathcal{C}$, for each $s \in S \cup S^{-1}$ there is a $k_0 \in \mathcal{C}$ such that $c(\bullet_{vs}) = k_0$ for every $\bullet_v$ colored k. In particular, $\bullet_{Js}$ must consist of vertices all the same color since $\bullet_J = c^{-1}(c(\bullet_e))$. But that means every vertex in $\bullet_{Js}$ is in the same left coset, which must be $\bullet_{sJ}$ since $s \in sJ, Js$. So $\bullet_{Js} \subseteq \bullet_{sJ}$. Similarly $\bullet_{(sJ)s^{-1}}$ must consist of vertices of all the same color since $\bullet_{sJ}$ is a left coset of a particular color. Therefore all edges labeled s entering $\bullet_{sJ}$ originate from the same left coset, which we have already identified as $\bullet_J$. So $\bullet_{sJ} \subseteq \bullet_{Js}$. Then $\bullet_{sJ} = \bullet_{Js}$, i.e., $sJ = Js$ for every $s \in S$. Let $g \in G$, $g = s_1 s_2 ... s_n$ $s_i \in S$. Finally we see, $Jg = Js_1 s_2 ... s_n = s_1 s_2 ... s_n J = gJ$.

$\therefore J \trianglelefteq G$.

1.2 The Graph Quotient and the Quotient Group

1.2.1 The graph quotient

Consider a Cayley graph $\Gamma_{G,S}$ with a coloring given by c. The natural equivalence relation induced by c allows us to define a *graph quotient*. For each color in $\mathcal{C}$ there is a unique vertex in the graph quotient $\Gamma_{G,S}/c$, and for that reason vertices in the quotient will be referred to by their color. For $s \in S$, an edge labeled s exists leaving k_1 and entering k_2 in the quotient if and only if $c(\bullet_{vs}) = k_2$ for every $\bullet_v$ colored k_1 and $c(\bullet_{ws^{-1}}) = k_1$ for every $\bullet_w$ colored k_2. If the quotient graph contains all edge labels in S at every vertex then it is called *full*. Consider the following corollaries to Lemma 1.1.

Corollary 1.2 Let $J \leq G$. Then $J \trianglelefteq G$ if and only if $\Gamma_{G,S}/c_J$ is full.

Proof. This follows directly Lemma 1.1 since $\Gamma_{G,S}/c_J$ is full if and only if given any color $k \in \mathcal{C}$, for each $s \in S \cup S^{-1}$ there is a $k_0 \in \mathcal{C}$ such that $c(\bullet_{vs}) = k_0$ for every $\bullet_v$ colored k.

Corollary 1.3. Let c be a coloring of $\Gamma_{G,S}$ where some color-equivalence class in $V(\Gamma_{G,S})$ of color k is a left coset of J with $J \trianglelefteq G$. If $\Gamma_{G,S}/c$ is full then $c = c_J$.

Proof. Cayley graphs are vertex-transitive so without loss of generality we may assume $c(\bullet_J) = k$. Proceed by induction. Let $W_n \subset G$ be all elements of word-length n with respect to generators S. Take $g \in W_1$. $\bullet_{gJ} = \bullet_{Jg}$ is monocolored since $g \in S$ and $\Gamma_{G,S}/c$ is full. Fix n and assume $\bullet_{gJ}$ is monocolored for every $g \in W_n$. Let $h \in W_{n+1}$ and write h as $g's$ where $g' \in W_n, s \in S$. Then $\bullet_{g'J}$ is monocolored by our assumption. So $\bullet_{hJ} = \bullet_{g'sJ} = \bullet_{(g'J)s}$ must be monocolored since $\Gamma_{G,S}/c$ is full. Thus each left coset of J is monocolored in $V(\Gamma_{G,S})$. Since $\bullet_J$ comprises an entire color-equivalence class, in each induction step we receive left cosets that are not only monocolored, but also comprise the entirety of a color-equivalence class. So, each left coset of J in $V(\Gamma_{G,S})$ represents a distinct color-equivalence class.

$\therefore c = c_J$.

1.2.2 Sabidussi's theorem

Sabidussi's theorem was originally introduced as a necessary and sufficient condition for a graph to be a Cayley graph. It is phrased in such a way that it applies only to finite graphs, and correspondingly only to Cayley graphs of finite groups. However, over time Sabidussi's original statement and proof have been slightly tweaked so that it may apply to Cayley graphs for any group. Sabidussi's theorem will be crucial to our main result in Theorem 1.4, so it is stated here and phrased very similarly to his original statement.

Sabidussi's Theorem. [S] Let Γ be graph. Then there exists a group G and generating set S such that $\Gamma \cong \Gamma_{G,S}$ if and only if $\mathrm{Aut}(\Gamma)$ contains a subgroup G_0 which acts simply transitively on $V(\Gamma)$. In that case $G \cong G_0$.

It is important to note that Sabidussi's statement deals with unlabeled, undirected graphs. However, any label- and direction-respecting graph automorphism group lives inside the larger group of automorphisms which do not necessarily respect labels and edge directions. So if Γ_ℓ is the graph Γ with some edge directions and labels then $\mathrm{Aut}(\Gamma_\ell) \leq \mathrm{Aut}(\Gamma)$, and if G_0 is a subgroup of $\mathrm{Aut}(\Gamma_\ell)$ then it is necessarily a subgroup of $\mathrm{Aut}(\Gamma)$.

1.2.3 Relation between the graph quotient and quotient group

Theorem 1.4. Let c be a coloring of $\Gamma_{G,S}$ for which $\Gamma_{G,S}/c$ is full and $J \subseteq G$ such that $\bullet_J = c^{-1}(c(\bullet_e))$. Then $J \trianglelefteq G$ if and only if $\Gamma_{G,S}/c$ is a Cayley graph. Moreover, if $J \trianglelefteq G$ then $\Gamma_{G,S}/c$ is a Cayley graph of G/J.

Proof. Assume $J \trianglelefteq G$. Then $c = c_J$ by Corollary 1.3. The colors in $\mathcal{C}$ are in one-to-one correspondence with vertices in $V(\Gamma_{G,S}/c)$ so it will be convention to refer to the vertices by their color.
Let G/J act on $V(\Gamma_{G,S}/c)$ as such:

$$gJ(v) \mapsto c(gc^{-1}(v))$$

This is well-defined since,

$$\begin{aligned} c(gc^{-1}(v)) \quad &= c(g\bullet_{g_0J}) \text{ for some } g_0 \in G \\ &= c(\bullet_{gg_0J}) \text{ which is the unique color of a left coset of } J. \end{aligned}$$

Notice that g only serves as a coset representative for gJ, though any element in gJ will do. Furthermore this is indeed an action,

$$\begin{aligned} J(v) \quad &= c(ec^{-1}(v)) = c(c^{-1}(v)) = v \text{ and} \\ hJ(gJ(v)) \quad &= hJ(c(\bullet_{gg_0J})) = c(hc^{-1}(c(\bullet_{gg_0J}))) \\ &= c(h\bullet_{gg_0J}) = c(\bullet_{hgg_0J}) \\ &= c(hg\bullet_{g_0J}) = hgJ(v) \\ &= hJgJ(v) \end{aligned}$$

Now we will show that G/J acts simply transitively on $V(\Gamma_{G,S}/c)$. Let $\alpha, \beta \in V(\Gamma_{G,S}/c)$ with $c^{-1}(\alpha) = \bullet_{aJ}$ and $c^{-1}(\beta) = \bullet_{bJ}$. Consider the element $ba^{-1}J \in G/J$,

$$\begin{aligned} (ba^{-1}J)(\alpha) \quad &= c(ba^{-1}c^{-1}(\alpha)) = c(ba^{-1}\bullet_{aJ}) \\ &= c(\bullet_{ba^{-1}aJ}) = c(\bullet_{bJ}) \\ &= \beta \end{aligned}$$

So the action is indeed transitive. Now suppose two elements $xJ, yJ \in G/J$ take α to β,

$$\begin{aligned} \text{then} \quad & c(\bullet_{xaJ}) = c(\bullet_{yaJ}) \\ & xaJ = yaJ \text{ since each coset has a distinct color} \\ & xJa = yJa \text{ since } J \trianglelefteq G \\ \text{and finally} \quad & xJ = yJ \text{ by cancellation.} \end{aligned}$$

Thus the action is simply transitive. Now we will show that G/J acts by graph automorphisms. Once again, take $\alpha, \beta \in V(\Gamma_{G,S}/c)$ with $c^{-1}(\alpha) = \bullet_{aJ}$ and $c^{-1}(\beta) = \bullet_{bJ}$, and suppose there is an edge labeled s going from α to β. Then $c(\bullet_{aJs}) = \beta = c(\bullet_{bJ})$. Take any $gJ \in G/J$,

$$\begin{aligned} gJ(\alpha) \quad &= c(g\bullet_{aJ}) = c(\bullet_{gaJ}) \text{ and} \\ gJ(\beta) \quad &= c(\bullet_{bJ}) = c(g\bullet_{aJs}) = c(\bullet_{gaJs}) \end{aligned}$$

And since $e \in J$ we know in particular that,

$$\begin{aligned} gJ(\alpha) \quad &= c(\bullet_{gaJ}) = c(\bullet_{ga}) \text{ and} \\ gJ(\beta) \quad &= c(\bullet_{gaJs}) = c(\bullet_{gas}) \end{aligned}$$

$\bullet_{ga}$ is connected to $\bullet_{gas}$ by an edge labeled s in $\Gamma_{G,S}$. So, $c(\bullet_{ga})$ must be connected to $c(\bullet_{gas})$ by an edge labeled s in $\Gamma_{G,S}/c$. Finally we can see that an edge labeled s goes from $gJ(\alpha)$ to $gJ(\beta)$ in the graph quotient. Incidence is preserved, so G/J acts via graph automorphisms, i.e., $G/J \leq \text{Aut}(\Gamma_{G,S}/c)$. So, $\text{Aut}(\Gamma_{G,S}/c)$ has a subgroup G/J that acts simply transitively on $\text{V}(\Gamma_{G,S}/\text{c})$.

$\therefore$ by Sabidussi's theorem $\Gamma_{G,S}/c$ is a Cayley graph of G/J.

Conversely, assume $\Gamma_{G,S}/c$ is a Cayley graph. Define $\varphi : G \to \mathcal{C}$ so that $\varphi(g) = c(\bullet_g)$.

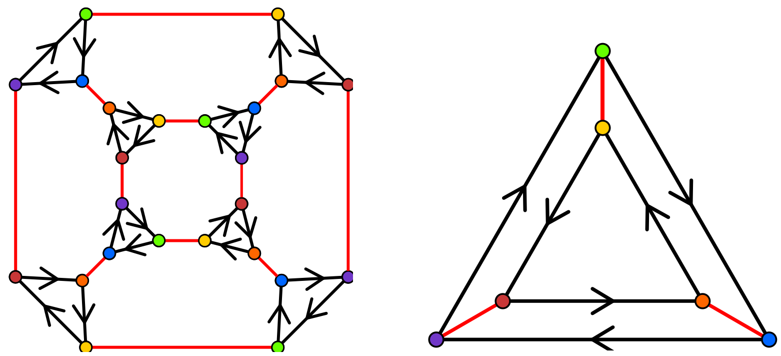
Any word from the generating set S, $s_1 s_2 ... s_l$, naturally defines a path in $\Gamma_{G,S}$ from $\bullet_e$ to $\bullet_{s_1 s_2 ... s_l}$. As the graph quotient is defined, and since $\Gamma_{G,S}/c$ is full, there is a corresponding path from $c(\bullet_e)$ to $c(\bullet_{s_1 s_2 ... s_l})$ acheived by following edges labeled $s_1, s_2, ..., s_l$ in sequence. Since $\Gamma_{G,S}/c$ is a Cayley graph, $\mathcal{C}$ is actually a group with operation given by composition of paths from $c(\bullet_e)$. Then following an edge labeled s in $\Gamma_{G,S}/c$ is right-multiplication (via path composition) by $c(\bullet_s) = \varphi(s)$ in the group $\mathcal{C}$. So we have,

$$\varphi(s_1 s_2 ... s_l) = c(\bullet_{s_1 s_2 ... s_l}) = c(\bullet_{s_1}) c(\bullet_{s_2}) ... c(\bullet_{s_l}) = \varphi(s_1)\varphi(s_2)...\varphi(s_l)$$

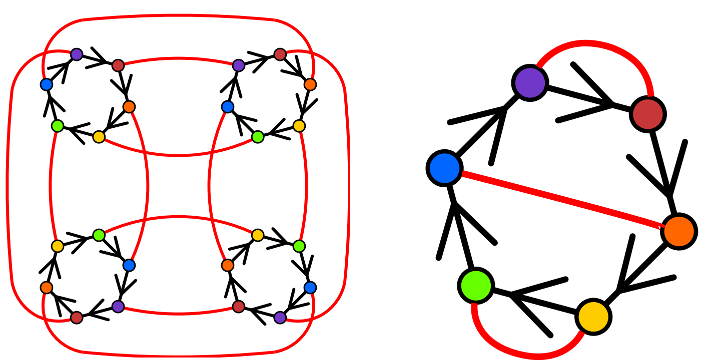
Thus φ is a homomorphism. Now we will show that it is surjective. Take any $k \in \mathcal{C}$. The coloring c is surjective, so there exists a $\bullet_g$ such that $c(\bullet_g) = k$. Then it is clear $\varphi(g) = k$. So φ is indeed surjective. Notice $\ker \varphi = J$ since the identity in G maps to the identity $c(\bullet_e)$ in $\mathcal{C}$ and $\bullet_J$ is the color-equivalence class containing $\bullet_e$.

$\therefore$ by the first isomorphism theorem $J \trianglelefteq G$. And furthermore $\mathcal{C} \cong G/J$ since φ is surjective, i.e., $\Gamma_{G,S}/c$ is a Cayley graph of G/J.

Example 1.5. Here on the left we have a 6-coloring of a Cayley graph for S_4 which results in a full quotient, seen on the right. We recognize the quotient as a Cayley graph of D_3, so the coloring corresponds to c_N for some subgroup N of order 4 normal in S_4. We will later return to this example with the goal of obtaining a presentation for N in order to describe which normal subgroup of S_4 we have identified.



Example 1.6. Here on the left we have a 6-coloring of a Cayley graph for $C_3 \rtimes D_4$ which results in a full quotient, seen on the right. However, the quotient clearly isn't a Cayley graph (notice the red involution is sometimes equal to the order-6 element and sometimes equal to its third power). Correspondingly, no color-equivalence class comprises a normal subgroup.



1.3 Coloring for Normal Subgroups

Here we propose some rules to follow when coloring the cosets of a supposed normal subgroup $J \trianglelefteq G$ inside a Cayley graph $\Gamma_{G,S}$. Because J is normal, we know from Corollary 1.3 that our coloring should be c_J. These rules result from Theorem 1.4, based upon the fact that moving along the labeled edges in $\Gamma_{G,S}/c_J$ is equivalent to multiplying by group elements in G/J since the graph quotient must be a Cayley graph. To be more specific about the generators represented by edges in the graph quotient, consider the following remark.

Remark 1.7. Let $J \trianglelefteq G$ and define $S_J = \{sJ \mid s \in S\}$. Then $\Gamma_{G,S}/c_J = \Gamma_{G/J,S_J}$.

And now for the rules.

Rule 1. The cycles of a generating element s in $\Gamma_{G,S}$ must be colored in a consistent pattern that respects the order of the element in G/J represented by following edges labeled s in the graph quotient. If the element is order n, then every cycle must have d colors where d divides n.

Rule 2. When a color is placed on a vertex v in $\Gamma_{G,S}$, if that color has also been placed on some other vertex w, then the colors of vertices neighboring w are duplicated around v, respecting edge labels and directions. This is a direct result of Lemma 1.1.

Rule 3. A result of Rule 2 is that left cosets of subgroups in $\Gamma_{G,S}$ must have disjoint or identical colorings. Each left coset is isomorphic in $\Gamma_{G,S}$ since the natural action on the left acts by group automorphisms. So if two left cosets contain a vertex of the same color, then the entire coloring of one left coset is duplicated on the other. We may use this fact to classify or *name left cosets* based upon their coloring, and recognize that two of the cosets have the same name if and only if they share a single common color.

Rule 4. Let us define a new type of graph which we will call a *GSH-graph*, which is defined with respect to a subgroup $H \leq G$ and Cayley graph $\Gamma_{G,S}$. The vertices in the GSH-graph are left cosets of H. An edge with label s runs from the vertex associated with aH to the vertex associated with bH if and only if $aH \neq bH$ and there exist $h_1, h_2 \in H$ such that an edge with label s runs from $\bullet_{ah_1}$ to $\bullet_{bh_2}$ in $\Gamma_{G,S}$. When a name is given to a coset aH, if the same name been given to some other coset bH, then the names of cosets neighboring bH are duplicated around aH in the GSH-graph. This follows from Rule 2 and Rule 3.

Rule 5. When looking at a planar GSH-graph and H is a cycle, each vertex can be given an orientation based upon the edge-direction of the cycle. Given some coset named **a**, the names of cosets neighboring any coset named **a** will always appear in order with respect to the orientation placed on the vertex for that coset. This again follows from Rule 2 and Rule 3.

And we will end with an important corollary to Theorem 1.4 related to normal subgroup colorings that leads into the next section.

Corollary 1.8. A group G is simple if and only if it permits exactly two colorings on $\Gamma_{G,S}$ so that the graph quotient is a Cayley graph. Either each vertex in $V(\Gamma_{G,S})$ has its own color or $V(\Gamma_{G,S})$ is monocolored.

Proof. Assume some group G is simple. Then its only normal subgroups are $\{e\}$ and G. Let c be a coloring of $\Gamma_{G,S}$ so that the graph quotient $\Gamma_{G,S}/c$ is a Cayley graph and let $\bullet_J = c^{-1}(c(e)) \subseteq V(\Gamma_{G,S})$. Then by Theorem 1.4 $J \trianglelefteq G$, meaning $J = G$ or $\{e\}$. $\bullet_J$ comprises a color-equivalence class and the graph quotient is full, so by Corollary 1.3 $c = c_J$. So, if $J = \{e\}$ then c assigns each vertex its own color-equivalence class. And if $J = G$ then c assigns all vertices to one color-equivalnce class.

$\therefore$ each vertex in $V(\Gamma_{G,S})$ has its own color or $V(\Gamma_{G,S})$ is monocolored.

Conversely, assume some group G permits exactly two colorings on $\Gamma_{G,S}$ so that the graph quotient is a Cayley graph. We know that $G \trianglelefteq G$ and $\{e\} \trianglelefteq G$, thus by Corollary 1.3 and Theorem 1.4 c_G and $c_{\{e\}}$ are the two distinct colorings that induce graph quotients of $\Gamma_{G,S}$ which are Cayley graphs. Suppose some other subgroup H were normal in G. Then c_H would be another such coloring, which is a contradiction. $\rightarrow\leftarrow$

$\therefore G$ is simple.

1.4 The Simplicity of the Alternating Groups

In this section we will begin by working with the alternating group A_5. To make useful the machinery built up to this point, we must first choose a Cayley graph of A_5 to work with. Take the generating set $S = \{(12345), (12)(34)\}$. The resulting Cayley graph $\Gamma_{A_5,S}$ is found in Figure 1 below. The red directed edges correspond to the generator (12345), and the black undirected edges correspond to the generator (12)(34), which is an involution.

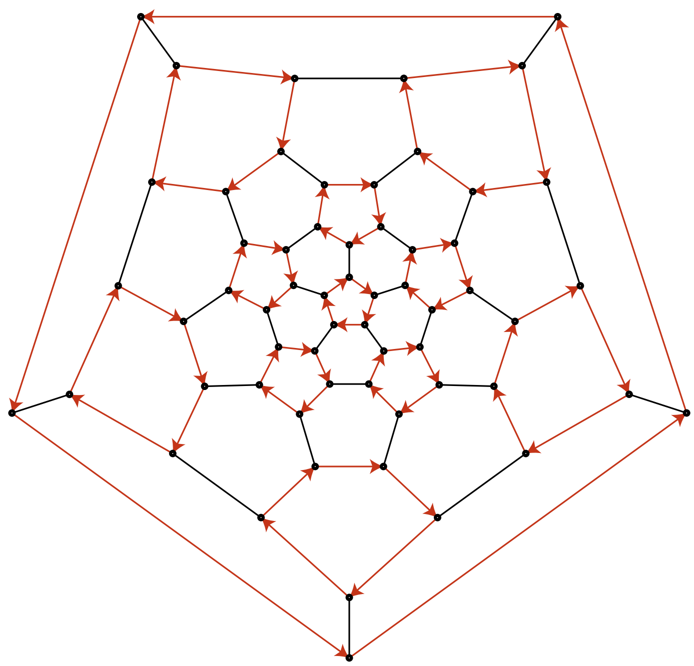


Figure 1: A Cayley graph of A_5 with generating set $S = \{(12345), (12)(34)\}$

The Cayley graph resembles the skeleton of a truncated icosahedron, each 5-cycle appearing where five triangular faces of an icosahedron would meet. So when we look at the GSH-graph for the subgroup generated by (12345), as in Figure 2, we indeed receive a graph of the skeleton of an icosahedron. It is a 5-regular graph on 12 vertices. The edges are undirected as they correspond to the involution (12)(34), but in accordance with the edge-directions in $\Gamma_{A_5,S}$ there is an implicit clockwise ordering of outgoing edges from any given vertex (as described in Rule 5, noting that the GSH-graph is planar).

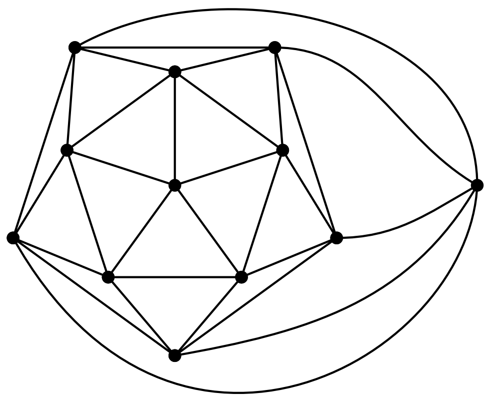


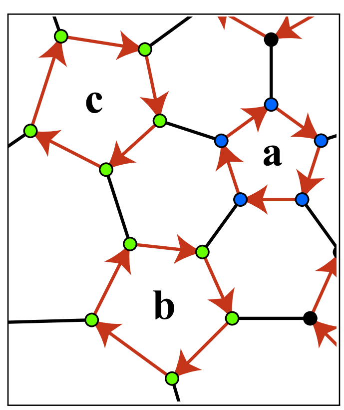
Figure 2: The GSH-graph of $\Gamma_{A_5,S}$ grouped by its generating 5-cycles, represented as vertices. Given a specific 5-cycle, there is an implicit clockwise ordering of neighboring 5-cycles.

Proposition 1.9. *A_5 is simple.*

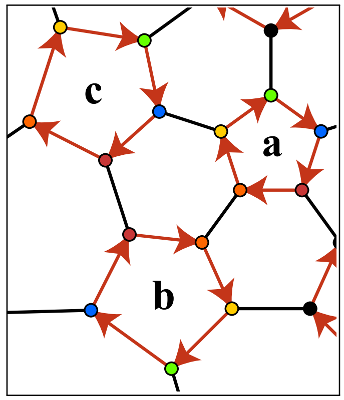
Proof. The generating 5-cycles in $\Gamma_{A_5,S}$ must be either 5-colored or monocolored by Rule 1. Similarly the generating involutions in $\Gamma_{A_5,S}$ must be either 2-colored or monocolored. By Rule 1, if a cycle given by a particular generating element is n-colored then all such cycles must also be n-colored. We are attempting to create a coloring c so that $\Gamma_{A_5,S}/c$ is a Cayley graph.

If both generators are monocolored then clearly $\Gamma_{A_5,S}$ becomes monocolored. This is one coloring that induces a quotient which is a Cayley graph.

If the 5-cycles are monocolored and the involutions are 2-colored then the involutions leading out of **a** go to **b** and **c**, which are adjacent via the involution yet have the same color. Thus the involution is forced to be monocolored between **b** and **c**. $\rightarrow\leftarrow$



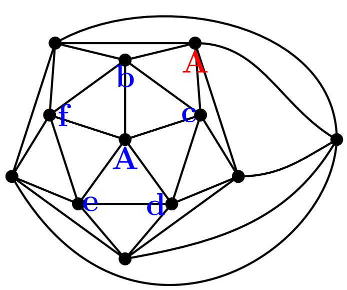
If the 5-cycles are 5-colored and the involutions are monocolored we obtain the picture below. Cycle **a** was 5-colored, the involution into **b** was colored, then Rule 2 was used to color **b**. Next, the involution from **b** into **c** was colored then Rule 2 was used again to color **c**. We find that the involution leading from **a** into **c** is forced to be 2-colored blue and yellow. →←

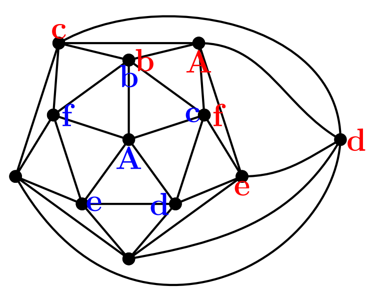


So, we attempt to color $\Gamma_{A_5,S}$ with 5-colored 5-cycles and 2-colored involutions. It will now be useful to use the GSH-graph in Figure 2 to talk about $\Gamma_{A_5,S}$ grouped by its 5-cycles. We will use named cosets as described in Rule 4. In this case the cosets are 5-cycles in $\Gamma_{A_5,S}$, or vertices in the GSH-graph. Notice in the GSH-graph each vertex has exactly five others distance 1 away, five distance 2 away, and exactly one "opposite" vertex distance 3 away. *Note, sometimes names will be seen in different colors, which is simply a device to distinguish multiple cosets with the same name.*

Case 1. Two 5-cycles with the same name are neighbors. Then the involution leading from one 5-cycle into the other must be colored with two distinct colors from that 5-cycle. But then an order 2 element must be equivalent some power an order 5 element in the supposed quotient $\Gamma_{A_5,S}/c$. Thus $\Gamma_{A_5,S}/c$ cannot be a Cayley graph for any group. →←

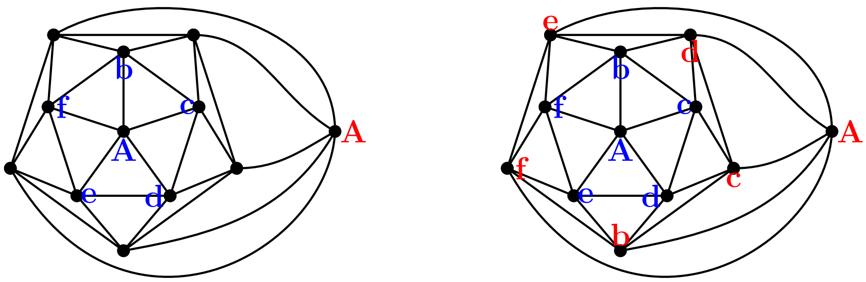
Case 2. Two 5-cycles with the same name, **A**, are distance 2 away. **A** is neighbored by cycles named **b**, **c**, **d**, **e**, and **f**, not all necessarily unique. Following Rule 5 the cycles are duplicated clockwise around **A** starting with the previously-placed **b**. We see that **c**=**f**, as one cycle shares both names. But **c** and **f** are found adjacent to each other. So two cycles with the same name are neighbors despite Case 1. →←





Case 3. Two 5-cycles with the same name, **A**, are distance 3 away. **A** is neighbored by cycles named **b**, **c**, **d**, **e**, and **f**. Following Rule 5 the cycles are filled in around **A**, starting

by giving the name **b** to the only cycle not distance 2 or 1 from **b**. We see that **c** and **c** are adjacent despite Case 1. $\rightarrow\leftarrow$



No two vertices in the GSH-graph are more than distance 3 away from each other, so Cases 1 through 3 show that no two cycles have the same name. Thus no two vertices have the same color in $\Gamma_{A_5,S}$, i.e., each vertex composes an entire color-equivalence class. This is a second coloring which induces a quotient which is a Cayley graph.

It has been demonstrated that $\Gamma_{A_5,S}$ permits exactly two colorings so that the induced graph quotient is a Cayley graph.

$\therefore$ A_5 is simple by Corollary 1.8.

Now we will show how useful the GSH-graph can be while working with a family of groups whose generating sets are defined inductively. Using the fact that there is a nice family of generating sets for the alternating groups and that A_5 is simple, we can finally show that the alternating groups A_n are simple for $n \geq 5$ by induction.

Proposition 1.10. A_n is simple for $n \geq 5$.

Proof. Noting that A_5 is simple by the previous Claim, we will proceed by induction. Let $n > 5$ and consider $G = A_n$. We will think of G as a permutation group of the set $\{1, ..., n\}$. Let G_x be the stabilizer of any element x in that set. First we must provide generating sets for the alternating groups. Mitsuhashi points out in [M] that A_n has a generating set $\Sigma_m = \{(1,2)(i, i+1) \mid 2 < i < n\}$, so we will work with the Cayley graph Γ_{A_n,Σ_n}. Before proceeding, notice that $G_n = A_{n-1}$ generated by Σ_{n-1}. We will study the GSH-graph of Γ_{A_n,Σ_n} with respect to subgroup G_n in order to show that Γ_{A_n,Σ_n} permits exactly two colorings.

Let h be the last generator $(1,2)(n-1,n)$ in Σ_n. It is the only generator not in G_n, so to understand the GSH-graph we only need to see where h sends elements in the cosets of G_n. Let $x \in G_n$ and notice that $hxh^{-1} \in G_n$ if and only if x fixes $n-1$. This is true because $n-1$ in the cycle form of x would be replaced by n under conjugation by h. Every such x comprises the subgroup fixing both $n-1$ and n, $G_{n-1,n} = A_{n-2}$. The generator h is an involution, so we may ignore inverses and say $G_{n-1,n}h \in hG_n$. And more generally, $aG_{n-1,n}h \in ahG_n$ for any $a \in G_n$.

Take two left cosets $aG_{n-1,n}$, $bG_{n-1,n}$ in G_n (note, $a, b \in G_n$). Then edges labeled h leave these cosets and enter ahG_n, bhG_n respectively. If $ahG_n = bhG_n$ then $h^{-1}b^{-1}ah \in G_n$.

But $b^{-1}a \in G_n$, so by the previous paragraph $b^{-1}a \in G_{n-1,n}$, i.e., $aG_{n-1,n}$, $bG_{n-1,n}$ are the same coset. So edges labeled h leaving each coset of $G_{n-1,n}$ in G_n lead to different cosets of G_n in G. The index of G_n in G is clearly n, and the index of $G_{n-1,n}$ in G_n is correspondingly $n-1$. So the $n-1$ cosets of $G_{n-1,n}$ are being sent to all $n-1$ non-trivial cosets of G_n. It is worth note that no coset of $G_{n-1,n}$ in G_n can be sent to the trivial coset G_n precisely because h is not in the generating set of G_n. Since all cosets of G_n sit in Γ_{A_n,Σ_n} isomorphically, we may say the same for the cosets of $G_{n-1,n}$ within any other coset of G_n. Thus the GSH-graph is a complete graph on n vertices.

Assume $N \trianglelefteq G$. Then $N \cap G_n \trianglelefteq G_n$. But $G_n = A_{n-1}$ is simple by the induction hypothesis, so a coloring c_N of Γ_{A_n,Σ_n} must have the left cosets of G_n each monocolored or each $|G_n|$-colored.

Case 1. The cosets of G_n are monocolored. Name the coset G_n **A**. Then by Rule 4 all other cosets must have identical names **B** since they each have a vertex colored $c_N(\bullet_{G_n h})$. But the GSH-graph is complete, so it certainly contains a triangle with two cosets named **B** and one named **A**. Then by the same reasoning, with respect to one of the cosets named **B**, the other two should have identical names. Then **A**=**B**. We see that every coset must have an identical name and each coset is monocolored, so all of Γ_{A_n,Σ_n} must be monocolored.

Case 2. The cosets of G_n are $|G_n|$-colored. Assume two cosets have the same name, and without loss of generality presume that one of them is the trivial coset G_n. These two cosets have identical colorings, so assume $\bullet_e$ and $\bullet_a$ have the same color with $a \notin G_n$. Since a doesn't fix n, assume it does the following: $x \mapsto n \mapsto y$. If $x = y$ choose $g \in G_n$ to be the element (xuv) with x, u, v, n all distinct (it exists since $n \geq 4$). Then $(ga)^2$, g^2 both fix n and $\bullet_{gaga}$, $\bullet_{g^2}$ are the same color, but they are not the same element since $gaga$ takes $x \mapsto u$ and g^2 takes $x \mapsto v$. This is a contradiction, as $\bullet_{G_n}$ is $|G_n|$-colored. If $x \neq y$ choose $g \in G_n$ to be the element (yxz) with x, y, z, n all distinct. Then $(ag)^2$, g^2 both fix n and $\bullet_{agag}$, $\bullet_{g^2}$ are the same color, but they are not the same element since $agag$ takes $y \mapsto y$ and g^2 takes $y \mapsto z$. Again this is a contradiction since $\bullet_{G_n}$ is $|G_n|$-colored.

So we see that no two cosets may have the same name. Then each coset is $|G_n|$-colored and no two cosets may share a single color by Rule 4, so each vertex of Γ_{A_n,Σ_n} must have a unique color. It has been demonstrated that Γ_{A_n,Σ_n} permits exactly two colorings.

$\therefore$ A_n is simple by Corollary 1.8, which completes the proof by induction.

Chapter 2

Cayley Graphs as Covers

2.1 Subgroup Actions on Cayley Graphs

It was mentioned much earlier that Cayley graphs are vertex-transitive, and that $\Gamma_{G,S}$ in fact has an automorphism group isomorphic to G itself. So there must be some action of G on $\Gamma_{G,S}$ permuting the vertices- which correspond directly to elements in G- in such a way that given any two vertices $\bullet_g, \bullet_h$ of $\Gamma_{G,S}$, there is exactly one element of G taking $\bullet_g$ to $\bullet_h$. This sounds a lot like G acting naturally on itself by group multiplication, which is exactly the case: hg^{-1} is the unique element in G taking g to h by left-multiplication. We see, then, that G may act on $\Gamma_{G,S}$ as such: $g\bullet_v \mapsto \bullet_{gv}$ with edges mapping correspondingly. Edges and their labels are preserved: given $s \in S$ and $(\bullet_v, \bullet_{vs}) \in E(\Gamma_{G,S})$, $g(\bullet_v, \bullet_{vs})$ maps to $(\bullet_{gv}, \bullet_{gvs})$, which is another edge in $E(\Gamma_{G,S})$ labeled s.

In general, the action coresponding to right-multiplication in G will not extend to an action on $\Gamma_{G,S}$ since edges would not be preserved: given $s \in S$ and $(\bullet_v, \bullet_{vs}) \in E(\Gamma_{G,S})$, $g(\bullet_v, \bullet_{vs})$ would map to $(\bullet_{vg}, \bullet_{vsg})$, which is not necessarily an edge in $E(\Gamma_{G,S})$ labeled s. Clearly if S were in the center of G then right-multiplication would extend to an action on $\Gamma_{G,S}$, but this is unexciting news since G would be abelian and the two actions would be equivalent. So from here on, we will refer only to the action of G on $\Gamma_{G,S}$ extending from left-multiplication.

In the proof of Theorem 1.4 the action of G on its Cayley graph was used colloquially. This chapter will require a slightly deeper understanding of the action of G as well as its subgroups. For $J \leq G$, recall that the coloring defined earlier c_J assigns a unique color to each left coset of J. These colorings sit very nicely in $\Gamma_{G,S}$ because the automorphisms of $\Gamma_{G,S}$ arise from left-multiplication, permuting the left cosets, and because for any two $\bullet_{gJ}$, $\bullet_{hJ}$ there is an automorphism given by hg^{-1} taking $\bullet_{gJ}$ to $\bullet_{hJ}$. However, when J acts on $\Gamma_{G,S}$ the vertex set is partitioned differently by the orbits of the action. Since the action arises from left-multiplication, the orbit of a vertex $\bullet_g$ consists of all elements of the form $\bullet_{jg}$, $j \in J$. Of course this set is $\bullet_{Jg}$, so we can see that the vertex orbits correspond to right cosets of J. In general the right cosets do not sit as "nicely" in $\Gamma_{G,S}$ since there will not typically be a graph automorphism sending $\bullet_{Jg}$ to $\bullet_{Jh}$. Notice, though, that such an automorphism will exist if hg^{-1} is in the normalizer of J: $hg^{-1}\bullet_{Jg} = \bullet_{hg^{-1}Jg} = \bullet_{Jhg^{-1}g} = \bullet_{Jh}$. We will find that when J is a normal subgroup its action has some special properties.

2.2 Some Regular Covers

In order to apply covering space theory to Cayley graphs we must be able to cleanly transition between thinking of $\Gamma_{G,S}$ as a combinatorial object, to which we have thus far attached edge labels and directions, and a topological object. A topological graph is constructed by attaching open intervals, the edges, to a discrete set of points, the vertices. This in turn may be considered as a quotient of the disjoint union of closed intervals (naturally having the quotient topology). Because the groups we work with are finitely generated, we may note that each vertex has finite valence. We see then that $\Gamma_{G,S}$ is a locally compact Hausdorff space (whereas any vertex of infinite valence would not have a compact neighborhood).

The automorphism group $\mathrm{Aut}_{\mathrm{T}}(\Gamma_{G,S})$ when $\Gamma_{G,S}$ is thought of as a topological graph may be much larger than the automorphism group $\mathrm{Aut}(\Gamma_{G,S})$ when it is thought of as a combinatorial object with edge labels and directions that must be respected. Graph automorphisms will always translate nicely to self-homeomorphisms of the corresponding topological graph, so $\mathrm{Aut}(\Gamma_{G,S})$ is a subgroup of $\mathrm{Aut}_{\mathrm{T}}(\Gamma_{G,S})$.

In the proceeding section, though $\Gamma_{G,S}$ will be thought of as a topological graph, we may still associate it with its combinatorial edge labels and directions which will serve as ancillary instructions for the natural action of $J \leq G$ on $\Gamma_{G,S}$. From here on we will reference $p : \Gamma_{G,S} \to \Gamma_{G,S}/J$ where $x \mapsto Jx$. Since p is continuous by definition of the quotient topology, p induces a homomorphism $p_* : \pi_1(\Gamma_{G,S}) \to \pi_1(\Gamma_{G,S}/J)$.

Proposition 2.1. $p : \Gamma_{G,S} \to \Gamma_{G,S}/J$ is a regular cover.

Proof. Let $v \in \Gamma_{G,S}$. It is important that there exists an open set $U \ni v$ so that U and $j(U)$ are disjoint for every $j \in J$. Since J acts on the topological graph by automorphisms, homeomorphic copies of U in its orbit are identified. The condition described above implies that that U and its image in the orbit space are homeomorphic since p will be locally injective.

Say v is a vertex of $\Gamma_{G,S}$. Only the identity in G fixes a vertex of $\Gamma_{G,S}$ since G acts on the graph simply transitively. So only the identity in J fixes v in $\Gamma_{G,S}$. Since the vertex set is discrete, there exists $U \ni v$ so that U and jU are disjoint for every $j \in J$. Now say v is on one of the edges of $\Gamma_{G,S}$. The edges in $\Gamma_{G,S}$ are directed, so if an edge is sent to itself by some $j \in J$, its originating vertex must also be sent to itself by j. Again, this is only true for the identity in J, so no edge is sent to itself. Then it is easy to see that there exists $U \ni v$ so that U and jU are disjoint for every $j \in J$. Thus by the quotient topology, for all $v \in \Gamma_{G,S}$ there exists an open set $U \ni v$ such that U and $p(U)$ are homeomorphic. The map p is surjective as defined, so any point in $\Gamma_{G,S}/J$ has an open neighborhood $p(U)$ whose pre-image consists of disjoint open sets jU with $j \in J$, each mapping homeomorphically to $p(U)$. So p is a cover.

Let $x \in \Gamma_{G,S}/J$ and take $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$. $\tilde{x}_1, \tilde{x}_2$ are in the same orbit, so they may be written as $j_1(\tilde{x}), j_2(\tilde{x})$ for $j_1, j_2 \in J$ and $\tilde{x}$ some lift of x. $j_2 j_1^{-1}$ moves points of $\Gamma_{G,S}$ within their orbits, so it's action is a deck transformation. Furthermore $j_2 j_1^{-1}$ takes $j_1(\tilde{x})$ to $j_2(\tilde{x})$.

$\therefore$ p is a regular cover.

The following proposition will describe the orbit space combinatorially. We can then understand the graph covering p in terms of $\Gamma_{G,S}$ covering some other graph concretely related to G, S, and J.

Proposition 2.2. The orbit space $\Gamma_{G,S}/J$ is the Schreier coset graph of $\Gamma_{G,S}$ given subgroup J.

Proof. The Schreier coset graph of $\Gamma_{G,S}$ given subgroup J has a single vertex $\circ_{Jg}$ for each right coset Jg, and an edge from $\circ_{Jg}$ to $\circ_{Jgs}$ for each $s \in S$. This is related closely to the structure of $\Gamma_{G,S}$. Consider $\bullet_g, \bullet_h \in V(\Gamma_{G,S})$. If there's an edge in the coset graph from $\circ_{Jg}$ to $\circ_{Jh}$ then there exists $s \in S$ such that $Jh = Jgs$. Then it is clear that every vertex in $\bullet_{Jg} \subset V(\Gamma_{G,S})$ has an edge labeled s leaving it and terminating at a vertex in $\bullet_{Jh}$.

The action of J on $\Gamma_{G,S}$ naturally sends vertices to vertices and edges to edges, as it acts by graph-automorphisms which are also self-homeomorphisms. The orbit of a vertex $\bullet_g$ is clearly $\bullet_{Jg}$. The orbit of an edge from $\bullet_g$ to $\bullet_{gs}$ consists of edges going from $\bullet_{jg}$ to $\bullet_{jgs}$ for all $j \in J$. Then clearly there is one edge corresponding to s in $\Gamma_{G,S}/J$ from the orbit of $\bullet_g$ to the orbit of $\bullet_{gs}$. But that means that $\Gamma_{G,S}/J$ has a single vertex for each right coset Jg coresponding to the orbit of $\bullet_g$, and an edge from the vertex $\mathrm{Orb}(\bullet_g)$ to the vertex $\mathrm{Orb}(\bullet_{gs})$ for each $s \in S$.

$\therefore$ The orbit space $\Gamma_{G,S}/J$ is the Schreier coset graph of $\Gamma_{G,S}$ given subgroup J.

The Schreier coset graph and the GSH-graph defined earlier usually bear little relation to each other. Firstly, the Schreier coset graph deals with relations between right cosets, while the GSH-graph studies left cosets. And secondly, the Schreier coset graph contains an edge from $\circ_{Jg}$ to $\circ_{Jgs}$ for each $s \in S$, which means there is exactly *one* edge labeled s entering and one leaving each vertex of the graph; on the other hand, the GSH-graph contains an edge labeled $s \in S - J$ from $\circ_{gJ}$ to $\circ_{hJ}$ whenever hJ intersects gJs, so there may be *many* edges of the same label leaving a particular vertex. While the Schreier coset graph, as we have seen, is covered by its associated Cayley graph, it is clear that the GSH-graph will not typically be covered by its associated Cayley graph (simply by inspecting vertex degrees). In the case that $J \trianglelefteq G$, of course left and right cosets are one in the same, which changes the situation. In that case the Schreier coset graph has a special structure, which will be seen in the proof of the following corollary to Proposition 2.2.

Corollary 2.3. For $N \trianglelefteq G$ the orbit space $\Gamma_{G,S}/N$ is the graph quotient $\Gamma_{G,S}/c_N$.

Proof. By the previous result $\Gamma_{G,S}/N$ is the Schreier coset graph of $\Gamma_{G,S}$ given subgroup N, so we may talk about its structure combinatorially. It has a vertex $\circ_{Ng}$ for each right coset Ng of N. But $gN = Ng$ since $N \trianglelefteq G$, so it is equally valid to assert $\Gamma_{G,S}/N$ has a vertex $\circ_{gN} = \circ_{Ng}$ for each left coset of N. Similarly, in $\Gamma_{G,S}/c_N$ there is a vertex for each left coset of N by its definition.

Now we must simply check that both graphs have the same edgeset. Let k_1, k_2 be two colors (vertices) in $\Gamma_{G,S}/c_N$ with $c_N^{-1}(k_1) = \bullet_{g_1 N}$ and $c_N^{-1}(k_2) = \bullet_{g_2 N}$. Assume there is an edge labeled s from k_1 to k_2 in the quotient. That is true if and only if every vertex colored k_1 in $\Gamma_{G,S}$ has an edge labeled s leaving it and terminating at a vertex colored k_2, and each vertex colored k_2 is the terminal vertex of an edge labeled s originating from a vertex colored k_1. That is equivalent to $\bullet_{g_1Ns}$ being equal to $\bullet_{g_2N}$, i.e., $Ng_1s = Ng_2$ since N is normal. This is true if and only if there is an edge in $\Gamma_{G,S}/N$ from $\circ_{Ng_1}$ to $\circ_{Ng_2}$. So both graphs indeed have the same edgeset.

$\therefore$ The orbit space $\Gamma_{G,S}/N$ is the graph quotient $\Gamma_{G,S}/c_N$.

2.3 Presenting Subgroups of G

The goal of this chapter harkens back to the original exercise of finding normal subgroups of G. We saw in Theorem 1.4 that normal subgroups can be identified by coloring $V(\Gamma_{G,S})$ in such a way that its quotient by that coloring is actually a Cayley graph. Rules 1-5 describe a general way to go about finding such colorings. But upon finding a normal subgroup in $\Gamma_{G,S}$ we have not described a way to understand precisely which group we have identified. This section addresses the issue, and actually generalizes to find a presentation for any subgroup $J \leq G$ based upon how it acts on $\Gamma_{G,S}$ (which is intimately related, as was shown in Section 2.1, to its embedding in G). In the case of a normal subgroup N, we've seen that $\Gamma_{G,S}/c_N$ and $\Gamma_{G,S}/N$ are the same, so the original question of presenting a normal subgroup given c_N will be answered. We will begin with a few technical results related to $\Gamma_{G,S}$ covering $\Gamma_{G,S}/J$.

Proposition 2.4. *J is the group of deck transformations of the cover $p : \Gamma_{G,S} \to \Gamma_{G,S}/J$.*

Proof. First note by the unique lifting property that deck transformations are determined by where they send a single point because deck transformations $\Gamma_{G,S} \to \Gamma_{G,S}$ are precisely lifts of the cover $p : \Gamma_{G,S} \to \Gamma_{G,S}/J$. The unique lifting property requires that $\Gamma_{G,S}$ be connected, which it is since S generates G and traveling along an edge is analogous to multiplying by some $s \in S$; that is, $\Gamma_{G,S}$ is path-connected, so it is connected.

We know already that J acts by deck transformations on $\Gamma_{G,S}$, so it remains to be shown that every deck transformation of p corresponds to the action of an element in J. By virtue of the action of J, the orbit of a vertex $\bullet_g$ in $\Gamma_{G,S}$ is the set of vertices $\bullet_{Jg}$ corresponding to the elements in some right coset of J. A deck transformation f necessarily sends points of $\Gamma_{G,S}$ within their orbits, so assume f takes $\bullet_{j_1g}$ to $\bullet_{j_2g}$. The element $j_2j_1^{-1} \in J$ also sends $\bullet_{j_1g}$ to $\bullet_{j_2g}$. As noted, deck transformations are determined by where they send a point, so f is realized by the action of $j_2j_1^{-1}$. Thus every deck transformation of p corresponds to the action of an element in J.

$\therefore$ J is the group of deck transformations of the cover p.

Proposition 2.5. The group of deck transformations of the path-connected, regular cover $q : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is isomorphic to $\dfrac{\pi_1(X, x_0)}{q_*(\pi_1(\tilde{X}, \tilde{x}_0))}$. Thus $J \cong \dfrac{\pi_1(\Gamma_{G,S}/J)}{p_*(\pi_1(\Gamma_{G,S}))}$.

Proof. Call the deck transformation group of the cover $D(\tilde{X})$. For any homotopy class of loops $[\lambda]$ (whose basepoint is x_0) of X, the lift $\tilde{\lambda}$ of λ will be a path from $\tilde{x}_0 \in q^{-1}(x_0)$ to some $\tilde{x}_1 \in q^{-1}(x_0)$. These endpoints are determined by $[\lambda]$ since all representatives of this homotopy class are naturally homotopic, and therefore lift to a homotopy of paths with fixed endpoints thanks to the path lifting property (this is well-explained in [H, pp. 60-61]).

Define the map $\phi : \pi_1(X, x_0) \to D(\tilde{X})$ so that $[\lambda]$ maps to the deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$, the endpoints of $\tilde{\lambda}$. Such a deck transformation exists because q is a regular cover and these endpoints are lifts of the point $x_0 \in X$. Let the deck transformation associated to $[\lambda]$ be τ_λ.

The map ϕ is a homomorphism (note, the basic idea here comes from [H, p. 71]). Let $[\lambda], [\lambda'] \in \pi_1(X, x_0)$ where $\tilde{\lambda}$ is a path from $\tilde{x}_0$ to $\tilde{x}_1$ and $\tilde{\lambda}'$ is a path from $\tilde{x}_0$ to $\tilde{x}_1'$. The lift of $\lambda \cdot \lambda'$ will be $\tilde{\lambda}$ with basepoint at $\tilde{x}_0$ composed with the lift of λ' whose basepoint is at $\tilde{x}_1$, the terminal endpoint of $\tilde{\lambda}$. In other words, the lift of $\lambda \cdot \lambda'$ is $\tilde{\lambda} \cdot \tau_\lambda(\tilde{\lambda}')$. So, the lift of $\lambda \cdot \lambda'$ has initial endpoint $\tilde{x}_0$ and terminal endpoint $\tau_\lambda(\tilde{x}_1') = \tau_\lambda \tau_{\lambda'}(\tilde{x}_0)$. Finally we can see that ϕ is indeed a homomorphism,

$$\phi([\lambda][\lambda']) = \phi([\lambda \cdot \lambda']) = \tau_\lambda \tau_{\lambda'} = \phi([\lambda])\phi([\lambda'])$$

The map ϕ is surjective. Let $f \in D(\tilde{X})$ be some deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$. The space $\tilde{X}$ is path-connected, so there exists a path from $\tilde{x}_0$ to $\tilde{x}_1$. Then the image of the path under q is a loop ℓ in X based at $x_0 = q(\tilde{x}_0) = q(\tilde{x}_1)$, which as defined naturally lifts to the path from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\phi([\ell])$ is the unique deck transformation f taking $\tilde{x}_0$ to $\tilde{x}_1$. So ϕ is surjective.

The kernel of ϕ consists of homotopy classes in $\pi_1(X, x_0)$ mapping to the trivial deck transformation taking $\tilde{x}_0$ to itself. So any such representative loop in X lifts to a path in $\tilde{X}$ whose initial and terminal endpoint is $\tilde{x}_0$. That is, it lifts to a loop based at $\tilde{x}_0$. These loops in X comprise precisely the homotopy classes $q_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Thus $\ker \phi = q_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

$\therefore$ by the first isomorphism theorem $D(\tilde{X}) \cong \pi_1(X, x_0)/q_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Since $\Gamma_{G,S}$ is path-connected and p is a regular cover whose deck transformation group is J, we may conclude that $J \cong \dfrac{\pi_1(\Gamma_{G,S}/J)}{p_*(\pi_1(\Gamma_{G,S}))}$.

In this final result we have set ourselves up fairly well to find a presentation for J. The idea is that $\pi_1(\Gamma_{G,S}/J)$ is a free group generated by some some homotopy classes, and the image of p_* provides exactly the trivial combinations of those homotopy classes required to describe J. Elements in the image of p_* are precisely loops in $\Gamma_{G,S}/J$ lifting to loops in $\Gamma_{G,S}$.

But loops in $\Gamma_{G,S}$ correspond to trivial words in G, so if $G =< S \mid R >$ then all loops in $\Gamma_{G,S}$ are paths corresponding to words in the conjugate closure $< R^{F_S} >$ of R in the free group on S, F_S. Naively we may present J by assigning a generator to each homotopy class in $\pi_1(\Gamma_{G,S}/J)$ and a relation to the image of each homotopy class in $\pi_1(\Gamma_{G,S})$, but the following theorem will use this fact associating $< R^{F_S} >$ and the homotopy classes of $\Gamma_{G,S}$ to provide a much more efficient presentation of J.

Theorem 2.6. Let $G =< S \mid R >$ be finitely generated and $J \leq G$. Then J has a presentation with generators in one-to-one correspondence with generating loops of $\pi_1(\Gamma_{G,S}/J)$ and relations in one-to-one correspondence with pairs in $V(\Gamma_{G,S}/J) \times R$.

Proof. By the previous result, $J \cong \dfrac{\pi_1(\Gamma_{G,S}/J)}{p_*(\pi_1(\Gamma_{G,S}))}$. We have already noted that elements in the image of p_* are loops in $\Gamma_{G,S}/J$ which lift to loops in $\Gamma_{G,S}$, and that elements in $< R^G >$ give words that correspond to all loops in $\Gamma_{G,S}$ (i.e., trivial words in G). We will fix basepoints $\bullet_e \in V(\Gamma_{G,S})$ and $p(\bullet_e) = \circ_J \in V(\Gamma_{G,S}/J)$.

First, let us identify the generating loops of $\pi_1(\Gamma_{G,S}/J)$. $\Gamma_{G,S}/J$ is connected since it is the continuous image under p of a Cayley graph, which is connected. So $\Gamma_{G,S}/J$ contains a maximal tree T. Associate with each edge e_i not in T a label a_i and a direction inherited from the combinatorial description of $\Gamma_{G,S}$. T is contractible, so the portion of any loop in $\Gamma_{G,S}/J$ passing through T does not contribute to its homotopy class; we may simply pay attention to how loops pass through the edges of $\Gamma_{G,S}/J$ not in T. Then these edges $\{e_i\}$ correspond to the generating loops $\{a_i\}$ of $\pi_1(\Gamma_{G,S}/J)$.

Our goal now is to determine a reasonable subset of the image of p_* in terms of the generators $\{a_i\}$ whose conjugate closure will be the whole image. We will interchangably talk about elements of $< R^{F_S} >$ and homotopy classes in $\pi_1(\Gamma_{G,S})$. Definitionally, then, $\pi_1(\Gamma_{G,S})$ is generated by $R^{F_S} = \{frf^{-1} \mid f \in F_S, r \in R\}$. So the image of p_* must be generated by $p_*(R^{F_S})$.

Define a function $\varphi_{Jh} : F_S \to \pi_1(\Gamma_{G,S}/J)$ for each right coset of J where $f \in F_S$ is expressed as a finite word in terms of S; this word corresponds to a path in $\Gamma_{G,S}$ based at any vertex of $\bullet_{Jh}$ whose image under p is a path in $\Gamma_{G,S}/J$ based at $\circ_{Jh}$; and finally, that path in $\Gamma_{G,S}/J$ is sent to its corresponding word in terms of the generators in $\{a_i\}$ which it sequentially passes through. Let $f_1, f_2 \in F_S$ and notice that $\varphi_J(f_1 f_2) = \varphi_J(f_1)\varphi_{Jf_1}(f_2)$. Additionally, when f corresponds to a loop in $\Gamma_{G,S}$ notice that as defined, $p_*(f) = \varphi_J(f)$. Now take any element $frf^{-1} \in R^{F_S}$, and bearing in mind that r corresponds to a closed loop we see,

$$p_*(frf^{-1}) = \varphi_J(f)\varphi_{Jf}(r)\varphi_{Jf}(f^{-1})$$

But clearly $\varphi_J(f)$ and $\varphi_{Jf}(f^{-1})$ are inverses in $\pi_1(\Gamma_{G,S}/J)$ since they trace opposite paths in $\Gamma_{G,S}/J$ corresponding to inverse words in $\pi_1(\Gamma_{G,S}/J)$. Thus, $p_*(frf^{-1})$ is conjugate to $\varphi_{Jh}(r)$ whenever f (word-reduced to an element in G) and h are in the same coset of J. $\varphi_{Jh}(r)$ is in the image of p_*, acheived by $p_*(frf^{-1})$ when f is a path in the lift of T at $\bullet_e$

from $\bullet_e$ to the lift of $\circ_{Jh}$, a vertex in $\Gamma_{G,S}$ corresponding to some coset representative of Jh. In this case $\varphi_J(f)$ is trivial since f projects to a path in T corresponding to a trivial word in $\pi_1(\Gamma_{G,S}/J)$. Then $p_*(frf^{-1}) = \varphi_{Jh}(r)$.

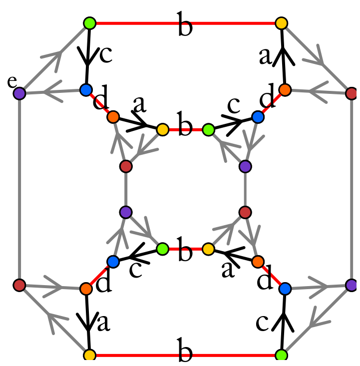
Let C be a set of words in F_S which are paths in the lift of T from $\bullet_e$ to the lift of each $\circ_{Jh}$, h ranging over right coset representatives, and let $\pi_J = \pi_1(\Gamma_{G,S}/J)$. Then we have found that,

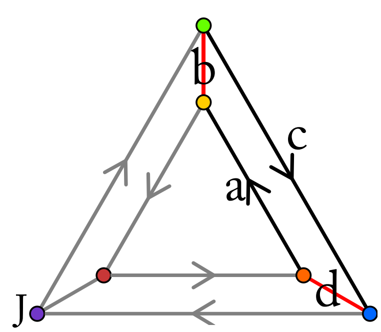
$$p_*(< R^{F_S} >) =< p_*(R^{F_S}) >=< p_*(R^C)^{\pi_J} >$$

$\therefore$ We may form the presentation $J =< \{a_i\} \mid p_*(R^C) >$. Elements of C are in one-to-one correspondence with vertices in $V(\Gamma_{G,S}/J)$ (there is exactly one path in C for each vertex within the unique lift of T at $\bullet_e$), so it is clear that relations in this presentation are in one-to-one correspondence with pairs in $V(\Gamma_{G,S}/J) \times R$. And as defined, generators $\{a_i\}$ are in one-to-one correspondence with generating loops of $\pi_1(\Gamma_{G,S}/J)$.

Example 2.7 Let us return to the to the example of the subgroup J of the symmetric group on four letters $G =< k, r \mid k^3, r^2, (kr)^4 >$ given by the coloring in the Cayley graph $\Gamma_{G,S}$ below on the left (in this case the generating set is $S = \{k, r\}$, where k is the label of the black edges and r is the label of the red ones). We wish to find a presentation for J based upon its action on $\Gamma_{G,S}$, i.e., its embedding in G. The orbit space $\Gamma_{G,S}/J$ can be seen on the right. It is the Schreier coset graph of $\Gamma_{G,S}$ given J, and also happens to be a Cayley graph for D_3 since J is normal in G, which was discerned in Example 1.5. The greyed edges are the maximal tree T in $\Gamma_{G,S}/J$ and the labels $\{a, b, c, d\}$ correspond to the generating loops of the graph's fundamental group.

The labels and maximal tree are lifted to $\Gamma_{G,S}$ where relations for J will be computed. Choose the unique lift of T at the purple vertex $\bullet_e$ and compute the relations $k^3, r^2, (kr)^4$ by paths in $\Gamma_{G,S}$ at each vertex along the lift of T in terms of the lifted generators $\{a, b, c, d\}$. We obtain the set of relations $\{a, c, b^2, d^2, (cdab)^2, (abcd)^2\}$, which reduces to $\{b^2, d^2, (bd)^2\}$. So, $J =< b, d \mid b^2, d^2, (bd)^2 >$, which is the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then in general we can say $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \trianglelefteq S_4$.





Example 2.8 The Nielsen-Schreier theorem states that every subgroup J of a free group $G =< S \mid R >$ is itself free. It is a direct result of the previous theorem since relations for J are in one-to-one correspondence with pairs in $V(\Gamma_{G,S}/J) \times R$, but of course there

are no such pairs since R is null. The following corollary to Theorem 2.6 will impose some modest conditions which imply that J is actually a finitely generated free group. This will be discussed in the next section.

Corollary 2.9. Let $G =< S \mid R >$ be finitely presented and J be a finite index subgroup of G. Then J is finitely presented with at most $[G:J](|S|-1)+1$ generators and $[G:J]|R|$ relations.

Proof. By Theorem 2.6, the size of the generating set for J is the number of generating loops in $\Gamma_{G,S}/J$, which is the same as the rank of the graph's fundamental group π_J. But for any finite graph Y the rank of $\pi_1(Y)$ is intimately connected to the Euler characteristic of the graph by the formula,

$$\text{rank}(\pi_1(Y)) = 1 - \chi(Y)$$

This can be seen by collapsing a maximal tree of Y, whose Euler characteristic is known, and noticing that what remains is a single vertex plus a certain number of edges, each corresponding to a generator of the fundamental group. The graph $\Gamma_{G,S}/J$ is finite since J is finite index, so it is very easy to calculate its Euler characteristic as $V - E$. It has one vertex for each right coset of J, so $V = [G : J]$. The number edges may be counted by simply counting the number of directed edges leaving each vertex and multiplying by the number of vertices (as $\Gamma_{G,S}/J$ is regular). Since every directed edge leaves a vertex and enters another, no edge will be counted twice. Of course there are $|S|$ edges leaving each vertex, so $E = [G : J]|S|$. Then we see,

$$\text{rank}(\pi_J) = 1 - \chi(\Gamma_{G,S}/J) = 1 - (V - E) = [G : J](|S| - 1) + 1$$

Again by Theorem 2.6, the relations in the presentation of J are in one-to-one correspondence with pairs in $V(\Gamma_{G,S}/J)\times R$. Of course $|V(\Gamma_{G,S}/J)| = [G : J]$, so there are $[G : J]|R|$ relations.

$\therefore$ J has a finite presentation with $[G : J](|S| - 1) + 1$ generators and $[G : J]|R|$ relations.

2.4 A (Near) Converse to the Schreier Index Formula

Corollary 2.9 is a generalization of the Schreier index formula, which states that a finite index subgroup of a finite rank free group is finitely generated with $[G : J](|S|-1)+1$ generators. The following results will allow us to provide a near converse to the Schreier index formula. We wish to begin with a *normal* finitely generated non-trivial subgroup of a free group and show that it must be of finite index. The virtue of the following lemma is that it can be applied in situations when one of the regular covers described in previous sections has a deck transformation group normal in G; in that case the orbit space is vertex-transitive. As we have seen, it is in fact a Cayley graph for the quotient group by Theorem 1.4.

Lemma 2.10. Let X be a connected, vertex-transitive graph of finite valence with $\pi_1(X)$ non-trivial. Then $\pi_1(X)$ is finitely generated if and only if $|V(X)|$ is finite.

Proof. Suppose $\pi_1(X)$ is a finitely generated free group on m generators, $m > 0$ since the group is non-trivial. Assume $|V(X)|$ is infinite and let T be a maximal tree of X. Then there exist $e_1, ..., e_m \in E(X-T)$ and at most $2m$ vertices in $V(X-T)$. T is connected and contains every vertex of X, so in T there already exists a path between any two vertices of X. Replacing any edge not in $X - T$ would therefore create a cycle, as there would be two disjoint paths between the two vertices of such an edge. So each $e_1, ..., e_m$ is in a cycle of X.

Fix vertex $v \in e_1 \in E(C)$, C some cycle in X. Let w be the vertex in $X - T$ farthest from v (noting that $|V(X - T)|$ is finite and in fact $\leq 2m$) and choose any v' distance $> \mathrm{d}(v, w) + \mathrm{diam}(C)$ from v. We may do this, as there exist vertices arbitrarily far away from v since $|V(X)|$ is infinite yet no vertex degree is infinite.

Any graph automorphism taking v to v' induces an automorphism of C to C', and such an automorphism exists because X is vertex-transitive. But there can be no vertex in $V(X - T) \bigcap V(C')$ since v' is more than $\mathrm{diam}(C) = \mathrm{diam}(C')$ away from the farthest vertex from v in $V(X - T)$. Then certainly no edge is in $E(X - T) \bigcap E(C')$. Thus C' is fully contained in T, which is a contradiction since T is a tree and cannot contain cycles. $\rightarrow\leftarrow$

$\therefore$ $|V(X)|$ must be finite.

Conversely, assume $|V(X)| = n$ is finite. X is vertex-transitive (implying regularity) and of finite valence, so assume each vertex has degree k. Then X cannot have more than kn edges. Since $|E(X)|$ is finite any maximal tree of X will contain all but some finite number of edges less than kn.

$\therefore$ $\pi_1(X)$ is a finitely generated free group with fewer than kn generators.

Now we have built all the necessary machinery to prove the final theorem in fairly succinct terms.

Theorem 2.11. Let N be a non-trivial finitely generated normal subgroup of free group of rank m, F_m. Then N is finite index in F_m.

Proof. F_m is finitely generated by m elements $\{a_1, ..., a_m\} = M$, i.e., $F_m =< M >$. Then F_m has a Cayley graph $\Gamma_{F_m,M}$, which is a $2m$-valent infinite tree (each vertex has an edge entering it and an edge leaving it for each generator in M). Since it is a tree, its fundamental group is trivial.

By Proposition 2.5, $N \cong \dfrac{\pi_1(\Gamma_{F_m,M}/N)}{p_*(\pi_1(\Gamma_{F_m,M}))}$. But $\Gamma_{F_m,M}$ has trivial fundamental group and by Corollary 2.3 $\Gamma_{F_m,M}/N$ is the same graph as $\Gamma_{F_m,M}/c_N$, so we have $N \cong \pi_1(\Gamma_{F_m,M}/c_N)$. N is finitely generated and non-trivial, so $\pi_1(\Gamma_{F_m,M}/c_N)$ must also be finitely generated and non-trivial. In fact it is a graph, so it has to be a finitely generated free group. We know the valence of $\Gamma_{F_m,M}/c_N$ is $2m$, the same as $\Gamma_{F_m,M}$, and that it is vertex-transitive because it is a Cayley graph by Theorem 1.4. So by Lemma 2.10 $|V(\Gamma_{F_m,M}/c_N)|$ must be finite. But the size of the vertex set of the Cayley graph for F_m/N is the same as the index of N in F_m since the vertices are in one-to-one correspondence with left cosets of N.

$\therefore$ N is finite index in F_m.

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